

Harmonic analysis for semigroups on commutative and noncommutative L^p spaces

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(specialité mathématiques)

par

Christoph Kriegler

Harmonic analysis for semigroups on commutative and noncommutative L^p spaces

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Abstract

The results in this memoir are threefold. We consider a c_0-semigroup of operators acting on either a classical $L^p(\Omega)$ space, a Bochner $L^p(\Omega, Y)$ space or a noncommutative $L^p(M)$ space. First we are interested in functional calculus of the generator A, a question that has a long standing history since the fundamental works of Stein [Ste70] and Cowling [Cow] and is well-known to be of great importance among others in spectral theory, maximal regularity and control theory. Our spectral multipliers are then bounded holomorphic functions on a sector in the complex plane ($\mathrm{H}^\infty(\Sigma_\omega)$ calculus) or defined on the positive half-line and coming with derivatives satisfying the Hörmander Mihlin condition, i.e. are weighted bounded in a certain way ($\mathcal{H}_2^\alpha$ calculus). We show boundedness of the H^∞ calculus for markovian and submarkovian semigroups on weighted $L^2(\Omega, wd\mu)$ space. Moreover, we obtain boundedness of $\mathcal{H}_2^\alpha$ calculus on $L^p(\Omega)$ for semigroups with Poisson estimates and for the Dunkl heat semigroup on the Bochner space $L^p(\mathbb{R}^d, h_\kappa^2, Y)$ where Y is a UMD lattice, generalising work by Bonami-Clerc, Dai-Wang and Dai-Xu. Then we consider selfadjoint semigroups satisfying classical or generalised Gaussian estimates on spaces of homogeneous type and obtain $\mathcal{H}_2^\alpha$ calculus again on the UMD lattice valued Bochner space $L^p(\Omega, Y)$.

In a second line, we are interested in maximal operators acting on $L^p(\Omega, Y)$, in its well-known spatial form of Hardy-Littlewood type or in spectral multiplier form $\sup_{t>0} |m(tA)f|$. We obtain that the Fefferman-Stein inequalities hold with a dimension free bound, on the range $1 < p < \infty$, and with $Y = \ell^q$, $1 < q < \infty$ or more generally, $Y =$ a UMD lattice. We also obtain that the Hardy-Littlewood maximal operator is bounded on $L^p(\Omega, Y)$ with Ω a space of homogeneous type. For the spectral multiplier maximal operators, we obtain bounds on $L^p(\Omega, Y)$ for symbols m decreasing at ∞, under the hypothesis of Hörmander calculus of A on $L^p(\Omega, Y)$.

In a third part we investigate noncommutative Fourier and Schur multipliers and show among others dimension free bisectorial H^∞ calculus of Hodge-Dirac operators stemming from markovian multiplier semigroups. This is built on previous work on Riesz transform estimates initiated by Junge, Mei and Parcet. These estimates also allow to prove that the Hodge-Dirac operators give rise to objects from noncommutative geometry: spectral triples and quantum (locally) compact metric spaces. Finally, in an independent section, we also investigate so-called decomposable operators acting on noncommutative L^p spaces, which generalise pointwise boundedly dominated operators from classical L^p theory. Then we consider again noncommutative Fourier and Schur multipliers and express their decomposability.

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2 List of publications presented in this memoir

This habilitation thesis is based on the following publications.

1. Hörmander Functional Calculus for Poisson Estimates, *Int. Equ. Op. Theory*, 80(3):379–413, 2014. Preprint on HAL, arXiv.org.

2. (with L. Deléaval) Dunkl spectral multipliers with values in UMD lattices, *Journal of Functional Analysis,* 272(5):2132–2175, 2017. Preprint here and on HAL.

3. (with C. Arhancet) Complementation of the subspace of radial multipliers in the space of Fourier multipliers on $\mathbb{R}^n$, *Archiv der Mathematik,* 112(1):93–100, 2018. Preprint on HAL, arXiv.org.

4. (with L. Deléaval) Dimension free bounds for the vector-valued Hardy-Littlewood maximal operator, *Revista Matemática Iberoamericana,* 35(1):101–123, 2019. Preprint on HAL, arxiv.org.

5. (with L. Deléaval and M. Kemppainen) Hörmander functional calculus on UMD lattice valued L^p spaces under generalised Gaussian estimates, accepted in Journal d'Analyse Mathématique, Preprint here, on HAL and arXiv.org.

It also contains parts of the following submitted articles which are available online.

1. (with C. Arhancet) Decomposable Schur multipliers and Fourier multipliers on noncommutative L^p-spaces. Submitted. Preprint on arXiv.org.

2. (with K. Domelevo and S. Petermichl) Diffusion semigroups on weighted L^2 spaces. Submitted. Preprint on HAL and arXiv.org.

3. (with C. Arhancet) Riesz transforms, Hodge-Dirac operators and functional calculus for multipliers I. Preprint on arXiv.org.

4. (with L. Deléaval) Maximal and q-variational Hörmander Functional Calculus. Submitted. Preprint here.

3 Introduction

Since the fundamental works of Stein [Ste70] and Cowling [Cow], the spectral theory for semigroups has become a wide mathematical field and a lot of mathematicians work in that field today. Much progress has been achieved over the last four decades, many beautiful connections have been proven to be fruitful in solving problems inside and outside harmonic analysis. We recall that a c_0-semigroup is a family T_t of bounded operators over a Banach space X indexed by $t \geqslant 0$ such that $T_{t+s} = T_t T_s$, $T_0 = \mathrm{Id}_X$ (semigroup property) and $T_t x \to x \quad (t \to 0+)$ (c_0, i.e. strong continuity property). It can also be described by a single (usually unbounded) operator, the (negative) generator A such that $\frac{d}{dt} T_t = -AT_t$. The aims in this habilitation thesis are to contribute answers to the following three meta questions, which arise in the context of spectral theory, functional calculus, harmonic analysis, singular integrals and abstract partial differential equations:

1. Under which circumstances does the generator A admit an H^∞ [KW1] or Hörmander(-Mihlin) functional calculus [DuOS]? The answer will depend e.g. on the underlying Banach space and we are interested in classical $L^p(\Omega)$ spaces, Bochner $L^p(\Omega, Y)$ spaces and noncommutative $L^p(M)$ spaces. Then kernel estimates of the semigroup, contractivity on the L^p scale together with selfadjointness on L^2, geometric properties of Ω, Banach space geometric properties of Y and - in case of noncommutative L^p spaces - algebraic structures and identities such as tensor products and $*$-homomorphisms are the starting point of our assumptions. In some parts but not all, the Banach space geometric notion of R-boundedness will be an appropriate vehicle to abstract these assumptions in a convient way to achieve the functional calculus.

2. For which cases is a maximal operator bounded, in the most classical, spatial form of Hardy-Littlewood type $M_{HL} f = \sup_{r>0} \frac{1}{V(x,r)} \int_{B(x,r)} |f(y)| dy$, and also in spectral multiplier form $\sup_{t>0} |m(tA)f|$ (e.g. $m(\lambda) = e^{-\lambda}$, so that $\sup_{t>0} |m(tA)f| = \sup_{t>0} |T_t f|$)? It is well-known and will also become apparent from the results in Sections 7 and 8 that Questions 1. and 2. are interrelated. Namely, our results show the passage of boundedness in 1. and 2. in both ways.

3. What kind of operators on noncommutative L^p spaces (see Definition 10.4) have a bounded H^∞ calculus, or yield bounded and completely bounded (see Subsection 10.1) maps? Here we focus in this memoir on prominent examples of such mappings important in harmonic analysis which are noncommutative Fourier multipliers (see Definitions 10.6, 12.6 and 12.12), Schur multipliers (see Definition 10.9), and operations stemming from second quantization (see Subsection 10.1) and constructed out of the former (see (10.33) and (10.41), and (10.73) and (10.75)).

Let us turn to an overview of the results and the methods that lead to them.

3.1 H^∞ and Hörmander-Mihlin functional calculus on Bochner spaces $L^p(\Omega, Y)$

The first main objective is to establish and characterize functional calculus of the generator A of the semigroup T_t on Bochner spaces $L^p(\Omega, Y)$, where Y is a further Banach space. This requires in most of the cases that Y has the UMD property, a property shared e.g. by commutative and noncommutative L^p spaces as long as $1 < p < \infty$. Also its Rademacher-type and -cotype as well as related notions such as p-convexity and q-concavity if Y is moreover a lattice play

a rôle here. The motivation for Bochner spaces comes from their importance in applications to abstract Cauchy problems, where Y takes over the rôle of a spatial variable (see e.g. the application in Subsection 5.4), whereas the time variable is the parameter t of the semigroup T_t; for square function estimates, where $Y = \ell^2$ (then the interesting functional calculus question involves a sequence of spectral multipliers $(f_k)_k$ – see Remark 5.5, (5.7), (5.11), Remark 7.12 and (8.7)); and lastly for descriptions of abstract function spaces associated with A, such as Sobolev and Triebel-Lizorkin spaces, where $Y = \ell^q$ [KuUl1]. Let us recall here the function classes in which the spectral multipliers of the functional calculus usually live in. For an angle $\omega \in (0, \pi)$, put $\Sigma_\omega = \{z \in \mathbb{C}^* : |\arg z| < \omega\}$ the sector around the positive half axis in the complex plane with half opening angle ω. Then we define

$$\mathrm{H}^\infty(\Sigma_\omega) = \{f : \Sigma_\omega \to \mathbb{C} : \ f \ \text{ analytic and bounded}\}$$

and equip this space with the norm $\|f\|_{\infty,\omega} = \sup_{z\in\Sigma_\omega} |f(z)|$, with respect to which $\mathrm{H}^\infty(\Sigma_\omega)$ becomes a Banach algebra. The spaces $\mathrm{H}^\infty(\Sigma_\omega)$ are contained in each other for different angles, the space becoming larger(!) when the angle becomes smaller. On the other hand, for parameters $1 < p < \infty$ and $\alpha > \frac{1}{p}$, we define the Hörmander(-Mihlin) norm by

$$\|f\|^p_{\mathcal{H}^\alpha_p} = |f(0)|^p + \max_{k=0,\dots,\alpha} \sup_{r>0} \int_r^{2r} \left| t^k \frac{d^k}{dt^k} f(t) \right|^p \frac{dt}{t},$$

and let $\mathcal{H}^\alpha_p = \{f : (0,\infty) \to \mathbb{C} : \ f \ \alpha\text{-times continuously differentiable and } \|f\|_{\mathcal{H}^\alpha_p} < \infty\}$. There is also a definition of $\mathcal{H}^\alpha_p$ for non-integer α in which case the integral above is replaced by a local W^α_p Sobolev norm. Then again $\mathcal{H}^\alpha_p$ is a Banach algebra and lowering α (resp. p) results in a larger space. It contains $\mathrm{H}^\infty(\Sigma_\omega)$ for any choice of parameters α, p and ω. We will mainly pick $p = 2$. Then a (H^∞ resp. Hörmander $\mathcal{H}^\alpha_p$) functional calculus means

$$\|f(A)\|_{B(X)} \leqslant C\,(|f(0)| + \|f\|_{\infty,\omega}) \quad (f \in \mathrm{H}^\infty(\Sigma_\omega)) \tag{3.1}$$

or

$$\|f(A)\|_{B(X)} \leqslant C\|f\|_{\mathcal{H}^\alpha_p} \quad (f \in \mathcal{H}^\alpha_p), \tag{3.2}$$

where the operator $f(A)$ is constructed in Subsection 4.3, respecting hereby ad-hoc definitions for rational functions f lying in the classes $\mathrm{H}^\infty(\Sigma_\omega)$ and $\mathcal{H}^\alpha_p$. It is clear from the above that (3.2) implies (3.1), and in a similar manner, lowering the parameter ω (resp. α, p) results in a stronger functional calculus. Let us recall that up to now, (3.1) and (3.2) have been studied in the following situations, mainly in the case $X = L^p(\Omega, \mathbb{C})$ (we have a by far non-exhaustive list of contributions):

1. Symmetric contraction and diffusion semigroups [Ste70, Cow, CoWe2, BlDo, BlDT, HiPr, CaDr1, DPW, Wro1, Wro2],
2. Regular contractive semigroups on a single L^p space [We01bis, Xu15],
3. Semigroups admitting Gaussian estimates [KW1, DuOS, GoY],
4. Concrete differential and pseudodifferential operators [BSS, CSS, DSS, GHT, KW2, SSe],
5. Sublaplacians on Lie groups of polynomial growth [Ale, Mul, MuSt, ACMM],
6. Recently (assymptotic) hyperbolic manifolds [ChHa],

7. The Grushin operator on flat space and sphere [CCM],
8. Semigroups acting on a strict subscale $(p_0, p_1) \subseteq (1, \infty)$ of L^p spaces and admitting generalised Gaussian estimates [KuU2, KuUl2],
9. Semigroups admitting Davies-Gaffney estimates and Stein-Thomas restrictions [COSY, CDLWY, SYY],
10. Bisectorial Hodge-Dirac [AHLLMT, AHLMT, AuTc, AKM, HMP1, HMP2, FMP, AuSt1, AuSt2, EHDT1, EHDT2] and strip-type operators [BMV],
11. Laplacians, i.e. Fourier multipliers, on Bochner spaces [GiWe, Hy04, Hy06, Hy10, Krol, LiVe, RoVe],
12. H^∞ calculus in connection with dilations [AFLM, Arh3, Arh4, Arh5, Fen1, Fen2, FrWe],
13. Ritt operators and other particular geometric forms of the spectrum [ArhM, ArnM, ArrM, BHM, CMM, GoTo, LaMe2, LeM3, Schw],
14. Noncommutative L^p spaces [JMX, JMP1, JMP2, GPJP, Arh3, Arh4, Arh5].

For an overview of H^∞ calculus we refer to [CDMY, Haas, HvNVW1, HvNVW2, KW1]. As a rule of thumb, whenever a Hörmander $\mathcal{H}^\alpha_2$ functional calculus (3.2) is available in one of the above situations, and Ω carries a notion of dimension d, then α will depend linearly on d, ideally $\alpha = \frac{d}{2}$. In preliminary work, I have established in collaboration with L. Weis several abstract criteria when an operator A has a Hörmander-Mihlin functional calculus [KrW2, KrW3]. This was in terms of R-boundedness of one of the following well-established spectral multiplier families: the semigroup T_t for complex times $t \in \mathbb{C}_+$, the imaginary powers A^{it}, $t \in \mathbb{R}$, the wave operators $(1+A)^{-\delta}\exp(itA)$, $t \in \mathbb{R}$, the Bochner-Riesz means $(1-tA)^\nu_+$ and partly also resolvents $(\lambda - A)^{-1}$ with $\lambda \in \mathbb{C}\backslash[0,\infty)$. Hereby, a family $\tau \subseteq B(X)$ is called R-bounded if

$$\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k T_k x_k\right\|_X \leqslant C\mathbb{E}\left\|\sum_{k=1}^n \varepsilon_k x_k\right\|_X \tag{3.3}$$

for some $C < \infty$, and any $x_k \in X$ and $T_k \in \tau$. The ε_k form a family of independent random variables such that $\mathbb{P}(\varepsilon_k = \pm 1) = \frac{1}{2}$. R-boundedness is a powerful tool to encode properties of an operator plus its Banach space in an abstract way.

Gaussian estimates (see (4.9) below) for a semigroup is a by now well-established property after far-reaching works of Coulhon, Grigor'yan, Saloff-Coste and others. On the other hand, UMD spaces are now standard in the context of vector valued Fourier series, multipliers (see references in point 11 above) and vector-valued singular integrals. In Section 7 we combine these two aspects and show that Gaussian and generalised Gaussian estimates (7.4) yield the R-boundedness from (3.3) for T_k the complex time semigroup on $X = L^p(\Omega, Y)$ in a quantified manner (see Theorem 4.11) suitable to deduce $\mathcal{H}^\alpha_2$ calculus for A on $L^p(\Omega, Y)$ where Y is a UMD lattice. For this first line of Hörmander multiplier results, we refer to Theorems 7.1 and 7.5.

In Section 5, we show new Hörmander functional calculus for two particular examples: First, semigroups admitting Poisson type estimates (see (5.1) below for this polynomial decay, weaker than Gaussian estimates) for their semigroup kernels and acting on scalar spaces $L^p(\Omega)$ (see Corollary 5.4). Second, the Dunkl heat semigroup acting on the weighted Bochner space $L^p(\mathbb{R}^d, h^2_\kappa, Y)$ (see Theorem 5.10). Also here, R-boundedness plays an important rôle, for the complex time semigroup in the first example and for Cesàro means (certain smoothed indicator

function spectral multipliers, see (5.8)) in the second example. Note that our method allows to include square function estimates for free (see Remark 5.5, (5.7), (5.11), Remark 7.12 and (8.7)). Thus our results in Subsection 5.1 on Poisson estimates are even new for the classical Poisson semigroup and yield R-boundedness, in other words square function estimates, for Fourier multipliers on $\mathbb{R}^d$ of Hörmander $\mathcal{H}_2^\alpha$ type to the order $\alpha > \frac{d}{2}$ which is optimal when seen over the whole scale $1 < p < \infty$.

Moreover, in Section 9 we consider the important classes of markovian and submarkovian semigroups (see Subsection 4.1 for these classical notions), and show bounded H^∞ calculus on weighted $L^2(\Omega, wd\mu)$ space (see Theorems 9.3 and 9.9) for weights belonging to a sort of Muckenhoupt class which is defined via the semigroup (see (9.2)). Here, we achieved the goal by a method independent of R-boundedness, but by means of a Bellman function. This method of dominating a bilinear functional, also called a weak square function/decomposition of unity, by means of a functional with both the correct convexity and upper bound, has been proved recently to be surprisingly powerful in application for H^∞ calculus of submarkovian and other semigroups [CaDr1, CaDr2]. It is well-known over several decades in other fields of harmonic analysis.

3.2 Maximal operators on Bochner spaces $L^p(\Omega, Y)$

From a general point of view, the importance of standard maximal operators in several branches of harmonic and real analysis (singular integrals, multipliers, Littlewood-Paley theory...) no longer needs to be demonstrated and justifies their extensions in numerous and very different settings. We refer to the survey [DGM] and the references therein for an overview of maximal operators with respect to various bodies. We recall that if (Ω, dist) is a metric space equipped with some Borel measure μ taking strictly positive values on balls $B(x,r) = \{y \in \Omega : \mathrm{dist}(x,y) < r\}$, then we can consider the Hardy-Littlewood maximal operator

$$M_{HL} f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| d\mu(y) \quad (x \in \Omega).$$

This formula makes perfect sense for $f \in L^p(\Omega, Y)$ a function living in a (UMD) lattice valued Bochner space. In Section 6, we obtain two boundedness results for M_{HL} on $L^p(\Omega, Y)$: First the dimension free bounds on $L^p(\mathbb{R}^d, \ell^q)$ i.e. the famous Fefferman-Stein inequalities [FeSt] hold for any $1 < p, q < \infty$ with a constant $C(p,q) < \infty$ independent of d and $(f_n)_n$:

$$\left\| \left(\sum_{n=1}^\infty |M_{HL} f_n|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)} \leqslant C(p,q) \left\| \left(\sum_{n=1}^\infty |f_n|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)}$$

(more generally, for ℓ^q replaced by any UMD lattice). Second the (dimension dependent) bound of M_{HL} on $L^p(\Omega, Y)$ for Ω a space of homogeneous type (see Subsection 4.4 for this notion) and again Y any UMD lattice. For the first result, Theorem 6.3, we shall use the spherical maximal operator

$$\mathcal{M}_S f(x) = \sup_{r>0} \left| \int_{S^{d-1}} f(x - ry) d\sigma(y) \right| \quad (x \in \mathbb{R}^d)$$

(this time, a priori well defined for $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$) where $d\sigma$ denotes the normalised Haar measure on the sphere $S^{d-1} \subseteq \mathbb{R}^d$, and provide dimension dependent bounds on $L^p(\mathbb{R}^d, Y)$. The bound on $\mathcal{M}_S$ interesting in itself, is in turn obtained via a Littlewood-Paley type decomposition of its Fourier multiplier. For the second result on spaces of homogeneous type, Theorem 6.10,

we shall use an argument of covering of Ω by "dyadic cubes" due to Hytönen and Kairema, see Theorem 6.16, together with Bourgain's characterisation of UMD property in terms of martingale maximal bounds over the unit interval, see Theorem 6.12.

We shall also be interested in maximal operators of spectral multiplier type

$$M_m f(x) = \sup_{t>0} |m(tA)f(x)| \quad (x \in \Omega)$$

where A is a sectorial operator and $m(tA)$ is the Hörmander functional calculus investigated in Subsection 3.1 above, with some fixed function m. Such maximal operators M_m for non-special multipliers have been considered e.g. in [RdF2, See, MaMe, Choi, CGHS, Wro2]. Note that if $m(\lambda) = e^{-\lambda}$, then $M_m f(x) = \sup_{t>0} |T_t f(x)|$ is the semigroup maximal operator. The boundedness of the latter on $L^p(\Omega, Y)$ is a consequence (resp. necessary condition) for the boundedness of M_{HL} in case that T_t has an integral kernel satisfying Gaussian upper estimates (4.9) (resp. lower estimates (4.10)). Such semigroup maximal operators on Bochner spaces have been recently studied in [BlDT, Xu15, HoMa1, HoMa2]. Independently of the existence of any integral kernel nor contractivity on the L^p scale, lattice positivity nor selfadjointness, we derive in Section 8 solely from a Hörmander calculus of A on $L^p(\Omega, Y)$ maximal estimates for M_m, for certain Hörmander spectral multipliers m decreasing at ∞ (Theorem 8.1 and Proposition 8.3). This is one evidence that the questions from Subsections 3.1 and 3.2 are linked. For a second instance that Hörmander spectral multiplier theorems can give new bounds for maximal operators, we refer to Remark 6.6 for an alternative proof of the boundedness of the spherical maximal operator above. Moreover, in the other way around, we use in Section 7 the boundedness of the Hardy-Littlewood maximal operator on a space of homogenous type, to obtain the Hörmander functional calculus for semigroups with Gaussian upper estimates.

3.3 Noncommutative Fourier and Schur multipliers

This third part is contained in the Sections 10, 11 and 12. Here we are interested in Schur multipliers (see Definition 10.9) and in noncommutative Fourier multipliers acting on group von Neumann algebras $\mathrm{VN}(G)$ associated with discrete groups, and in the last Subsection 12.5 also on locally compact groups. A Fourier multiplier on $\mathbb{R}^d$ transforms a character function $e^{is(\cdot)} \mapsto \phi(s)e^{is(\cdot)}$. A group von Neumann algebra is generated by the noncommutative characters $\lambda_s \in B(L^2(G))$, $\lambda_s(f)(t) = f(s^{-1}t)$, and a noncommutative Fourier multiplier maps $\lambda_s \mapsto \phi(s)\lambda_s$, where $\phi\colon G \to \mathbb{C}$ is the symbol. Functional calculi of multiplier semigroups have been investigated in [JMX, JMP1, JMP2, GPJP, Arh3, Arh4, Arh5, PRS]. It is a non-trivial task to define the notion of a dimension on the group G / the operator semigroup usable for the Hörmander $\mathcal{H}_2^\alpha$ calculus exponent α, which is solved in different ways in these works. Not all groups are manageable up to date.

In a first part, we are interested in Riesz transforms associated with markovian semigroups of Fourier and Schur multipliers. As often the case in noncommutative harmonic analysis, a considerable effort is spent here in the correct formulation of the mathematical problem in order to obtain a correct result. The Riesz transform equivalence, in the classical form written as

$$\left\| (-\Delta)^{\frac{1}{2}} f \right\|_{L^p(\mathbb{R}^d)} \cong \|\partial f\|_{L^p(\mathbb{R}^d, \ell_d^2)},$$

aims at expressing the L^p norm of $A^{\frac{1}{2}} f$, where A(hereabove $= -\Delta$) generates a semigroup, in terms of some gradient ∂. Here, the ambient space for $A^{\frac{1}{2}} f$ will be the noncommutative L^p space (see Definition 10.4) $L^p(\mathrm{VN}(G))$ and our gradient ∂ will be defined in terms of a Gaussian process. It will live in another noncommutative L^p space, defined over some von

Neumann algebra which extends both the probability space of the Gaussians and the group von Neumann algebra $\mathrm{VN}(G)$. Then the Riesz transform equivalence has been obtained in [JMP2] for noncommutative Fourier multipliers and ordinary Gaussians; we aim at the extension of this result to q-Gaussians (see Subsection 10.1 for this notion) and also to markovian semigroups of Schur multipliers. To achieve this goal, we prove a Khintchine type inequality for q-Gaussians in crossed product spaces, and use the UMD property of noncommutative L^p spaces in form of the Hilbert transform, together with a transference principle for bounded c_0-groups and an intertwining formula between $A^{\frac{1}{2}}$ and ∂.

We shall also extend the Riesz transform equivalence to boundedness of the bisectorial H^∞ calculus of Hodge-Dirac operators, which are a differential square root of (an extension of) the markovian semigroup generator A of Fourier or Schur multipliers. We answer a question from [JMP2, Problem C.5] in giving moreover dimension free bounds of this H^∞ calculus. The above results are contained in Section 10. Then in Section 11, we are able to prove that the Hodge-Dirac operators give rise to objects from noncommutative geometry: spectral triples and quantum (locally) compact metric spaces.

In a last Section 12, we study so-called decomposable mappings on noncommutative L^p spaces. As an origin, we refer to the Akcoglu-Sucheston theory for regular contractive L^p operators. Such operators arise e.g. from a submarkovian semigroup $(T_t)_{t\geqslant 0}$ defined on a classical $L^p(\Omega)$ space. For such an operator $T = T_t$, one has a pointwise domination $|Tf(x)| \leqslant |T|f(x)$, where $|T|$ is a contractive (linear) positive operator. Then Akcoglu-Sucheston's result [AcS] states that T can be dilated to a surjective isometry $U : L^p(\Omega') \to L^p(\Omega')$. That is, there exists an injection $J : L^p(\Omega) \to L^p(\Omega')$ and a contraction $P : L^p(\Omega') \to L^p(\Omega)$ such that $T^n = PU^nJ$ for all $n \geqslant 0$. From there, one can deduce further properties of T and $(T_t)_t$ such as functional calculus. Thus positive contractions (and also contractively dominated operators) on classical L^p spaces are well-behaved operators and share often the same nice properties as contractions on Hilbert spaces. From [JMe, Corollary 4.4], it is known that no Akcoglu-Sucheston result holds in general for (completely) positive contractions on noncommutative L^p spaces.

We have now undergone a thorough study of the noncommutative situation in [HDR6]. The pointwise domination property as considered above is replaced by decomposability of an operator $T\colon L^p(M) \to L^p(N)$ meaning that T is the corner of a completely positive (see Subsection 10.1) mapping

$$\begin{bmatrix} v_1 & T \\ T^\circ & v_2 \end{bmatrix} : S_2^p(L^p(M)) \to S_2^p(L^p(N)), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} v_1(a) & T(b) \\ T^\circ(c) & v_2(d) \end{bmatrix}.$$

On the other hand, it is well-known that pointwise dominated operators on classical L^p spaces admit bounded tensor extensions $T\otimes \mathrm{Id}_X : L^p(\Omega, X) \to L^p(\Omega, X)$ for any Banach space X. This lead Pisier [Pis2] to introduce the notion of a regular operator $T : L^p(M) \to L^p(N)$ between noncommutative L^p spaces being such that $T \otimes \mathrm{Id}_E : L^p(M, E) \to L^p(N, E)$ is completely bounded for any operator space E. Note that as already mentioned above, noncommutative harmonic analysis requires more effort to formulate, here the definitions of $S_2^p(L^p(M))$, the vector valued Bochner spaces $L^p(M, E)$ and complete positivity and complete boundedness. Moreover, the Bochner spaces require the von Neumann algebra M to be approximately finite-dimensional. We refer to Subsections 10.1 and 12.1. Using approximation structure of approx. finite-dim. von Neumann algebras, we show then that regular operators $L^p(M) \to L^p(N)$ on these algebras are precisely the decomposable ones, and moreover, we obtain an isometric statement after having defined the decomposable and regular norms (Theorem 12.2).

Then we turn our attention again to noncommutative Fourier and Schur multipliers and characterise their decomposability. As an interesting by-product, we obtain a projection result

(Theorem 12.7) from the space of completely bounded operators $\mathrm{CB}(L^p(\mathrm{VN}(G)))$ onto the subspace of Fourier multipliers (or Schur multipliers) based on work by Haagerup. For this last result, topological properties of the group G become important. Theorem 12.7 initially holds for discrete groups and does not hold for all locally compact groups. For an extension to l.c. groups, we develop an appropriate notion of approximation by discrete subgroups, see Theorem 12.14.

3.4 Explanation of the setup

Section 4 contains the preliminaries of the subsequent Sections 5 – 12. Note that most of the sections can be read independently. Only Section 7 uses the maximal bound $M_{HL} : L^p(\Omega, Y) \to L^p(\Omega, Y)$ from Section 6, Section 11 uses Section 10, and Section 12 uses some preliminaries from Section 10. Here is a table of correspondence of the sections with the publications.

Section	Publication
5	[HDR1, HDR2]
6	[HDR4], [HDR5, Section 3]
7	[HDR5]
8	[HDR9]
9	[HDR7]
10	[HDR8, Sections 2-4]
11	[HDR8, Section 5]
12	[HDR3, HDR6]

Each of the Subsections 5.1, 5.3, 6.1, 6.5 and then Sections 7, 8, 9 start by putting the mathematical problem into its context and then announce the main result. Then the proof of the main result(s) is sketched in its several steps. Afterwards, the theorem is illustrated by corollaries, extensions or comparisons with the literature; examples and further applications are given and we conclude with open questions. Thus the reader can choose according to his/her interests among proof ideas at the beginnings of the sections or applications in separate subsections. The memoir is intended to be largely self-contained.

The noncommutative results in Section 10 and 11, and partly 12 are certainly those which demand the most of technical effort in this memoir, to define the mathematical context and to prove the results afterwards. We have thus put the supplementary preliminaries in Subsection 10.1 apart, and decided to present only few proof sketches of important results (i.e. the theorems), or sketches of those proofs demanding a lot of machinery and results from different areas of analysis in Banach spaces. Other proofs, depending on the result, can be rather easy, or however technically involved. Afterwards, we refer to [HDR6, HDR8] containing all the details.

4 Preliminaries

This section contains some common background of the following ones.

4.1 Semigroups

Definition 4.1 *Let X be a Banach space. A family of bounded operators $(T_t)_{t\geqslant 0}$ in $B(X)$ is called a c_0-semigroup, if*

1. *$T_0 = \mathrm{Id}_X$ is the identity operator over X.*

2. $T_{t+s} = T_t \circ T_s$ *for* $t, s \geqslant 0$.

3. $x = \lim_{t\to 0+} T_t(x)$ *for all* $x \in X$.

If $(T_t)_{t\geqslant 0}$ is a c_0-semigroup, then

$$\mathrm{dom}(A) = \{x \in X : \lim_{t\to 0+} \frac{x - T_t(x)}{t} \text{ exists}\}$$

with $A(x) = \lim_{t\to 0+} \frac{x-T_t(x)}{t}$ is called the (negative) generator of $(T_t)_{t\geqslant 0}$. Then A is densely defined and closed [EnN]. In some cases which will be in our interest, the mapping $t \mapsto T_t$ extends to an analytic mapping on some sector $\Sigma_\omega \ni z \mapsto T_z$. Here, $\omega \in (0, \frac{\pi}{2}]$, and we let for $\omega \in (0, \pi)$, $\Sigma_\omega = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \omega\}$. If moreover, the semigroup is uniformly bounded on each subsector $\Sigma_{\omega'}$ for $\omega' \in (0, \omega)$ and for $x \in \mathrm{dom}(A)$, the limit $A(x) = \lim_{z\to 0} \frac{x-T_z(x)}{z}$ holds where z approaches to 0 on each such subsector, then we call $(T_z)_{z\in\Sigma_\omega}$ an analytic semigroup. An important case is when $\omega = \frac{\pi}{2}$, the maximal reasonable sector, in which case we also write $\mathbb{C}_+ = \Sigma_{\frac{\pi}{2}}$.

Definition 4.2 *Let (Ω, μ) be a σ-finite measure space and let $(T_t)_{t\geqslant 0}$ be a c_0-semigroup on $L^2(\Omega)$.*

1. *Then $(T_t)_{t\geqslant 0}$ is called a submarkovian semigroup, if*

   (a) *T_t extends boundedly to an operator on $L^p(\Omega)$ for all $p \in [1, \infty]$ and we have $\|T_t\|_{p\to p} \leqslant 1$ for any $t \geqslant 0$ and $p \in [1, \infty]$.*

   (b) *T_t is selfadjoint for any $t \geqslant 0$.*

   (c) *$T_t(f) \geqslant 0$ for any $t \geqslant 0$ whenever $f \in \bigcup_{p\in[1,\infty]} L^p(\Omega)$ with $f \geqslant 0$.*

2. *$(T_t)_{t\geqslant 0}$ is called a markovian semigroup, if $(T_t)_{t\geqslant 0}$ is submarkovian and in addition, $T_t(1) = 1$ for any $t \geqslant 0$.*

4.2 R-boundedness and related geometric properties of Banach spaces

Definition 4.3 *Let X, Y be Banach spaces and $\tau \subseteq B(X, Y)$. Then τ is called R-bounded if there is some $C < \infty$ such that for any $n \in \mathbb{N}$, any $x_1, \ldots, x_n \in X$ and any $T_1, \ldots, T_n \in \tau$, we have*

$$\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k T_k x_k\Big\|_Y \leqslant C\mathbb{E}\Big\|\sum_{k=1}^n \varepsilon_k x_k\Big\|_X,$$

where the ε_k are i.i.d. Rademacher variables on some probability space, that is, $\mathbb{P}(\varepsilon_k = \pm 1) = \frac{1}{2}$. The least admissible constant C is called R-bound of τ and is denoted by $R(\tau)$.

Remark 4.4 *Clearly, $R(\{T\}) = \|T\|_{B(X,Y)}$ if $\tau = \{T\}$ is a singleton. In general, we have $R(\tau) \geqslant \sup_{T\in\tau} \|T\|_{B(X,Y)}$ above. If X and Y are (isomorphic to) Hilbert spaces, then any family $\tau \subseteq B(X, Y)$ is R-bounded if and only if τ is bounded. The converse is also true if $X = Y$. We refer to [KW1, HvNVW2] for basics on R-boundedness. In particular, in case that X is a Banach lattice with finite cotype (see below), then R-boundedness is equivalent to*

$$\Big\|\Big(\sum_{k=1}^n |T_k x_k|^2\Big)^{\frac{1}{2}}\Big\|_X \leqslant C\mathbb{E}\Big\|\Big(\sum_{k=1}^n |x_k|^2\Big)^{\frac{1}{2}}\Big\|_X. \tag{4.1}$$

Definition 4.5 *Let X be a Banach space and $(\varepsilon_n)_n$ be a sequence of independent Rademacher variables.*

1. *We say that X has Pisier's property (α) if there are constants $c_1, c_2 > 0$ such that for any array $(x_{n,k})_{n,k=1}^N$ in X,$(\varepsilon'_k)_k$ a second sequence of independent Rademacher variables independent of $(\varepsilon_n)_n$, and $(\varepsilon''_{n,k})_{n,k}$ a doubly indexed sequence of independent Rademacher variables, the following equivalence holds:*

$$c_1 \mathbb{E}\mathbb{E}' \left\| \sum_{k,n=1}^N \varepsilon_n \varepsilon'_k x_{n,k} \right\|_X \leqslant \mathbb{E}'' \left\| \sum_{k,n=1}^N \varepsilon''_{n,k} x_{n,k} \right\|_X \leqslant c_2 \mathbb{E}\mathbb{E}' \left\| \sum_{k,n=1}^N \varepsilon_n \varepsilon'_k x_{n,k} \right\|_X .$$

2. *Let $p \in [1,2]$ and $q \in [2,\infty]$. We say that X has type p if for some constant $c > 0$ and any sequence $(x_n)_{n=1}^N$ in X, we have*

$$\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n x_n \right\|_X \leqslant c \left(\sum_{n=1}^N \|x_n\|^p \right)^{\frac{1}{p}} .$$

In this case, we write $\mathrm{type}(X) = p$ (not uniquely determined value). We say that X has cotype q if for some constant $c > 0$ and any sequence $(x_n)_{n=1}^N$ in X, we have

$$\left(\sum_{n=1}^N \|x_n\|^q \right)^{\frac{1}{q}} \leqslant c\mathbb{E} \left\| \sum_{n=1}^N \varepsilon_n x_n \right\|_X .$$

In this case, we write $\mathrm{cotype}(X) = q$ (not uniquely determined value).

Note that for $1 \leqslant p < \infty$, a classical $L^p(\Omega)$ space has property (α) and $\mathrm{type}(L^p(\Omega)) = \min(p,2)$, $\mathrm{cotype}(L^p(\Omega)) = \max(p,2)$. Also a UMD lattice (see Subsection 4.5) has property (α).

4.3 Sectorial operators and functional calculus

Definition 4.6 *Let X be a Banach space, $\omega \in (0,\pi)$ and $A : \mathrm{dom}(A) \subseteq X \to X$ an operator. A is called ω-sectorial if*

1. *A is closed and densely defined on X.*
2. *The spectrum $\sigma(A)$ is contained in $\overline{\Sigma_\omega}$.*
3. *For any $\omega' > \omega$, we have $\sup_{\lambda \in \mathbb{C} \setminus \overline{\Sigma_{\omega'}}} \|\lambda(\lambda - A)^{-1}\| < \infty$.*

We also call A 0-sectorial if A is ω-sectorial for all (small) $\omega \in (0,\pi)$.

The notion of sectoriality is linked to analytic semigroups from Subsection 4.1. Namely, let $\omega \in [0, \frac{\pi}{2})$. Then an operator A on some Banach space X is ω-sectorial if and only if it generates an analytic semigroup $(T_z)_{z \in \Sigma_\theta}$, where $\theta = \frac{\pi}{2} - \omega$ [EnN].

If X is reflexive, which will always be our case in this memoir and A is ω-sectorial, then A admits a canonical decomposition

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & 0 \end{bmatrix} : X = \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A) \to \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A) \tag{4.2}$$

such that $A_0 : \mathrm{dom}(A_0) \subseteq \overline{\mathrm{Ran}(A)} \to \overline{\mathrm{Ran}(A)}$ is again ω-sectorial and in addition injective and has dense range [KW1, Proposition 15.2]. Here, $\mathrm{Ran}(A)$ stands for the range of A and $\mathrm{Ker}(A)$ for its kernel. The operator A_0 is called the injective part of A. For $\theta \in (0,\pi)$, let

$$\mathrm{H}^\infty(\Sigma_\theta) = \big\{ f : \Sigma_\theta \to \mathbb{C} : \ f \text{ analytic and bounded} \big\}$$

equipped with the uniform norm $\|f\|_{\infty,\theta} = \sup_{z\in\Sigma_\theta} |f(z)|$. Let further

$$\mathrm{H}^\infty_0(\Sigma_\theta) = \big\{ f \in \mathrm{H}^\infty(\Sigma_\theta) : \ \exists\, C, \varepsilon > 0 : \ |f(z)| \leqslant C \min(|z|^\varepsilon, |z|^{-\varepsilon}) \big\}.$$

For an ω-sectorial operator A and $\theta \in (\omega,\pi)$, one can define a functional calculus $\mathrm{H}^\infty_0(\Sigma_\theta) \to B(X)$, $f \mapsto f(A)$ extending the ad hoc rational calculus, by using the Cauchy integral formula

$$f(A) = \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta'}} f(\lambda) R(\lambda, A) d\lambda, \tag{4.3}$$

where $\theta' = \frac{1}{2}(\omega+\theta)$ and $\partial\Sigma_{\theta'}$ is the boundary of a sector oriented counterclockwise. If moreover, there exists a constant $C < \infty$ such that $\|f(A)\| \leqslant C\|f\|_{\infty,\theta}$ for any $f \in \mathrm{H}^\infty_0(\Sigma_\theta)$, then A is said to have a (bounded) $\mathrm{H}^\infty(\Sigma_\theta)$ calculus. If X is reflexive and A has a bounded $\mathrm{H}^\infty(\Sigma_\theta)$ calculus, then so does A_0 and $f(A) = f(A_0) \oplus 0 : \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A) \to \overline{\mathrm{Ran}(A)} \oplus \mathrm{Ker}(A)$ for $f \in \mathrm{H}^\infty_0(\Sigma_\theta)$. Moreover, the functional calculus defined for $f \in \mathrm{H}^\infty_0(\Sigma_\theta)$ can be extended to a bounded Banach algebra homomorphism $\mathrm{H}^\infty(\Sigma_\theta) \to B(\overline{\mathrm{Ran}(A)})$, $f \mapsto f(A_0)$. If X is reflexive, A has a bounded $\mathrm{H}^\infty(\Sigma_\theta)$ calculus, and $f \in \mathrm{H}^\infty(\Sigma_\theta)$ such that $f(0) \in \mathbb{C}$ is defined, then $f(A) = f(A_0) \oplus f(0) P_{\mathrm{Ker}(A)} \in B(X)$ is a well-defined operator, where $P_{\mathrm{Ker}(A)}$ stands for the bounded projection onto $\mathrm{Ker}(A)$ annihilating $\overline{\mathrm{Ran}(A)}$.

For further information on the H^∞ calculus, we refer e.g. to the classical work [CDMY] or the more recent [KW1, HvNVW2]. We now turn to Hörmander function classes and their calculi.

Definition 4.7 *Let $\alpha > \frac{1}{2}$. We define the Hörmander class by*

$$\mathcal{H}^\alpha_2 = \big\{ f : [0,\infty) \to \mathbb{C} \text{ is bounded and continuous on } (0,\infty), \underbrace{|f(0)| + \sup_{R>0} \|\phi f(R\,\cdot)\|_{W^\alpha_2(\mathbb{R})}}_{=:\|f\|_{\mathcal{H}^\alpha_2}} < \infty \big\}.$$

Here ϕ is any $C^\infty_c(0,\infty)$ function different from the constant 0 function (different choices of functions ϕ resulting in equivalent norms) and $W^\alpha_2(\mathbb{R})$ is the classical Sobolev space.

Sometimes we also use the class for $1 \leqslant q \leqslant \infty$ and $\alpha > \frac{1}{q}$,

$$\mathcal{H}^\alpha_q = \big\{ f : [0,\infty) \to \mathbb{C} \text{ is bounded and continuous on } (0,\infty), \underbrace{|f(0)| + \sup_{R>0} \|\phi f(R\,\cdot)\|_{W^\alpha_q(\mathbb{R})}}_{=:\|f\|_{\mathcal{H}^\alpha_q}} < \infty \big\}.$$

Note that there is no continuity assumption of f in 0. The Hörmander classes have the following properties.

Lemma 4.8 *1. Assume that $\alpha \in \mathbb{N}$. Then a locally integrable function $f : (0,\infty) \to \mathbb{C}$ belongs to the Hörmander class $\mathcal{H}^\alpha_2$ if and only if*

$$|f(0)|^2 + \sum_{k=0}^{\alpha} \sup_{R>0} \int_R^{2R} \Big| t^k \frac{d^k}{dt^k} f(t) \Big|^2 \frac{dt}{t} < \infty,$$

and the above quantity is equivalent to $\|f\|^2_{\mathcal{H}^\alpha_2}$.

2. *If $\alpha > \frac{1}{2}$, then $\mathcal{H}^\alpha_2$ is a Banach algebra for the pointwise multiplication.*

3. *One has the continuous injections $\mathcal{H}^\alpha_q \hookrightarrow \mathcal{H}^\beta_r \hookrightarrow \mathcal{H}^\beta_q$ for $\alpha > \frac{1}{q} > \frac{1}{r}$ and $\alpha \geqslant \beta + \frac{1}{q} - \frac{1}{r}$.*

Proof : 1., 2. See [KrW3, Section 3], [KrPhD, Section 4.2.1] for the case that $\|f\|_{\mathcal{H}^\alpha_2}$ does not contain the summand $|f(0)|$. The present case is deduced immediately.

3. See [KrW3, Lemma 3.2]. ■

We can base a Hörmander functional calculus on the H^∞ calculus by the following procedure.

Definition 4.9 *Let X be a reflexive Banach space and $\alpha > \frac{1}{2}$. We say that a 0-sectorial operator has a bounded $\mathcal{H}^\alpha_2$ calculus if for some $\theta \in (0,\pi)$ and any $f \in \mathrm{H}^\infty(\Sigma_\theta)$, $\|f(A)\|_{B(\overline{\mathrm{Ran}(A)})} \leqslant C\|f\|_{\mathcal{H}^\alpha_2} (\leqslant C'(|f(0)| + \|f\|_{\infty,\theta}))$.*

In this case, the $\mathrm{H}^\infty(\Sigma_\theta)$ calculus can be extended to a bounded Banach algebra homomorphism $\mathcal{H}^\alpha_2 \to B(X)$ in the following way. Let

$$\mathcal{W}^\alpha_2 = \big\{ f : (0,\infty) \to \mathbb{C} : \ f \circ \exp \in W^\alpha_2(\mathbb{R}) \big\}$$

equipped with the norm $\|f\|_{\mathcal{W}^\alpha_2} = \|f \circ \exp\|_{W^\alpha_2(\mathbb{R})}$. Note that for any $\theta \in (0,\pi)$, the space $\mathrm{H}^\infty(\Sigma_\theta) \cap \mathcal{W}^\alpha_2$ is dense in $\mathcal{W}^\alpha_2$ [KrW3]. Since $\mathcal{W}^\alpha_2 \hookrightarrow \mathcal{H}^\alpha_2$, by the above density, we get a bounded mapping $\mathcal{W}^\alpha_2 \to B(\overline{\mathrm{Ran}(A)})$, $f \mapsto f(A_0)$ extending the H^∞ calculus.

Definition 4.10 *Let $(\phi_k)_{k\in\mathbb{Z}}$ be a sequence of functions in $C^\infty_c(0,\infty)$ with the properties that $\operatorname{supp}\phi_k \subseteq [2^{k-1}, 2^{k+1}]$, $\phi_k(t) = \phi_0(2^{-k}t)$ and $\sum_{k\in\mathbb{Z}} \phi_k(t) = 1$ for all $t > 0$. Then $(\phi_k)_{k\in\mathbb{Z}}$ is called a dyadic partition of unity.*

Let $(\phi_k)_{k\in\mathbb{Z}}$ be a dyadic partition of unity. For $f \in \mathcal{H}^\alpha_2$, we have that $\phi_k f \in \mathcal{W}^\alpha_2$, hence $(\phi_k f)(A_0) \in B(\overline{\mathrm{Ran}(A)})$ is well-defined. Then it can be shown that for any $x \in \overline{\mathrm{Ran}(A)}$, $\sum_{k=-n}^n (\phi_k f)(A_0)x$ converges as $n \to \infty$ and that it is independent of the choice of $(\phi_k)_{k\in\mathbb{Z}}$. This defines the operator $f(A_0) \in B(\overline{\mathrm{Ran}(A)})$, which in turn yields a bounded Banach algebra homomorphism $\mathcal{H}^\alpha_2 \to B(\overline{\mathrm{Ran}(A)})$, $f \mapsto f(A_0)$. Then extend by reflexivity to $f(A) = f(A_0) \oplus f(0)P_{\mathrm{Ker}(A)}$ and get a bounded Banach algebra homomorphism $\mathcal{H}^\alpha_2 \to B(X)$. This is the Hörmander functional calculus. This procedure also works for the $\mathcal{H}^\alpha_q$ class, $1 \leqslant q < \infty$, and in case that an estimate $\|f(A)\|_{B(\overline{\mathrm{Ran}(A)})} \leqslant C\,\|f\|_{\mathcal{H}^\alpha_q}$ holds for all $f \in \mathrm{H}^\infty(\Sigma_\theta)$ it gives a bounded homomorphism $\mathcal{H}^\alpha_q \to B(X)$. For details, we refer to [KrW3, Section 4], [KrPhD, Sections 4.2.3 - 4.2.6].

Note that in the literature, the Hörmander functional calculus on $L^p(\Omega)$ for some selfadjoint operator A on $L^2(\Omega)$ is often defined via the spectral theorem for selfadjoint operators on $L^2(\Omega)$ and then extended by density of $L^2(\Omega) \cap L^p(\Omega)$ in $L^p(\Omega)$. In this case, one recovers the same calculus as in our procedure above (see also [KrW3, Lemma 4.6]).

Assume that $\frac{1}{2} < \alpha < \beta$ and $0 < \theta < \omega < \pi$. Then $\mathrm{H}^\infty(\Sigma_\omega) \subseteq \mathrm{H}^\infty(\Sigma_\theta)$ and $\mathcal{H}^\beta_2 \subseteq \mathcal{H}^\alpha_2$ are continuous injections. Moreover, if a function $f \in \mathrm{H}^\infty(\Sigma_\theta)$ is defined in 0, then $f \in \mathcal{H}^\beta_2$ and $\|f\|_{\mathcal{H}^\beta_2} \lesssim |f(0)| + \|f\|_{\infty,\theta}$. Thus, in view of the above, we have the chain of implications: A has a ...

$$\mathcal{H}^\alpha_2 \text{ functional calculus} \Longrightarrow \mathcal{H}^\beta_2 \text{ f.c.} \Longrightarrow \mathrm{H}^\infty(\Sigma_\theta) \text{ f.c.} \Longrightarrow \mathrm{H}^\infty(\Sigma_\omega) \text{ f.c.} \tag{4.4}$$

We shall access the Hörmander functional calculus by the following Theorem from [KrW3, Theorem 7.1] or [KrPhD]. Here we say that a $\mathcal{H}^\beta_2$ (resp. $\mathrm{H}^\infty(\Sigma_\omega)$) functional calculus is R-bounded provided that

$$\Big\{ f(A) : \ \|f\|_{\mathcal{H}^\beta_2} \leqslant 1 \,(\text{resp. } \ \|f\|_{\infty,\omega} \leqslant 1) \Big\}$$

is R-bounded.

Theorem 4.11 (K.-Weis) *Let A be a generator of an analytic semigroup on some reflexive Banach space X with property (α). Assume that A has an $\mathrm{H}^\infty(\Sigma_\sigma)$ calculus to some angle $\sigma \in (0,\pi)$ and that*

$$\big\{\exp(-te^{i\theta}2^k A) : \ k \in \mathbb{Z}\big\} \tag{4.5}$$

is R-bounded in $B(X)$ for all $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$ and $t>0$, with R-bound $\leqslant C(\cos(\theta))^{-\alpha}$. This is clearly the case if

$$\big\{\big(\cos(\arg z)\big)^\alpha \exp(-zA) : \ z \in \mathbb{C}_+\big\} \tag{4.6}$$

is R-bounded in $B(X)$. Then A has a Hörmander $\mathcal{H}^\beta_2$ functional calculus on X with $\beta > \alpha + \frac{1}{2}$. Moreover, this functional calculus is R-bounded.

Proof : For a proof, we refer to [KrW3, Theorem 7.1] and [HDR5, Theorem 2.7]. ■

As an important consequence of a $\mathcal{H}^\beta_2$ functional calculus, we state the following.

Lemma 4.12 (K.-Weis) *Let A be a 0-sectorial operator with $\mathcal{H}^\alpha_2$ calculus for some $\alpha > \frac{1}{2}$. Let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of $\mathbb{R}_+$ in the sense of Definition 4.10. Then we have the following so-called Paley-Littlewood decomposition for $x \in \overline{\mathrm{Ran}(A)}$:*

$$\|x\|_X \cong \mathbb{E}\left\|\sum_{n\in\mathbb{Z}} \varepsilon_n \phi_n(A)x\right\|_X, \tag{4.7}$$

where the series $\sum_{n\in\mathbb{Z}}\phi_n(A)x$ converges unconditionally in X. In particular, if X is a Banach lattice with finite cotype, (4.7) reads as

$$\|x\|_X \cong \left\|\left(\sum_{n\in\mathbb{Z}} |\phi_n(A)x|^2\right)^{\frac{1}{2}}\right\|_X.$$

Proof : See [HDR9, Lemma 2.18], [KrW2, Theorem 4.1]. ■

Bisectorial operators and functional calculus We shall also need in Sections 10 and 11 bisectors, bisectorial operators and their H^∞ calculus. To this end, for $\omega \in (0,\frac{\pi}{2})$, we let $\Sigma^\pm_\omega = \Sigma_\omega \cup (-\Sigma_\omega)$ be a bisector, and $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ the algebra of bounded holomorphic functions on the bisector. The definition of a bisectorial operator D/the Cauchy integral formula to define $m(D)$ for $m \in \mathrm{H}^\infty_0(\Sigma^\pm_\omega)$/the notion of bounded $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ calculus is then analogous to the case of a sectorial operator. Note that the maximal angle for the bisectorial situation is $\frac{\pi}{2}$, and not π as in the sectorial case. We refer to [HvNVW2, Subsection 10.6]. Of particular interest of us will be the following result.

Theorem 4.13 *[HvNVW2, Theorem 10.6.7] Let D be a bisectorial operator of angle $\theta \in (0,\frac{\pi}{2})$ and $\omega \in (\theta,\frac{\pi}{2})$. Assume that moreover D is R-bisectorial, meaning that $\{\lambda R(\lambda,D) : \lambda \in \mathbb{C}\backslash\overline{\Sigma^\pm_\theta}\}$ is R-bounded.*

1. *If D admits a bounded $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ calculus, then D^2 admits a bounded $\mathrm{H}^\infty(\Sigma_{2\omega})$ calculus.*

2. *If D^2 admits a bounded $\mathrm{H}^\infty(\Sigma_{2\omega})$ calculus, then D admits a bounded $\mathrm{H}^\infty(\Sigma^\pm_\vartheta)$ calculus for all $\omega < \vartheta < \frac{\pi}{2}$.*

Similarly to the sectorial case, if D admits a bounded $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ calculus, then a priori, the spectral multipliers $m(D)$ for $m \in \mathrm{H}^\infty(\Sigma^\pm_\omega)$ are well-defined operators only $\overline{\mathrm{Ran}(D)} \to \overline{\mathrm{Ran}(D)}$. Note however that an R-bisectorial operator D on a reflexive space gives rise to the so-called Hodge Decomposition of the Banach space X on which D acts. That is, $X = \overline{\mathrm{Ran}(D)} \oplus \mathrm{Ker}(D)$ with corresponding bounded projections. Then if D has a bounded $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ calculus, one can define, in the analogous way to the sectorial case discussed above, for $m \in \mathrm{H}^\infty(\Sigma^\pm_\omega)$ such that $m(0)$ is well-defined,

$$m(D) = m(D|\overline{\mathrm{Ran}(D)}) \oplus m(0)P_{\mathrm{Ker}(D)} \in B(X),$$

where $D|\overline{\mathrm{Ran}(D)}$ is injective with dense range, defined on the subspace $\overline{\mathrm{Ran}(D)}$, and $P_{\mathrm{Ker}(D)}$ is the bounded projection onto the kernel of D as above.

4.4 Spaces of homogeneous type and kernel estimates

We recall here the definition of a space of homogeneous type in the sense of Coifman and Weiss [CoWe1] (we restrict to classical metric spaces). Such spaces are frequently used in recent work on spectral multipliers.

Definition 4.14 *Let $(\Omega, \mathrm{dist}, \mu)$ be a metric measure space, that is,* dist *is a metric on Ω and μ is a Borel measure on Ω. We denote $B(x,r) = \{y \in \Omega : \mathrm{dist}(x,y) \leqslant r\}$ the closed balls of Ω. We assume that $\mu(B(x,r)) \in (0,\infty)$ for any $x \in \Omega$ and $r > 0$. Then Ω is said to be a space of homogeneous type if there exists a constant $C < \infty$ such that the doubling condition holds:*

$$\mu(B(x,2r)) \leqslant C\mu(B(x,r)) \quad (x \in \Omega,\ r > 0).$$

We write in short $V(x,r) = \mu(B(x,r))$. It is well-known that there exists some finite $d \in (0,\infty)$ such that $V(x,\lambda r) \leqslant C\lambda^d V(x,r)$ for any $x \in \Omega$, $r > 0$ and $\lambda \geqslant 1$. Such a d is called (homogeneous) dimension of Ω.

Definition 4.15 *Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type. Let $(T_t)_{t\geqslant 0}$ be a semigroup acting on $L^2(\Omega)$. Assume that*

$$T_t f(x) = \int_\Omega p_t(x,y) f(y)\, dy \tag{4.8}$$

for any $f \in L^2(\Omega)$, $x \in \Omega$, $t > 0$ and some measurable functions $p_t : \Omega \times \Omega \to \mathbb{C}$. Let $m \geqslant 2$. Then $(T_t)_t$ is said to satisfy upper Gaussian estimates (of order m) if there exist constants $C, c > 0$ such that

$$|p_t(x,y)| \leqslant C\frac{1}{V(x,r_t)} \exp\left(-c\left(\frac{\mathrm{dist}(x,y)}{r_t}\right)^{\frac{m}{m-1}}\right) \quad (x,y \in \Omega,\ t > 0), \tag{4.9}$$

where $r_t = t^{\frac{1}{m}}$. Moreover, $(T_t)_t$ is said to satisfy lower Gaussian estimates (of order m) if $p_t(x,y) > 0$ for all $x, y \in \Omega$ and $t > 0$, and there exist some other constants $C, c > 0$ such that

$$p_t(x,y) \geqslant C\frac{1}{V(x,r_t)} \exp\left(-c\left(\frac{\mathrm{dist}(x,y)}{r_t}\right)^{\frac{m}{m-1}}\right) \quad (x,y \in \Omega,\ t > 0). \tag{4.10}$$

4.5 UMD lattices

In several results of this memoir, UMD lattices, i.e. Banach lattices which enjoy the UMD property, play a prevalent rôle. For a general treatment of Banach lattices and their geometric properties, we refer the reader to [LTz, Chapter 1]. We recall now definitions and some useful properties. A Banach space Y is called UMD space if the Hilbert transform

$$H : L^p(\mathbb{R}) \to L^p(\mathbb{R}),\ Hf(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \geqslant \varepsilon} \frac{1}{x-y} f(y)\, dy$$

extends to a bounded operator on $L^p(\mathbb{R}, Y)$, for some (equivalently for all) $1 < p < \infty$ [HvNVW1, Theorem 5.1]. The importance of the UMD property in harmonic analysis was recognised for the first time by Burkholder [Bur81, Bur83], see also his survey [Bur01]. He settled a geometric characterization via a convex functional [Bur81] and together with Bourgain [Bou83], they showed that the UMD property can be expressed by boundedness of Y-valued martingale sequences. A UMD space is super-reflexive [Ald], and hence (almost by definition) B-convex. As a survey for UMD lattices and their properties in connection with results in harmonic analysis, we refer the reader to [RdF1].

A Köthe function space Y is a Banach lattice consisting of equivalence classes of locally integrable functions on some σ-finite measure space (Ω', μ') with the additional properties

1. If $f : \Omega' \to \mathbb{C}$ is measurable and $g \in Y$ is such that $|f(\omega')| \leqslant |g(\omega')|$ for almost every $\omega' \in \Omega'$, then $f \in Y$ and $\|f\|_Y \leqslant \|g\|_Y$.
2. The indicator function 1_A is in Y whenever $\mu'(A) < \infty$.
3. Moreover, we will assume that Y has the Fatou property: If a sequence $(f_k)_k$ of non-negative functions in Y satisfies $f_k(\omega') \nearrow f(\omega')$ for almost every $\omega' \in \Omega'$ and $\sup_k \|f_k\|_Y < \infty$, then $f \in Y$ and $\|f\|_Y = \lim_k \|f_k\|_Y$.

Note that for example, any $L^p(\Omega')$ space with $1 \leqslant p \leqslant \infty$ is such a Köthe function space.

Assumption 4.16 *In this memoir, the UMD lattice Y will always be assumed to be a Köthe function space.*

Let E be any Banach space. We can consider the vector valued lattice $Y(E) = \{F : \Omega' \to E :\ F \text{ is strongly measurable and } \omega' \mapsto \|F(\omega')\|_E \in Y\}$ with norm $\|F\|_{Y(E)} = \big\|\|F(\cdot)\|_E\big\|$. From [RdF1, Corollary page 214], we know that if Y is UMD and E is UMD, then also $Y(E)$ is UMD. Moreover, we shall consider specifically in Section 8 spaces $L^p(\Omega, Y(E))$. For the natural identity $L^p(\Omega, Y)(E) = L^p(\Omega, Y(E))$ guaranteed e.g. by reflexivity of Y, we refer to [Lin, Sections B.2.1, B.2.2, Theorem B.2.7].

Lemma 4.17 *Let Y be a UMD lattice. Then it has the σ-Levi property: any increasing and norm-bounded sequence $(x_n)_n$ in Y has a supremum in Y. It also has the Fatou-property and hence the σ-Fatou property: any increasing sequence $(x_n)_n$ in Y with a supremum $x \in Y$ satisfies $\|x\|_Y = \sup_n \|x_n\|_Y$. Note that if $1 < p < \infty$ and (Ω, μ) is a σ-finite measure space, then $L^p(\Omega, Y)$ is again a UMD lattice, so has the above σ-Levi and σ-Fatou properties.*

Proof : Note that a UMD lattice is reflexive. Then we refer to [Lin, Proposition B.1.8]. ■

For the following definition, we refer e.g. to [LTz, Definition 1.d.3], [RdF1, TJ].

Definition 4.18 *Let Y be a Banach lattice and $1 \leqslant p, q \leqslant \infty$. Then Y is called p-convex if there exists a constant $C < \infty$ such that for any $x_1, \dots, x_n \in Y$, we have*

$$\left\| \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \right\|_Y \leqslant C \left(\sum_{i=1}^n \|x_i\|_Y^p \right)^{\frac{1}{p}}.$$

Similarly, Y is called q-concave if there exists a constant $C > 0$ such that for any $x_1, \dots, x_n \in Y$, we have

$$\left\| \left(\sum_{i=1}^n |x_i|^q \right)^{\frac{1}{q}} \right\|_Y \geqslant C \left(\sum_{i=1}^n \|x_i\|_Y^q \right)^{\frac{1}{q}}$$

(obvious modifications if $p = \infty$ or $q = \infty$). We shall write p_Y for a p-convexity and q_Y for a q-concavity exponent of Y.

5 Hörmander calculus via R-boundedness: Two examples

The results of this section are contained in the publications [HDR1, HDR2]. We let $(T_t)_{t \geqslant 0}$ be a c_0-semigroup as in Definition 4.1, with negative generator A. We assume A to be 0-sectorial and are interested in its functional calculus. This section presents two independent situations where we prove a new Hörmander functional calculus as in Definition 4.9, based on R-boundedness for particular spectral multipliers. For the first situation in Subsection 5.1, this will be achieved by means of Theorem 4.11, where the spectal multipliers are the complex time semigroup. The underlying Banach space will be $L^p(\Omega)$ for some $1 < p < \infty$. For the second situation in Subsection 5.3, we will use R-boundedness of certain Cesàro mean spectral multipliers, whose spectral multiplier functions are "smoothed indicator functions" of an interval $[0, n]$. The underlying Banach space will be $L^p(\Omega, Y)$ for some $1 < p < \infty$ and Y a UMD lattice (see Subsection 4.5).

5.1 Hörmander calculus for Poisson estimates

We describe the first situation. Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type as in Subsection 4.4. Let moreover $(T_t)_{t \geqslant 0}$ be a semigroup on $L^2(\Omega)$ having an integral kernel as in (4.8). There is by now a big literature established in the last twenty years, under which assumptions on the kernel, the generator of the semigroup has a functional calculus on the space $L^p(\Omega)$, $1 < p < \infty$ [Ale, Bl, CCMS, Chr, CDY, CO, COSY, Duo, DuOS, DuRo, KrW3, KuU1, KuU2, Mar, Ouh05, SYY].

We state as a pars pro toto the following important result.

Theorem 5.1 (Duong-Ouhabaz-Sikora) *[DuOS, Theorem 3.1] Assume that $(T_t)_{t \geqslant 0}$ consists of selfadjoint operators and that its generator A is selfadjoint positive definite. Assume that $(T_t)_{t \geqslant 0}$ satisfies the Gaussian upper estimate (4.9) for some order $m \geqslant 2$. Let $\alpha > \frac{d}{2}$, where we denote d a dimension of the space Ω. Let $1 < p < \infty$. Then $(T_t)_{t \geqslant 0}$ extends to a uniformly bounded c_0-semigroup on $L^p(\Omega)$, and its generator A has a bounded $\mathcal{H}^\alpha_\infty$ calculus in the sense of Subsection 4.3. That is, there is a constant $C > 0$ such that for any $f \in \mathcal{H}^\alpha_\infty$,*

$$\|f(A)\|_{L^p(\Omega) \to L^p(\Omega)} \leqslant C \, \|f\|_{\mathcal{H}^\alpha_\infty}.$$

This theorem covers a large class of operators such as Laplacian operators on Lie groups of polynomial growth, Schrödinger operators and elliptic operators on Riemannian manifolds.

See e.g. the list of examples in [DuOS, Section 7], or in Subsection 7.1. Note that there are semigroups satisfying a weaker type of decay than the Gaussian upper estimate (4.9) that we describe in (5.1) below. This already occurs for the Poisson semigroup $T_t = \exp(-t(-\Delta)^{\frac{1}{2}})$ on $L^p(\mathbb{R}^d)$.

Definition 5.2 *Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type. Let $(T_t)_{t\geqslant 0}$ be a semigroup acting on $L^2(\Omega)$. Assume that it extends to an analytic semigroup (see Subsection 4.1) $(T_z)_{z\in\mathbb{C}_+}$ such that the T_z have an integral kernel $p_z(x,y)$ as in (4.8). We say that the $(T_z)_{z\in\mathbb{C}_+}$ satisfy Poisson estimates provided that*

$$|p_z(x,y)| \leqslant C(\cos\arg z)^{-\beta} \frac{1}{V(x,|z|)} \frac{1}{\left|1+\frac{\mathrm{dist}(x,y)^2}{z^2}\right|^{\frac{d+1}{2}}} \quad (z\in\mathbb{C}_+,\ x,y\in\Omega) \tag{5.1}$$

for some $C, \beta \geqslant 0$ and d a doubling dimension of Ω.

We give several examples in Subsection 5.2 below. The name Poisson estimate here is due to the fact that in case $\Omega = \mathbb{R}^d$, the term

$$\frac{1}{V(x,|z|)} \frac{1}{\left(1+\frac{\mathrm{dist}(x,y)^2}{z^2}\right)^{\frac{d+1}{2}}} = c(\arg z) \frac{z}{(z^2+\mathrm{dist}(x,y)^2)^{\frac{d+1}{2}}}$$

equals a constant times the kernel of the Poisson semigroup $T_z = \exp(-z(-\Delta)^{\frac{1}{2}})$. Then the additional factor $(\cos\arg z)^{-\beta}$ appears naturally in some examples (see Subsection 5.2 below). Note that in (5.1), we assume estimates for complex and not only real times as in the Gaussian estimates (4.9). This is due to the fact that for Gaussian estimates, if the semigroup is selfadjoint, then there is some extrapolation procedure from real to complex times [CaCoOu, Proposition 4.1], using the Phragmen-Lindelöf Theorem and rapid decay of the Gaussian function. To our knowledge, such an extrapolation is not known for the milder decay of (5.1). However, in the proofs of Theorem 5.1 and also Theorem 5.3 below, precise knowledge on the behaviour of the kernel for z close to the boundary $i\mathbb{R}$ of $\mathbb{C}_+$ is vital.

Then our result for semigroups with Poisson estimates reads as follows. Note that our space Ω has to satisfy more volume hypotheses than ordinary spaces of homogeneous type.

Theorem 5.3 *[HDR1, Theorem 3.2] Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type. Denote the diameter $\mathrm{diam}(\Omega) = \sup\{\mathrm{dist}(x,y) :\ x,y\in\Omega\}$. Assume that*

$$\begin{aligned} &V(x,r) \cong r^d \text{ if } \mathrm{diam}(\Omega) = \infty, \\ &V(x,r) \cong \min(r^d, 1) \text{ if } \mathrm{diam}(\Omega) < \infty \end{aligned} \tag{5.2}$$

and

$$\mu(B(x,R)\backslash B(x,r)) \leqslant C(R^d - r^d) \quad (x\in\Omega,\ R>r>0) \text{ if } \mathrm{diam}(\Omega) = \infty,$$

$$\mu(B(x,R)\backslash B(x,r)) \leqslant C(R-r)\min(R^{d-1},1) \quad (x\in\Omega,\ R>r>\frac{1}{2}R>0) \text{ if } \mathrm{diam}(\Omega) < \infty.$$

Let $(T_t)_{t\geqslant 0}$ be a c_0-semigroup which acts on all $L^p(\Omega)$, $1<p<\infty$. Assume that $(T_z)_{z\in\mathbb{C}_+}$ is analytic and that T_z has an integral kernel satisfying the Poisson estimate (5.1). Assume that for $z\in\mathbb{C}_+$, $\|T_z\|_{B(L^2(\Omega))} \lesssim (\cos(\arg z))^{-\frac{d-1}{2}-\beta}(1+|\log(\cos(\arg z))|)^2$, which is the case e.g. when A is selfadjoint. Then the semigroup $T_z = \exp(-zA)$ satisfies on $X = L^p(\Omega)$ for any $1<p<\infty$ the R-bound estimate

$$R\left(\exp(-e^{i\theta}2^j tA) :\ j\in\mathbb{Z}\right) \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta}(1+|\log(\cos(\theta))|)^2.$$

Corollary 5.4 *Let the assumptions of Theorem 5.3 be satisfied. Assume moreover that A has a bounded $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^2(\Omega)$ for some $\omega \in (0,\pi)$ (e.g. A is selfadjoint). Then A has a bounded $\mathcal{H}_2^\alpha$ calculus on $L^p(\Omega)$ for any $1 < p < \infty$ and $\alpha > \frac{d}{2} + \beta$. Moreover this calculus is an R-bounded mapping, i.e.*

$$R(f(A) : \|f\|_{\mathcal{H}_2^\alpha} \leqslant 1) < \infty.$$

Proof : The $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^2(\Omega)$ extends to an $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega)$ by [DuRo, Theorem 3.1], see also [HDR1, Proposition 3.1]. Then the corollary follows from Theorems 4.11 and 5.3. ∎

Remark 5.5

1. *Note that [DuRo, Theorem 3.1] (H^∞ functional calculus for complex time kernel estimates of Poisson type) in combination with [CDMY, Theorem 4.10] (estimates for the H^∞ functional calculus norm for small angles yields a Hörmander functional calculus), also yields a Hörmander functional calculus for selfadjoint semigroups with Poisson kernel estimates (5.1), but with a worse differentiation parameter.*

2. *In case $\beta = 0$, the differentiation parameter $\alpha > \frac{d}{2}$ that we obtain is essentially sharp for the range $1 < p < \infty$ (cf. the one for the Poisson semigroup on $L^p(\mathbb{R}^d)$). Note that if one replaces the hypothesis of Poisson upper estimates (5.1) by Gaussian upper estimates (4.9) and selfadjointness of the semigroup, then Corollary 5.4 is false for $\alpha = \frac{d}{2} + \frac{1}{6}$ in dimension $d = 1$ [Than]. Thus, Poisson estimates are not a consequence of Gaussian estimates; the behaviour of Poisson estimates (see the examples in Subsection 5.2 below) is somehow better suitable in complex times than that of Gaussian estimates. The drawback however is that for Poisson estimates, one has to assume a priori complex times in (5.1).*

3. *Note that the R-boundedness in Corollary 5.4, by virtue of (4.1) yields square function estimates for Hörmander spectral multipliers.*

4. *We shall give in Proposition 5.7 a non selfadjoint example to which Corollary 5.4 applies.*

Question 5.6 *It would be interesting to know under which hypotheses Poisson type estimates for real times extrapolate to complex time as in (5.1).*

In the following, we explain the ideas of the proof of Theorem 5.3 in several steps.

Proof of Theorem 5.3: 1st step: Reduction of R-boundedness of operator family to boundedness of a single operator Let $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $j \in \mathbb{Z}$, $t_0 \in [1,2]$ and $x, y \in \Omega$. Write in short $T_j = \exp(-e^{i\theta} 2^j t_0 A)$. Recall from (4.1) that $\{T_j : j \in \mathbb{Z}\}$ is R-bounded on $L^p(\Omega)$ with R-bound $C < \infty$, if and only if

$$\left\| \left(\sum_{j \in F} |T_j f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)} \leqslant C \left\| \left(\sum_{j \in F} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega)}$$

for any finite index set $F \subseteq \mathbb{Z}$ and $f_j \in L^p(\Omega)$, $j \in F$. To prove the theorem, it thus suffices to show that

$$\left\| T : \begin{cases} L^p(\Omega, \ell^2(F)) & \to L^p(\Omega, \ell^2(F)) \\ (f_j)_{j \in F} & \mapsto (T_j f_j)_{j \in F} \end{cases} \right\| \lesssim (\cos(\theta))^{-\frac{d-1}{2} - \beta} (1 + |\log(\cos(\theta))|)^2 \tag{5.3}$$

independently of F.

2nd step: Reduction to a vector-valued singular integral Note that the operator T in (5.3) is given by

$$T(f_j)_j(x) = \left(\int_\Omega p_{e^{i\theta}2^j t_0}(x,y) f_j(y) d\mu(y) \right)_j = \int_\Omega P_{\theta,t_0}(x,y)(f_j)_j(y) d\mu(y),$$

where $P_{\theta,t_0}(x,y) \in B(\ell^2(F))$ is the operator valued kernel given by the diagonal matrix $p_{e^{i\theta}2^j t_0}(x,y)$. Boundedness of integral operators with operator valued kernels on Bochner spaces is treated in [MoLu], based on [DuMc].

3rd step: Vector-valued singular integrals In [MoLu, Theorem 2.3], boundedness of an integral operator such as T is proved, provided that T is bounded on $L^2(\Omega, \ell^2(F))$ and that a certain integral cancellation condition (5.4) holds. In contrast to classical cancellation where smoothness of the kernel of T in space is needed, in [DuMc, MoLu], the kernel of T is tested against an "approximate identity", which amounts in our case in testing against the kernel of the semigroup itself with a time shift $e^{i\theta}2^j t_0 \mapsto e^{i\theta}2^j t_0 + t$. Denoting then for $t > 0$, $Q_{\theta,t_0,t}(x,y)$ the operator valued kernel given by the diagonal matrix $p_{e^{i\theta}2^j t_0 + t}$, the condition is [MoLu, Definition 2.1]

(5.4)
$$\int_{\mathrm{dist}(x,y) \geqslant c_1 t} \| P_{\theta,t_0}(x,y) - Q_{\theta,t_0,t}(x,y) \|_{B(\ell^2(F))} \, d\mu(y) \lesssim (\cos(\theta))^{-\frac{d-1}{2}-\beta} (1 + |\log(\cos(\theta))|)^2$$

for all $x \in \Omega$ and $t > 0$.

4st step: Reduction of the cancellation condition (5.4) to the kernel condition (5.1) Note that our assumption in Theorem 5.3 on boundedness of the semigroup on $L^2(\Omega)$ carries over one-to-one to that of T on $L^2(\Omega, \ell^2(F))$. In view of [MoLu, Theorem 2.3], we are reduced to check (5.4). For this in turn, we use that both P_{θ,t_0} and $Q_{\theta,t_0,t}$ arise from the kernel of an analytic semigroup, which is itself an analytic function for fixed $x, y \in \Omega$. Therefore, the Cauchy integral formula is at hand and also the mean value theorem, so that the difference $P_{\theta,t_0} - Q_{\theta,t_0,t}$ can be expressed as an integral over a (non-cancellating) term of the semigroup kernel. Then one is reduced to check that the Poisson estimate (5.1) yields indeed the integral condition (5.4). This is a highly delicate task in splitting the space Ω into annuli and using hereby the annuli volume condition (5.2) from the hypotheses of the theorem. Also other splittings and considerations by cases depending on values of $\theta, t_0, t, 2^j, \mathrm{dist}(x,y)$ are necessary [HDR1, pages 7-14, 22-34].

5.2 Examples of semigroups satisfying Poisson estimates (5.1)

Note first that in the most classical example, namely $A = (-\Delta)^{\frac{1}{2}}$ and $\exp(-zA)$ the Poisson semigroup on $L^p(\mathbb{R}^d)$ for some $1 < p < \infty$ and $d \in \mathbb{N}$, Corollary 5.4 gives the (sharp when the range $1 < p < \infty$ is allowed) order of derivation of the classical Hörmander multiplier theorem and strengthens it in that it includes the R-boundedness of spectral multipliers whose associated functions have bounded $\mathcal{H}^\alpha_2$ norm.

For a generalization, we consider the situation in [MMMM]. Let M be a positive integer. Consider the constant coefficient second order, $M \times M$ system, differential operator

$$Lu = \sum_{\gamma=1}^{M} \sum_{r,s=1}^{d+1} (\partial_r (a^{\alpha\gamma}_{rs} \partial_s u_\beta))_{1 \leqslant \alpha \leqslant M},$$

where $a_{rs}^{\alpha\beta}$ are real coefficients for $r,s=1,\dots,d+1$ and $\alpha,\gamma=1,\dots,M$. Here, u is a function defined on the upper half space $\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times [0,\infty)$. Further, we assume as in [MMMM] the ellipticity condition

$$\sum_{\alpha,\gamma=1}^{M}\sum_{r,s=1}^{d+1} \mathrm{Re}\,[a_{rs}^{\alpha\gamma}\xi_r\xi_s\overline{\eta_\alpha}\eta_\beta] \geqslant \kappa_0|\xi|^2|\eta|^2$$

for every $(\xi_r)_{1\leqslant r\leqslant d+1}\in\mathbb{R}^{d+1}$, $(\eta_\alpha)_{1\leqslant\alpha\leqslant M}\in\mathbb{C}^M$ and some $\kappa_0>0$. Then in [MMMM] the following Dirichlet problem on $\mathbb{R}^{d+1}_+$ is considered:

$$\begin{cases} Lu=0 \text{ in } \mathbb{R}^{d+1}_+ \\ u|^{n.t.}_{\partial\mathbb{R}^{d+1}_+} = f\in L^p(\mathbb{R}^d,\mathbb{C}^M), \end{cases}$$

where $\partial\mathbb{R}^{d+1}_+ = \mathbb{R}^d\times\{0\}$, *n.t.* means non-tangential trace of u, and f is a given function in $L^p(\mathbb{R}^d,\mathbb{C}^M)$, $1<p<\infty$. If $\mathfrak{A}_L^{dis}\neq\emptyset$, a certain condition, see [MMMM, (3.12)], which will be satisfied in our example, then this problem is well-posed in $L^p(\mathbb{R}^d,\mathbb{C}^M)$ [MMMM, Theorem 4.1], so it possesses a unique solution u. As moreover the coefficients defining L are constant, we have $\partial_r[u(\cdot+(0,t))]|_{\cdot=x} = (\partial_r u)(x+(0,t))$, so that the expression $T_tf(x):=u(x,t)$, $x\in\mathbb{R}^d$, $t\geqslant 0$ defines a semigroup on $L^p(\mathbb{R}^d,\mathbb{C}^M)$. In the sequel, we are interested in the Hörmander functional calculus of the negative generator of that semigroup. Note that for some cases, this semigroup is given by a convolution kernel. We now restrict to the following specific example.

Lamé system of elasticity. Assume that $M=d+1$ above. The so-called Lamé operator in $\mathbb{R}^{d+1}$ has the form

$$Lu=\mu\Delta u+(\lambda+\mu)\nabla\,\mathrm{div}\,u,\quad u=(u_1,\dots,u_{d+1}), \tag{5.5}$$

where the constants $\lambda,\mu\in\mathbb{R}$ (typically called Lamé moduli) are assumed to satisfy $\mu>0$ and $2\mu+\lambda>0$. Then according to [MMMM, Theorem 5.2], $T_tf(x)=u(x,t)$ is given by

$$\begin{aligned}(T_tf)_\alpha(x) &= \frac{4\mu}{3\mu+\lambda}\frac{1}{\omega_d}\int_{\mathbb{R}^d}\frac{t}{(|x-y|^2+t^2)^{\frac{d+1}{2}}}f_\alpha(y)dy \\ &+\frac{\mu+\lambda}{3\mu+\lambda}\frac{2(d+1)}{\omega_d}\sum_{\gamma=1}^{d+1}\int_{\mathbb{R}^d}\frac{t(x-y,t))_\alpha(x-y,t)_\gamma}{(|x-y|^2+t^2)^{\frac{d+3}{2}}}f_\gamma(y)dy\quad(\alpha=1,\dots,d+1),\end{aligned} \tag{5.6}$$

where ω_d is the area of the unit sphere S^d in $\mathbb{R}^{d+1}$, and $(x-y,t)_\alpha = x_\alpha-y_\alpha$ if $\alpha\in\{1,\dots,d\}$ and $(x-y,t)_{d+1}=t$.

Proposition 5.7 *Let A be the negative generator of the Lamé semigroup given in (5.6) on $L^p(\mathbb{R}^d,\mathbb{C}^{d+1})$ for some $1<p<\infty$. Then A has an $\mathcal{H}^\alpha_2$ calculus for any $\alpha>\frac{d}{2}+1$. Moreover, $\{f(A):\ \|f\|_{\mathcal{H}^\alpha_2}\leqslant 1\}$ is R-bounded on $L^p(\mathbb{R}^d,\mathbb{C}^{d+1})$ for these α.*

Proof : See [HDR1, Proposition 4.1, Corollary 4.2]. ∎

Remark 5.8 *There are other, selfadjoint "Lamé operators" in the literature [KuU1, Theorem 5.1] of quite different nature. Note that our Lamé operator A is not selfadjoint.*

We also obtain Hörmander $\mathcal{H}^\alpha_2$ functional calculus of the Dirichlet-to-Neumann operator which satisfies Poisson type estimates according to [OtE]. Also pseudodifferential operators of order 1, acting on compact closed Riemannian manifolds with non-negative spectral bound

satisfy Poisson estimates [GiGr, Theorem 3.14] and can be covered by Corollary 5.4. Note that in these two examples, we obtain a non-optimal differentiation parameter α (see [OtE] and [SeSo] for better parameters), however our method includes square function estimates like

$$\left\| \left(\sum_{j=1}^{n} |g_j(A) f_j|^2 \right)^{\frac{1}{2}} \right\|_p \lesssim \max_{j=1}^{n} \|g_j\|_{\mathcal{H}_2^\alpha} \left\| \left(\sum_{j=1}^{n} |f_j|^2 \right)^{\frac{1}{2}} \right\|_p . \tag{5.7}$$

We refer to [HDR1, End of Section 4].

5.3 Spectral multipliers for the Dunkl operator

In the second situation, the 0-sectorial operator will be a so-called Dunkl Laplacian Δ_κ acting on $L^p(\mathbb{R}^d, h_\kappa^2, Y)$, where h_κ^2 is a weight defined below and Y is a UMD lattice. We recall some basic concepts of Dunkl operators which will be needed in this subsection. For more details on Dunkl's analysis, the reader may especially consult [DuXu, Ros] and the references therein. We can all the same point out that the theory was originally developed by Dunkl to bypass the classical approach to the construction of orthogonal polynomials as spherical functions on homogeneous spaces. The introduction of the differential-difference operators (and their related objects) now called Dunkl operators turned out to be a powerful tool in harmonic analysis associated with Coxeter groups, in the theory of multivariable special functions, in the theory of stochastic processes with values in Weyl chambers or in the theory of integrable quantum many body systems of Calogero-Moser-Sutherland type, for instance.

Let $d \in \mathbb{N} \setminus \{0\}$. Let $W \subseteq O(\mathbb{R}^d)$ be a finite reflection group associated with a reduced root system $\mathcal{R}$ (not necessarily crystallographic) and let $\kappa : \mathcal{R} \to [0, \infty)$ be a multiplicity function, that is, a W-invariant function. The (rational) Dunkl operators D_ξ^κ on $\mathbb{R}^d$, introduced in [Dun], are the following κ-deformations of directional derivatives ∂_ξ by reflections

$$D_\xi^\kappa f(x) = \partial_\xi f(x) + \sum_{\alpha \in \mathcal{R}_+} \kappa(\alpha) \frac{f(x) - f(\sigma_\alpha(x))}{\langle x, \alpha \rangle} \langle \xi, \alpha \rangle, \quad x \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean inner product, σ_α denotes the reflection with respect to the hyperplane orthogonal to α and $\mathcal{R}_+$ denotes a positive subsystem of $\mathcal{R}$. The definition is of course independent of the choice of the positive subsystem since κ is W-invariant.

The Dunkl Laplacian is then $\Delta_\kappa f = \sum_{i=1}^d (D_{e_i}^\kappa)^2 f$, where $(e_i)_{1 \leqslant i \leqslant d}$ is the canonical basis of $\mathbb{R}^d$, and can be written explicitly as follows (see [Dun])

$$\Delta_\kappa f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathcal{R}_+} \kappa(\alpha) \left(\frac{\partial_\alpha f(x)}{\langle \alpha, x \rangle} - \frac{\|\alpha\|^2}{2} \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right).$$

It generates a semigroup $(\mathcal{H}_t^\kappa)_{t \geqslant 0}$ on $L^p(\mathbb{R}^d, h_\kappa^2)$, $1 \leqslant p < \infty$, which is a markovian semigroup in the sense of Definition 4.2 [Del, Theorem 2.6], where the weight h_κ^2 defined on $\mathbb{R}^d$ by

$$h_\kappa^2(x) = \prod_{\alpha \in \mathcal{R}_+} |\langle x, \alpha \rangle|^{2\kappa(\alpha)}$$

is invariant under the action of W and homogeneous of degree $2\gamma_\kappa$, with

$$\gamma_\kappa = \sum_{\alpha \in \mathcal{R}_+} \kappa(\alpha).$$

As regards the harmonic analysis of Dunkl operators and their related objects, the subjacent analytic structure has a rich analogy with the Fourier analysis. However, there are still many problems to be solved and the theory is still at its infancy. One of the main obstructions is the lack of an explicit formula for the operator V_κ which intertwines the commutative algebra of Dunkl operators D_ξ^κ with the algebra of standard differential operators ∂_ξ with constant coefficients. Apart from the case $W = \mathbb{Z}_2^d$ where the known formula for V_κ allows to tackle and bypass some difficulties, many tools of harmonic analysis are not accessible. However, we do not restrict ourselves to this particular reflection group. We recall the following result.

Theorem 5.9 (Dai-Wang) *[DaWa, Theorem 4.1] Let $1 < p < \infty$ and $\alpha \in \mathbb{N}$ such that $\alpha > \frac{d}{2} + \gamma_\kappa + \frac{1}{2}$. Then the Dunkl operator $-\Delta_\kappa$ has a bounded $\mathcal{H}_1^\alpha$ calculus on $L^p(\mathbb{R}^d, h_\kappa^2)$.*

Then the main result in this subsection is the following extension.

Theorem 5.10 *[HDR2, Theorem 3.13]. Let $1 < p < \infty$ and $Y = Y(\Omega')$ be a UMD lattice. Let $\alpha \in \mathbb{N}$ be an integer such that $\alpha > \frac{d}{2} + \gamma_\kappa + \frac{1}{2}$. Then the Dunkl operator $-\Delta_\kappa$ has a bounded $\mathcal{H}_1^\alpha$ calculus on $L^p(\mathbb{R}^d, h_\kappa^2, Y)$.*

Remark 5.11

1. *By injectivity of $-\Delta_\kappa$, here there is no need of the term $|f(0)|$ at the right hand side of the functional calculus estimate $\|f(-\Delta_\kappa)\|_{B(L^p(\mathbb{R}^d, h_\kappa^2, Y))} \leqslant C \|f\|_{\mathcal{H}_1^\alpha}$.*
2. *Note that the class $\mathcal{H}_1^\alpha$ is strictly larger than the $\mathcal{H}_2^\alpha$ class.*
3. *Already in the simplest case of $\kappa = 0$ (so that $\gamma_\kappa = 0$), a part from the restriction to integer values, the order of α is optimal in Theorem 5.10 when relying on the $\mathcal{H}_1^\alpha$ class. Indeed, this can be seen from the fact that m_δ belongs to $\mathcal{H}_1^\alpha$ on the one hand for the value $\delta > \alpha - 1$, and the failure of boundedness of the Bochner-Riesz multiplier $m_\delta(-\Delta) = (1+\Delta)_+^\delta$ on the other hand for the correct choice of δ. See [HDR2, Remark 3.14].*
4. *One of the features of the vector valued character of this theorem is that an operator of the form $\mathcal{A} = \mathrm{Id}_{L^p} \otimes \mathcal{A}_0$ will commute with Δ_κ (or powers of it) and therefore, spectral theory of a sum $(-\Delta_\kappa)^\beta + \mathcal{A}$ is at hand. Consequently, we apply Theorem 5.10 to existence, uniqueness and (maximal) regularity of solutions of Cauchy problems or time independent problems involving $-\Delta_\kappa + \mathcal{A}$, see Subsection 5.4.*
5. *The heat Dunkl semigroup $(\mathcal{H}_t^\kappa)_{t \geqslant 0}$ does not satisfy Gaussian estimates in general [ABSDH, Remark 2.4], so that Theorem 5.10 is not a consequence of the results in Section 7.*

We divide the proof of Theorem 5.10 into several steps.

Proof of Theorem 5.10: 1st step: Reduction of multipliers on $L^p(\mathbb{R}^d, h_\kappa^2, Y)$ to spherical multipliers on $L^p(S^d, h_{\kappa'}^2, Y)$ This step follows the transference principle from zonal multipliers on S^d to radial multipliers on $\mathbb{R}^d$ developed by Bonami-Clerc [BoCl] and adapted recently to the context of Dunkl operators by Dai-Wang [DaWa]. We denote $S^d \subseteq \mathbb{R}^{d+1}$ the d-dimensional sphere. We extend our finite reflection group $W \subseteq O(d)$ to $W' \subseteq O(d+1)$ in one dimension higher, by letting for $g \in W$, $g'(x, x_{d+1}) = (gx, x_{d+1})$ and $W' = \{g' : g \in W\}$. It is associated with the reduced root system $R' = \{(\alpha, 0) : \alpha \in R\}$. We also let $\kappa' : R' \to \mathbb{R}_+$, $(\alpha, 0) \mapsto \kappa(\alpha)$ and associate with it the weight $k_{\kappa'}^2$. Then we consider the Bochner space $L^p(S^d, h_{\kappa'}^2, Y)$ where the measure is $h_{\kappa'}(x)$ times surface measure on S^d. If P is a homogeneous polynomial of degree n then P is called h-harmonic provided that $\Delta_{\kappa'} P = 0$. Then we denote $\mathrm{proj}_n^{\kappa'} : L^2(S^{d-1}, h_{\kappa'}^2) \to L^2(S^{d-1}, h_{\kappa'}^2)$ the orthogonal projection onto the L^2-closed span of such P.

In the following, we consider spectral multipliers with respect to the sequence of operators $\mathrm{proj}_n^{\kappa'}$, $n \geqslant 0$. These are operators of the form $M_\mu = \sum_{n=0}^\infty \mu_n \,\mathrm{proj}_n^{\kappa'}$ where $(\mu_n)_{n\geqslant 0}$ is a scalar sequence. First we have the following reduction result.

Proposition 5.12 *[HDR2, Theorem 3.12]. Let Y be a Banach space. Let $m\colon (0,\infty) \to \mathbb{R}$ be a continuous and bounded function. For $\varepsilon > 0$ and $n \geqslant 0$, let $\mu_n^\varepsilon = m(\varepsilon n)$. Let $M_\varepsilon(f) = \sum_{n=0}^\infty m(\varepsilon n)\,\mathrm{proj}_n^{\kappa'}(f)$. Assume that for some $1 < p < \infty$ and any $f \in L^p(S^d, h^2_{\kappa'}, Y)$,*

$$\sup_{\varepsilon>0} \|M_\varepsilon(f)\|_{L^p(S^d,h^2_{\kappa'},Y)} \leqslant C\,\|f\|_{L^p(S^d,h^2_{\kappa'},Y)}\,.$$

Then $\tilde m(-\Delta_\kappa)$ is a bounded radial Dunkl spectral multiplier on $L^p(\mathbb{R}^d, h^2_\kappa, Y)$, where $\tilde m(t) = m(\sqrt{t})$.

It is shown in [HDR2, Theorem 3.13] that an $\mathcal{H}_1^\alpha$ spectral multiplier m yields an appropriate multiplier M_ε for Proposition 5.12, that is the estimates (5.9) and (5.10) below will hold for such m. We are thus reduced to find sufficient criteria to have bounded spectral multipliers on the sphere as in Proposition 5.12, which will be developed in the subsequent steps.

2nd step: Spherical multipliers and Littlewood-Paley g-functions We define particular spherical multipliers by letting for $\delta > 0$ and $n \in \mathbb{N}_0$,

$$S_n^\delta(f) = \sum_{j=0}^\infty a_j^{\delta,n}\,\mathrm{proj}_j^\kappa(f) \tag{5.8}$$

where $a_j^{\delta,n} = \frac{1}{A_n^\delta} A_{n-j}^\delta \chi_{0\leqslant j\leqslant n}$ and $A_l^\delta = \frac{\Gamma(l+\delta+1)}{\Gamma(l+1)\Gamma(\delta+1)}$. The operator S_n^δ is a "smoothed indicator function" spectral multiplier and called Cesàro mean (of order δ).

Then we define the functional $g_\delta(f)$ for given $f \in L^p(S^{d-1}, h^2_{\kappa'}, Y)$ by

$$g_\delta(f) = \left(\sum_{n=1}^\infty \left|S_n^{\delta+1}f - S_n^\delta f\right|^2 \frac{1}{n}\right)^{\frac12},$$

and its modification, subjet to a sequence $(\nu_k)_{k\geqslant 1}$ of nonnegative numbers such that $\sup_{n\geqslant 1}\frac1n\sum_{k=1}^n \nu_k = M < \infty$,

$$g_\delta^*(f) = \left(\sum_{n=1}^\infty \left|S_n^{\delta+1}f - S_n^\delta f\right|^2 \frac{\nu_n}{n}\right)^{\frac12},$$

These functionals are in the spirit of Littlewood-Paley g-functions, where derivatives of a c_0-semigroup are replaced by differences of our Cesàro means. Then we have the following proposition.

Proposition 5.13 *Let δ be the smallest integer strictly larger than $\lambda_{\kappa'} = \frac{d}{2} + \gamma_{\kappa'} - 1$. Denote Δ the difference operator acting on sequences $\Delta(x_j)_j = (x_j - x_{j+1})_j$, and iteratively $\Delta^{n+1} = \Delta\circ\Delta^n$. Let $(\mu_j)_{j\geqslant 0}$ be a sequence satisfying the conditions*

$$\sup_{j\geqslant 0} |\mu_j| \leqslant M < \infty, \tag{5.9}$$

$$\sup_{j\geqslant 0} 2^{j\delta} \sum_{l=2^j}^{2^{j+1}} |\Delta^{\delta+1}\mu_j| \leqslant M < \infty. \tag{5.10}$$

Then we have for $M_\mu = \sum_{n=0}^{\infty} \mu_n \operatorname{proj}_n^{\kappa'}$,

$$\|g_\delta(M_\mu f)\|_{L^p(S^d,h^2_{\kappa'},Y)} \leqslant C \|g^*_\delta(f)\|_{L^p(S^d,h^2_{\kappa'},Y)},$$

where the sequence $(\nu_k)_{k\geqslant 1}$ *is* $\nu_k = 1 + \sum_{j=1}^{\delta+1} |\Delta^j \mu_k| k^j$ *which satisfies* $\sup_{n\geqslant 1} \frac{1}{n} \sum_{j=1}^{n} \nu_j \leqslant cM$.

Proof : See [HDR2, Lemma 3.11], see also [DaXu, (4.4.2), page 47]. ■

In view of Proposition 5.13, we need to find conditions ensuring that the g-functions behave as often: being equivalent in L^p norm to the original element f itself. This is the content of the next step.

3rd step: Equivalence of Littlewood-Paley g-functions and R-boundedness of the Cesàro means

Proposition 5.14 *Let* $1 < p < \infty$, $\delta \geqslant 0$ *and* $Y = Y(\Omega')$ *be a UMD lattice. Then for* $f \in L^p(S^d, h^2_{\kappa'}, Y)$ *such that* $\int_{S^d} f(y) h^2_{\kappa'}(y) dy = 0$, *we have*

$$\|f\|_{L^p(S^d,h^2_{\kappa'},Y)} \leqslant c_{p,\delta} \|g_\delta(f)\|_{L^p(S^d,k^2_{\kappa'},Y)}.$$

Moreover, if the Cesàro means are R*-bounded on* $L^p(S^{d-1}, h^2_{\kappa'}, Y)$, *then*

$$\|g^*_\delta(f)\|_{L^p(S^d,h^2_{\kappa'},Y)} \leqslant c_{p,\delta} M \|f\|_{L^p(S^d,h^2_{\kappa'},Y)},$$

where $M = \sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} \nu_k$.

Proof : See [HDR2, Proposition 3.9]. ■

Proposition 5.15 *Let* $1 < p < \infty$, $Y = Y(\Omega')$ *be a UMD lattice. Assume that* $\delta > \frac{d}{2} + \gamma_{\kappa'} - 1$. *Then the Cesàro means* $(S^\delta_n)_{n\geqslant 0}$ *are* R*-bounded on* $L^p(S^d, h^2_{\kappa'}, Y)$.

Proof : See [HDR2, Lemma 3.5] for the proof which is based on the Hopf-Dunford-Schwartz maximal operator associated with the generalised heat semigroup $H_s^{\kappa'} = \sum_{n=0}^{\infty} e^{-n(n+2\lambda_{\kappa'})s} \operatorname{proj}_n^{\kappa'}$ on $(S^d, h^2_{\kappa'}(y)dy)$. ■

Combining the three steps proves the Hörmander functional calculus for the Dunkl operator $-\Delta_\kappa$ from Theorem 5.10.

5.4 Maximal regularity for problems related to the Dunkl operator

We present some application of Theorem 5.10. First note that it entails that Bochner-Riesz means $(1+\Delta_\kappa/R)^\beta_+$ are uniformly bounded in $R > 0$ for $\beta > \frac{d}{2} + \gamma_\kappa$, on the space $L^p(\mathbb{R}^d, h^2_\kappa, Y)$. This is a simple computation of the $\mathcal{H}^\alpha_1$ norm of the spectral multiplier function $(1 - t/R)^\beta_+$.

Next we note that the Hörmander calculus also entails a Paley-Littlewood spectral decomposition. Let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity as in Definition 4.10. Then for any $f \in L^p(\mathbb{R}^d, h^2_\kappa, Y)$, according to Lemma 4.12 we have the norm description

$$\|f\|_{L^p(\mathbb{R}^d,h^2_\kappa,Y)} \cong \left\| \left(\sum_{n\in\mathbb{Z}} \left|\phi_n(-\Delta_\kappa) f\right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^d,h^2_\kappa,Y)}. \tag{5.11}$$

Note hereby that since Δ_κ is injective and $L^p(\mathbb{R}^d, h^2_\kappa, Y)$ is reflexive, $\overline{\operatorname{Ran}(\Delta_\kappa)} = L^p(\mathbb{R}^d, h^2_\kappa, Y)$ is the whole space.

Finally, as an illustration, we obtain maximal regularity (see Corollary 5.16 for this notion) for a partial differential equation involving a first order time derivative (acting on the variable t), the Dunkl Laplacian (acting on s) and a second order differential operator (acting on a spatial variable $x \in \mathbb{R}_+^{n+1}$). For simplicity, we restrict to the one dimensional case $d = 1$ (that is $W = \mathbb{Z}_2$) and $\beta = 1$ for the Dunkl Laplacian. In this case, Δ_κ takes the form

$$\Delta_\kappa u = \left(\frac{d}{ds} + \kappa \frac{\mathrm{Id} - \sigma}{s}\right)^2 u, \quad u \in L^p(\mathbb{R}, |s|^{2\kappa} ds), \tag{5.12}$$

where $\kappa \geqslant 0$ is the only value (still denoted by κ) taken by the multiplicity function (since there is only one class of conjugation), $\sigma(u)(x) = u(-x)$, and $(\mathbb{R}^d, h_\kappa^2(s)ds) = (\mathbb{R}, |s|^{2\kappa} ds)$. Let us introduce $\mathcal{A}$ being the operator A_B from [KW1, page 156]. That is, we take $Y = L^q(\mathbb{R}_+^{n+1})$, $1 < q < \infty$, where $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times [0, \infty)$. Further, we define the homogeneous second order differential operator acting on $\mathbb{R}_+^{n+1}$,

$$\mathcal{A} = \sum_{|\alpha|=2} a_\alpha D^\alpha,$$

subject to the homogeneous first order boundary operator

$$\mathcal{B} = \sum_{|\beta|=1} b_\beta D^\beta,$$

where a_α, $b_\beta \in \mathbb{C}$ and for $\alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{N}_0^{n+1}$, we let

$$D^\alpha = (-i)^{\alpha_1 + \ldots + \alpha_{n+1}} \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{n+1}}}{\partial x_{n+1}^{\alpha_{n+1}}}.$$

Let $\omega_0 \in [0, \pi)$ and assume that

$$\mathcal{A}(\xi) := \sum_{|\alpha|=2} a_\alpha (-i)^{\alpha_1 + \ldots + \alpha_{n+1}} \xi_1^{\alpha_1} \cdot \ldots \cdot \xi_{n+1}^{\alpha_{n+1}} \in \overline{\Sigma_{\omega_0}} \quad (\xi \in \mathbb{R}^{n+1}),$$

and $\mathcal{A}(\xi) \geqslant \frac{1}{M}\|\xi\|^2$ for some $M > 0$. For the operator $\mathcal{B}$, we assume the Lopatinskij-Shapiro condition [KW1, (7.4)]. That is, we set $b_0 = b_{(0,0,\ldots,0,1)}$, $a_k(\xi') = \sum_{|\beta|=k} a_{(\beta,2-k)}(\xi')^\beta$ for $k = 0, 1, 2$ and $\xi' \in \mathbb{R}^n$. We assume $b_0 \neq 0$ for simplicity, and that the characteristic polynomial

$$a_0 \mu^2 + a_1(\xi')\mu + a_2(\xi') + \lambda = 0$$

has two distinct roots $\mu_\pm$ with $\operatorname{Im} \mu_+ > 0 > \operatorname{Im} \mu_-$, for any $\xi' \in \mathbb{R}^n$ and $\lambda \in \Sigma_{\omega_0}$. Now we obtain the following existence, unicity and regularity results on partial differential equations involving Δ_κ and the differential operator $\mathcal{A}$.

Corollary 5.16 *Let $1 < p, q, r < \infty$ and $0 < T \leqslant \infty$. Let Δ_κ acting on $L^p(\mathbb{R}, |s|^{2\kappa} ds)$ and $\mathcal{A}$ acting on $L^q(\mathbb{R}_+^{n+1})$ be as above.*

1. *Assume that $\omega_0 < \frac{\pi}{2}$. Then the Cauchy problem*

$$\begin{cases} \frac{d}{dt}u(t,s,x) - \left(\frac{d}{ds} + \kappa\frac{\mathrm{Id}-\sigma}{s}\right)^2 u(t,s,x) + \sum_{|\alpha|=2} a_\alpha D^\alpha u(t,s,x) & = f(t,s,x) \quad \left(t > 0,\ s \in \mathbb{R},\ x \in \mathbb{R}_+^{n+1}\right) \\ u(0,s,x) & = 0 \quad \left(s \in \mathbb{R},\ x \in \mathbb{R}_+^{n+1}\right) \\ \sum_{|\beta|=1} b_\beta D^\beta u(t,s,x) & = 0 \quad \left(t > 0,\ s \in \mathbb{R},\ x \in \partial\mathbb{R}_+^{n+1}\right) \end{cases}$$

for given $f \in L^r((0,T), L^p(\mathbb{R}, |s|^{2\kappa}ds, L^q(\mathbb{R}^{n+1}_+)))$ *has a unique solution*

$$u \in L^r_{loc}((0,T), L^p(\mathbb{R}, |s|^{2\kappa}ds, L^q(\mathbb{R}^{n+1}_+)))$$

which is almost everywhere differentiable in t*, and there exists a constant* $C < \infty$ *such that*

$$\begin{aligned}&\left\|\frac{d}{dt}u\right\| + \left\|\left(\frac{d}{ds} + \kappa\frac{\mathrm{Id}-\sigma}{s}\right)^2 u\right\| + \left\|\sum_{|\alpha|=2} a_\alpha D^\alpha u\right\| \\ &\leqslant C\|f\|,\end{aligned}$$

where the four norms here are all in $L^r((0,T), L^p(\mathbb{R}, |s|^{2\kappa}ds, L^q(\mathbb{R}^{n+1}_+)))$.

2. *Assume that* $\omega_0 < \pi$*, and that* $a_0 > 0$ *above. Then the problem*

$$\begin{cases} -\left(\frac{d}{ds} + \kappa\frac{\mathrm{Id}-\sigma}{s}\right)^2 u(s,x) + \sum_{|\alpha|=2} a_\alpha D^\alpha u(s,x) & = f(s,x) \quad (s \in \mathbb{R},\ x \in \mathbb{R}^{n+1}_+) \\ \sum_{|\beta|=1} b_\beta D^\beta u(s,x) & = 0 \quad (s \in \mathbb{R},\ x \in \partial\mathbb{R}^{n+1}_+) \end{cases}$$

for given $f \in L^p(\mathbb{R}, |s|^{2\kappa}ds, L^q(\mathbb{R}^{n+1}_+))$ *has a unique solution* $u \in L^p(\mathbb{R}, |s|^{2\kappa}ds, L^q(\mathbb{R}^{n+1}_+))$ *and there exists a constant* $C < \infty$ *such that*

$$\|u\| + \left\|\left(\frac{d}{ds} + \kappa\frac{\mathrm{Id}-\sigma}{s}\right)^2 u\right\| + \left\|\sum_{|\alpha|=2} a_\alpha D^\alpha u\right\| \leqslant C\|f\|,$$

where the four norms here are all in $L^p(\mathbb{R}, |s|^{2\kappa}ds, L^q(\mathbb{R}^{n+1}_+))$.

Proof : We refer to [HDR2, Proposition 4.4]. ∎

6 Maximal operators on UMD lattices

This section presents two results on the Hardy-Littlewood maximal acting on UMD lattice valued L^p spaces. It is based on [HDR4] and [HDR5, Section 3].

6.1 Dimension free bounds for the Fefferman-Stein inequalities

At the beginning of the 1980s, Elias Stein proved in [Ste82] (the complete detailed proof is in the paper of Stein-Strömberg [StSt]) that the standard Hardy-Littlewood maximal operator, that is associated with Euclidean balls, satisfies $L^p(\mathbb{R}^d)$ estimates with constant independent of the dimension d for every $p > 1$. More precisely, if we denote by M_{HL} the Hardy-Littlewood maximal operator, initially defined for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ by

$$M_{HL}f(x) = \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|dy, \quad x \in \mathbb{R}^d,$$

with $B(x,r)$ the Euclidean ball centered at x of radius $r > 0$ and $|X|$ the Lebesgue measure of a Borel subset X of $\mathbb{R}^d$, then Stein's result reads as follows.

Theorem 6.1 (Stein) *Let* $1 < p \leqslant \infty$*. If* $f \in L^p(\mathbb{R}^d)$*, then we have*

$$\|M_{HL}f\|_{L^p(\mathbb{R}^d)} \leqslant C(p)\|f\|_{L^p(\mathbb{R}^d)}, \tag{6.1}$$

where $C(p)$ *is a constant independent of* d*.*

The importance of such dimension free results has been pointed out by Stein in [Ste83]. This result, which improves in a spectacular fashion the behavior previously known, has opened the way to the following programme: is it possible to bound uniformly in dimension the constant appearing in Hardy-Littlewood type estimates for maximal operators associated with symmetric convex bodies? This topic has been studied by various authors during the period 1986-1990 (see the papers of Bourgain [Bou86, Bou86ter, Bou85], Carbery [Car] and Müller [Mul]), and has been recently renewed by further advances, especially due to Bourgain, Mirek, Stein and Wróbel [Bou14, BMSW1, BMSW2]. For a thorough exposition of this subject, we refer the reader to the recent survey [DGM]. In fact, Stein's result has opened the way, beyond the case of maximal functions, of proving fundamental estimates in harmonic analysis in $\mathbb{R}^d$ with formulations with bounds independent of the dimension. We note that the optimal constants $C(p)$ in (6.1) are unknown today; for the particular case of dimension 1 and the corresponding weak $(1,1)$ estimate, Melas [Mel] has found the optimal bound $\frac{11+\sqrt{61}}{12}$. This seems to be the only explicitly known optimal bound for the Hardy-Littlewood operator on its diverse spaces.

The first main result of this section will be the dimensionless behavior of the constant in the vector-valued extensions of the Hardy-Littlewood maximal theorem, the so-called Fefferman-Stein inequalities [FeSt]. Let us first recall these inequalities.

Theorem 6.2 (Fefferman-Stein) *Let $1 < p, q < \infty$ and let $(f_n)_{n\geqslant 1}$ be a sequence of measurable functions defined on $\mathbb{R}^d$. If $\left(\sum_{n=1}^{\infty} |f_n(\cdot)|^q\right)^{\frac{1}{q}} \in L^p(\mathbb{R}^d)$, then we have*

$$\left\| \left(\sum_{n=1}^{\infty} |M_{HL} f_n(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)} \leqslant C(d,p,q) \left\| \left(\sum_{n=1}^{\infty} |f_n(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)},$$

where $C(d,p,q)$ is a constant independent of $(f_n)_{n\geqslant 1}$.

The proof given by Fefferman and Stein for their inequalities, mainly based on the Calderón-Zygmund decomposition (for a weak-type result), the Marcinkiewicz interpolation theorem and a suitable weighted inequality, leads to a constant which grows exponentially with d. Another approach, based on Banach-space valued singular integrals [GCRdF] (see also [Gr]), does not achieve this dimensionless goal either. We succeeded in proving the following dimensionless result.

Theorem 6.3 *[HDR4, Theorem 1.1] Let $1 < p, q < \infty$ and let $(f_n)_{n\geqslant 1}$ be a sequence of measurable functions defined on $\mathbb{R}^d$. If $\left(\sum_{n=1}^{\infty} |f_n(\cdot)|^q\right)^{\frac{1}{q}} \in L^p(\mathbb{R}^d)$, then we have*

$$\left\| \left(\sum_{n=1}^{\infty} |M_{HL} f_n(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)} \leqslant C(p,q) \left\| \left(\sum_{n=1}^{\infty} |f_n(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)},$$

where $C(p,q)$ is a constant independent of d and $(f_n)_{n\geqslant 1}$.

As in the proof of the dimensionless result by Stein for the Hardy-Littlewood maximal operator, the main tool in our proof will be the following spherical maximal operator $\mathcal{M}_S$, initially defined for $f \in \mathcal{S}(\mathbb{R}^d)$ (Schwartz class) by

$$\mathcal{M}_S f(x) = \sup_{r>0} \left| \int_{S^{d-1}} f(x - ry) d\sigma(y) \right|, \quad x \in \mathbb{R}^d,$$

where $d\sigma$ denotes the normalised Haar measure on S^{d-1}, and for which we will prove in particular the following vector-valued estimates.

Theorem 6.4 *[HDR4, Theorem 1.3] Let $d \geqslant 3$ and let $\frac{d}{d-1} < p, q < d$. Let $(f_n)_{n\geqslant 1}$ be a sequence of measurable functions defined on $\mathbb{R}^d$. If $\left(\sum_{n=1}^{\infty} |f_n(\cdot)|^q\right)^{\frac{1}{q}} \in L^p(\mathbb{R}^d)$, then we have*

$$\left\| \left(\sum_{n=1}^{\infty} |\mathcal{M}_S f_n(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)} \leqslant C(d,p,q) \left\| \left(\sum_{n=1}^{\infty} |f_n(\cdot)|^q \right)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^d)},$$

where $C(d,p,q)$ is a constant independent of $(f_n)_{n\geqslant 1}$.

We point out that vector-valued estimates for $\mathcal{M}_S$ have been recently proved by Manna in [Man], for the range $\frac{2d}{d-1} < p, q < \infty$, by use of a convenient weighted inequality for $\mathcal{M}_S$. We believe that the range $\frac{d}{d-1} < p, q < \infty$ is optimal for $d \geqslant 3$, and, also in the case $d = 2$, this might be true, as in the scalar case, see [Bou86bis]. We shall now give some parts of the proof of Theorems 6.4 and 6.3, and then illustrate by further extensions of these results to related contexts. At the end, we will give some open questions that arise from our results.

6.2 Proof of Theorem 6.4

We present a sketch of the proof of the spherical maximal theorem. We do not follow Stein's ideas for the scalar case, but rather those of Rubio de Francia in [RdF2]. Let

$$A_t f(x) = \int_{S^{d-1}} f(x - ty) d\sigma(y), \tag{6.2}$$

so that $\mathcal{M}_S f = \sup_{t>0} |A_t f|$.

1st step: Decomposition of $\mathcal{M}_S$ into dyadic frequency pieces First note that the operator A_t is translation invariant, so can be written under the form of a Fourier multiplier. Since A_t is moreover rotation invariant, the Fourier multiplier function is radial. In fact, one can give this function explicitly. One obtains $A_t f(x) = \left(\hat{f}(\cdot) m(t\cdot)\right)^{\vee}(x)$, where

$$m(x) = \widehat{d\sigma}(x) = \frac{2\pi}{|x|^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(2\pi|x|),$$

and J_α denotes the Bessel function of order α. We decompose m into radial pieces with localised frequencies. To this end, we let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity as in Definition 4.10 and group together $\varphi_0(x) = \sum_{n\leqslant 0} \phi_n(|x|)$, as well as $\varphi_l(x) = \phi_l(|x|)$ $(l \geqslant 1)$. So we have $\operatorname{supp} \varphi_0 \subseteq B(0,2)$ and $\operatorname{supp} \varphi_l \subseteq B(0,2^{l+1}) \backslash B(0,2^{l-1})$ for $l \geqslant 1$. We decompose then $m_l = \varphi_l m$, $l \geqslant 0$, into dyadic radial pieces. Since $\sum_{l=0}^{\infty} \varphi_l = 1$, we have that $m = \sum_{l=0}^{\infty} m_l$. We obtain the following pointwise inequality:

$$\mathcal{M}_S f(x) \leqslant \sum_{l=0}^{\infty} M_{m_l} f(x) \quad (x \in \mathbb{R}^d), \tag{6.3}$$

with $M_{m_l} f(x) = \sup_{t>0} \left| \left(\hat{f}(\cdot) m_l(t\cdot)\right)^{\vee}(x) \right|$. In the further steps, we shall estimate the pieces M_{m_l} for fixed $l \geqslant 0$.

2nd step: decaying estimate on $L^2(\mathbb{R}^d,\ell^2)$ We let $l\geqslant 1$ and claim that

$$\Big\|\Big(\sum_{n=1}^{\infty}|M_{m_l}f_n(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}\leqslant\frac{C(d)}{2^{\frac{l(d-2)}{2}}}\Big\|\Big(\sum_{n=1}^{\infty}|f_n(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}, \tag{6.4}$$

where $C(d)$ is a constant independent of l and $(f_n)_{n\geqslant 1}$. Indeed, using $F^2(r)=\int_0^r\frac{d}{dt}(F^2(t))dt=2\int_0^r F(t)F'(t)dt$ for the function $F(t)=\Big(\hat{f}_n(\cdot)m_l(t\cdot)\Big)^{\vee}(x)$ which vanishes at $t=0$, we obtain

$$\big|\big(\hat{f}_n m_l(r\cdot)\big)^{\vee}(x)\big|^2\leqslant 2\int_0^{\infty}\big|\big(\hat{f}_n(m_l(t\cdot))^{\vee}(x)\big|\big|\big(\hat{f}_n\tilde{m}_l(t\cdot)\big)^{\vee}(x)\big|\frac{dt}{t},$$

where we have set $\tilde{m}_l(x)=\langle x,\nabla m_l(x)\rangle$. Taking the supremum over all $r>0$ yields

$$(M_{m_l}f_n(x))^2\leqslant 2\int_0^{\infty}\big|\big(\hat{f}_n(m_l(t\cdot))^{\vee}(x)\big|\big|\big(\hat{f}_n\tilde{m}_l(t\cdot)\big)^{\vee}(x)\big|\frac{dt}{t}.$$

Now sum over n and integrate over $x\in\mathbb{R}^d$. Apply then Cauchy-Schwarz inequality in the Hilbert space $L^2(\mathbb{R}^d,\ell^2(L^2(\mathbb{R}_+,\frac{dt}{t})))$ to bound

$$\Big\|\Big(\sum_{n=1}^{\infty}|M_{m_l}f_n(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|^2_{L^2(\mathbb{R}^d)}\leqslant 2\Big\|\Big(\sum_{n=1}^{\infty}|g_{m_l}(f_n)(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}\Big\|\Big(\sum_{n=1}^{\infty}|g_{\tilde{m}_l}(f_n)(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}.$$

Here, we denote for a function $\omega\in\mathcal{S}(\mathbb{R}^d)$,

$$g_{\omega}(f)(x)=\left(\int_0^{\infty}\left|\Big(\hat{f}\omega(t\cdot)\Big)^{\vee}(x)\right|^2\frac{dt}{t}\right)^{\frac{1}{2}}.$$

Then one checks with the dyadic support of m_l and of $\tilde{m}_l$ together with Plancherel that

$$2\Big\|\Big(\sum_{n=1}^{\infty}|g_{m_l}(f_n)(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}\Big\|\Big(\sum_{n=1}^{\infty}|g_{\tilde{m}_l}(f_n)(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}.\leqslant 2\left(\frac{C_1(d)\sqrt{\ln 4}}{2^{\frac{l(d-1)}{2}}}\Big\|\Big(\sum_{n=1}^{\infty}|f_n(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}\right)\left(\frac{C_2(d)\sqrt{\ln 4}}{2^{\frac{l(d-3)}{2}}}\Big\|\Big(\sum_{n=1}^{\infty}|f_n(\cdot)|^2\Big)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^d)}\right).$$

We deduce the decay (6.4).

3rd step: estimate on $L^p(\mathbb{R}^d,\ell^q)$ First we claim that for any $l\geqslant 1$, and $1<p,q<\infty$, we have the (growing and non-summable in l) estimate

$$\Big\|\Big(\sum_{n=1}^{\infty}|M_{m_l}f_n(\cdot)|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^d)}\leqslant C(d,p,q)\,2^l\Big\|\Big(\sum_{n=1}^{\infty}|f_n(\cdot)|^q\Big)^{\frac{1}{q}}\Big\|_{L^p(\mathbb{R}^d)}.$$

Indeed, one can prove with the Funk-Hecke formula that $|m_l^{\vee}(x)|\leqslant C(d)\frac{2^l}{(1+|x|)^{d+1}}$. Then we deduce thanks to [Gr, Corollary 2.1.12., page 84], that

$$\sup_{r>0}\big|(f_n*(m_l^{\vee})_r)(x)\big|\leqslant\tilde{C}(d)2^l M_{HL}f_n(x),$$

and the standard Fefferman-Stein inequalities for the Hardy-Littlewood maximal operator M_{HL} allow us to conclude.

Next we put the above estimate together with the $L^2(\mathbb{R}^d, \ell^2)$ estimate form the 2nd step, by means of complex interpolation. Namely, we have

$$\Big\| \Big(\sum_{n=1}^{\infty} |M_{m_l} f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)} = \Big\| A_{m_l}\big((f_n(\cdot))_{n\geqslant 1}\big) \Big\|_{L^p(\mathbb{R}^d, \ell^q(L^\infty(\mathbb{R}_+)))},$$

where we have set

$$A_{m_l} : \begin{cases} L^p(\mathbb{R}^d, \ell^q) \to L^p(\mathbb{R}^d, \ell^q(L^\infty(\,]0, +\infty[\,))) \\ (f_n(\cdot))_{n\geqslant 1} \mapsto \big(r \mapsto \big(\hat{f}_n m_l(r\cdot) \big)^\vee \big)_{n\geqslant 1}. \end{cases}$$

The 2nd step yields that

$$\|A_{m_l}\|_{L^2(\mathbb{R}^d,\ell^2)\to L^2(\mathbb{R}^d,\ell^2(L^\infty(\mathbb{R}_+)))} \leqslant C(d) 2^{-\frac{l(d-2)}{2}},$$

whereas the above esimate gives

$$\|A_{m_l}\|_{L^{p_0}(\mathbb{R}^d,\ell^{q_0})\to L^{p_0}(\mathbb{R}^d,\ell^{q_0}(L^\infty(\mathbb{R}_+)))} \leqslant C(d, p_0, q_0) 2^l$$

for any $1 < p_0, q_0 < +\infty$. Complex interpolation between these two estimates (think of p_0, q_0 close to either 1 or ∞ and p (resp. q) sitting in between p_0 and 2 (resp. q_0 and 2)) yields that

$$\|A_{m_l}\|_{L^p(\mathbb{R}^d,\ell^q)\to L^p(\mathbb{R}^d,\ell^q(L^\infty(\mathbb{R}_+)))} \leqslant C(d, p, q) 2^{l(1-\frac{d}{2}\eta)},$$

where $\eta \in (0,1)$ is the correct parameter, depending on the choices of p_0, q_0, p, q. One checks that for given $\frac{d}{d-1} < p, q < d$, one can choose $\eta \in (0,1)$ such that in the above exponent, $1 - \frac{d}{2}\eta < 0$. Thus, we have with (6.3),

$$\begin{aligned} \Big\| \Big(\sum_{n=1}^{\infty} |\mathcal{M}_S f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)} \leqslant \Big(&\|A_{m_0}\|_{L^p(\mathbb{R}^d,\ell^q)\to L^p(\mathbb{R}^d,\ell^q(L^\infty(\mathbb{R}_+)))} + \\ &\sum_{l=1}^{+\infty} \|A_{m_l}\|_{L^p(\mathbb{R}^d,\ell^q)\to L^p(\mathbb{R}^d,\ell^q(L^\infty(\mathbb{R}_+)))} \Big) \Big\| \Big(\sum_{n=1}^{\infty} |f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)}, \end{aligned}$$

and finally, noting that the series is then summable in l and A_{m_0} is equally bounded,

$$\Big\| \Big(\sum_{n=1}^{\infty} |\mathcal{M}_S f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)} \leqslant C(d, p, q) \Big\| \Big(\sum_{n=1}^{\infty} |f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)}.$$

The proof of Theorem 6.4 is thus complete.

Remark 6.5 *It is easy to check that the condition $\frac{d}{d-1} < p$ in Theorem 6.4 is necessary. See also [HDR4]. However, the condition $p < d$ is not optimal [Man].*

Remark 6.6 *For non-optimal constants $r_d < p, q < s_d$ such that $r_d \to 1$ and $s_d \to \infty$ as $d \to \infty$, there is an interesting alternative proof of Theorem 6.4, using the results from Section 8. Namely, the radial spectral multiplier for the spherical maximal function that we have*

encountered before, $m(x) = \widehat{d\sigma}(x) = \frac{2\pi}{|x|^{\frac{d-2}{2}}} J_{\frac{d-2}{2}}(2\pi|x|)$*, is a Hörmander spectral multiplier satisfying the assumptions of Proposition 8.3 for the right choice of exponent* c *there. Moreover, the operator* $A = -\Delta$ *does have a* $\mathcal{H}^\alpha_2$ *calculus on* $L^p(\mathbb{R}^d, \ell^q)$*, again for the right choice of exponent* α *depending on* d, p, q*. This in turn is a consequence e.g. of Theorem 7.1. Fortunately, one can choose* c *on the one hand sufficently large compared to* α *as needed in Proposition 8.3, and on the other hand sufficently small compared to the decay of the particular spectral multiplier* m *above, so that Proposition 8.3 does apply for* $\mathcal{M}_S$ *! This proof generalises to the case where* ℓ^q *is replaced by a UMD lattice. See also Proposition 6.8 below.*

6.3 Proof of Theorem 6.3

The proof of the dimension free Fefferman-Stein theorem is based on the above vector valued spherical maximal estimate of Theorem 6.4 together with the so-called method of rotations, which is also used to prove Stein's Theorem 6.1 (see [DGM, StSt]). We mention the main ingredients. Let us first introduce the following weighted maximal operator, depending on a parameter $k \in \mathbb{N}$,

$$\mathcal{M}_{d,k} f(x) = \sup_{r>0} \frac{\int_{|y|\leqslant r} |f(x-y)|\,|y|^k dy}{\int_{|y|\leqslant r} |y|^k dy}, \quad x \in \mathbb{R}^d.$$

It is enough to take polar coordinates in the definition of $\mathcal{M}_{d,k}$ in order to obtain the following pointwise inequality

$$\mathcal{M}_{d,k} f(x) \leqslant \mathcal{M}_S |f|(x), \quad x \in \mathbb{R}^d.$$

Therefore, if we apply Theorem 6.4, we get that for $d \geqslant 3$, $d/(d-1) < p, q < d$ and every sequence $(f_n)_{n\geqslant 1}$ of measurable functions defined on $\mathbb{R}^d$ such that $\big(\sum_{n=1}^\infty |f_n(\cdot)|^q\big)^{\frac{1}{q}} \in L^p(\mathbb{R}^d)$

$$\Big\| \Big(\sum_{n=1}^\infty |\mathcal{M}_{d,k} f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)} \leqslant C(d,p,q) \Big\| \Big(\sum_{n=1}^\infty |f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)}, \tag{6.5}$$

where $C(d,p,q)$ is a constant independent of k and $(f_n)_{n\geqslant 1}$.

Now, we shall obtain Theorem 6.3 by lifting inequality (6.5) in lower dimension d' into $\mathbb{R}^d$ (with $d' \leqslant d$ and $k = d - d'$) by integrating over the Grassmannian of d'-planes in $\mathbb{R}^d$. This method of descent is in the spirit of the Calderón-Zygmund method of rotations. We therefore decompose $\mathbb{R}^d$ as follows: $\mathbb{R}^d = \mathbb{R}^{d'} \times \mathbb{R}^{d-d'}$ and for $x \in \mathbb{R}^d$, we write $x = (x_{d'}, x_{d-d'})$ with $x_{d'} \in \mathbb{R}^{d'}$ and $x_{d-d'} \in \mathbb{R}^{d-d'}$. Besides, for all $\theta \in O(d) = \{\theta' \in \mathbb{R}^{d\times d} : |\theta'(x)| = |x| \text{ for all } x \in \mathbb{R}^d\}$ the orthogonal group, we introduce the following auxiliary operator

$$\mathcal{M}^\theta_{d'} f(x) = \sup_{r>0} \frac{\int_{|y_{d'}|\leqslant r} \big|f\big(x - \theta(y_{d'}, 0)\big)\big|\,|y_{d'}|^{d-d'} dy_{d'}}{\int_{|y_{d'}|\leqslant r} |y_{d'}|^{d-d'} dy_{d'}}, \quad x \in \mathbb{R}^d.$$

We shall need the following lemma, which provides us Fefferman-Stein inequalities for $\mathcal{M}^\theta_{d'}$ with bound independent of θ and d.

Lemma 6.7 *Let* $d' \geqslant 3$ *and* $d'/(d'-1) < p, q < d'$*. Let* $(f_n)_{n\geqslant 1}$ *be a sequence of measurable functions defined on* $\mathbb{R}^d$*. If* $\big(\sum_{n=1}^\infty |f_n(\cdot)|^q\big)^{\frac{1}{q}} \in L^p(\mathbb{R}^d)$*, then we have*

$$\Big\| \Big(\sum_{n=1}^\infty |\mathcal{M}^\theta_{d'} f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)} \leqslant C(d',p,q) \Big\| \Big(\sum_{n=1}^\infty |f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^d)},$$

where $C(d',p,q)$ *is a constant independent of* d*,* θ *and* $(f_n)_{n\geqslant 1}$*.*

Then Theorem 6.3 follows for large dimensions d such that Theorem 6.4 is applicable, from the pointwise inequality

$$M_{HL}f(x) \leqslant \int_{O(d)} \mathcal{M}^{\theta}_{d'} f(x) d\mu(\theta), \quad (x \in \mathbb{R}^d),$$

where μ is normalised Haar measure on $O(d)$ and $d' \leqslant d$ is fixed. For small dimensions, it suffices to apply the ordinary Fefferman-Stein inequalities from Theorem 6.2.

6.4 Extensions and concluding remarks

Theorem 6.3 can be extended to the case of a UMD lattice Y taking over the rôle of ℓ^q. One obtains

Proposition 6.8 *Let $1 < p < +\infty$ and $Y = Y(\Omega')$ be a UMD Banach lattice. We have with notations $x \in \mathbb{R}^d$ and $\omega \in \Omega$,*

$$\Big\| \Big\| M_{HL}\big(f(\cdot,\omega)\big)(x) \Big\|_Y \Big\|_{L^p(\mathbb{R}^d)} \leqslant C(p,Y) \|f\|_{L^p(\mathbb{R}^d,Y)},$$

where $C(p,Y)$ is a constant independent of d and $f \in L^p(\mathbb{R}^d, Y)$.

For a proof this result, we refer to [HDR4].

Next, Theorem 6.3 generalises to maximal operators associated with the Grushin operator. Let us recall the setting. The Grushin operator is given by

$$\Delta_G = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} + |x|^2 \frac{\partial^2}{\partial u^2} = \sum_{i=1}^{d} (X_i^2 + U_i^2)$$

on the space $\mathbb{R}^{d+1} = \mathbb{R}^d_x \times \mathbb{R}_u$, with

$$|x|^2 = \sum_{i=1}^{d} x_i^2, \quad X_i = \frac{\partial}{\partial x_i}, \quad U_i = x_i \frac{\partial}{\partial u},$$

where the smooth vector fields $\{X_i, U_i\}_{1 \leqslant i \leqslant d}$ satisfy the Hörmander condition. We point out that the operator Δ_G is related to the Heisenberg group $\mathbb{H}_d$, since it is actually the image of a Sublaplacian associated with $\mathbb{H}_d$ under a representation acting on functions on $\mathbb{R}^{d+1}$. Let d_{CC} denote the Carnot-Carathéodory distance associated with $\{X_1, \ldots, X_d, U_1, \ldots, U_d\}$ (see for example [VSCC]). Then $(\mathbb{R}^{d+1}, d_{CC}, dm)$ is a space of homogeneous type, where dm stands for the Lebesgue measure, which is not, however, translation invariant. We define a further pseudo-metric on $\mathbb{R}^{d+1}$. Namely, for $g = (x,u)$ and $g' = (x',u')$ belonging to $\mathbb{R}^d_x \times \mathbb{R}_u$, we let

$$d_K(g,g') = \sqrt{\sqrt{(|x|^2 + |x'|^2)^2 + (2|u-u'|)^2} - 2\langle x, x' \rangle},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard Euclidean scalar product. Then d_K is a pseudo-distance on $\mathbb{R}^{d+1}$ (which is, in fact, equivalent to d_{CC} [Li]) related to the fundamental solution of Δ_G (that is to say Green's function). We denote balls with respect to these two (pseudo)-distances by

$$B_{CC}(g,r) = \{g' \in \mathbb{R}^{d+1} : \, d_{CC}(g,g') \leqslant r\}$$

and

$$B_K(g,r) = \{g' \in \mathbb{R}^{d+1} : \, d_K(g,g') \leqslant r\}.$$

This gives rise to the following Hardy-Littlewood maximal operators $\mathcal{M}_{CC}$ and $\mathcal{M}_K$, respectively and naturally given for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^{d+1})$ by

$$\mathcal{M}_{CC} f(g) = \sup_{r>0} \frac{1}{|B_{CC}(g,r)|} \int_{B_{CC}(g,r)} |f(g')| dg', \qquad g \in \mathbb{R}^{d+1},$$

$$\mathcal{M}_K f(g) = \sup_{r>0} \frac{1}{|B_K(g,r)|} \int_{B_K(g,r)} |f(g')| dg', \qquad g \in \mathbb{R}^{d+1}.$$

Then we obtain the following result.

Proposition 6.9 *Let $1 < p, q < +\infty$. Then $\mathcal{M}_{CC}$ and $\mathcal{M}_K$ extend to bounded operators on $L^p(\mathbb{R}^{d+1}, \ell^q)$ and there exists a constant $C = C(p,q)$ independent of d such that*

$$\Big\| \Big(\sum_{n=1}^{\infty} |\mathcal{M}_{CC} f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^{d+1})} \leqslant C(p,q) \Big\| \Big(\sum_{n=1}^{\infty} |f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^{d+1})}$$

and

$$\Big\| \Big(\sum_{n=1}^{\infty} |\mathcal{M}_K f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^{d+1})} \leqslant C(p,q) \Big\| \Big(\sum_{n=1}^{\infty} |f_n(\cdot)|^q \Big)^{\frac{1}{q}} \Big\|_{L^p(\mathbb{R}^{d+1})}.$$

In the same manner, if Y is a UMD Banach lattice, then $\mathcal{M}_{CC}$ and $\mathcal{M}_K$ extend to bounded operators on $L^p(\mathbb{R}^{d+1}, Y)$ with norm $C = C(p,Y)$ independent of d.

For a proof of Proposition 6.9, we refer to [HDR4].

We end this subsection with some concluding remarks and open questions. Note that in Theorem 6.3, we obtained bounds for the standard Hardy-Littlewood maximal operator associated with the euclidean ball. One can replace this ball by a convex symmetric body, that is to say, a bounded, symmetric and convex subset of $\mathbb{R}^d$ with non-empty interior, and then ask again for dimension free bounds as in Theorem 6.3. We refer to the survey [DGM] for a thorough discussion of choices of convex bodies as well as dimension dependence questions. Moreover, very recently the study of dimension independent estimates of averaging operators has been extended to other related settings [BMSW1, BMSW2]. Here, averaging operators (of scalar valued functions) in the sense of e.g. (6.2) associated with general convex symmetric bodies are considered. Then in [BMSW1], estimates independent of the dimension and the convex body for the q-variation (see Subsection 8.1 below for this notion) of the averaging operators are proved. In [BMSW2], maximal estimates of discrete averaging operators are considered; their bounds are proved to be sometimes dimension free, sometimes not, depending on the particular operator (choice of covex body, range of radii over which the supremum is taken, and the value of p).

Going back to our result of Theorem 6.3, we believe that dimension free vector-valued estimates for general symmetric convex bodies should be true as well, but certainly not in full generality for both p and the body. Sharp vector-valued estimates on maximal operators associated with (radial) Fourier multipliers might be a key step, among others, to obtain such dimension free bounds.

A related question of generalisation of our dimension free result is whether one may replace $(\mathbb{R}^d, dx)$ by the locally compact Heisenberg group H^n, that is, whether for a UMD lattice Y

$$\|M_{HL} : L^p(H^n, Y) \to L^p(H^n, Y)\| \leqslant C(p,Y)$$

for all $n \in \mathbb{N}$. A result in this direction is the dimension free estimate of $M_{HL} : L^p(H^n, \mathbb{C}) \to L^p(H^n, \mathbb{C})$ due to Zienkiewicz [Zien].

Going back to the Stein-Strömberg result, we recall that Theorem 6.1 comes for each $1 < p < \infty$ with a constant $C(p)$ independent of d. It is known that the Hardy-Littlewood maximal operator is also weak $(1,1)$ bounded, with a bound $C(1,d)$ in dependence of the dimension. It is not known whether $\sup_{d\in\mathbb{N}} C(1,d) < \infty$ and tendency goes to conjecture it to be false. Stein-Strömberg's method of proof from Theorem 6.1 does not seem to be adaptable. In a vector valued setting, one can ask the following question: is it true or false that for a UMD lattice Y, $\left\|M_{HL} : L^1(\mathbb{R}^d, Y) \to L^{(1,\infty)}(\mathbb{R}^d, Y)\right\| \leqslant C(d, Y)$ with $C(d, Y) \leqslant C(Y) \cdot d$?

6.5 The vector-valued Hardy-Littlewood operator on spaces of homogeneous type

In this subsection we let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type and $Y = Y(\Omega')$ be a Banach lattice (mainly a UMD lattice). Moreover, we consider the Hardy-Littlewood lattice maximal function

$$M_{HL}(f)(x,\omega') = \sup_{r>0} \frac{1}{V(x,r)} \int_{B(x,r)} |f(y,\omega')|\, d\mu(y), \quad x \in \Omega, \quad \omega' \in \Omega'. \tag{6.6}$$

where $B(x,r)$ stands for the closed ball centered in x of radius r and $V(x,r)$ stands for the volume of that ball.

Then the main result of this subsection is the following.

Theorem 6.10 *[HDR5, Theorem 3.1] M_{HL} is bounded on $L^p(\Omega, Y)$ for any $p \in (1,\infty)$ and for every UMD lattice Y.*

The rest of the subsection is devoted to the proof of Theorem 6.10 and some concluding remarks. Boundedness of such vector-valued maximal operators originates in the case $Y = \ell^q$ and $\Omega = \mathbb{R}^d$ in the work of Fefferman and Stein [FeSt] that we studied in Subsection 6.1. More recently, the case $Y = \ell^q$ and Ω a space of homogeneous type has been thoroughly solved in [GLY]. The boundedness of lattice maximal operators is commonly abstracted in the following Banach space property [GMT1, GMT2]:

Definition 6.11 *Let $Y = Y(\Omega')$ be a Banach lattice and denote by $\mathcal{D}$ the family of dyadic intervals on the unit interval $[0,1)$. The space Y is said to have the Hardy-Littlewood property if the dyadic lattice maximal function*

$$M_d(f)(x,\omega') = \sup_{\substack{I \ni x \\ I \in \mathcal{D}}} \frac{1}{|I|} \left| \int_I f(y,\omega')\, dy \right|, \quad x \in [0,1), \quad \omega' \in \Omega', \tag{6.7}$$

defines a bounded operator on $L^p([0,1), Y)$ for one (or, equivalently, for all) $p \in (1,\infty)$.

Note that the definition in [GMT1] refers directly to M_{HL} on $\mathbb{R}^d$. The point of this subsection is to extend this property to M_{HL} on spaces of homogeneous type, and it is useful to begin with dyadic maximal operators. It is immediately clear that M_{HL} dominates any dyadic maximal operator on $\mathbb{R}^d$. Conversely, using the well-known Euclidean version of Theorem 6.16 below, we see that dyadic maximal operators dominate M_{HL}. The equivalence between the definition in [GMT1] and the one above will thereby quickly follow from our considerations.

The UMD property was connected with the Hardy-Littlewood property by Bourgain in [Bou84, Lemma 1], see also [RdF1, Theorem 3].

Theorem 6.12 (Bourgain) *Let Y be a Banach lattice. Then Y is UMD if and only if Y and Y' have the Hardy-Littlewood property.*

The proof of Theorem 6.10 is based on the following transference result:

Lemma 6.13 *Let $Y = Y(\Omega')$ be a Banach lattice. Further, let $\mathcal{F} = (\mathcal{F}_k)_{k\in\mathbb{Z}}$ be a filtration on a σ-finite measure space (Ω, μ) and denote by E_k the corresponding conditional expectation operators. If Y has the Hardy-Littlewood property, then the lattice maximal function*

$$M_{\mathcal{F}}(f)(x,\omega') = \sup_{k\in\mathbb{Z}} |E_k f(x,\omega')|, \quad x \in \Omega, \quad \omega' \in \Omega', \tag{6.8}$$

defines a bounded operator on $L^p(\Omega, Y)$ for all $p \in (1,\infty)$. Moreover, the operator norm of $M_{\mathcal{F}}$ is not greater than the operator norm of M_d on $L^p([0,1), Y)$.

The proof of Lemma 6.13 is based on a concave function argument originating from the work of Burkholder [Bur81] (see also [Bur01]). We follow closely the argument presented in [Kem, Section 7]. We begin by observing that, given a filtration $(\mathcal{F}_k)_{k\in\mathbb{N}}$ on (Ω,μ), the inequality

$$\int_\Omega \Big\| \sup_{0\leqslant k\leqslant n} |E_k f(x,\cdot)| \Big\|^p \, d\mu(x) \leqslant C \int_\Omega \|E_n f(x)\|^p \, d\mu(x), \quad f \in L^p(\Omega, Y), \quad n \in \mathbb{N}, \tag{6.9}$$

where C is a fixed constant, is equivalent with

$$\int_\Omega V_p\Big(\{E_k f(x)\}_{k=0}^n, E_n f(x)\Big)\, d\mu(x) \leqslant 0, \quad f \in L^p(\Omega, Y), \quad n \in \mathbb{N}, \tag{6.10}$$

where

$$V_p(S,y) = \Big\| \sup_{y'\in S} |y'(\cdot)| \Big\|^p - C\|y\|^p, \quad S \subset Y \text{ finite}, \quad y \in Y. \tag{6.11}$$

Proposition 6.14 *Suppose that Y is a Banach lattice and let $1 < p < \infty$. The following conditions are equivalent:*

1. *(6.10) holds for the dyadic filtration on the unit interval (with the Lebesgue measure).*
2. *There exists a real-valued function $U : \{\text{finite subsets of } Y\} \times Y \to \mathbb{R}$ such that*
   - *$U(S,y) \geqslant V_p(S,y)$*
   - *$U(S \cup \{y\}, y) = U(S,y)$*
   - *$U(\emptyset, y) \leqslant 0$*
   - *$U(S,\cdot)$ is concave*

   for finite subsets S of Y and $y \in Y$.
3. *(6.10) holds for any filtration on any σ-finite measure space.*

Proof : For a proof we refer to [HDR5, Lemma 3.5]. See also [Kem, Section 7]. ∎

Proof of Lemma 6.13 : First note that we can reduce the case of a filtration indexed by $\mathbb{Z}$ to the case of a filtration indexed by $\mathbb{N}$. Indeed, if we can bound $\sup_{k\geqslant k_0} |E_k f(x,\omega')|$ uniformly in $L^p(\Omega, Y)$ norm, then letting $k_0 \to -\infty$ and using the Fatou property from Lemma 4.17, we can also bound $\sup_{k\in\mathbb{Z}} |E_k f(x,\omega')|$. Considering f a function which is $\mathcal{F}_n$ measurable for a fixed

level n, we see that (6.9) holds for $\Omega = [0,1)$ and the usual dyadic filtration. Then it follows by Proposition 6.14 that (6.9) also holds for Ω a general measure space and $\mathcal{F}$ a general filtration. Letting $n \to \infty$, (6.9) implies then the boundedness of $M_{\mathcal{F}}$ from (6.8) on $L^p(\Omega, Y)$. ∎

Fortunately, for spaces of homogeneous type Ω, the existence of sufficiently rich filtrations is known in order that M_{HL} can be controlled by the filtration maximal operator $M_{\mathcal{F}}$ from (6.8). We explain the details now.

Definition 6.15 *Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type. A dyadic system $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$ consists of measurable subsets of Ω, where each collection $\mathcal{D}_k$ consists of pairwise disjoint sets of positive measure, the dyadic cubes, with the following properties:*

- $\Omega = \bigcup_{Q\in\mathcal{D}_k} Q$,
- *if $Q \in \mathcal{D}_k$ and $R \in \mathcal{D}_l$ with $l \geqslant k$, then either $R \subset Q$ or $Q \cap R = \emptyset$,*

Theorem 6.16 (Hytönen-Kairema) *Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type. There exists a finite collection of dyadic systems $\mathcal{D}^m$, $m = 1, \dots, M$, on Ω, so that for every ball B one can find a dyadic cube Q_B in one of the systems such that $B \subset Q_B$ and $\mu(Q_B) \lesssim \mu(B)$.*

Proof : See [HyKa, 2.21 and Theorem 4.1]. ∎

Let now $\mathcal{D} = \bigcup_{k\in\mathbb{Z}} \mathcal{D}_k$ be a dyadic system on Ω as in Definition 6.15. Denote by $\mathcal{F}_k$ the σ-algebra generated by $\mathcal{D}_k$ and note that the corresponding conditional expectation is

$$E_k f(x) = \sum_{Q\in\mathcal{D}_k} \frac{1_Q(x)}{\mu(Q)} \int_Q f(y)\, d\mu(y). \tag{6.12}$$

The maximal function associated with the increasing filtration $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ is therefore given by

$$M_{\mathcal{F}}(f)(x,\omega') = \sup_{\substack{Q\ni x \\ Q\in\mathcal{D}}} \frac{1}{\mu(Q)} \left| \int_Q f(y,\omega')\, d\mu(y) \right|, \quad x \in \Omega, \quad \omega' \in \Omega'. \tag{6.13}$$

Theorem 6.16 allows us to control the Hardy-Littlewood lattice maximal function M_{HL} by its dyadic counterparts. Indeed, we see that for any ball B,

$$\frac{1}{\mu(B)} \int_B |f(y,\omega')|\, d\mu(y) \lesssim \frac{1}{\mu(Q_B)} \int_{Q_B} |f(y,\omega')|\, d\mu(y). \tag{6.14}$$

Therefore,

$$M_{HL}(f)(x,\omega') \lesssim \sum_m M_{\mathcal{F}^m}(|f|)(x,\omega'), \quad x \in \Omega, \quad \omega' \in \Omega', \tag{6.15}$$

where $\mathcal{F}^m$ are the filtrations arising from the finite collection of dyadic systems $\mathcal{D}^m$.

We are now in a position to prove Theorem 6.10.

Proof of Theorem 6.10 : The result is an immediate consequence of the considerations above. Indeed, if Y is a UMD lattice, it has the Hardy-Littlewood property by Theorem 6.12. By Theorem 6.16, we may construct dyadic filtrations $\mathcal{F}^m$ so that M_{HL} is dominated pointwise by the sum of $M_{\mathcal{F}^m}$ as in (6.15). By Lemma 6.13, the latter maximal operators are bounded on $L^p(\Omega, Y)$ for any $p \in (1,\infty)$, and therefore so is M_{HL}. ∎

Remark 6.17 *In the case $Y = \ell^s(\ell^2)$, $1 < s < \infty$, Theorem 6.10 can also be proved by a similar method as [GLY, Theorem 1.2, Corollary 2.9]. Such a space Y is relevant for applications to square function estimates in Section 7.*

At the end of this subsection, let us remark that the case of a weak $(1,1)$ estimate of

$$M_{HL}\colon L^1(\Omega, Y) \to L^{(1,\infty)}(\Omega, Y)$$

is still open for Ω being a space of homogeneous type and Y a UMD lattice. This question and the related problem of weak $(1,1)$ boundedness of maximal spectral multipliers

$$M_m f(x,\omega) = \sup_{t>0} |m(tA) f(x,\omega')|$$

(think first of $m(tA) = \exp(-tA)$ being the semigroup generated by the sectorial operator A) on the same spaces, is ongoing work of the author. To illustrate that not so much is known on weak $(1,1)$ bounds for semigroups, we pose another question: Does a submarkovian semigroup admit a weak type $(1,1)$ bound of $M_{\exp(-\cdot)}$ on $L^1(\Omega, Y)$? Even the case $Y = \mathbb{C}$ is open here.

7 Hörmander calculus on vector valued L^p spaces via maximal estimates

The results of this section are contained in [HDR5]. Consider as a guiding example and starting point for further considerations in this section the Laplacian operator $A = -\Delta$ on $\mathbb{R}^d$. It is a classical result by Hörmander [Hor] that $-\Delta$ has a $\mathcal{H}^\alpha_2$ calculus on $L^p(\mathbb{R}^d)$ with $\alpha > \frac{d}{2}$ and $1 < p < \infty$. It has been a deeply studied question over the last years to know to what extent one can replace the ordinary Laplacian subjacent to Hörmander's theorem by other operators A acting on some $L^p(\Omega)$ space. A theorem of Hörmander type holds true for many elliptic differential operators A, including sub-Laplacians on Lie groups of polynomial growth, Schrödinger operators and elliptic operators on Riemannian manifolds (see the references in Subsection 3.1). More recently, spectral multipliers have been studied for operators acting on $L^p(\Omega)$ only for a strict subset of $(1,\infty)$ of exponents. We refer to Subsections 7.1 and 7.3 for examples.

Another generalisation of Hörmander's result is to consider a further Banach space Y and the tensor extension $A = -\Delta \otimes \mathrm{Id}_Y$ acting on the Bochner space $L^p(\mathbb{R}^d, Y)$, and then ask again if this operator has some Hörmander calculus. One cannot take any Banach space Y but one is restricted to take a UMD space Y [KW1, 10.3 Remark], for a definition of the UMD property see Subsection 4.5. The programme of Hörmander $\mathcal{H}^\alpha_2$ calculus for $-\Delta \otimes \mathrm{Id}_Y$ on $L^p(\mathbb{R}^d, Y)$ and related questions has been pursued in the last two decades by many authors [ArB1, ArB2, Bou, GiWe, Hy04, Hy10, McC, SW, We01, Zim]. Then the Fourier type [GiWe], and Rademacher type/cotype [Hy10] of Y play a rôle when one strives for better or best possible derivation order α.

In this section, we will generalise this programme from the pure Laplacian to a selfadjoint operator A acting on some $L^2(\Omega)$ space, which generates a semigroup $(T_t)_{t\geqslant 0}$ satisfying Gaussian estimates (4.9) or so called generalised Gaussian estimates, see (7.4) for this notion. Then the main topic of the present section is to determine in which cases an a priori H^∞ calculus of A (see Proposition 7.9 below) improves to a Hörmander functional calculus in the sense of Definition 4.9 on the space $L^p(\Omega, Y)$, i.e. when is $f(A) \otimes \mathrm{Id}_Y$ bounded on $L^p(\Omega, Y)$ for f a Hörmander spectral multiplier. In case that $f(A)$ is bounded on $L^p(\Omega)$, it is well-known that if Y is a Hilbert space, or if $f(A)$ is lattice positive, or if $f(A)$ is both bounded on $L^\infty(\Omega)$ and

on $L^1(\Omega)$, then this tensor extension is possible, but in general, this is a difficult task, e.g. for a multiplier $f(A)$ with singular integral kernel having a cancellation effect. As a motivation for this question, take the following abstract hyperbolic PDE

$$\begin{cases} \partial_t^2 u(x,y,t) &= -A_x u(x,y,t) \quad (x \in \Omega,\ y \in \Omega',\ t > 0) \\ u(x,y,0) &= f(x,y) \qquad\qquad (x \in \Omega,\ y \in \Omega'), \end{cases}$$

which is solved formally by $u(t) = u(x,y,t) = \exp(it\sqrt{A})(f)(x,y)$. Noting that

$$f(\lambda) = (1+\lambda)^{-\delta} \exp(it\sqrt{\lambda})$$

belongs to the class $\mathcal{H}^\beta_2$ for $\delta \geqslant \frac{\beta}{2}$ [KrW3, Lemma 3.9], [KrPhD, Prop 4.8 (4)] yields that $\|u(t)\|_X \leqslant C(1+|t|)^{2\delta} \|(1+A)^\delta f\|_X$ provided that A has a $\mathcal{H}^\beta_2$ calculus. One thus obtains a norm estimate of the solution u in terms of fractional domain space norms $\mathrm{dom}((1+A)^\delta) \subseteq X$ of the initial value f. As an example, we can take $X = L^p(\Omega, L^s(\Omega'))$. We refer to [HiPr, Sections 5 and 6] and Corollary 5.16 in Subsection 5.4 for further applications of the functional calculus to differential equations on such X.

Our strategy of proof is to use once again Theorem 4.11, thus we have to establish R-boundedness of the complex time semigroup on $L^p(\Omega, Y)$ and H^∞ calculus for $A \otimes \mathrm{Id}_Y$ on $L^p(\Omega, Y)$.

To establish R-bounds, which by virtue of (4.1) amounts in square function estimates such as

$$\Big\| \Big(\sum_k |T_{t_k} f_k|^2 \Big)^{\frac{1}{2}} \Big\|_X \leqslant C \Big\| \Big(\sum_k |f_k|^2 \Big)^{\frac{1}{2}} \Big\|_X, \tag{7.1}$$

where T_{t_k} is a spectral multiplier of A (in case of Theorem 4.11, T_{t_k} is the semigroup generated by A), we shall use maximal estimates. In the simplest form, they state as

$$|T_t f| \leqslant c M_{HL}(f) \quad (t \geqslant 0), \tag{7.2}$$

where we refer to (6.6) for the definition of the Hardy-Littlewood maximal operator M_{HL}. By the pointwise nature of the estimate in (7.2), a natural framework for us will be that $Y = Y(\Omega')$ is a *UMD lattice* over some measure space Ω'. In some of our results, the convexity and concavity exponents of Y will take over the rôle of the above mentioned type/cotype of Y. Moreover, Ω will be a space of homogeneous type, see Subsection 4.4. We will then prove square function estimates (7.1) first for T_{t_k} the semigroup, and then a posteriori also for general Hörmander spectral multipliers, both on the space $X = L^p(\Omega, Y)$, see Remark 7.12.

The following is the first result of this section. The additional hypotheses of volume growth and dispersive estimate allow for a proof being not too technical and containing nevertheless the main ideas. We refer to Subsection 7.2 for examples where the theorem applies. Note that from now on, we do not distinguish the notations A and $A \otimes \mathrm{Id}_Y$.

Theorem 7.1 *[HDR5, Corollary 4.23] Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type of dimension d, let Y be any UMD lattice and let $1 < p < \infty$. Assume that A generates the selfadjoint semigroup $(T_t)_t$ on $L^2(\Omega)$ satisfying Gaussian upper estimates (4.9) with $m = 2$. Assume that A has a bounded $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega, Y)$ for some $\omega \in (0, \pi)$. Additionally, assume that the volume of balls in Ω satisfies a polynomial growth*

$$V(x,r) \leqslant C|r|^d$$

for some fixed $C, d > 0$ and any $x \in \Omega$ and $r > 0$. Assume finally that the semigroup satisfies the dispersive estimate

$$\| \exp(itA) \|_{L^1(\Omega) \to L^\infty(\Omega)} \leqslant C|t|^{-\frac{d}{2}}$$

for some $C > 0$ and all $t \in \mathbb{R}\backslash\{0\}$. Then the $\mathrm{H}^\infty(\Sigma_\omega)$ calculus for A improves to a Hörmander $\mathcal{H}^\beta_2$ functional calculus on $L^p(\Omega, Y)$ for any exponent $\beta > \frac{d}{2} + \frac{1}{2}$.

Proof : The proof is divided into three steps. Our goal is to show R-boundedness of $\{\exp(-e^{i\theta}tA) : t > 0\}$ over the Banach space $L^p(\Omega, Y)$ for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, with control of the R-bound $\lesssim (\cos(\theta))^{-\frac{d}{2}}$. Then apply Theorem 4.11.

1st step: Extrapolation of Gaussian estimates to complex time As already observed in Subsection 5.1, we could extrapolate Gaussian estimates from real to complex time by [CaCoOu, Proposition 4.1]. However, since we are interested in precise blow-up of the semigroup kernel close to the boundary $i\mathbb{R}$ of $\mathbb{C}_+$, we need an improved extrapolation, which is available in presence of the dispersive estimate. Observe that rescaling the time $t \rightsquigarrow ct$ in the semigroup if necessary, we have a semigroup satisfying [CouSi, (3.1), (3.2)], where we use [CouSi, Lemma 3.2]. According to [CouSi, pages 521-522], the dispersive assumption and the Gaussian upper estimate (on the diagonal $x = y$) imply for $z \in \mathbb{C}_+$

$$|p_z(x,y)| \leqslant C|z|^{-\frac{d}{2}} \exp\left(-\operatorname{Re}\left[\frac{\operatorname{dist}^2(x,y)}{4z}\right]\right) \leqslant C' \frac{1}{V(x, \sqrt{|z|})} \exp\left(-\operatorname{Re}\left[\frac{\operatorname{dist}^2(x,y)}{4z}\right]\right). \tag{7.3}$$

2nd step: Estimation of complex time maximal operator In this step, we introduce the following maximal operator associated with the semigroup. We let for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$$M_\theta(f)(x,\omega') = \sup_{t>0} \left| T_{te^{i\theta}}(f(\cdot,\omega'))(x) \right|.$$

The goal of this step will be to bound M_θ in terms of M_{HL} from (6.6). We estimate with a time shift $t \mapsto t\cos(\theta)$, and the doubling condition

$$\begin{aligned}
M_\theta f(x,\omega') &= \sup_{t>0} \left| T_{te^{i\theta}}(f(\cdot,\omega'))(x) \right| \\
&\overset{(7.3)}{\lesssim} \sup_{t>0} \int_\Omega \frac{1}{\sqrt{V(x, |te^{i\theta}|)}} \exp\left(-\operatorname{Re}\left[\frac{\operatorname{dist}(x,y)^2}{4e^{i\theta}t}\right]\right) |f(y,\omega')|\, dy \\
&= \sup_{t>0} \int_\Omega \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{\operatorname{dist}(x,y)^2}{4t}\cos(\theta)\right) |f(y,\omega')|\, dy \\
&= \sup_{t>0} \int_\Omega \frac{1}{V(x,\sqrt{t\cdot\cos(\theta)})} \exp\left(-\frac{\operatorname{dist}(x,y)^2}{4t}\right) |f(y,\omega')|\, dy \\
&\lesssim (\cos(\theta))^{-\frac{d}{2}} \sup_{t>0} \int_\Omega \frac{1}{V(x,\sqrt{t})} \exp\left(-\frac{\operatorname{dist}(x,y)^2}{4t}\right) |f(y,\omega')|\, dy.
\end{aligned}$$

Now decompose the space Ω into annular regions $A_n = B(x, 2\sqrt{t})$ if $n = 0$ and $A_n = B(x, 2^{n+1}\sqrt{t})\backslash B(x, 2^n\sqrt{t})$ if $n \geqslant 1$. Then the above calculation continues

$$= (\cos(\theta))^{-\frac{d}{2}} \sup_{t>0} \sum_{n=0}^\infty \frac{1}{V(x,\sqrt{t})} \int_{A_n} \exp\left(-\frac{\operatorname{dist}(x,y)^2}{4t}\right) |f(y,\omega')|\, dy$$

$$
\begin{aligned}
&\lesssim (\cos(\theta))^{-\frac{d}{2}} \sup_{t>0} \sum_{n=0}^{\infty} 2^{(n+1)d} \frac{1}{V(x, 2^{n+1}\sqrt{t})} \int_{A_n} \exp\left(-\frac{2^{2n}t}{4t}\right) |f(y,\omega')|\, dy \\
&\leqslant (\cos(\theta))^{-\frac{d}{2}} \sup_{t>0} \sum_{n=0}^{\infty} 2^{(n+1)d} \frac{1}{V(x, 2^{n+1}\sqrt{t})} \int_{B(x,2^{n+1}\sqrt{t})} \exp\left(-\frac{2^{2n}}{4}\right) |f(y,\omega')|\, dy \\
&\leqslant (\cos(\theta))^{-\frac{d}{2}} \sum_{n=0}^{\infty} 2^{(n+1)d} \exp\left(-\frac{2^{2n}}{4}\right) M_{HL}(f)(x,\omega') \\
&\lesssim (\cos(\theta))^{-\frac{d}{2}} M_{HL}(f)(x,\omega').
\end{aligned}
$$

We have used that the exponential term is decreasing in $\mathrm{dist}(x,y)$ and that the series is summable. All these estimates were pointwise in $\omega' \in \Omega'$.

3rd step: Passing from maximal estimate to R-boundedness In view of (4.1), the R-boundedness of the complex time semigroup to prove is a square function estimate. Noting that $Y(\ell^2)$ is again a UMD lattice if Y is a UMD lattice, the operator M_{HL} is bounded on $L^p(\Omega, Y(\ell^2))$ according to Theorem 6.10, and we deduce

$$
\begin{aligned}
\left\| \left(\sum_i |T_{t_i e^{i\theta}} f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,Y)} &\leqslant \left\| \left(\sum_i \big(M_\theta(f_i)\big)^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,Y)} \\
&\lesssim \big(\cos(\theta)\big)^{-\frac{d}{2}} \left\| \left(\sum_i \big(M_{HL}(f_i)\big)^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,Y)} \\
&\lesssim \big(\cos(\theta)\big)^{-\frac{d}{2}} \left\| \left(\sum_i |f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,Y)}.
\end{aligned}
$$

Thus, $\{T_{te^{i\theta}} : t > 0\}$ is R-bounded in $L^p(\Omega, Y)$ with R-bound $\lesssim \big(\cos(\theta)\big)^{-\frac{d}{2}}$. Then use the H^∞ calculus assumption and Theorem 4.11 to deduce that A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\Omega, Y)$ for $\beta > \frac{d+1}{2}$. ∎

Note that the volume growth condition and the dispersive assumption in Theorem 7.1 can be omitted. Then, in the first step of the proof, the extrapolation of Gaussian estimates to complex time will be achieved by means of [CaCoOu, Proposition 4.1]. This estimate is cruder and one obtains at the end a $\mathcal{H}^\alpha_2$ calculus for A on $L^p(\Omega, Y)$ to the exponent $\alpha > d + \frac{1}{2}$ [HDR5, Corollary 4.23 1.]. Theorem 7.1 admits a great generalisation, see Theorem 7.5. Namely, in recent years, spectral multipliers have also been proved for semigroups without an integral kernel, but merely a type of non-pointwise local estimate, which are called generalised Gaussian estimate. For examples of such semigroups, we refer to Subsection 7.3.

Definition 7.2 *Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type. Let A be a selfadjoint operator on $L^2(\Omega)$ generating the semigroup $(T_t)_{t\geqslant 0}$. Let $p_0 \in [1,2)$ and $m \in [2,\infty)$. We say that $(T_t)_{t\geqslant 0}$ satisfies generalised Gaussian estimates (with parameters p_0, m) if there exist $c, C < \infty$ such that*

$$
\begin{aligned}
(7.4)\quad &\big\| 1_{B(x,r_t)} T_t 1_{B(y,r_t)} \big\|_{L^{p_0}(\Omega)\to L^{p_0'}(\Omega)} \\
&\qquad \leqslant C|V(x,r_t)|^{-(\frac{1}{p_0}-\frac{1}{p_0'})} \exp\left(-c\left(\frac{\mathrm{dist}(x,y)}{r_t}\right)^{\frac{m}{m-1}}\right) \quad (x,y\in\Omega,\ t>0),
\end{aligned}
$$

where $r_t = t^{\frac{1}{m}}$.

Remark 7.3 *According to [BK02, Proposition 2.9] and [BK05, Proposition 2.1], Gaussian upper estimates (4.9) with parameter* $m \geqslant 2$ *for a semigroup imply generalised Gaussian estimates (7.4) with parameter* $p_0 = 1$ *and* m. *Moreover, according to [BK05, Proposition 2.1], generalised Gaussian estimates with parameters* $p_0 \in [1, 2)$ *and* $m \geqslant 2$ *imply generalised Gaussian estimates with parameters* $p_1 \in [p_0, 2)$ *and* m.

The Hörmander differentiation parameter for semigroups in Theorem 7.5 below is governed by the following quantity.

Definition 7.4 *Let* $p \in (1, \infty)$, $p_Y \in (1, 2]$ *and* $q_Y \in [2, \infty)$. *We put*

$$\alpha(p, p_Y, q_Y) = \max\left(\frac{1}{p}, \frac{1}{p_Y}, \frac{1}{2}\right) - \min\left(\frac{1}{p}, \frac{1}{q_Y}, \frac{1}{2}\right) \in (0, 1). \tag{7.5}$$

Informally spoken, this is the length of the segment, which is the convex hull of the points $\frac{1}{p}, \frac{1}{p_Y}, \frac{1}{q_Y}$ *and* $\frac{1}{2}$ *sitting on the real line.*

In the following theorem, we use the notion of p-convex and q-concave Banach lattices, see Definition 4.18. Note that if $1 \leqslant r \leqslant \infty$, $L^r(\Omega')$ is r-convex and r-concave and for $1 \leqslant p < r$, the p-convexification is $L^r(\Omega')^p = L^{\frac{r}{p}}(\Omega')$, which is again a UMD lattice. Note that every UMD lattice has some non-trivial convexity and concavity [RdF1, (c.3)], [TJ, pages 218-219], [HDR5, Lemma 2.11].

Theorem 7.5 *[HDR5, Theorem 4.2, Theorem 4.10] Let* $(\Omega, \mathrm{dist}, \mu)$ *be a space of homogeneous type with a dimension* d. *Let* A *be a selfadjoint operator on* $L^2(\Omega)$ *generating the semigroup* $(T_t)_{t \geqslant 0}$. *Let* $p_0 \in [1, 2)$ *and* $m \in [2, \infty)$. *Assume that* $(T_t)_{t \geqslant 0}$ *satisfies generalised Gaussian estimates with parameters* p_0, m. *Let* Y *be a UMD lattice which is* p_Y*-convex and* q_Y*-concave for some* $p_Y \in (p_0, 2]$ *and* $q_Y \in [2, p_0')$. *Assume that the convexifications* Y^{p_Y} *and* $(Y')^{q_Y'}$ *are also UMD lattices. Finally, assume that* A *has a bounded* $\mathrm{H}^\infty(\Sigma_\omega)$ *calculus on* $L^p(\Omega, Y)$ *for some fixed* $p \in (p_0, p_0')$ *and* $\omega \in (0, \pi)$.

Then A *has a Hörmander* $\mathcal{H}_2^\beta$ *calculus on* $L^p(\Omega, Y)$ *with*

$$\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2}$$

and $\alpha(p, p_Y, q_Y)$ *from (7.5).*

For a proof, we refer to [HDR5, Theorems 4.2, Theorem 4.10]. Note that compared to Theorem 7.1, we need neither the dispersive estimate nor the volume growth condition any more. Indeed, they were only used in the proof of Theorem 7.1 through the enhanced complex time Gaussian upper estimate (7.3). Instead, the proof of Theorem 7.5 uses the machinery on real and complex time estimates for generalised Gaussian estimates as in (7.4) and the division of the space Ω into annuli (cf. proof of Theorem 7.1), developed mainly by Blunck and Kunstmann [BK05, Bl07]. It also uses the following "convexified" versions of means of functions over balls and the corresponding maximal operator.

Definition 7.6 *Let* $f : \Omega \to Y$ *locally integrable,* $q \in [1, \infty]$ *and* $r > 0$.

1. *We put*

$$N_{q,r}(f)(x,\omega') = \frac{1}{V(x,r)^{\frac{1}{q}}}\left(\int_{B(x,r)} |f(y,\omega')|^q \, d\mu(y)\right)^{\frac{1}{q}}, \tag{7.6}$$

(obvious modification if $q=\infty$).

2. *Furthermore, we put*

$$M^q_{HL}(f)(x,\omega') = \sup_{r>0} N_{q,r}(f)(x,\omega'). \tag{7.7}$$

The parameter q will take a value $\frac{1}{q} < \min\left(\frac{1}{p}, \frac{1}{q_Y}, \frac{1}{2}\right)$ for $N_{q,r}$ and $\frac{1}{q} > \max\left(\frac{1}{p}, \frac{1}{p_Y}, \frac{1}{2}\right)$ for M^q_{HL}. Upper and lower square function estimates of the non-linear $N_{q,r}$ and boundedness of M^q_{HL} on $L^p(\Omega, Y)$ are proved in [HDR5] and play an important rôle in the proof of Theorem 7.5. Note that Theorems 7.1 and 7.5 assume a priori that A has a bounded H^∞ calculus on $L^p(\Omega, Y)$. Now we gather several situations, in which this is the case.

Proposition 7.7 *Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type and E a Banach space. Let A generate the selfadjoint semigroup $(T_t)_t$ on $L^2(\Omega)$ satisfying Gaussian estimates (4.9) of order $m \geqslant 2$. Assume that A has an $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^{p_0}(\Omega, E)$ for some $\omega \in (0,\pi)$ and some $p_0 \in (1,\infty)$. Then for any $p \in (1,\infty)$, A has an $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega, E)$.*

Proof : This is an adaptation of the extrapolation procedure by Calderon-Zygmund decomposition of an H^∞ spectral multiplier from [DuRo, Theorem 3.1]. See [HDR5, Theorem 4.6]. ■

Proposition 7.8 (Kunstmann-Ullmann) *Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type and A generate a selfadjoint semigroup $(T_t)_t$ on $L^2(\Omega)$ satisfying generalised Gaussian estimates (7.4) with parameters $p_0 \in [1,2)$ and $m \in [2,\infty)$. Let $Y = L^s(\Omega')$ with $s \in (p_0, p_0')$. Then A has a bounded $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega, L^s(\Omega'))$ for any $p \in (p_0, p_0')$ and $\omega \in (0,\pi)$.*

Proof : See [KuUl2, Theorem 2.1, Theorem 2.3]. See also [HDR5, Theorem 4.7]. ■

Proposition 7.9 (Fendler/Hieber-Prüss/Taggart) *Let (Ω, μ) be a σ-finite measure space.*

1. *Let $(T_t)_t$ be a semigroup acting on $L^p(\Omega)$ for some fixed $p \in (1,\infty)$, such that the T_t are regular contractive, that is, there exist S_t positive and contractive operators on $L^p(\Omega)$ such that $|T_t f| \leqslant S_t |f|$ for all $t > 0$. Assume that $p \neq 2$ or that T_t are themselves positive. Then the generator A of $(T_t)_t$ has an $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega, Y)$ for any $\omega \in (\frac{\pi}{2}, \pi)$ and any UMD space Y.*

2. *Let $(T_t)_t$ be a semigroup which is contractive on $L^p(\Omega)$ for all $p \in [1,\infty]$ (strong continuity only for finite p). Then the generator A of $(T_t)_t$ has an $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega, Y)$ for any $p \in (1,\infty)$, $\omega \in (\frac{\pi}{2}, \pi)$ and any UMD space Y.*

3. *Let $(\Omega, \mathrm{dist}, \mu)$ be a space of homogeneous type and Y a UMD space. Suppose that the selfadjoint semigroup $(T_t)_t$ on $L^2(\Omega)$ satisfies the Gaussian estimates (4.9) and that $(T_t)_t$ is moreover (lattice) positive, i.e. $p_t(x,y) \geqslant 0$ for any $t > 0$ and $x, y \in \Omega$, where $p_t(x,y)$ is the integral kernel as in (4.9). Then for any $1 < p < \infty$, the generator A has an $\mathrm{H}^\infty(\Sigma_\omega)$ calculus on $L^p(\Omega, Y)$ for any $\omega \in (\frac{\pi}{2}, \pi)$.*

Proof : 1. We refer to [Fen2, Theorem 4.2.1 & page 45] and [HiPr, Theorem 5]. See also [HDR5, Proposition 4.8].

2. We refer to [Tagg, Theorem 2.2.1]. See also [HDR5, Proposition 4.8].
3. We refer to [HDR5, Corollary 4.9]. ■

Remark 7.10 *1. Spectral multiplier theorems under generalised Gaussian estimates have been obtained in the last 5 years by different methods, in the scalar case $Y = \mathbb{C}$. We note that in this case, $p_Y = q_Y = 2$ and our Hörmander functional calculus exponent from Theorem 7.5 becomes*

$$\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2} = \left(\max\left(\frac{1}{p}, \frac{1}{2}\right) - \min\left(\frac{1}{p}, \frac{1}{2}\right)\right) d + \frac{1}{2} = \left|\frac{1}{p} - \frac{1}{2}\right| d + \frac{1}{2}.$$

Let us compare this result with those scalar valued Hörmander type spectral multiplier theorems obtained in the literature, sometimes under stronger hypotheses. In the literature, the definition of $\mathcal{H}^\beta_q$ is extended for values $q \neq 2$ in an obvious manner as stated after Definition 4.7. We denote $(GGE_{p_0,m})$ for $p_0 \in [1,2)$ and $m \geqslant 2$ our generalised Gaussian estimate hypothesis (7.4), and refer to the sources below for the definition of other hypotheses. In all cases, the semigroup is moreover assumed to be selfadjoint, acting on $L^2(\Omega)$ with Ω a space of homogeneous type. Finally, in the last two sources, there is an autoimprovement of the calculus by selfadjointness of T_t possible.

Resource	*Hypotheses*	$\mathcal{H}^\beta_q$ *calculus on* $L^p(\Omega, \mathbb{C})$ *with* $p_0 < p < p_0'$
This memoir, Theorem 7.5	$(GGE_{p_0,m})$	$\beta > \lvert\frac{1}{p} - \frac{1}{2}\rvert d + \frac{1}{2}$, $q = 2$
[Bl, Theorem 1.1]	$(GGE_{p_0,m})$	$\beta > \frac{d}{2} + \frac{1}{2}$, $q = 2$
[KuU2, Theorem 5.4 a)]	$(GGE_{p_0,m})$	$\beta > \lvert\frac{1}{p} - \frac{1}{2}\rvert (d+1)$, $\frac{1}{q} < \lvert\frac{1}{p} - \frac{1}{2}\rvert$
[KuU2, Theorem 5.4b)]	$(GGE_{p_0,m})$	$\beta > \lvert\frac{1}{p} - \frac{1}{2}\rvert d$, $q = \infty$
[COSY, Theorem 4.1]	$(FS) + (ST^q_{p_0,2})$	$\beta > \max(d(\frac{1}{p_0} - \frac{1}{2}), \frac{1}{q})$
[SYY, Theorem 5.1]	$(DG_m) + (ST^q_{p_0,2,m})$	$\beta > \max(d(\frac{1}{p_0} - \frac{1}{2}), \frac{1}{q})$

2. *Let us turn back to the toy case $A = -\Delta$. In [Hy04], Hytönen investigates operator valued Fourier multipliers on $L^p(\mathbb{R}^d, Y)$ for Y a UMD space with Fourier type. If $Y = L^q(\Omega')$ for some $q \in (1, \infty)$, then his order of differentiation in the Hörmander condition on the multiplier symbol is essentially $\frac{d}{\min(q,q^*)}$, which is comparable to our $\frac{d}{\min(q,q^*)} + \frac{1}{2}$ (Hytönen takes the rounding up number, but needs derivatives in each of the d directions only of order 1 or less). Predecessors of [Hy04] are [McC, Zim, GiWe].*

3. *In the case that $\Omega = \mathbb{R}^d$ and $(T_t)_t$ satisfying classical Gaussian estimates (4.9), a combination of [ALV] and [GoY] also yields UMD lattice valued spectral multipliers. Indeed, in [ALV] it is shown that if $m(A)$ satisfies weighted estimates $L^p(\mathbb{R}^d, w) \to L^p(\mathbb{R}^d, w)$ for any weight $w \in A_{p/r_0}$ in the classical Muckenhoupt class, and any $r_0 < p < \infty$, then it extends boundedly to $L^p(\mathbb{R}^d, Y) \to L^p(\mathbb{R}^d, Y)$ for $r_0 < p < \infty$ (in fact, even to $L^p(\mathbb{R}^d, w, Y) \to L^p(\mathbb{R}^d, w, Y)$ for such weights w), where $r_0 = p_Y$ is the convexity exponent of Y. On the other hand, [GoY] establishes such scalar weighted estimates $m(A) : L^p(\mathbb{R}^d, w) \to L^p(\mathbb{R}^d, w)$. Going into the parameter calculations in [GoY, ALV], one obtains that A has a bounded $\mathcal{H}^{\beta'}_\infty$ calculus on $L^p(\mathbb{R}^d, Y)$ for $\beta' > \frac{d}{p_Y}$ and $p_Y < p < \infty$ and for $\beta' > \frac{d}{q_Y'}$ and $1 < p < q_Y$. This result and ours from Theorem 7.5 are incomparable, since this $\mathcal{H}^{\beta'}_\infty$ class and our $\mathcal{H}^\beta_2$ class are not contained in each other, also due to the fact that we take into account the concavity exponent q_Y in addition to the convexity exponent p_Y. Moreover, we also obtain square function estimates in Theorem 7.5, see*

Remark 7.12 below. On the other hand, [GoY, ALV] obtain weighted UMD lattice valued estimates.

Remark 7.11 *We note that if $Y(\Omega') = [Z(\Omega'), L^2(\Omega')]_\theta$ is a complex interpolation space with Z a further UMD lattice and $\theta \in (0,1)$, then one can apply complex interpolation to improve the derivation exponent in the Hörmander calculus of Theorem 7.5. Note however that one passes from an exponent which is maybe not optimal to another one again not optimal. One obtains a $\mathcal{H}^{\beta_\theta}_{q_\theta}$ Hörmander calculus for A on $L^p(\Omega, Y)$. Numerically, one gets in case $\frac{1}{p_Y} = 1 - \frac{\theta}{2}$ and $\frac{1}{q_Y} = \frac{\theta}{2}$ that $\beta_\theta > 2d|\frac{1}{p} - \frac{1}{2}| + |\frac{1}{p} - \frac{1}{2}|$ for $p \in (1, p_Y)$ or $p \in (q_Y, \infty)$ and a certain $q_\theta \in (2, \infty)$. For p close to p_Y, the differentiation index β_θ is close to $\frac{1}{2} - |\frac{1}{p_Y} - \frac{1}{2}|$ better than what gives Theorem 7.5.*

Remark 7.12 *Note that since the proofs of Theorem 7.1 and 7.5 go through an application of Theorem 4.11, the statements of these Theorems can be strengthened in that they contain also square function estimates. That is, under the assumptions of Theorem 7.1 resp. 7.5 with the corresponding choice of the differentiation parameter β, one gets for a bounded family $(m_i)_{i=1}^k$ in $\mathcal{H}^\beta_2$ that*

$$\left\| \left(\sum_{i=1}^k |m_i(A) f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,Y)} \lesssim \max_{i=1}^k \|m_i\|_{\mathcal{H}^\beta_2} \left\| \left(\sum_{i=1}^k |f_i|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\Omega,Y)}.$$

7.1 Gaussian estimates

In this subsection, we show that for many examples of differential operators in different contexts, Gaussian estimates and H^∞ calculus on $L^p(\Omega, Y)$ are available, and thus Theorem 7.5 on the Hörmander calculus on $L^p(\Omega, Y)$ applies for $1 < p < \infty$ and Y a UMD lattice. We recall that the derivation exponent $\alpha = \alpha(p, p_Y, q_Y) \in (0,1)$ is given in (7.5) and p_Y resp. q_Y define the convexity and concavity exponents in $(1,\infty)$ of the lattice Y to which Y^{p_Y} and $(Y')^{q_Y'}$ are still UMD lattices. For example, if $Y = L^s(\Omega')$ for some $s \in (1,\infty)$, then any $p_Y \in (1,s)$ and any $q_Y \in (s,\infty)$ are admissible.

Manifolds Let $\Omega = M$ be a complete Riemannian manifold with non-negative Ricci curvature. Then the heat semigroup (associated with the Laplace-Beltrami operator) is a symmetric contraction semigroup with Gaussian estimates (4.9) of order $m = 2$. See [LY], [GriTel, page 3/70 (1.3)], [Sal]. Hence on these manifolds, according to Theorem 7.5 and Proposition 7.9, the heat semigroup has a Hörmander $\mathcal{H}^\beta_2$ calculus on $L^p(\Omega, Y)$ for $1 < p < \infty$, for any UMD lattice Y and $\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2}$.

Schrödinger and differential operators We show now that our main results apply for several Schrödinger operators.

Start with the case that $\Omega = M$ is a connected and complete Riemannian manifold with non-negative Ricci curvature. Consider a potential $V : \Omega \to \mathbb{R}$ such that $V \geqslant 0$ and $V \in L^1_{loc}(\Omega)$. Then $A = -\Delta + V$, defined by the quadratic form technique, generates a selfadjoint semigroup $(T_t)_t$ on $L^2(\Omega)$, and moreover, as a consequence of the Trotter-Kato product formula, $|T_t f(x)| \leqslant S_t|f|(x)$, where $S_t = \exp(t\Delta)$ is the heat semigroup [DuOS, Section 7.4]. According to the preceding paragraph on manifolds, S_t is L^1 and L^∞ contractive, so according to Proposition 7.9, A has an H^∞ calculus on $L^p(\Omega, Y)$ for $1 < p < \infty$ and Y any UMD lattice. Moreover, $(T_t)_t$

has Gaussian estimates (4.9) of order $m = 2$ [DuOS, (7.8)], so that according to Theorem 7.5, A has a $\mathcal{H}_2^\beta$ calculus on $L^p(\Omega, Y)$ with $\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2}$.

Now consider the case that $\Omega \subseteq \mathbb{R}^d$ is an open subset of homogeneous type. Take the following selfadjoint differential operator defined on $L^2(\Omega)$ [Ouh06, (1)]:

$$A = -\sum_{k,j=1}^{d} \frac{\partial}{\partial x_j}\Big(a_{kj}\frac{\partial}{\partial x_k}\Big)$$

where $a_{kj} = a_{jk} \in L^\infty(\Omega, \mathbb{R})$, $1 \leqslant k, j \leqslant d$ and a_{kj} satisfy the standard ellipticity condition $\eta I \leqslant (a_{kj})_{kj} \leqslant \mu I$ for some constants $0 < \eta < \mu < \infty$. We assume Dirichlet boundary conditions. Then according to [Ouh06, Theorem 1], the semigroup $(T_t)_t$ generated by A is positive and according to [Ouh06, (4)], satisfies Gaussian estimates 4.9 with $m = 2$ (note that $V(x, \sqrt{t}) \leqslant Ct^{\frac{d}{2}}$ there). Thus according to Proposition 7.9, A has an H^∞ calculus on $L^p(\Omega, Y)$ for Y any UMD lattice, and according to Theorem 7.5, A has a $\mathcal{H}_2^\beta$ calculus on $L^p(\Omega, Y)$ for $\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2}$.

Now consider the case that $\Omega = \mathbb{R}^d$ and a potential $V : \Omega \to \mathbb{R}$ such that $V = V^+ - V^-$, $V^+, V^- \geqslant 0$ and V^+, V^- belong to the Kato class (see [Sim]). Then according to [Ouh06, Corollary 3], $A = -\Delta + V$, the selfadjoint Schrödinger operator with potential V, generates a positive semigroup $(T_t)_t$ (even with a certain lower Gaussian estimate), and according to [Ouh06, Theorem 1], the shifted semigroup generated by $A - s(A) + \varepsilon$ for an $\varepsilon > 0$ has Gaussian estimates (4.9). Here, $s(A) = \inf \sigma(A)$ is the spectral bound of A in $L^2(\mathbb{R}^d)$. Thus, according to Proposition 7.9 and Theorem 7.5, $A - s(A) + \varepsilon$ has a $\mathcal{H}_2^\beta$ calculus on $L^p(\Omega, Y)$ for $\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2}$.

Now consider for $\lambda > 0$ the Bessel operator

$$A = \Delta_\lambda = -x^{-\lambda}\frac{d}{dx}x^{2\lambda}\frac{d}{dx}x^{-\lambda} = -\frac{d^2}{dx^2} + \lambda(\lambda - 1)x^{-2}$$

on $\Omega = (0, \infty)$ [BCRM, page 343]. Then according to [Ouh95], the semigroup $(T_t)_t$ generated by A satisfies Gaussian estimates (4.9) provided that the potential $\lambda(\lambda - 1)x^{-2}$ is positive, i.e. $\lambda \geqslant 1$. According to [BCRM, Theorem 1.6], A has an H^∞ calculus on $L^p(\Omega, Y)$ for $1 < p < \infty$ and any UMD space Y that is a complex interpolation space between a Hilbert space and another UMD space. According to [TJ, pages 218-219, Theorem 28.1] and [Lor, Proposition 3.3.6], any UMD lattice is of this form. Thus, for the particular case Y being a UMD lattice and $\lambda \geqslant 1$, we can strengthen [BCRM, Theorem 1.6] and deduce for Δ_λ a $\mathcal{H}_2^\beta$ calculus on $L^p(\Omega, Y)$ for $\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2}$.

There are other Schrödinger and differential operators, where Gaussian estimates are available and the semigroup is positive, hence Proposition 7.9 and Theorem 7.5 apply. We refer to [Ouh06], [Ouh05, Section 6.4, in particular Theorems 6.10, 6.11] for upper Gaussian estimates, and for lower Gaussian estimates [Ouh06, Section 7.8].

Lie groups of polynomial volume growth Consider $\Omega = G$ a Lie group having polynomial volume growth. Then Ω is a space of homogeneous type. Consider moreover $A = -\sum_{k=1}^N X_k^2$, where $\{X_1, \ldots, X_N\}$ is a family of left invariant vector fields having the Hörmander property. For example, $G = \mathbb{R}^{2n+1}$ is the Heisenberg group, and $A = -\sum_{k=1}^n X_k^2 + Y_k^2$ is the standard Sublaplacian. Then according to [Sal, Theorem 4.2, Example 2], [Gri], the semigroup $(T_t)_t$ satisfies two-sided Gaussian estimates (4.9) and (4.10) with $m = 2$. Therefore, according to Proposition 7.9 and Theorem 7.5, A has a $\mathcal{H}_2^\beta$ calculus on $L^p(\Omega, Y)$ for $\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2}$.

Fractals There are several fractals $\Omega \subseteq \mathbb{R}^n$ on which there exists a heat semigroup satisfying upper and lower Gaussian estimates. Namely, one first turns Ω into a metric measure space by choosing a metric, e.g. the intrinsic metric inherited from $\mathbb{R}^n$ and a Hausdorff measure. Then, the heat generator A is defined using the form method, often by means of a Brownian motion Dirichlet form [GriTel, preprint version, pages 3-4]. The heat kernel $p_t(x,y)$ satisfies [GriTel, (1.4)]

$$p_t(x,y) \cong \frac{C}{t^{\alpha/\beta}} \exp\left(-c\left(\frac{d^\beta(x,y)}{t}\right)^{\frac{1}{\beta-1}}\right)$$

for certain $\alpha > 0$ and $\beta > 1$ ($\beta \geqslant 2$ according to [GHL, Abstract]), and the implied constant c may be different between upper and lower estimate. The parameter β is called walk dimension. In case of volume comparability $V(x,t) \cong t^\alpha$, e.g. if A is the Laplace operator on the Sierpinski Gasket [DuOS, Section 7.11], we can apply our Proposition 7.9 and Theorem 7.5 to deduce that A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\Omega, Y)$ for $1 < p < \infty$ and Y a UMD lattice, with $\beta > \alpha(p, p_Y, q_Y) \cdot d + \frac{1}{2}$.

For a discussion of many further examples where Gaussian estimates as in (4.9) are satisfied, we refer to [DuOS, Section 7]. Hence in all these cases, Proposition 7.8 and Theorem 7.5 are applicable and we obtain for the operators A a bounded $\mathcal{H}^\beta_2$ calculus on $L^p(\Omega, L^s(\Omega'))$ with $1 < p, s < \infty$ and $\beta > \big(\max(\frac{1}{p}, \frac{1}{s}, \frac{1}{2}) - \min(\frac{1}{p}, \frac{1}{s}, \frac{1}{2})\big) \cdot d + \frac{1}{2}$.

7.2 Dispersive estimates

In this subsection, we indicate in which situations of the preceding subsection there is a dispersive estimate

$$\|\exp(itA)\|_{L^1(\Omega)\to L^\infty(\Omega)} \leqslant C|t|^{-\frac{d}{2}} \quad (t \in \mathbb{R}\backslash\{0\}) \tag{7.8}$$

available, so that Theorem 7.1 is applicable, and we can deduce a $\mathcal{H}^\beta_2$ calculus on $L^p(\Omega, Y)$ for $\beta > \frac{d}{2} + \frac{1}{2}$. This is a smaller differentiation order, hence a better result than what we had obtained in the preceding subsection, in case that $\alpha(p, p_Y, q_Y) > \frac{1}{2}$, e.g. if p is close to ∞ and Y is an $L^s(\Omega')$ space with s close to 1.

Schrödinger operators Throughout the paragraph, we assume $\Omega = \mathbb{R}^d$ and $A = -\Delta + V$ a Schrödinger operator with positive locally integrable potential.

First, consider the case $d = 1$. Then if $\int_{\mathbb{R}} V(x)(1+|x|)\,dx < \infty$, if there is no resonance at zero energy and if there are no bound states (which implies that the spectral projection $P_{ac}(A)$ onto the absolutely continuous spectral subspace is the identity), then according to [GoS, Theorem 1], A satisfies (7.8). Consequently, A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\mathbb{R}, Y)$ for any $\beta > \frac{1}{2} + \frac{1}{2} = 1$.

Second, consider the case $d = 3$. Then if $V(x) \leqslant C(1+|x|)^{-b}$ for some $b > 3$ and all $x \in \mathbb{R}^3$, if 0 is neither an eigenvalue of A nor a resonance, and if there are no bound states, then according to [GoS, Theorem 2], (7.8) holds. Consequently, A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\mathbb{R}^3, Y)$ for any $\beta > \frac{3}{2} + \frac{1}{2} = 2$.

Next, consider $d \in \mathbb{N}$ an arbitrary odd value. Then if $V \in C^{\frac{d-3}{2}}(\mathbb{R}^d)$ for $d \in \{5, 7\}$, if $V(x) \leqslant c(1+|x|)^{-b}$ for some $b > \frac{3d+5}{2}$ and for $1 \leqslant j \leqslant \frac{d-3}{2}$, $|\nabla^j V(x)| \leqslant c(1+|x|)^{-a}$ for some $a > 3$ for $d = 5$ and for some $a > 8$ for $d = 7$, if 0 is not an eigenvalue of A and if there are no bound states, then according to [ErGr, Theorem 1.1], (7.8) holds. Consequently, A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\mathbb{R}^d, Y)$ for any $\beta > \frac{d}{2} + \frac{1}{2}$.

Now, if V is of the form $V(x_1,\dots,x_d) = W(x_1) + W(x_2) + \dots + W(x_d)$ with $W : \mathbb{R} \to \mathbb{R}_+$ such that $\int_{\mathbb{R}} W(x)(1+|x|)^2\,dx < \infty$, then according to [Pier, Corollary 1.6], (7.8) holds. Consequently, A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\mathbb{R}^d, Y)$ for any $\beta > \frac{d}{2} + \frac{1}{2}$.

Stratified Lie groups We refer to the recent work [BFG] for a study when (7.8) or a stronger estimate holds in the case that $\Omega = G$ is a 2-step stratified Lie group with further properties and $A = -\Delta$ is the Laplace-Beltrami operator.

7.3 Generalised Gaussian estimates

In the recent past, several operators with generalised Gaussian estimates (7.4) for some $p_0 > 1$ have been studied. In these cases we will obtain according to Proposition 7.8 and Theorem 7.5 that A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\Omega, L^s(\Omega'))$ for $p_0 < p, s < p_0'$ and

$$\beta > \left(\max\left(\frac{1}{p}, \frac{1}{s}, \frac{1}{2}\right) - \min\left(\frac{1}{p}, \frac{1}{s}, \frac{1}{2}\right)\right) \cdot d + \frac{1}{2}. \tag{7.9}$$

Elliptic operators in divergence form Suppose that $\Omega = \mathbb{R}^d$ and A is given by

$$Af = \sum_{|\gamma|,|\delta|=m} (-1)^{|\delta|} \partial^\delta (a_{\gamma\delta} \partial^\gamma f),$$

where $a_{\gamma\delta} \in L^\infty(\Omega, \mathbb{R})$. We suppose that the form $\mathfrak{a}$ associated with A, given by

$$\mathfrak{a}(f,g) = \int \sum_{|\gamma|,|\delta|=m} a_{\gamma\delta}(x) \partial^\gamma f(x) \overline{\partial^\delta g(x)}\,dx$$

gives rise to a selfadjoint operator and satisfies the ellipticity condition

$$\mathfrak{a}(f,f) \geqslant \eta \big\| (-\Delta)^{\frac{m}{2}} f \big\|_2^2 \quad (f \in W^{m,2}(\mathbb{R}^d))$$

for some $\eta > 0$. Then according to [KuUl2, Section 3 a) (iii)], (7.4) holds with m replaced by $2m$ and $p_0 = p_1'$, where $p_1 = \frac{2d}{d-2m}$ for $d > 2m$, and $p_1 = \infty$ if $d < 2m$. Consequently, A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\mathbb{R}^d, L^s(\Omega'))$ with $p_0 < p, s < p_0'$ and β given by (7.9).

Schrödinger operators with singular potentials Suppose again that $\Omega = \mathbb{R}^d$, with $d \geqslant 3$, and that $A = -\Delta + V$ is a Schrödinger operator. We suppose that $V = V^+ - V^-$ with $V^+, V^- : \mathbb{R}^d \to \mathbb{R}_+$ and that V^+ is locally integrable and V^- belongs to the pseudo-Kato class [KPS]. A typical example is $V(x) = -\frac{c}{|x|^2}$ for a certain range of $c > 0$ [KPS, KuUl2]. Then A is selfadjoint, and according to [KuUl2, Section 3 (c) (ii)], (7.4) holds for some $p_0 > 1$. Consequently, A has a $\mathcal{H}^\beta_2$ calculus on $L^p(\mathbb{R}^d, L^s(\Omega'))$ for any $p_0 < p, s < p_0'$ and β as in (7.9).

We refer to [Bl, Section 2],[KuUl2, Section 3] and the references therein for detailed explanations of the two preceding paragraphs and more examples.

7.4 Concluding remarks

In Propositions 7.7, 7.8 and 7.9, we gave some sufficient conditions, when A has an H^∞ calculus on $L^p(\Omega, Y)$. Nevertheless, it would be interesting to know whether generalised Gaussian

estimates and selfadjointness of the semigroup T_t imply already themselves that A has an H^∞ calculus (and thus a $\mathcal{H}^\beta_2$ calculus) on $L^p(\Omega, Y)$ provided that Y is a p_0-convex and p_0'-concave UMD lattice. Already the case of classical Gaussian estimates and selfadjointness is open here (then the convexity and concavity assumption on Y is void).

Another question is whether Theorems 7.1 and 7.5 hold for Y being an intermediate UMD space, that is, $Y = [L^2(\Omega'), Z]_\theta$ for some further UMD space Z and $\theta \in (0,1)$, or even for Y being any UMD space. Then, we suspect that the convexity and concavity notions, which only make sense for lattices, have to be replaced by Rademacher type and cotype (see Definition 4.5). For an $\mathrm{H}^\infty(\Sigma_\omega)$ functional calculus result on $L^p(\Omega, Y)$ with Y an intermediate UMD space and an estimate for the angle $\omega < \frac{\pi}{2}$, we refer to [BCRM, Theorem 1.6] with a particular Bessel operator A, and [Xu15, Theorem 4] for regular contractive and analytic semigroups.

A further question is whether a version of Proposition 7.7 holds for generalised instead of classical Gaussian estimates.

Spectral multiplier theorems such as the $\mathcal{H}^\beta_q$ calculus (see after Definition 4.7) are particularly powerful in case that the optimal exponents β and q are known. In this respect, it would be interesting to determine how the convexity and concavity exponents of Y precisely contribute to the optimal values of β and q. Hoewever, the question about optimal exponents β, q is even open to our knowledge in the scalar case $L^p(\Omega)$ within the class of all selfadjoint semigroups with Gaussian estimates.

8 Maximal and q-variational Hörmander calculus

The results of this section are contained in [HDR9]. In this section, we let A be a 0-sectorial operator acting on $L^p(\Omega, Y)$, where Ω is a σ-finite measure space and $Y = Y(\Omega')$ is a UMD Banach lattice as in Subsection 4.5. In Section 7, particularly the proof of Theorem 7.1 we have seen that in order to establish a Hörmander functional calculus for A, boundedness of maximal operators

$$M(f)(x,\omega') = \sup_{t>0} |m(tA)f(x,\omega')| \tag{8.1}$$

plays an important rôle (in Theorem 7.1, we had $m(\lambda) = \exp(-e^{i\theta}\lambda)$ for some $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$). In the present section, we will go *the other way around* and show that to some extent, a known Hörmander functional calculus for A yields in turn a bound for the maximal operator in (8.1) for *certain* Hörmander spectral multipliers m (see Theorem 8.1 and Proposition 8.3):

$$\Big\| \sup_{t>0} |m(tA)f| \Big\|_{L^p(\Omega,Y)} \leqslant C \|f\|_{L^p(\Omega,Y)}. \tag{8.2}$$

One of the early results of the type (8.2) in the euclidean case for non-special spectral multipliers m is due to Rubio de Francia [RdF2]. It is known from [CGHS] that already in the euclidean case and A being the Laplacian operator, (8.2) cannot hold for all $\mathcal{H}^\alpha_2$ Hörmander multipliers m, even for a large prescribed derivation order α. Other assumptions are therefore needed.

In this direction, if $\Omega = G$ is a stratified Lie group and A is a left invariant Sublaplacian, Mauceri and Meda proved in [MaMe, Theorem 2.6] that (8.2) holds provided that

$$\sum_{n\in\mathbb{Z}} \|m(2^n\cdot)\phi_0\|_{W^c_2(\mathbb{R})} < \infty \tag{8.3}$$

where $\phi_0 \in C^\infty_c(0,\infty)$ satisfies $\phi_0(t) = 1$ for $t \in (1,2)$, $W^c_2(\mathbb{R})$ stands for the usual Sobolev space with derivation exponent $c > Q(\frac{1}{p} - \frac{1}{2}) + \frac{1}{2}$ $(1 < p \leqslant 2)$ or $c > (Q-1)(\frac{1}{2} - \frac{1}{p}) + \frac{1}{2}$

$(2 \leqslant p \leqslant \infty)$, Q denoting the homogeneous dimension of the group. Note that it is well-known that if the sum over n in (8.3) is replaced by the supremum, then one obtains the Hörmander $\mathcal{H}^c_2$ norm. A more recent result for the usual Laplace operator on euclidean space is that of [CGHS, Theorem 1.2, Corollary 1.3], telling that (8.2) holds with summation (8.3) relaxed to

$$\sum_{n\in\mathbb{Z}} \frac{1}{|n|+1} \|m(2^n\cdot)\phi_0\|_{W_2^{\frac{d}{\min(p,2)}}(\mathbb{R})} < \infty$$

(even relaxed to a certain rearrangement $m(2^n\cdot) \rightsquigarrow m(2^{k_n}\cdot)$ that minimizes the sum). Moreover, in [Choi], Choi extends this result again to the Mauceri-Meda setting of left-invariant subplacians on stratified Lie groups under a slightly more restrictive summation condition involving a supplementary $\log(|n|+2)$ factor. Also, [Wro2] obtained recently an estimate as in (8.2) under another variant of the summability condition (8.3), for the quite general case of symmetric contraction semigroup generators A.

We provide in this section a quite general result of maximal estimates as in (8.2) for multipliers m as e.g. in (8.3), under the only hypothesis of A having a Hörmander calculus on $L^p(\Omega, Y)$. Our approach is again Banach space geometric. Note that the expression on the left hand side of (8.2) is a norm in the space $L^p(\Omega, Y(L^\infty(\mathbb{R}_+)))$, so in a Bochner space. However, the lattice $Y(L^\infty(\mathbb{R}_+))$ looses the UMD property (and other nice geometric properties) that had Y. Our approach in Theorem 8.1 starts with the observation from [MaMe], that there is a Hilbert space $\Lambda^\beta = \Lambda^\beta_{2,2}(\mathbb{R}_+)$ which is continuously embedded in $C_0(\mathbb{R}_+) \subseteq L^\infty(\mathbb{R}_+)$ and which will still carry the function $t \mapsto m(tA)(f)(x,\omega')$ in our context. The space is $\Lambda^\beta_{2,2}(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{C},\ f \circ \exp \in W^\beta_2(\mathbb{R})\}$, so the usual Sobolev space transferred to $\mathbb{R}_+$ via the exponential function, and $\|f\|_{\Lambda^\beta_{2,2}(\mathbb{R}_+)} = \|f \circ \exp\|_{W^\beta_2(\mathbb{R})}$. We shall obtain an estimate

$$\left\| \sup_{t>0} |m(tA)f| \right\|_{L^p(\Omega,Y)} \lesssim_\beta \|t \mapsto m(tA)f\|_{L^p(\Omega, Y(\Lambda^\beta_{2,2}(\mathbb{R}_+)))} \lesssim_m \|f\|_{L^p(\Omega,Y)}\,, \tag{8.4}$$

where the first holds in case $\beta > \frac{1}{2}$.

Then the strategy of the proof of Theorem 8.1, valid for any semigroup generator A on $L^p(\Omega, Y)$ with Hörmander calculus, is to exploit the abstract approach of that functional calculus due to the author and Weis [KrW3]. It allows to expand the norm in $L^p(Y(\Lambda^\beta_{2,2}(\mathbb{R}_+)))$ in (8.4) via the Littlewood-Paley decomposition (see Lemma 4.12) into pieces of compactly supported spectral multipliers. Here the property of $\Lambda^\beta_{2,2}(\mathbb{R}_+)$ being a Hilbert space, thus having much nicer geometrical properties than $C_0(\mathbb{R}_+)$, is appropriate for our Banach space geometrical proof. A second part is to use for these compactly supported spectral multipliers a representation formula (8.8) below which in turn allows to transfer the R-boundedness from certain wave spectral multipliers to the Littlewood-Paley pieces. Everything boils then down to a calculation of the $\Lambda^\beta_{2,2}(\mathbb{R}_+)$ norm of a function h_s defined in terms of the Fourier transform of m (see also the 3rd step of the proof).

Henceforth, we write in short $L^p(Y) = L^p(\Omega, Y)$. Note that type and cotype of a Banach space had been defined in Definition 4.5. The notation $W^c_2(\mathbb{R})$ stands for the usual Sobolev space. The space $Y(\Lambda^\beta_{2,2}(\mathbb{R}_+))$ is well-defined since Y is a Banach lattice which can be expressed as a space of functions living on some measure space Ω' (see Subsection 4.5). The type / cotype below take values $\operatorname{type} L^p(Y) \in (1,2]$ and $\operatorname{cotype} L^p(Y) \in [2,\infty)$, and in case $Y = \mathbb{C}$, $\max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p} - \frac{1}{\operatorname{cotype} L^p}\right) = \frac{1}{2}$. We have the following first result in this section.

Theorem 8.1 *[HDR9, Theorem 3.1] Let Y be a UMD lattice, $1 < p < \infty$ and (Ω, μ) a σ-finite measure space. Let $\beta \geqslant 0$. Let A be a 0-sectorial operator on $L^p(Y)$. Assume that A*

has a $\mathcal{H}^\alpha_2$ calculus on $L^p(Y)$ for some $\alpha > \frac{1}{2}$. Let $m \in W^c_2(\mathbb{R})$ be a spectral multiplier with $\operatorname{supp}(m) \subseteq [\frac{1}{2}, 2]$, *with*

$$c > \alpha + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right) + \frac{1}{2} + \beta. \tag{8.5}$$

Then

$$\|t \mapsto m(tA)f\|_{L^p(Y(\Lambda^\beta_{2,2}(\mathbb{R}_+)))} \leqslant C\|m\|_{W^c_2(\mathbb{R})}\|f\|_{L^p(Y)}. \tag{8.6}$$

Moreover, let $(m_k)_{k\in\mathbb{N}}$ be a family of spectral multipliers in $W^c_2(\mathbb{R})$ with $\operatorname{supp}(m_k) \subseteq [\frac{1}{2}, 2]$. *Then*

$$\left\| \left(\sum_{k\in\mathbb{N}} \|t \mapsto m_k(tA)f_k\|^2_{\Lambda^\beta_{2,2}(\mathbb{R}_+)} \right)^{\frac{1}{2}} \right\|_{L^p(Y)} \leqslant C \sup_{k\in\mathbb{N}} \|m_k\|_{W^c_2(\mathbb{R})} \left\| \left(\sum_{k\in\mathbb{N}} |f_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(Y)}. \tag{8.7}$$

Proof : We shall now explain the main ideas of the proof of Theorem 8.1.

1st step: Reduce the estimate (8.6) to the R-boundedness of certain spectral multipliers Since $\Lambda^\beta = \Lambda^\beta_{2,2}(\mathbb{R}_+)$ is a Hilbert space, for any $n \in \mathcal{H}^\alpha_2$, the operator $n(A) \otimes \mathrm{Id}_{\Lambda^\beta}$ extends to a bounded operator on $L^p(Y(\Lambda^\beta))$ (see also [HDR9, Lemma 2.9]). Thus, A has a bounded $\mathcal{H}^\alpha_2$ calculus on $L^p(Y(\Lambda^\beta))$ and therefore satisfies the Paley-Littlewood decomposition from Lemma 4.12 on $L^p(Y(\Lambda^\beta))$ and on $L^p(Y)$. We express, where $(\phi_n)_{n\in\mathbb{Z}}$ is a dyadic partition of $\mathbb{R}_+$ in the sense of Definition 4.10,

$$\begin{aligned}
\|t \mapsto m(tA)f\|_{L^p(Y(\Lambda^\beta))} &\cong \left\| \left(\sum_{n\in\mathbb{Z}} \|t \mapsto \phi_n(A)m(tA)f\|^2_{\Lambda^\beta} \right)^{\frac{1}{2}} \right\|_{L^p(Y)} \\
&= \left\| \left(\sum_{n\in\mathbb{Z}} \|t \mapsto \phi_n(A)m(2^{-n}tA)f\|^2_{\Lambda^\beta} \right)^{\frac{1}{2}} \right\|_{L^p(Y)} \\
&= \left\| \left(\sum_{n\in\mathbb{Z}} \|t \mapsto \psi(t)m(2^{-n}tA)\phi_n(A)f\|^2_{\Lambda^\beta} \right)^{\frac{1}{2}} \right\|_{L^p(Y)} \\
&\lesssim R\left(\{t \mapsto \psi(t)m(2^ntA) : \ n \in \mathbb{Z}\}_{L^p(Y)\to L^p(Y(\Lambda^\beta))} \right) \left\| \left(\sum_{n\in\mathbb{Z}} |\phi_n(A)f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(Y)} \\
&\cong R\left(\{t \mapsto \psi(t)m(2^ntA) : \ n \in \mathbb{Z}\}_{L^p(Y)\to L^p(Y(\Lambda^\beta))} \right) \|f\|_{L^p(Y)}.
\end{aligned}$$

Here, we have used the Paley-Littlewood decomposition in the space $L^p(Y(\Lambda^\beta))$ from Lemma 4.12 in the first line and the dilation invariance of the Λ^β norm ($\|g\|_{\Lambda^\beta} = \|g(c\cdot)\|_{\Lambda^\beta}$ for $c > 0$) in the second line. Moreover, we have used that $\phi_n(A)m(2^{-n}tA)f = 0$ for $t \notin [2^{-2}, 2^2]$ (supports of the spectral multipliers!) and thus introduced a function $\psi \in C^\infty_c(\mathbb{R}_+)$ with support in $[2^{-3}, 2^3]$ and $\psi(t) = 1$ for $t \in [2^{-2}, 2^2]$ in the third line. Finally, in the fourth line, we used R-boundedness together with the square function equivalence to Rademacher sums (cf. (4.1)) and in the fifth line, the Paley-Littlewood decomposition in the space $L^p(Y)$ from Lemma 4.12.

It remains to estimate the R-bound of the family $\{t \mapsto \psi(t)m(2^ntA) : \ n \in \mathbb{Z}\}_{L^p(Y)\to L^p(Y(\Lambda^\beta))}$.

2nd step: Expressing a general spectral multiplier as an integral over wave spectral multipliers Let $\gamma > 0$ be a value determined below. We write with [HDR9, Lemma 2.17] the operator valued Fourier inversion formula

$$\begin{aligned}\psi(t)m(2^n tA)f &= \psi(t)m(2^n tA)\phi(2^n A)f \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{t}\hat{m}\left(\frac{s}{t}\right)\psi(t)(1+2^n A)^{-\gamma}\exp(i2^n sA)(1+2^n A)^{\gamma}\phi(2^n A)f\,ds,\end{aligned} \tag{8.8}$$

where $\phi \in C_c^\infty(\mathbb{R}_+)$ with $\phi(s) = 1$ for $s \in \operatorname{supp}(m(t\cdot)) \subseteq [2^{-5}, 2^5]$ where $t \in \operatorname{supp}(\psi)$, so that $m(ts) = m(ts)\phi(s)$ for such t. According to [HDR9, Proposition 2.14], the bounded $\mathcal{H}_2^\alpha$ calculus of A on $L^p(Y)$ improves to an R-bounded $\mathcal{H}_2^\gamma$ calculus, where

$$\gamma > \alpha + \max\left(\frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}, \frac{1}{2}\right).$$

Here we need that $L^p(Y)$ has property (α) from Definition 4.5. This means that $\{n(A) : \|n\|_{\mathcal{H}_2^\gamma} \leqslant 1\}$ is R-bounded over $L^p(Y)$. As $\|(1+(\cdot))^\gamma\phi\|_{\mathcal{H}_2^\gamma} < \infty$ and the $\mathcal{H}_2^\gamma$ norm is dilation invariant, the set

$$\{(1+2^n A)^\gamma \phi(2^n A) : \ n \in \mathbb{Z}\}_{L^p(Y)\to L^p(Y)}$$

is R-bounded. Since the composition $\tau \circ \sigma = \{T \circ S : \ T \in \tau, S \in \sigma\}$ of two R-bounded sets τ and σ is again R-bounded, in view of (8.8) and the last sentence of the 1st step, it remains to estimate the R-bound of the following family from $B(L^p(Y), L^p(Y(\Lambda^\beta)))$:

$$\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{t}\psi(t)\hat{m}\left(\frac{s}{t}\right)\langle s\rangle^{\gamma+\delta}\langle s\rangle^{-(\gamma+\delta)}(1+2^n A)^{-\gamma}\exp(i2^n sA)ds : \ n \in \mathbb{Z}\right\}, \tag{8.9}$$

where we pick any $\delta > 1$.

3rd step: R-boundedness of the operator valued integral (8.9) This is the most technical part. It uses the R-boundedness of the operator valued factor in (8.9), which is

$$\left\{\langle s\rangle^{-(\gamma+\delta)}(1+2^n A)^{-\gamma}\exp(i2^n sA) : \ n \in \mathbb{Z}\right\}.$$

This is a consequence of the R-bounded $\mathcal{H}_2^\gamma$ calculus. Moreover, for the scalar valued factor which is $h_s(t) = \frac{1}{t}\hat{m}\left(\frac{s}{t}\right)\psi(t)\langle s\rangle^{\gamma+\delta}$, it uses the compact support assumption on m and the fact that m belongs to $W_2^c(\mathbb{R})$ to obtain weighted integrability of its Fourier transform. It is this part, which explains the summands $\frac{1}{2} + \beta$ in the formula for the derivation parameter c, (8.5). On the other hand, the summands $\alpha + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right)$ come from the $\mathcal{H}_2^\alpha$ Hörmander calculus assumption and the enhancement to the R-bounded $\mathcal{H}_2^\gamma$ Hörmander calculus with $\gamma = \alpha + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right)$. We refer to [HDR9, End of the proof of Theorem 3.1], where the Λ^β norm of h_s is calculated.

Altogether, we have proved the Λ^β estimate for single spectral multipliers, (8.6). With essentially the same method, since A has an R-bounded $\mathcal{H}_2^\gamma$ calculus on $L^p(Y)$, one can enhance to the square function estimate involving a family of spectral multipliers, (8.7). Again we refer to [HDR9, Theorem 3.1] for the details. ∎

Using the dilation invariance structure of the $\Lambda^\beta_{2,2}(\mathbb{R}_+)$ norm, we can easily generalise Theorem 8.1 in the following corollary, where the compact support condition on the spectral multiplier m is replaced by a summability condition of norms of dilates of m.

Corollary 8.2 *Let Y be a UMD lattice, $1 < p < \infty$ and (Ω, μ) a σ-finite measure space. Let $\beta \geqslant 0$. Let A be a 0-sectorial operator on $L^p(Y)$. Assume that A has a $\mathcal{H}^\alpha_2$ calculus on $L^p(Y)$. Pick as in Theorem 8.1, (8.5)*

$$c > \alpha + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right) + \frac{1}{2} + \beta.$$

Let m be a spectral multiplier with $m(0) = 0$ such that for some dyadic partition $(\phi_n)_{n\in\mathbb{Z}}$ of $\mathbb{R}_+$, we have $\sum_{n\in\mathbb{Z}} \|m(2^n\cdot)\phi_0\|_{W^c_2(\mathbb{R})} < \infty$. Then

$$\|t \mapsto m(tA)f\|_{L^p(Y(\Lambda^\beta_{2,2}(\mathbb{R}_+)))} \leqslant C \sum_{n\in\mathbb{Z}} \|m(2^n\cdot)\phi_0\|_{W^c_2(\mathbb{R})} \|f\|_{L^p(Y)}. \tag{8.10}$$

Proof : Let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity as in Definition 4.10. Write simply $m(\lambda) = \sum_{n\in\mathbb{Z}} m(\lambda)\phi_n(\lambda)$ and accordingly $\|t \mapsto m(tA)f\|_{L^p(Y(\Lambda^\beta))} \leqslant \sum_{n\in\mathbb{Z}} \|t \mapsto (m\phi_n)(tA)f\|_{L^p(Y(\Lambda^\beta))}$. Use now the fact that the cut spectral multipliers $m\phi_n$ have compact support in $[2^{n-1}, 2^{n+1}]$ and the above mentioned dilation invariance of Λ^β to use Theorem 8.1 for each piece. We refer to [HDR9, Corollary 3.3] for the details. ∎

Note that there is also a square function version of Corollary 8.2 for a family of spectral multipliers, see [HDR9, Corollary 3.3].

As a simple example, a function $m\colon (0,\infty) \to \mathbb{C}$ satisfies the assumptions of Corollary 8.2 provided that m is of class $C^c(0,\infty)$ with $c \in \mathbb{N}$ satisfying (8.5), m vanishes on $(0,1]$ and for some $\varepsilon > 0$,

$$\left|\lambda^k \frac{d^k}{d\lambda^k} m(\lambda)\right| \leqslant C_m \lambda^{-\varepsilon} \quad (k = 0, 1, \ldots, c,\ \lambda > 1).$$

The Λ^β vectorial estimate in Theorem 8.1 and Corollary 8.2 is particularly interesting in case that $\beta > \frac{1}{2}$. Indeed, in this case we have the continuous Sobolev embedding

$$\Lambda^\beta_{2,2}(\mathbb{R}_+) \hookrightarrow C_0(\mathbb{R}_+), \tag{8.11}$$

where $C_0(\mathbb{R}_+)$ is as usual equipped with the supremum norm. Thus, Theorem 8.1 and Corollary 8.2 yield maximal estimates for $\beta > \frac{1}{2}$. Second, observe that if $m : [0,\infty) \to \mathbb{C}$ is a Hörmander spectral multiplier such that $\lim_{t\to 0+} m(t) \neq 0$, then the sum on the right hand side of (8.10) is infinite, so that Corollary 8.2 is not applicable to yield a bounded $L^p(Y(\Lambda^\beta))$ estimate. However, for a maximal estimate

$$\|t \mapsto m(tA)f\|_{L^p(Y(L^\infty(\mathbb{R}_+)))} \leqslant C_m \|f\|_{L^p(Y)}, \tag{8.12}$$

the case $m(0) \neq 0$ does not always cause a problem. Indeed, taking $m(\lambda) = e^{-\lambda}$, so that $m(0) = 1$, then (8.12) holds true for submarkovian semigroups, see [Xu15, Theorem 2].

Using this observation together with (8.11), we then obtain the following.

Proposition 8.3 *[HDR9, Proposition 3.9] Let $Y = Y(\Omega')$ be a UMD lattice, $1 < p < \infty$ and (Ω, μ) a σ-finite measure space. Let A be a 0-sectorial operator on $L^p(Y)$. Assume that A has a $\mathcal{H}^\alpha_2$ calculus on $L^p(Y)$. In this proposition, we also assume that A is of the form $A = A_0 \otimes \mathrm{Id}_Y$, where A_0 is a generator of a submarkovian semigroup. Choose some integer c with*

$$c > \alpha + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right) + 1.$$

Let $m \in C^{c+1}[0,\infty)$ be a spectral multiplier satisfying

$$\max_{k=0}^{c} \sup_{n\leqslant 0} \int_{2^{n-1}}^{2^{n+1}} |2^{nk} m^{(k+1)}(\lambda)|^2 \frac{d\lambda}{\lambda} + \sum_{n\geqslant 0} \int_{2^{n-1}}^{2^{n+1}} |2^{nk} m^{(k)}(\lambda)|^2 \frac{d\lambda}{\lambda} \leqslant D.$$

Then for almost every $(x,\omega) \in \Omega \times \Omega'$, $t \mapsto m(tA)f(x,\omega)$ belongs to $L^\infty(\mathbb{R}_+)$ and

$$\left\| \sup_{t>0} |m(tA)f| \right\|_{L^p(Y)} \leqslant C\,(|m(0)| + D)\,\|f\|_{L^p(Y)}. \tag{8.13}$$

Proof : Decompose the spectral multiplier into $m(\lambda) = m(0)e^{-\lambda} + (m(\lambda) - m(0)e^{-\lambda})$ and accordingly,

$$\left\| \sup_{t>0} |m(tA)f| \right\|_{L^p(Y)} \leqslant |m(0)| \left\| \sup_{t>0} |e^{-tA}f| \right\|_{L^p(Y)} + \left\| \sup_{t>0} |(m(tA) - m(0)e^{-tA})(f)| \right\|_{L^p(Y)}.$$

Use then the result [Xu15, Theorem 2] for the part $m(0)e^{-tA}$. Moreover, the spectral multiplier function $n(\lambda) = m(\lambda) - m(0)e^{-\lambda}$ will satisfy the assumptions of Corollary 8.2, and the choice of the differentiation parameter c in the proposition allows to use the embedding (8.11). We refer to [HDR9, Proposition 3.9] for the details. ∎

Note the original source [HDR9, Proposition 3.9] allows a slightly more general context and non-integer c, and also yields a square function result [HDR9, (3.14)]. We spell out the particular case of two important Hörmander spectral multipliers: wave operators and Bochner-Riesz means.

Corollary 8.4 *Assume that the hypotheses of Proposition 8.3 hold. Let c as in this proposition and*

$$\delta > c > \alpha + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right) + 1.$$

Then the wave operators associated with A satisfy the maximal estimate

$$\left\| \sup_{t>0} \left|(1+tA)^{-\delta} \exp(itA) f\right| \right\|_{L^p(Y)} \leqslant C\|f\|_{L^p(Y)}.$$

Moreover, let

$$\gamma > c - \frac{1}{2} > \alpha + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right) + \frac{1}{2}.$$

Then the Bochner-Riesz means associated with A satisfy the maximal estimate

$$\left\| \sup_{t>0} \left| \left(1 - \frac{A}{t}\right)_+^{\gamma} f \right| \right\|_{L^p(Y)} \leqslant C\|f\|_{L^p(Y)}.$$

Note that since the embedding (8.11) goes into $C_0(\mathbb{R}_+)$, the functions $t \mapsto m(tA)f(x,\omega)$ that we consider converge to 0 as $t \to 0+$ and $t \to \infty$ under the correct hypotheses. In summary, we obtain

Remark 8.5 *Let the hypotheses of Proposition 8.3 hold. Then we obtain a pointwise convergence for $f \in L^p(Y)$ and a.e. $(x,\omega) \in \Omega \times \Omega'$:*

$$m(tA)f(x,\omega) \to m(0)f(x,\omega) \quad (t \to 0+),$$
$$m(tA)f(x,\omega) \to m(0)Pf(x,\omega) \quad (t \to \infty),$$

where $P : L^p(Y) \to L^p(Y)$ denotes the projection onto the null-space of A. In particular, we have with δ and γ as in Corollary 8.4, for $f \in L^p(Y)$ and a.e. $(x,\omega) \in \Omega \times \Omega'$:

$$(1+tA)^{-\delta}\exp(itA)f(x,\omega) \to f(x,\omega) \quad (t \to 0+),$$
$$(1+tA)^{-\delta}\exp(itA)f(x,\omega) \to Pf(x,\omega) \quad (t \to \infty),$$
$$(1-tA)_+^{\gamma} f(x,\omega) \to f(x,\omega) \quad (t \to 0+),$$
$$(1-tA)_+^{\gamma} f(x,\omega) \to Pf(x,\omega) \quad (t \to \infty).$$

Remark 8.6 *In Proposition 8.3 above, in case that $A_0 \otimes \mathrm{Id}_Z$ has a Hörmander calculus $\mathcal{H}^\alpha_2$ on $L^q(Z)$ for any $1 < q < \infty$ and any UMD lattice Z with a uniform value of α, then the parameter c in Proposition 8.3 can be improved by complex interpolation. We refer to [HDR9, Remark 3.7] for the details.*

Remark 8.7 *In [Wro2, Theorem 3.1]. there is an easier proof of (8.13) applying in most of the interesting cases of operators A (the statement there concerns only scalar valued $L^p(\Omega)$ spaces, but seems verbatim to translate to the UMD lattice valued space case). Wróbel's proof uses imaginary powers in place of the wave operators $\exp(isA)$ as they appear in our proof of Theorem 8.1, and he imposes a different norm on m, which is slightly bigger in its derivation exponent, that is $c = \alpha + 2$, (note that in case of scalar valued $L^p(\Omega)$ spaces, we always have $\max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(\Omega)} - \frac{1}{\operatorname{cotype} L^p(\Omega)}\right) = \frac{1}{2}$, so that our $c > \alpha + \frac{3}{2}$). The exact norm in [Wro2] is uncomparable to our expression in (8.13), since the integration exponent in [Wro2, $C(n,\beta)$, page 146] is $q = 1$, whereas we have $q = 2$ in our space $W^c_q(\mathbb{R}) = W^c_2(\mathbb{R})$. Note that [Wro2] does not yield the pointwise convergence from Remark 8.5 nor square function estimates as we do in Proposition 8.3; but a sort of Paley-Littlewood equivalence in [Wro2, Section 4].*

Remark 8.8 *In the special cases of euclidean Laplacian [CGHS] and Sublaplacian on a stratified Lie group [MaMe, Choi], in these sources, for the scalar case $Y = \mathbb{C}$, somewhat bigger classes of maximal multipliers m than in Proposition 8.3 are obtained.*

8.1 q-variation of Hörmander spectral multipliers

In this subsection, we shortly indicate how the results from Section 8 can be strengthened, replacing the supremum from Proposition 8.3 to a larger quantity, the q-variation. We refer to the recent papers [LMX], [HoMa2], [BMSW1] for the use of q-variation for spectral multipliers.

Definition 8.9 *Let $q \in [2,\infty)$. For a function $a : \mathbb{R}_+ \to \mathbb{C}$, we define the q-variation*

$$\|a\|_{V^q} = \sup\left\{\left(|a_{t_0}|^q + \sum_{k=1}^{\infty} |a_{t_k} - a_{t_{k-1}}|^q\right)^{\frac{1}{q}}\right\},$$

where the supremum runs over all increasing sequences $(t_k)_{k\in\mathbb{N}}$ in $\mathbb{R}_+$. Then the space V^q consisting of all functions with finite q-variation is a Banach space [LMX, Section 1].

The following simple observation is at the heart of the proof of Proposition 8.11 below.

Lemma 8.10 *Let $q \in [2,\infty)$ and $\beta > \frac{1}{2}$. Then $\Lambda^\beta_{2,2}(\mathbb{R}_+) \hookrightarrow V^q$, where we take for the equivalence class of an element $a \in \Lambda^\beta_{2,2}(\mathbb{R}_+)$ the (unique) continuous representative.*

Proof : See [HDR9, Lemma 4.2]. ∎

Thanks to Lemma 8.10, we have the following variant of Proposition 8.3, which yields finite q-variation spectral multipliers. Thus we extend [LMX, (1.3)] resp. [HoMa2, (1.3)] from the semigroup to general spectral multipliers, on scalar resp. UMD lattice valued L^p spaces in case that A has a Hörmander calculus.

Proposition 8.11 *Let $Y = Y(\Omega')$ be a UMD lattice, $1 < p < \infty$ and (Ω,μ) a σ-finite measure space. Let A be a 0-sectorial operator on $L^p(Y)$ having a $\mathcal{H}^\alpha_2$ calculus. Assume that A is of the form $A = A_0 \otimes \mathrm{Id}_Y$, where A_0 generates a submarkovian semigroup. Choose some integer c with*

$$c > \alpha + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right) + 1.$$

Let $m \in C^{c+1}[0,\infty)$ be a spectral multiplier satisfying

$$\max_{k=0}^{c} \sup_{n\leqslant 0} \int_{2^{n-1}}^{2^{n+1}} |2^{nk} m^{(k+1)}(\lambda)|^2 \frac{d\lambda}{\lambda} + \sum_{n\geqslant 0} \int_{2^{n-1}}^{2^{n+1}} |2^{nk} m^{(k)}(\lambda)|^2 \frac{d\lambda}{\lambda} \leqslant D.$$

Then for $2 < q < \infty$, any $f \in L^p(\Omega, Y)$ and a.e. $(x,\omega) \in \Omega \times \Omega'$, $t \mapsto m(tA)f(x,\omega)$ has finite q-variation, and

$$\| \|t \mapsto m(tA)f\|_{V^q}\|_{L^p(Y)} \leqslant C\,(|m(0)| + D)\,\|f\|_{L^p(Y)}.$$

Proof : See [HDR9, Proposition 4.3]. ∎

Again, under the assumptions of Proposition 8.11 above, we can also formulate a square function estimate for a family of spectral multipliers $(m_k(tA))_k$.

Remark 8.12 *The q-variation allows to obtain estimates for so-called jump inequalities. We refer to [HDR9, Corollary 4.5] for this notion and the corresponding result.*

8.2 Applications

We present two applications of the results in Section 8 and Subsection 8.1.

The q-variation of spherical and ball means We consider for $t > 0$, $d \in \mathbb{N}$, $1 < p < \infty$ and $f \in \mathcal{S}(\mathbb{R}^d)$ a Schwartz function,

$$A_t f(x) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x - ty)\, d\sigma(y), \tag{8.14}$$

$$M_t f(x) = \frac{1}{|B_d|} \int_{B_d} f(x - ty)\, dy = \frac{1}{|B(x,t)|} \int_{B(x,t)} f(y)\, dy, \tag{8.15}$$

where S^{d-1} (resp. B_d) denotes the unit sphere (resp. the unit euclidean ball) $\subseteq \mathbb{R}^d$, $|S^{d-1}|$ (resp. $|B_d|$) denotes the surface measure of S^{d-1} (resp. the Lebesgue measure of B_d) and $d\sigma(y)$ (resp. dy) denotes surface measure (resp. euclidean measure). The definitions (8.14) and (8.15) extend literally to $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$ when Y is a UMD lattice. They moreover extend by density to $f \in L^p(\mathbb{R}^d, Y)$ once a priori boundedness of A_t and M_t on $L^p(\mathbb{R}^d, Y)$ is clarified. In this section, we shall show q-variation results of A_t and M_t on $L^p(\mathbb{R}^d, Y)$. We recall the following results from the literature.

Proposition 8.13 *1. (See [JSW, Theorem 1.4(i)]) Let $2 < q < \infty$, $d \geqslant 2$, $\frac{d}{d-1} < p \leqslant 2d$ and $f \in \mathcal{S}(\mathbb{R}^d)$. There exists a constant $C_{p,q,d}$ independent of f such that*

$$\|t \mapsto A_t f\|_{L^p(\mathbb{R}^d, V^q)} \leqslant C_{p,q,d} \|f\|_{L^p(\mathbb{R}^d)}. \tag{8.16}$$

Moreover, if $p > 2d$ and $q > \frac{p}{d}$, then (8.16) holds and conversely, if (8.16) holds, then necessarily $q \geqslant \frac{p}{d}$.

2. (See [BMSW1, Theorem 1.3]) Let $2 < q < \infty$, $d \geqslant 1$, $1 < p < \infty$ and $f \in \mathcal{S}(\mathbb{R}^d)$. There exists a constant $C_{p,q}$ independent of f and the dimension d such that

$$\|t \mapsto M_t f\|_{L^p(\mathbb{R}^d, V^q)} \leqslant C_{p,q} \|f\|_{L^p(\mathbb{R}^d)}. \tag{8.17}$$

Moreover, the same result holds if the euclidean ℓ^2_d ball in the definition (8.15) is replaced by an ℓ^r_d-ball with $1 \leqslant r \leqslant \infty$.

3. (See [HHL, Theorem 1.1]) Let $2 < q < \infty$, $d \geqslant 1$, $1 < p < \infty$, Y a UMD lattice and $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$. Then there exists a constant $C_{p,q,Y}$ independent of f and the dimension d such that

$$\|t \mapsto M_t f\|_{L^p(\mathbb{R}^d, Y(V^q))} \leqslant C_{p,q,Y} \|f\|_{L^p(\mathbb{R}^d, Y)}. \tag{8.18}$$

We shall prove the following extension of (8.16) to the UMD lattice valued case. In order (8.16) to hold, we had a minimal dimension $d > d_1$ with $d_1 = \max\left(\frac{p}{2}, \frac{p}{p-1}\right)$ or $d_1 = \frac{p}{q}$. In Theorem 8.14, we also need some minimal dimension, but this time, it also depends on the geometry of the UMD lattice.

Theorem 8.14 *[HDR9, Theorem 5.2] Let $2 < q < \infty$, $1 < p < \infty$, Y be a UMD lattice and*

$$d > d_0 := \frac{2 + \max\left(\frac{1}{2}, \frac{1}{\operatorname{type} L^p(Y)} - \frac{1}{\operatorname{cotype} L^p(Y)}\right)}{\frac{1}{2} - \max\left(\left|\frac{1}{p} - \frac{1}{2}\right|, \left|\frac{1}{p_Y} - \frac{1}{2}\right|, \left|\frac{1}{q_Y} - \frac{1}{2}\right|\right)}, \tag{8.19}$$

where $p_Y \in (1, 2]$ and $q_Y \in [2, \infty)$ stand for the convexity and concavity exponents of Y (see Definition 4.18). Let $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$. Then there exists a constant $C_{p,q,d,Y}$ independent of f such that

$$\|t \mapsto A_t f\|_{L^p(\mathbb{R}^d(Y(V^q)))} \leqslant C_{p,q,d,Y} \|f\|_{L^p(\mathbb{R}^d, Y)}.$$

Proof : The idea of the proof is the following. The operator A_t commutes with translations in $\mathbb{R}^d$, so is a Fourier multiplier associated to some symbol $m : \mathbb{R}^d \to \mathbb{C}$. Since A_t is rotation invariant, also m is rotation invariant and thus, A_t is a spectral multiplier of the Laplacian $-\Delta$ on $\mathbb{R}^d$. For $-\Delta$, a Hörmander functional calculus on $L^p(\mathbb{R}^d, Y)$ space has been established in [Hy06, GiWe], see also Theorem 7.1 for the refined version where the Hörmander derivation exponent can be chosen in terms of p_Y and q_Y. So Proposition 8.11 is available. It will then suffice to check the finiteness of the norms of the specific spectral multiplier function m imposed in this Proposition 8.11. We refer to [HDR9, Theorem 5.2] for the details. ∎

Remark 8.15 *Let Y be a UMD lattice. Let $2 < q < \infty$, $1 < p < \infty$ and $d > d_0$ from (8.19).*

1. *Theorem 8.14 immediately extends to the same result for the ball means M_t from (8.15) in place of the spherical means A_t, with the same constraint on the minimal dimension d_0. That is, we have for any $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$ with a constant $C_{p,q,d,Y}$ independent of f that*
$$\|t \mapsto M_t f\|_{L^p(\mathbb{R}^d(Y(V^q)))} \leqslant C_{p,q,d,Y} \|f\|_{L^p(\mathbb{R}^d,Y)}.$$
The idea of proof is to write by Fubini $M_t f(x,\omega) = \frac{1}{|B(x,t)|} \int_0^t c_d r^{d-1} A_r f(x,\omega) dr$ and to pass the norm estimate from Theorem 8.14 inside the integral $\int_0^t$.

2. *With the method of rotations from Subsection 6.3, one can reprove (8.18) for large dimensions, by the alternative proof of using our method, more specifically Proposition 8.11. That is, there exists a constant $C_{p,q,Y}$ such that for any $d > d_0$ and $f \in \mathcal{S}(\mathbb{R}^d) \otimes Y$, we have*
$$\|t \mapsto M_t f\|_{L^p(\mathbb{R}^d,Y(V^q))} \leqslant C_{p,q,Y} \|f\|_{L^p(\mathbb{R}^d,Y)}.$$

Schrödinger and wave maximal estimates We shall show that Proposition 8.16 below, which is a variant of Theorem 8.1, gives to some extent maximal estimates for Schrödinger and wave operators, under rather general conditions on A, see Corollary 8.17. The underlying question is the following. Suppose that $A = -\Delta$ is the usual Laplacian operator on $L^p(\mathbb{R}^d)$ or that $A = \sqrt{-\Delta}$. Then $t \mapsto u(t) = \exp(it\Delta)u_0$ is the solution of the Schrödinger equation with initial data $u_0 \in L^p(\mathbb{R}^d)$, so
$$\begin{cases} i\frac{\partial}{\partial t}u(t) & = -\Delta u(t) \\ u(0) & = u_0 \end{cases}.$$
In the same way, $t \mapsto u(t) = \exp(it\sqrt{-\Delta})u_0$ is the solution of the wave equation with initial data $u(0) = u_0 \in L^p(\mathbb{R}^d)$ and $u'(0) = i\sqrt{-\Delta}u_0$, so
$$\begin{cases} \frac{\partial^2}{\partial t^2}u(t) & = \Delta u(t) \\ u(0) & = u_0 \\ u'(0) & = i\sqrt{-\Delta}u_0 \end{cases}.$$
In [RV1], the authors investigate the boundedness of the associated Schrödinger maximal operators
$$S^* f = \sup_{0<t<1} |\exp(it\Delta)f| \quad \text{and} \quad S^{**} f = \sup_{t\in\mathbb{R}} |\exp(it\Delta)f|.$$
Moreover, in [RV2, (4),(5)], these authors study the boundedness of the wave maximal operators
$$S^* f = \sup_{0<t<1} \left|\exp(\pm it\sqrt{-\Delta})f\right| \quad \text{and} \quad S^{**} f = \sup_{t\in\mathbb{R}} \left|\exp(it\sqrt{-\Delta})f\right|.$$
Here, typical bounds are between spaces $S^*, S^{**} : W_2^s(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ [RV1, RV2]. Note that it is well-known that $A = -\Delta$ and thus also $A = \sqrt{-\Delta}$ has a Hörmander $\mathcal{H}_2^\alpha$ functional calculus on $L^p(\mathbb{R}^d)$ for $\alpha > \frac{d}{2}$ [Hor]. Moreover, the Sobolev space admits a description $W_2^s(\mathbb{R}^d) = \{f \in L^2(\mathbb{R}^d),\ (1-\Delta)^{\frac{s}{2}} f \in L^2(\mathbb{R}^d)\}$, so that the above maximal operator estimates read as follows

$$\left\| \sup_{0<t<1} |\exp(\pm itA)f| \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)} + \|(1+A)^s f\|_{L^2(\mathbb{R}^d)} \tag{8.20}$$

$$\left\| \sup_{t\in\mathbb{R}} |\exp(itA)f| \right\|_{L^p(\mathbb{R}^d)} \lesssim \|f\|_{L^2(\mathbb{R}^d)} + \|(1+A)^s f\|_{L^2(\mathbb{R}^d)}, \tag{8.21}$$

for $A = -\Delta$ and $A = \sqrt{-\Delta}$, the right choices of s and p depending on the dimension d, thus on the order of the Hörmander $\mathcal{H}^\alpha_2$ calculus of A. In Corollary 8.17, we shall show that estimates with some similarity to (8.20), (8.21) hold for operators A having a Hörmander $\mathcal{H}^\alpha_2$ calculus. Our Sobolev space will be abstractly modelled with A as the right hand sides of (8.20) and (8.21) indicate. Note that the assumptions on A in Corollary 8.17 are that of Proposition 8.3, that is, we allow a UMD lattice valued space $L^p(\Omega, Y)$, $A = A_0 \otimes \mathrm{Id}_Y$ has a Hörmander calculus on $L^p(\Omega, Y)$ and $\exp(-tA_0)$ is submarkovian, so very general and in particular satisfied in the above examples from [RV1, RV2]. The price we have to pay for this generality is that instead of suprema over $0 < t < 1$ or $t \in \mathbb{R}$ as in (8.20) or (8.21), we can only handle $t \in [a, b] \subseteq (0, \infty)$ compactly (via a test function $\psi_0 \in C^\infty_c(\mathbb{R}_+)$). For further results on maximal estimates of Schrödinger and wave operators, we refer to [Rog, BeGo, DeGu].

The following Proposition is a variant of Theorem 8.1, where elements $f \in L^p(Y)$ are replaced by functions $f_t \in L^p(Y(\Lambda^\beta))$ depending also on the time variable t, so have a $\Lambda^\beta_{2,2}(\mathbb{R}_+)$ component. It has a different value for the derivation exponent c and involves the $W^c_1(\mathbb{R})$ norm in place of the $W^c_2(\mathbb{R})$ norm.

Proposition 8.16 *Let Y be a UMD lattice, $1 < p < \infty$ and (Ω, μ) a σ-finite measure space. Let $\beta \geqslant 0$. Let A be a 0-sectorial operator on $L^p(Y)$. Assume that A has a $\mathcal{H}^\alpha_2$ calculus on $L^p(Y)$ for some $\alpha > \frac{1}{2}$. Let $m \in W^c_1(\mathbb{R})$ be a spectral multiplier with $\mathrm{supp}(m) \subseteq [\frac{1}{2}, 2]$, with*

$$c > \alpha + \max\left(\frac{1}{2}, \frac{1}{\mathrm{type}\, L^p(Y)} - \frac{1}{\mathrm{cotype}\, L^p(Y)}\right) + 1 + \beta.$$

Then for $f_t \in L^p(Y(\Lambda^\beta_{2,2}(\mathbb{R}_+)))$, we have

$$\|t \mapsto m(tA) f_t\|_{L^p(Y(\Lambda^\beta_{2,2}(\mathbb{R}_+)))} \leqslant C \|m\|_{W^c_1(\mathbb{R})} \|t \mapsto f_t\|_{L^p(Y(\Lambda^\beta_{2,2}(\mathbb{R}_+)))}. \tag{8.22}$$

Proof : See [HDR9, Proposition 6.1]. ∎

Using Proposition 8.16 together with the two enhancements of allowing spectral multipliers m with full support in $(0, \infty)$ (as in Corollary 8.2) and of allowing a non-zero limit of m in zero (as in Proposition 8.3), then gives the following.

Corollary 8.17 *Let Y be a UMD lattice, $1 < p < \infty$ and (Ω, μ) a σ-finite measure space. Let $\beta > \frac{1}{2}$. Let A be a 0-sectorial operator on $L^p(Y)$. Assume that A has a $\mathcal{H}^\alpha_2$ calculus on $L^p(Y)$ for some $\alpha > \frac{1}{2}$. In this corollary, we assume that A is of the form $A = A_0 \otimes \mathrm{Id}_Y$ where A_0 generates a submarkovian semigroup. Let*

$$\delta > \alpha + \max\left(\frac{1}{2}, \frac{1}{\mathrm{type}\, L^p(Y)} - \frac{1}{\mathrm{cotype}\, L^p(Y)}\right) + 1 + \beta.$$

Let $\psi_0 \in C^\infty_c(\mathbb{R}_+)$, i.e. with support included in a compact interval $\subseteq (0, \infty)$. Then, with implied constants depending on ψ_0 and its support, we have

$$\begin{aligned} \|\sup_{t>0} |\psi_0(t) \exp(itA) f|\, \|_{L^p(Y)} &\lesssim \|t \mapsto \psi_0(t)(1 + tA)^\delta f\|_{L^p(Y(\Lambda^\beta))} \\ &\lesssim \max_{k=0,1,\ldots,\lceil\beta\rceil} \| \sup_{t \in \mathrm{supp}(\psi_0)} |A^k (1 + tA)^{\delta - k} f|\, \|_{L^p(Y)}. \end{aligned}$$

Moreover, if $p \geqslant 2$ and Y is 2-convex (e.g. $Y = \mathbb{C}$), then we also have

$$\|\sup_{t>0} |\psi_0(t) \exp(itA) f|\, \|_{L^p(Y)} \lesssim_{\psi_0} \max_{k=0,1,\ldots,\lceil\beta\rceil} \sup_{t \in \mathrm{supp}(\psi_0)} \|A^k (1 + tA)^{\delta - k} f\|_{L^p(Y)}.$$

8.3 Concluding remarks

Already for classical (i.e. not maximal/q-variational) Hörmander multiplier theorems, a nice description of the exact norm $\|m(A)\|_{L^p\to L^p}$ in terms of a function norm $\|m\|$ of the spectral multiplier is not known today. This problem is equally present for our maximal spectral multipliers in this section and only a step by step progression of sufficient conditions in the form $\|t\mapsto m(tA)(\cdot)\|_{L^p\to L^p(L^\infty(\mathbb{R}_+))}\lesssim\|m\|_{\dots}$ seems to be manageable. In this direction, it would be interesting to know whether in the context of Corollary 8.2 of semigroup generators, one can relax the summation condition to

$$\sum_{n\in\mathbb{Z}}\frac{\|m(2^n\cdot)\phi_0\|_{W_2^c(\mathbb{R})}}{1+|n|}<\infty$$

as in the euclidean case [CGHS], or with an additional factor $\log(|n|+2)$ as in [Choi]. As another possible relaxation, one can ask the question, whether

$$\sum_{n\in\mathbb{Z}}\|m(2^n\cdot)\phi_0\|^2_{W_q^c(\mathbb{R})}<\infty$$

is sufficient for a maximal estimate, which is known to be true for the euclidean Laplacian [CGHS, (1.3)]. Also maximal estimates for spectral multipliers that do not decay at ∞ are not well understood. Already the scalar case $Y=\mathbb{C}$ would be interesting.

As a partial result for radial spectral multipliers of the euclidean Laplacian, and on the radial part $L^p_{\rm rad}(\mathbb{R}^d)$, see [HNS] for a description of $\|m(-\Delta)\|_{L^p\to L^p}$ and [Kim] for a description of $\|t\mapsto m(-t\Delta)(\cdot)\|_{L^p\to L^p(L^\infty(\mathbb{R}_+))}$ in terms of the associated convolution kernel of $m(-\Delta)$.

In the context of Subsection 8.2, it would be interesting to get information on the norm $C_{p,q,d,Y}$ in Theorem 8.14 and on $C_{p,q,Y}$ in Remark 8.15 2., depending on $p,q(,d)$ and Y.

9 H^∞ calculus for submarkovian semigroups on weighted L^2 spaces

This section contains the main results from the article [HDR7]. It is well established by now that the H^∞ functional calculus of a sectorial operator has important applications in the spectral theory of partial differential operators and the theory of evolution equations, e.g., in determining the domain of fractional powers of a partial differential operator in the solution of Kato's problem (e.g. [AHLLMT, AuTc, DDHPV, Gig, Yag]), in connection with maximal regularity of parabolic evolution equations (e.g. [HiPr, LaLaMe, LaMe1, LeM1, PrSi, We01bis]) and certain estimates in control theory ([HaLe, HaOu, LeM2]). Today it is known that many systems of elliptic partial differential operators, Schrödinger operators and related important examples of semigroup generators do have an H^∞ calculus ([BK03, CDMY, DuOS, DSY, GCMMST, GoY, HvNP, KW1, NeP]). Also from an abstract point of view, a lot of effort has been achieved to establish, characterise and transfer H^∞ calculus ([AFLM, Haas, KaWe1, KaWe2, KaKuWe]).

We recall from Subsection 4.3 that H^∞ calculus is the question whether the Cauchy integral formula (4.3) can be reasonably extended to all $m\in\mathrm{H}^\infty(\Sigma_\theta)$ and whether one obtains the estimate

$$\|m(A)\|_{B(X)}\leqslant C\,(|m(0)|+\|m\|_{\infty,\theta})\quad(m\in\mathrm{H}^\infty(\Sigma_\theta)).\tag{9.1}$$

This is a difficult task and its solution, for more or for less concrete operators A, requires several fundamental tools from harmonic analysis such as square functions (see e.g. [CDMY, Section

6], [KaWe2]), bounded imaginary powers of A [CDMY, Section 5], bilinear embeddings [CDMY, Section 4] and transference principles [AFLM, CoWe2, Fen2, HiPr]. Note that a positive answer of (9.1) depends in general on θ and a smaller angle yields a more restrictive condition, since $\mathrm{H}^\infty(\Sigma_\theta) \subseteq \mathrm{H}^\infty(\Sigma_\sigma)$ if $\sigma \leqslant \theta$ by uniqueness of analytic continuation.

Let us give a brief overview of important operator theoretic results when an H^∞ calculus is known. Let (Ω, μ) be a σ-finite measure space. First suppose that the semigroup $(T_t)_{t\geqslant 0}$ is markovian (see Subsection 4.1 for the definition of this classical notion). The first universal multiplier theorem was proved by E. M. Stein, who showed that if m is of Laplace transform type, then $m(A)$ is bounded on $L^p(\Omega)$ for $1 < p < \infty$ [Ste70, Corollary 3, page 121]. This result was later extended to submarkovian semigroups (see again Subsection 4.1 for the definition) and for m belonging to $\mathrm{H}^\infty(\Sigma_\theta)$ by Cowling [Cow, Theorem 1] and Meda [Med, Theorem 3]. The angle of the functional calculus depends on p and by complex interpolation with the selfadjoint calculus on $L^2(\Omega)$, one obtains $\theta > \pi \left|\frac{1}{p} - \frac{1}{2}\right|$. Later on it was observed by Duong [Duo], (see also [HiPr] for $\theta > \frac{\pi}{2}$ and [KaWe1, Corollary 5.2] for $\theta < \frac{\pi}{2}$) that semigroups acting on a single $L^p(\Omega)$ space and consisting of positive and contractive operators, or even only regular contractive operators [CoWe2, Fen2, Ste70] suffices to obtain an $\mathrm{H}^\infty(\Sigma_\theta)$ calculus. A recent extension of [HiPr] and [KW1, Corollary 10.15] is [Xu15, Theorem 4] where the setting is a vector valued semigroup of the form $T_t = T_t^{(0)} \otimes \mathrm{Id}_Y$ acting on the Bochner space $X = L^p(\Omega, Y)$, where Y is an intermediate UMD space and where $T_t^{(0)}$ is an analytic semigroup consisting of regular contractive operators acting on a single $L^p(\Omega)$ space. Here the novelty compared to [KW1, Corollary 10.15] is an angle of $\mathrm{H}^\infty(\Sigma_\theta)$ calculus $\theta < \frac{\pi}{2}$. Concerning the optimality of the H^∞ calculus angle, the recent breakthrough result [CaDr1] yields $\theta > \theta_p = \arctan\left(\frac{|p-2|}{2\sqrt{p-1}}\right)$ on $X = L^p(\Omega)$, $1 < p < \infty$, within the class of submarkovian semigroups (or even the class of selfadjoint semigroups which are contractive on the $L^p(\Omega)$ scale). Here the angle θ_p is already optimal in the important example of the Ornstein-Uhlenbeck semigroup acting on $L^p(\mathbb{R}^d, \mu)$ where $d\mu(x) = (2\pi)^{-\frac{d}{2}} \exp\left(-\frac{|x|^2}{2}\right) dx$ is Gaussian measure and $A = -\Delta + x\cdot\nabla$ [GCMMST]. The norm of the $\mathrm{H}^\infty(\Sigma_\theta)$ calculus in [CaDr1] is universally bounded, independent of any dimension or whatsoever notion of Ω.

In the present section, we consider markovian and submarkovian semigroups, and add a weight w to the picture, so that $\|f\|_X = \|f\|_{L^2(\Omega, w d\mu)} = \left(\int_\Omega |f(x)|^2 w(x) d\mu(x)\right)^{\frac{1}{2}}$.

Definition 9.1 *A weight is a measurable function $w\colon \Omega \to (0, \infty)$ defined on a σ-finite measure space (Ω, μ).*

Weighted estimates for spectral multipliers have been recently studied by [DSY, Theorem 3.1 and Theorem 3.2] and [GoY, Theorem 4.1 and Theorem 4.2]. In the latter works, the space Ω is supposed to be of homogeneous type and the semigroup $(T_t)_{t\geqslant 0}$ is selfadjoint and has an integral kernel satisfying Gaussian upper bounds (see (4.9)). The multiplier function m is allowed to belong to a Hörmander class. Note that according to (4.4) the Hörmander class contains $\mathrm{H}^\infty(\Sigma_\theta)$ for any $\theta \in (0, \pi)$, so that [DSY, GoY] yield an H^∞ calculus to any angle on weighted L^p spaces. The weights that are allowed here belong to a certain (spatially defined) Muckenhoupt class, see also Remark 9.7 for a comparison with our results.

In this section, we settle the case of markovian and submarkovian semigroups without any dimension assumption on Ω nor integral kernel estimates of $(T_t)_{t\geqslant 0}$. Our underlying Banach space will always be $X = L^2(\Omega, w d\mu)$. Note that it is unrealistic to obtain an H^∞ calculus result for all weights w. The natural condition for w is the semigroup characteristic

$$Q_2^A(w) = \sup_{t>0} \text{ess-sup}_{x\in\Omega}\, T_t w(x) T_t\left(w^{-1}\right)(x) < \infty. \tag{9.2}$$

Following standard abuse of notation, we also write $w \in Q_2^A$ for condition (9.2). Our semigroups have to satisfy certain technical assumptions which we resume here.

Assumption 9.2 *Our markovian semigroups $(T_t)_{t\geqslant 0}$ in this section, acting on (Ω, μ), are assumed to satisfy one of the following alternative conditions.*

1. *The measure space is finite, $\mu(\Omega) < \infty$, or*
2. *For any $t > 0$, T_t maps $L^\infty(\Omega)$ into the domain $\mathrm{dom}(A_\infty)$ of the w^* L^∞ realization of A, or*
3. *$(T_t)_{t\geqslant 0}$ satisfies the following local diffusion: there exist $C, R > 0$ such that for all $w \in L^\infty(\Omega)$ with $w \geqslant 0$ and $0 \leqslant s \leqslant r \leqslant t$, we have $T_{t+s}w(x) \leqslant CT_{R(t+r)}w(x)$.*

In case 2. and 3. above, we assume moreover that for any $v \in L^\infty(\Omega)$, $T_t v(x) \to v(x)$ as $t \to 0+$ μ-almost everywhere.

We refer to [HDR7, Remarks 4.5, 4.6, 4.7] for a list of common properties of the semigroup $(T_t)_{t\geqslant 0}$ (Gaussian upper estimates, two-sided Poisson estimates, Feller semigroups) ensuring Assumption 9.2. Then our first main result reads as follows.

Theorem 9.3 *[HDR7, Theorem 1.1, Corollary 4.4] Let $(T_t)_{t\geqslant 0}$ be a markovian semigroup on (Ω, μ) satisfying Assumption 9.2. Let w be a weight on Ω such that $Q_2^A(w) < \infty$. Then the (negative) generator A of $(T_t)_{t\geqslant 0}$ is $\frac{\pi}{2}$-sectorial on $L^2(\Omega, wd\mu)$. Moreover, for any $\theta > \frac{\pi}{2}$, there exists a constant C_θ such that for any $m \in \mathrm{H}^\infty(\Sigma_\theta)$, we have*

$$\|m(A)\|_{L^2(\Omega,wd\mu)\to L^2(\Omega,wd\mu)} \leqslant C_\theta Q_2^A(w)\left(|m(0)| + \|m\|_{\infty,\theta}\right).$$

For a sketch of the proof, we refer to Subsection 9.1; for a detailed proof, see [HDR7, Section 4].

Remark 9.4 *Note that the constant C_θ in Theorem 9.3 is universal in the class of all markovian semigroups satisfying Assumption 9.2, independent of any dimension or whatsoever notion of Ω.*

Let us compare our weight characteristic with classical notions of weights.

Remark 9.5 *Suppose that $(\Omega, \mathrm{dist}, \mu)$ is a space of homogeneous type. In the following cases, the semigroup characteristic can be compared to the classical A_2 characteristic defined in terms of means over balls, that is*

$$Q_2^{class}(w) = \sup_{B \text{ ball in } \Omega} \frac{1}{\mu(B)}\int_B w(y)d\mu(y)\frac{1}{\mu(B)}\int_B \frac{1}{w(y)}d\mu(y).$$

Let $(T_t)_{t\geqslant 0}$ be a submarkovian semigroup acting on $L^2(\Omega)$. Suppose that T_t has an integral kernel $p_t(x, y)$. Recall the semigroup characteristic Q_2^A from (9.2).

1. *If $p_t(x, y)$ satisfies Gaussian upper estimates (4.9) of order $m = 2$, then there exists some $c < \infty$ such that for any weight $w : \Omega \to \mathbb{R}_+$, we have $Q_2^A(w) \leqslant cQ_2^{class}(w)$.*
2. *If $p_t(x, y)$ satisfies Gaussian lower estimates (4.10) of order $m = 2$, then there exists some $c < \infty$ such that for any weight $w : \Omega \to \mathbb{R}_+$, we have $Q_2^{class}(w) \leqslant cQ_2^A(w)$.*

Consequently, if the semigroup satisfies two-sided Gaussian estimates, then $Q_2^{class}(w) \cong Q_2^A(w)$.

Proof : See [HDR7, Remark 2.17]. ■

Recall that the notion of $\mathrm{H}^\infty(\Sigma_\theta)$ calculus depends on an angle $\theta \in (0,\pi)$. Of particular importance is an angle $\theta < \frac{\pi}{2}$, since it entails analyticity of the semigroup (see Subsection 4.1) and, on Hilbert spaces, also that the generator A has maximal regularity (see e.g. Subsection 5.4 for an application of this notion). Note that our Theorem 9.3 only gives $\mathrm{H}^\infty(\Sigma_\theta)$ calculus for $\theta > \frac{\pi}{2}$. However, in some cases, the angle can be reduced to $\theta < \frac{\pi}{2}$.

Proposition 9.6 *Let $(T_t)_{t\geqslant 0}$ be a markovian semigroup on (Ω,μ) satisfying Assumption 9.2. Assume that the weight w satisfies $w^\delta \in Q_2^A$ for some $\delta > 1$. Then A has an $\mathrm{H}^\infty(\Sigma_\theta)$ calculus on $L^2(\Omega, wd\mu)$ for some $\theta < \frac{\pi}{2}$ and in particular, the analytic semigroup T_z extends boundedly to $L^2(\Omega, wd\mu)$ for $|\arg z| < \frac{\pi}{2} - \theta$, and A has maximal regularity on $L^2(\Omega, wd\mu)$.*

Note that at least in case that $\Omega = \mathbb{R}^n$, and $Q_2^{class}(w) \cong Q_2^A(w)$ (e.g. in the situation of Remark 9.5), a weight $w \in Q_2^A$ automatically satisfies $w^\delta \in Q_2^A$ for some $\delta > 1$.

Proof : See [HDR7, Proposition 4.13, Remark 4.14], and also [GCRdF, Theorem 2.7, page 399] for the last sentence. ■

For specialists in H^∞ calculus, there is also a result for the sharp angle $\theta = \frac{\pi}{2}$. Namely, define for $J > 0$ the class $\mathrm{H}^\infty(\Sigma_{\frac{\pi}{2}}; J)$ to consist of those $\mathrm{H}^\infty(\Sigma_{\frac{\pi}{2}})$ functions such that the boundary function on $i\mathbb{R}$ lies in a certain Besov $\Lambda^J_{\infty,1}$ class. Then under the hypotheses of Theorem 9.3, $m(A)$ is also bounded on $L^2(\Omega, wd\mu)$ for $m \in \mathrm{H}^\infty(\Sigma_{\frac{\pi}{2}}; J)$ and $J > 1$. Note that this is interesting, since the semigroup spectral multiplier $m(\lambda) = e^{-\lambda}$ does not lie in any $\mathrm{H}^\infty(\Sigma_\theta)$ for $\theta > \frac{\pi}{2}$, however its resolvent regularisation $m(\lambda) = (1+\lambda)^{-J}e^{-\lambda}$ does lie in $\mathrm{H}^\infty(\Sigma_{\frac{\pi}{2}}; J)$. We refer to [HDR7, Corollary 4.4, Subsection 2.2] for the details.

Remark 9.7 *Assume that $(\Omega, \mathrm{dist}, \mu)$ is a space of homogeneous type and that the semigroup $(T_t)_{t\geqslant 0}$ satisfies Gaussian estimates (4.9) of order $m = 2$, and is selfadjoint on $L^2(\Omega)$. In [DSY, Theorem 3.2] [GoY, Theorem 4.2], a functional calculus for the (negative) generator A on weighted L^p spaces is proved. If $(T_t)_{t\geqslant 0}$ is in addition markovian, then one can compare these results to ours. On the one hand, the results in [DSY, GoY] are stronger in respect that L^p spaces with exponents $p \in (r_0, \infty)$ (for a certain $r_0 \in [1,2)$) different from 2 are allowed and that non-holomorphic Hörmander spectral multipliers $m \in \mathcal{H}^s_\infty$ are admitted. Here, $s > \frac{d}{2}$, where d is a doubling dimension of Ω, Note that the class $\mathcal{H}^s_\infty$ contains $\mathrm{H}^\infty(\Sigma_\theta)$ for all (small) angles $\theta \in (0,\pi)$. On the other hand, our result, Theorem 9.3 is stronger in respect that Q_2^A weights are admitted, whereas in [DSY, GoY], one has to take the smaller class of weights belonging to $Q^{class}_{2/r_0} \subsetneq Q_2^{class} \subsetneq Q_2^A$ (cf. Remark 9.5). Note that Assumption 9.2 is satisfied, so that Theorem 9.3 does apply.*

Note that in our setting (markovian semigroups, no kernel estimates in general), no Hörmander calculus result on weighted L^2 can be available in general, see the negative result from Theorem 9.10.

Theorem 9.3 admits also a variant for submarkovian semigroups, see Theorem 9.9 below. Let $(T_t)_{t\geqslant 0}$ be a submarkovian semigroup on (Ω,μ). Define $\Omega' = \Omega \cup \{\infty\}$ with some exterior cemetery point $\infty \notin \Omega$. Define moreover the measure $\mu'(A) = \mu(A\cap\Omega) + \delta_A(\infty)$ on Ω'. Then we put for $f' = f'|_\Omega + f'(\infty)\delta_\infty \in L^\infty(\Omega')$ and $t \geqslant 0$

$$S_t(f')(x) = \begin{cases} T_t(f'|_\Omega)(x) + f'(\infty)(1 - T_t(1))(x) & : x \in \Omega \\ f'(\infty) & : x = \infty. \end{cases} \tag{9.3}$$

It is easy to check that S_t is a positive semigroup[1] on $L^\infty(\Omega')$ and that moreover, $S_t(1) = 1$. Thus S_t are contractions on $L^\infty(\Omega')$. Note however that S_t is in general no longer selfadjoint

1. We do not need any continuity assumption of $t \mapsto S_t$.

or even defined on $L^p(\Omega')$ for $p < \infty$. If $w : \Omega \to (0, \infty)$ is a weight, we define the characteristic associated with S_t by

$$\tilde{Q}_2^A(w) = \sup_{t>0} \text{ess-sup}_{x\in\Omega'} S_t(w')(x) S_t(w'^{-1})(x),$$

where $w'(x) = w(x)$ for $x \in \Omega$ and $w'(\infty) = 1$. Note that even if w has support only in Ω, the characteristic $\tilde{Q}_2^A(w)$ is in general larger than $Q_2^A(w)$. For the statement of Theorem 9.9, we will again need technical assumptions on the semigroup.

Assumption 9.8 *Our submarkovian semigroups $(T_t)_{t\geqslant 0}$ in this section, acting on (Ω, μ), are assumed to satisfy one of the following alternative conditions.*

1. *The measure space is finite, $\mu(\Omega) < \infty$, or*
2. *For any $t > 0$, T_t maps $L^\infty(\Omega)$ into the domain $\text{dom}(A_\infty)$ of the w^* L^∞ realization of A, or*
3. *The amplified semigroup $(S_t)_{t\geqslant 0}$ satisfies the local diffusion from Assumption 9.2.*

In case 2. and 3. above, we assume moreover that for any $v \in L^\infty(\Omega)$, $T_t v(x) \to v(x)$ as $t \to 0+$ μ-almost everywhere.

Theorem 9.9 *[HDR7, Theorem 1.2, Corollary 5.3] Let $(T_t)_{t\geqslant 0}$ be a submarkovian semigroup on some σ-finite measure space (Ω, μ), satisfying Assumption 9.8. Let A be the (negative) generator of $(T_t)_{t\geqslant 0}$. Then for any $\theta > \frac{\pi}{2}$, there exists a constant C_θ such that for any weight $w : \Omega \to (0, \infty)$ with $\tilde{Q}_2^A(w) < \infty$, we have*

$$\|m(A)\|_{L^2(\Omega,wd\mu)\to L^2(\Omega,wd\mu)} \leqslant C_\theta \tilde{Q}_2^A(w) \left(|m(0)| + \|m\|_{\infty,\theta}\right).$$

Proof : We refer to [HDR7, Theorem 5.2, Corollary 5.3] for the details of the modifications of the proof of Theorem 9.3. In particular, the Bellman function (see next Subsection 9.1 below) has to satisfy more secundary properties. ∎

As in the case of markovian semigroups, there is also a variant of Theorem 9.9 for $\text{H}^\infty(\Sigma_{\frac{\pi}{2}}; J)$ functions defined on the right half-plane, and a variant for weights $w \in \tilde{Q}_2^A$ such that $w^\delta \in \tilde{Q}_2^A$ for some $\delta > 1$ to obtain $\text{H}^\infty(\Sigma_\theta)$ calculus with angle θ strictly less than $\frac{\pi}{2}$. We refer to [HDR7, Corollary 5.3, Proposition 5.4] for the details.

9.1 Proof of Theorem 9.3

The proof of Theorem 9.3 relies on a bilinear (or weak square function) estimate together with the Bellman function method. It owes a lot to the method in [CaDr1]; we divide it into several steps. We refer to [HDR7, Section 4] for more details.

1st step: Reduction of H^∞ calculus to a bilinear estimate Note that according to [CDMY, Theorem 4.4], in order that some sectorial operator A have an $\text{H}^\infty(\Sigma_\theta)$ calculus for $\theta > \frac{\pi}{2}$ on a Banach space X, it is sufficient to show the following bilinear estimate.

$$\int_0^\infty |\langle AT_t f, g\rangle| \, dt \leqslant C \|f\|_X \|g\|_{X'}.$$

We shall apply this with $X = L^2(\Omega, wd\mu)$, $X' = L^2(\Omega, w^{-1}d\mu)$ and the duality bracket $\langle f, g\rangle = \int_\Omega f(x)g(x)d\mu(x)$. Since T_t is selfadjoint (also with respect to bilinear duality bracket, since real part preserving) and using some standard density argument, we are reduced to show

$$\int_0^\infty |\langle AT_t f, T_t g\rangle|\, dt \leqslant CQ \, \|f\|_{L^2(\Omega,wd\mu)} \, \|g\|_{L^2(\Omega,w^{-1}d\mu)} \quad (f, g \in L^1(\Omega,\mu) \cap L^\infty(\Omega,\mu)), \tag{9.4}$$

where $Q = Q_2^A(w)$. Note that we do not have assumed our A, that is, the generator of the markovian semigroup, to be a priori sectorial on $L^2(\Omega, wd\mu)$. However, the bilinear estimate (9.4) which still makes sense is sufficiently strong to imply sectoriality. We refer to [HDR7, Proposition 2.9] for the details. Finally note that we can assume that the weight w is bounded above and below by some $\frac{1}{\varepsilon}$ and $\varepsilon > 0$, if the constant C in (9.4) to show will be independent of ε. The details of the cut-off procedure are in [HDR7, Lemma 2.16, beginning of proof of Theorem 4.3].

2nd step: From bilinear estimate to domination of Bellman functional In order to prove (9.4), we introduce a functional $\mathcal{E} : [0,\infty) \to \mathbb{R}_+$ given by

$$\mathcal{E}(t) = \int_\Omega B(T_t(f), T_t(g), T_t(w^{-1}), T_t(w))d\mu.$$

Here, the function $B\colon D_B \to \mathbb{R}_+$ together with its needed properties will be determined in the course of this proof. It is called a *Bellman function*. For the moment, we observe that by the property of the weight w belonging to Q_2^A (and its bounds from the 1st step), it is sufficient to define B on

$$D_B = \mathbb{C} \times \mathbb{C} \times \{(r,s) \in \mathbb{R}_+^2 : \varepsilon \leqslant r, s \leqslant \frac{1}{\varepsilon}, \, 1 \leqslant r \cdot s \leqslant Q\}. \tag{9.5}$$

Using the technical Assumption 9.2 that we imposed on the markovian semigroup, one can show that $\mathcal{E}$ is differentiable on $(0,\infty)$, provided that

$$B \text{ is a } C^1 \text{ function on its domain.} \tag{9.6}$$

Suppose that we know that

$$|\langle AT_t f, T_t g\rangle| \leqslant -CQ\mathcal{E}'(t) \quad (t > 0). \tag{9.7}$$

Then we can integrate over $t \in (0,\infty)$ and obtain, in case that $\mathcal{E}$ is continuous in 0 (we refer to [HDR7, end of proof of Theorem 4.3] together with Assumption 9.2 for this technical point here),

$$\int_0^\infty |\langle AT_t f, T_t g\rangle|\, dt \leqslant CQ(\mathcal{E}(0) - \mathcal{E}(\infty)) \leqslant CQ\mathcal{E}(0), \tag{9.8}$$

where we noted that $\mathcal{E}$ takes only positive values. Suppose that the Bellman function also satisfies

$$B(x,y,r,s) \leqslant C\left(\frac{|x|^2}{r} + \frac{|y|^2}{s}\right) \tag{9.9}$$

for (x,y,r,s) belonging to the domain D_B from (9.5). Then $\mathcal{E}(0) = \int_\Omega B(f,g,w^{-1},w)d\mu \leqslant C\left(\int_\Omega |f|^2 w d\mu + \int_\Omega |g|^2 w^{-1} d\mu\right)$. Inserting this into (9.8) and optimising $f \rightsquigarrow \lambda f$, $g \rightsquigarrow \lambda^{-1} g$ with $\lambda > 0$ then yields (9.4). We are left with showing (9.7).

3rd step: From (9.7) to Bellman function properties In order to show (9.7), one can explicitly calculate the derivative of $\mathcal{E}$. One gets with $v = w^{-1}$

$$-\mathcal{E}'(t) = \mathrm{Re}[\int_\Omega \partial_x B(T_t f, T_t g, T_t v, T_t w) A T_t f + \partial_y B(T_t f, T_t g, T_t v, T_t w)) A T_t g \\ + \partial_r B(T_t f, T_t g, T_t v, T_t w) A T_t v + \partial_s B(T_t f, T_t g, T_t v, T_t w) A T_t w d\mu]. \tag{9.10}$$

Upon replacing $T_t f$ (resp. $T_t g$, $T_t v$, $T_t w$) by a generic element f (resp. g, v_1, v_2), and using that $A = \lim_{t\to 0} \frac{1}{t}(\mathrm{Id} - T_t)$ on its domain, (9.7) would follow from

$$\left|\int_\Omega (\mathrm{Id} - T_t)(f) g d\mu\right| \leqslant CQ\,\mathrm{Re}\left[\int_\Omega \partial_x B(f, g, v_1, v_2)(\mathrm{Id} - T_t)(f) + \partial_y B(f, g, v_1, v_2)(\mathrm{Id} - T_t)(g) \right. \\ \left. + \partial_r B(f, g, v_1, v_2)(\mathrm{Id} - T_t)(v_1) + \partial_s B(f, g, v_1, v_2)(\mathrm{Id} - T_t)(v_2) d\mu\right] \tag{9.11}$$

(From this, divide on both sides by t and let $t \to 0+$, to get with (9.10) the line (9.7).) Let us explain the philosophy of the remaining part of the proof. If T_t has an integral kernel, then both sides of (9.11) can be expressed as a double integral over (Ω, μ). Then by Fubini, one can separate the double integration part, and show that (9.11) holds if it is true when one replaces $\mathrm{Id} - T_t$ and (Ω, μ) by the archetype of markovian semigroup generator, which is

$$\mathcal{G} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{9.12}$$

acting on $\Omega = \{a, b\}$ a two-point measure space with counting measure $\mu = \delta_a + \delta_b$. For the validity of (9.11) with $\mathcal{G}$, one can then show that it is sufficient to have a convexity property of the Bellman function,

$$B(W_1) - B(W_2) - dB(W_2) \cdot (W_1 - W_2) \geqslant \frac{c}{Q} |x_1 - x_2|\,|y_1 - y_2| \tag{9.13}$$

whenever $W_i = (x_i, y_i, r_i, s_i)$, $i = 1, 2$, belong to the domain D_B from (9.5). Here, dB stands for the first order differential (with respect to 6 real variables, see (9.5)). Even if T_t does not have an integral kernel, by the markovianity, it is not far from being so. In fact, then there is the (involutive, algebra) Gelfand homomorphism $\mathcal{F} : L^\infty(\Omega, \mu) \to L^\infty(\widehat{\Omega}, \hat{\mu})$ respecting moreover L^p norms which transfers the question (9.11) to a version on $\widehat{\Omega}$ with a new operator $\widehat{T}_t$ which this time does have an integral kernel.

4th step: Conclusion Resuming the steps 1-3 hereabove, we have to find a Bellman function having the domain (9.5), the differentiability (9.6), the pointwise majorisation (9.9) and the particular convexity property (9.13). All these and some secundary, technical properties hidden in the 3rd step can be realised by a single function, constructed out of several pieces! We refer to [HDR7, Section 3] and [DoPe] for this delicate task. The proof of Theorem 9.3 is complete.

9.2 Complements and open questions

Note that Theorems 9.3 and 9.9 gave us an $\mathrm{H}^\infty(\Sigma_\theta)$ calculus on $L^2(\Omega, w d\mu)$ for $\theta > \frac{\pi}{2}$. In the unweighted case $w = 1$, by selfadjointness of A, one can choose $\theta > 0$ arbitrarily small. Moreover, according to [CaDr1], one can choose an explicit angle $\theta_p < \frac{\pi}{2}$ for $\mathrm{H}^\infty(\Sigma_{\theta_p})$ calculus on $L^p(\Omega, \mu)$, $1 < p < \infty$. The question arises, if one can lower the angle in the weighted L^2

case also to a value $< \frac{\pi}{2}$. Here, we only have a partial negative result. Recall the spectral multiplier class $\mathcal{H}^s_\infty$ from Definition 4.7. Note that according to [CDMY, Theorem 4.10], the failure of an $\mathcal{H}^s_\infty$ calculus for all $s > 0$ is equivalent to the failure of a polynomial estimate $\|m(A)\| \leqslant C_s \theta^{-s} \|m\|_{\infty,\theta}$ for all $\theta \in (0,\pi)$, $m \in \mathrm{H}^\infty(\Sigma_\theta)$ and any $s > 0$.

Theorem 9.10 *[HDR7, Theorem 6.1] There exists a markovian semigroup $T_t = \exp(-tA)$ on a probability space and a Q_2^A weight w such that A does not have a Hörmander $\mathcal{H}^s_\infty$ calculus for any $s > 0$ on weighted $L^2(w)$ space, that is, for no $s > 0$ and no $C > 0$, the estimate*

$$\|m(A)\|_{L^2(\Omega,wd\mu)\to L^2(\Omega,wd\mu)} \leqslant C\|m\|_{\mathcal{H}^s_\infty} \quad (m \in \mathcal{H}^s_\infty) \tag{9.14}$$

holds. In fact, (9.14) does not even hold for all $m_z(\lambda) = \exp(-z\lambda)$ with $z \in \Sigma_{\frac{\pi}{2}}$.

We remind that the above theorem is in contrast with the positive result in Remark 9.7 for selfadjoint semigroups on spaces of homogeneous type satisfying Gaussian estimates and a restricted weight class.

The semigroup exhibiting the counter-example for the statement (9.14) is based on the two-point semigroup that we have already encountered in (9.12) in the proof of Theorem 9.3, together with a tensor power extension of the semigroup. So we consider a two point space $\Omega_0 = \{a, b\}$ equipped with counting measure $\mu_0 = \delta_a + \delta_b$. Consider moreover the operator

$$\mathcal{G} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} : L^2(\Omega_0,\mu_0) \to L^2(\Omega_0,\mu_0)$$

which generates the markovian semigroup

$$\exp(-t\mathcal{G}) = \frac{1}{2}\begin{bmatrix} 1+e^{-2t} & 1-e^{-2t} \\ 1-e^{-2t} & 1+e^{-2t} \end{bmatrix}. \tag{9.15}$$

Lemma 9.11 *For any $n \in \mathbb{N}$, the above semigroup admits a tensor power extension to a markovian semigroup in the following way. We let $w_1 = (u_1, v_1), \ldots, w_n = (u_n, v_n)$ be weights on Ω_0. Then we put*

$$\begin{aligned} \Omega &= \underbrace{\Omega_0 \times \Omega_0 \times \ldots \times \Omega_0}_{n \text{ factors}} = \Omega_0^n, \\ \mu &= \underbrace{\mu_0 \otimes \mu_0 \otimes \ldots \otimes \mu_0}_{n \text{ factors}} = \mu_0^{\otimes n}, \\ T_t &= \underbrace{e^{-t\mathcal{G}} \otimes e^{-t\mathcal{G}} \otimes \ldots \otimes e^{-t\mathcal{G}}}_{n \text{ factors}}, \\ w &= w_1 \otimes w_2 \otimes \ldots \otimes w_n, \end{aligned}$$

where $T_t(\sum_k f_1^{(k)} \otimes \ldots \otimes f_n^{(k)}) = \sum_k (e^{-t\mathcal{G}}(f_1^{(k)})) \otimes \ldots \otimes (e^{-t\mathcal{G}}(f_n^{(k)}))$ and $w(x_1,\ldots,x_n) = w_1(x_1)\cdot\ldots\cdot w_n(x_n)$. We have that T_t is a markovian semigroup on (Ω,μ). Moreover,

$$\|T_z\|_{L^2(\Omega,wd\mu)\to L^2(\Omega,wd\mu)} \geqslant \prod_{k=1}^n \|e^{-z\mathcal{G}}\|_{L^2(\Omega_0,w_k d\mu_0)\to L^2(\Omega_0,w_k d\mu_0)}, \tag{9.16}$$

$$Q_2^A(w) = \prod_{k=1}^n Q_2^{\mathcal{G}}(w_k) = \prod_{k=1}^n \frac{1}{4}\left(2 + \frac{u_k}{v_k} + \frac{v_k}{u_k}\right), \tag{9.17}$$

where Q_2^A stands for the weight characteristics with respect to the markovian semigroup T_t.

Proof : See [HDR7, Lemma 6.2]. ∎

Note that for a semigroup spectral multiplier $m_z(\lambda) = \exp(-z\lambda)$ for complex time $z \in \mathbb{C}_+$, the $\mathcal{H}^s_\infty$ norm becomes visible when z approches the imaginary axis. More precisely, we have for $z = te^{i\phi}$ with $t > 0$ and $\phi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $\|m_z\|_{\mathcal{H}^s_\infty} \lesssim (\cos(\phi))^{-s}$ (cf. [KrW3, Lemma 3.9 (1)]). Thus, in order to refute the Hörmander functional calculus, we will minorise the norm $\|T_z\|_{L^2(\Omega,wd\mu)\to L^2(\Omega,wd\mu)}$ for $z = te^{i\phi}$ with ϕ close to $\frac{\pi}{2}$, against a super-polynomially growing term in $(\cos(\phi))^{-1}$. To this end, we have

Lemma 9.12 *Consider the two-point semigroup from (9.15), and a weight $w = (1, v^2)$ with $v = 1 + \varepsilon$. Then we have for $z = te^{i\phi}$ with $t > 0$ and $\phi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,*

$$\|e^{-z\mathcal{G}}\|_{L^2(w)\to L^2(w)} = 1 + \frac{1}{32}\left(1 + \tan^2(\phi) + o_t(1)\right)\varepsilon^2 + o_\varepsilon(\varepsilon^2). \tag{9.18}$$

Proof : See [HDR7, Lemma 6.4]. ∎

Proof of Theorem 9.10 : Let $m_z(\lambda) = \exp(-z\lambda)$ be the spectral multiplier as considered above. The counterexample will be the direct sum of n-fold tensor powers of the two-point semigroup as in Lemma 9.11. Take some $n \in \mathbb{N}$. We pick the weight $w_k = \left(1, (1+\varepsilon)^2\right)$ as in Lemma 9.12. We note first that it is not hard to check that there exists a constant $C > 0$ such that for any $\varepsilon \geqslant 0$, we have $Q_2^{\mathcal{G}}(w_k) \leqslant 1 + C\varepsilon^2$. From this, we deduce for the tensor power weight $w = w_1 \otimes \ldots \otimes w_n$ from Lemma 9.11 that according to (9.17),

$$\log(Q_2^{A^{(n)}}(w)) \leqslant \sum_{k=1}^n \log(1 + C\varepsilon^2) \leqslant \sum_{k=1}^n C\varepsilon^2.$$

If $\varepsilon = \frac{1}{\sqrt{n}}$, we thus obtain that $Q_2^{A^{(n)}}(w) \leqslant Q$ with a constant Q independent of n. Take now the semigroup $T_t = T_t^{(n)}$ generated by $A^{(n)}$, and associated to Ω_0^n, $\mu_0^{\otimes n}$ and $w^{(n)}$ as in Lemma 9.11. We estimate with this lemma together with (9.18)

$$\|T_z^{(n)}\|_{L^2(w^{(n)})\to L^2(w^{(n)})} \geqslant \prod_{k=1}^n \left(1 + \frac{1}{32}(1 + \tan^2(\phi) + o_t(1))\varepsilon^2 + o_\varepsilon\left(\varepsilon^2\right)\right).$$

Choose t sufficiently close to 0 to have $o_t(1) \geqslant -1$ here above. Moreover, for given $z = te^{i\phi}$, choose n so large that $\tan^2(\phi)\varepsilon^2 \leqslant 1$, i.e. $n \geqslant \tan^2(\phi)$, and that $o_\varepsilon(\varepsilon^2)$ is in force. Then we obtain

$$\begin{aligned}
\log\left(\|T_z^{(n)}\|_{L^2(w^{(n)})\to L^2(w^{(n)})}\right) &\geqslant \sum_{k=1}^n \log\left(1 + \frac{1}{32}\tan^2(\phi)\varepsilon^2 + o_\varepsilon\left(\varepsilon^2\right)\right) \\
&\gtrsim \sum_{k=1}^n \frac{1}{32}\tan^2(\phi)\varepsilon^2 \\
&= \frac{1}{32}\tan^2(\phi). \qquad (9.19)
\end{aligned}$$

Take now the direct sum $\Omega = \bigsqcup_{n\in\mathbb{N}} \Omega_0^n$ with the sum (probability) measure $\mu = \bigoplus_{n\in\mathbb{N}} \frac{1}{2^{2n}}\mu_0^{\otimes n}$ and the weight $w = \bigoplus_{n\in\mathbb{N}} w^{(n)}$. Moreover, let $(T_t)_{t\geqslant 0}$ be the semigroup on $L^2(\Omega) = \bigoplus_{n\in\mathbb{N}} L^2(\Omega_0^n)$

acting "diagonally" by $T_t(f_n)_n = (T_t^{(n)} f_n)_n$, and let as usual A be its negative generator. Then $Q_2^A(w) \leqslant \sup_{n\in\mathbb{N}} Q_2^{A^{(n)}}(w^{(n)}) \leqslant Q$, and

$$\|T_z\|_{L^2(w)\to L^2(w)} \geqslant \sup_{n\in\mathbb{N}} \left\|T_z^{(n)}\right\|_{L^2(w^{(n)})\to L^2(w^{(n)})} \overset{(9.19)}{\geqslant} e^{c\tan^2(\phi)}.$$

The last quantity clearly grows faster when $\phi \to \pm\frac{\pi}{2}$ than any power of $(\cos(\phi))^{-1} \cong |\tan(\phi)|$. ■

Remark 9.13 *Theorem 9.10 gives some negative statement for* H^∞ *calculus with angle close to* 0. *The author also has some unpublished work of a negative result concerning some intermediate angle* $\theta_0 \in (0, \frac{\pi}{2})$. *Namely, for no constant* $C > 0$ *the estimate*

$$\|m(A)\|_{L^2(\Omega,wd\mu)\to L^2(\Omega,wd\mu)} \leqslant CQ_2^A \left(|m(0)| + \|m\|_{\infty,\theta_0}\right) \quad (m \in \mathrm{H}^\infty(\Sigma_{\theta_0}))$$

holds uniformly in the class of all markovian semigroups satisfying Assumption 9.2 and all Q_2^A *weights* w. *There is some numerical bound* $\theta_0 \geqslant \frac{\pi}{2} - \arctan(\sqrt{31}) \approx 0.113 \cdot \frac{\pi}{2}$.

In view of the above, we define

Definition 9.14 *Let* $\theta \in (0, \pi)$ *be an angle. We call* θ *a universal angle of weighted* L^2 *calculus if for any markovian semigroup satisfying the Assumption 9.2 and any* Q_2^A *weight* w, *there is a constant* $C_w > 0$ *such that*

$$\|m(A)\|_{L^2(\Omega,wd\mu)\to L^2(\Omega,wd\mu)} \leqslant C_w \left(|m(0)| + \|m\|_{\infty,\theta}\right) \quad (m \in \mathrm{H}^\infty(\Sigma_\theta)).$$

According to Theorem 9.3, any $\theta > \frac{\pi}{2}$ is a universal angle for weighted L^2 calculus.

Conjecture 9.15 *No angle* $\theta < \frac{\pi}{2}$ *is a universal angle for weighted* L^2 *calculus.*

Of course also the question arises whether markovian semigroups admit H^∞ functional calculus on weighted L^p for $1 < p < \infty$. To this end, the natural class of weights would be determined by

$$Q_p^A(w) = \sup_{t>0} \text{ess-sup}_{x\in\Omega}\, T_t(w)(x) \left[T_t(w^{-\frac{1}{p-1}})(x)\right]^{p-1} < \infty.$$

This is work in progress.

10 Riesz transforms, Hodge-Dirac operators and H^∞ calculus for multipliers

This section contains the main results from [HDR8, Sections 1-4]. The continuity of the Hilbert transform on $L^p(\mathbb{R})$ by Riesz [Ri] is known as one of the greatest discoveries in analysis of the twentieth century. This transformation is at the heart of many areas : complex analysis, harmonic analysis, Banach space geometry, martingale theory and signal processing. We refer to the thick books [HvNVW1], [HvNVW2], [Kin1] and [Kin2] and references therein for more information. Directional Riesz transforms R_j are higher-dimensional generalizations of the Hilbert transform defined by the formula

$$R_j \overset{\text{def}}{=} \partial_j \circ (-\Delta)^{-\frac{1}{2}}, \quad j = 1, \ldots, n \tag{10.1}$$

where Δ is the Laplacian on $\mathbb{R}^n$. Generalizing Riesz's result, Calderón and Zygmund proved in [CaZ] that these operators are bounded on $L^p(\mathbb{R}^n)$ if $1 < p < \infty$. It is known that the L^p norms do not depend on the dimension n by [IwG] and [BaW]. In [Ste83] (see also [IwG] and [BaW]), Stein showed that the vectorial Riesz transform $\partial(-\Delta)^{-\frac{1}{2}}$ satisfies

$$\left\|\partial(-\Delta)^{-\frac{1}{2}} f\right\|_{L^p(\mathbb{R}^n,\ell^2)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)} \tag{10.2}$$

with dimension free bound where $\partial f \overset{\text{def}}{=} (\partial_1 f, \dots, \partial_n f)$ is the gradient of a function f belonging to some suitable subspace of $L^p(\mathbb{R}^n)$. Furthermore, by duality we have by e.g. [CoD, Proposition 2.1] an equivalence of the form

$$\left\|(-\Delta)^{\frac{1}{2}} f\right\|_{L^p(\mathbb{R}^n)} \approx_p \|\partial f\|_{L^p(\mathbb{R}^n,\ell_n^2)}\,. \tag{10.3}$$

Note that this equivalence can be seen as a variant of the famous Kato square root problem solved in [AHLMT] and in [AKM], see also [Tch] and [HLM].

An important generalization was given by Meyer [Mey]. It consists in replacing the Laplacian $-\Delta$ by the L^p realization A_p of the negative infinitesimal generator A of a Markov semigroup $(T_t)_{t\geqslant 0}$ of operators acting on the L^p spaces of a measure space Ω and to replace the gradient ∂ by the "carré du champ" Γ introduced by Roth [Rot] (see also [Hir]) defined[2] by

$$\Gamma(f,g) \overset{\text{def}}{=} \frac{1}{2}\big[A(\overline{f})g + \overline{f}A(g) - A(\overline{f}g)\big]. \tag{10.4}$$

In the case of the Heat semigroup $(e^{t\Delta})_{t\geqslant 0}$ with generator Δ, we recover the gradient form $\langle \partial f, \partial g\rangle_{\ell_n^2}$. Meyer was interested by the equivalence

$$\left\|A_p^{\frac{1}{2}}(f)\right\|_{L^p(\Omega)} \approx_p \left\|\Gamma(f,f)^{\frac{1}{2}}\right\|_{L^p(\Omega)} \tag{10.5}$$

on some suitable subspace (ideally dom $A_p^{\frac{1}{2}}$) of $L^p(\Omega)$. Meyer proved such an equivalence for the Ornstein-Uhlenbeck semigroup. Nevertheless, with sharp contrast, if $1 < p < 2$ these estimates are surprisingly false for the Poisson semigroup on $L^p(\mathbb{R}^n)$ which is a Markov semigroup of Fourier multipliers, see [JMP2, Appendix D]. Actually, other examples of semigroups illustrating this phenomenon are already present in the papers of Lust-Piquard [Lus2, Proposition 2.9] and [Lus1, page 283] relying on an observation of Lamberton.

Of course, when something goes wrong with a mathematical problem it is rather natural to change slightly the formulation of the problem in order to obtain a natural positive statement. By introducing some gradients with values in a *noncommutative* space, Junge, Mei and Parcet obtained in [JMP2] dimension free estimates for Riesz transforms associated with arbitrary Markov semigroups $(T_t)_{t\geqslant 0}$ of Fourier multipliers acting on classical L^p spaces $L^p(\hat{G})$ where G is for example an abelian discrete group with (compact) dual group $\hat{G}$ (and more generally on the noncommutative L^p spaces $L^p(\mathrm{VN}(G))$ associated with a nonabelian group G). We denote by $\psi\colon G \to \mathbb{C}$ the symbol of the (negative) infinitesimal generator A of the semigroup. In the spirit of (10.3), the above authors proved in [JMP2] estimates of the form

$$\left\|A_p^{\frac{1}{2}}(f)\right\|_{L^p(\hat{G})} \approx_p \|\partial_{\psi,1,p}(f)\|_{L^p(L^\infty(\Omega)\rtimes_\alpha G)} \tag{10.6}$$

where $\partial_{\psi,1,p}$ is some kind of gradient defined on a dense subspace of the classical L^p space $L^p(\hat{G})$. It takes values in a *noncommutative* L^p space $L^p(L^\infty(\Omega) \rtimes_\alpha G)$ associated with some

2. Here, the domain of A must contain a suitable involutive algebra.

crossed product $L^\infty(\Omega) \rtimes_\alpha G$ where Ω is a probability space and where $\alpha\colon G \to \mathrm{Aut}(L^\infty(\Omega))$ is an action of G on $L^\infty(\Omega)$ determined by the semigroup. Let us explain the simplest case, i.e. the case where α is trivial. In this *non-crossed and very particular situation*, we have an identification of $L^p(L^\infty(\Omega) \rtimes_\alpha G)$ with the classical L^p space $L^p(\Omega \times \hat{G})$ and the map $\partial_{\psi,1,p}$ is defined on the span of characters $\langle s, \cdot \rangle_{G,\hat{G}}$ in $L^p(\hat{G})$ with values in $L^p(\Omega \otimes \hat{G})$. It is defined by

$$\partial_{\psi,1,p}\big(\langle s, \cdot \rangle_{G,\hat{G}}\big) \overset{\text{def}}{=} \mathcal{W}(b_\psi(s)) \otimes \langle s, \cdot \rangle_{G,\hat{G}}. \tag{10.7}$$

where $\mathcal{W}\colon H \to L^0(\Omega)$ is an H-isonormal *Gaussian process*[3] for some real Hilbert space H and where $b_\psi\colon G \to H$ is a specific function satisfying

$$\psi(s) = \|b_\psi(s)\|_H^2\,, \quad s \in G. \tag{10.8}$$

We refer to Subsection 10.1 for the (crossed) general situation where the action α is obtained by second quantization from an orthogonal representation $\pi\colon G \to B(H)$ associated to the semigroup, see (10.32).

The approach by Junge, Mei and Parcet highlights an intrinsic noncommutativity since $L^p(\Gamma_q(H) \rtimes_\alpha G)$ is in general a highly noncommutative object *although* the group G may be abelian. It is fair to say that this need of noncommutativity was first noticed and explicitly written by Lust-Piquard in [Lus1] and [Lus2] in some particular cases under a somewhat different but essentially equivalent form of (10.6). Moreover, it is remarkable that the estimates of [Lus2] were exploited in a decisive way by Naor [Naor] to understand subtle geometric phenomena. Finally, note that the existence of gradients suitable for arbitrary Markov semigroups of linear operators appears already in the work of Cipriani and Sauvageot, see [Sau1], [CiS] and the survey [Cip].

In the context of Riesz transforms, the authors of the classical and remarkable paper [AKM] were the first to introduce suitable Hodge-Dirac operators. The L^p boundedness of the H^∞ calculus of this unbounded operator allows everyone to obtain immediately the L^p boundedness of Riesz transforms. The authors of [JMP2] introduced a similar operator in the context of Markov semigroups $(T_t)_{t\geqslant 0}$ of Fourier multipliers acting on classical L^p spaces and more generally on noncommutative L^p spaces $L^p(\mathrm{VN}(G))$ associated with group von Neumann algebras $\mathrm{VN}(G)$ where $1 < p < \infty$ and where G is a discrete group. We refer to the papers [CGIS], [Cip, Definition 10.4], [HMP1], [HMP2], [MaN1] and [NeV] for Hodge-Dirac operators in related contexts.

The authors of [JMP2] define the Hodge-Dirac operator

$$\mathcal{D}_{\psi,1,p} \overset{\text{def}}{=} \begin{bmatrix} 0 & (\partial_{\psi,1,p})^* \\ \partial_{\psi,1,p} & 0 \end{bmatrix} \tag{10.9}$$

which is an unbounded densely defined operator on some suitable subspace of $L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)$. In [JMP2, Problem C.5], the authors ask for dimension free estimates for the operator $\operatorname{sgn} \mathcal{D}_{\psi,1,p} \overset{\text{def}}{=} \mathcal{D}_{\psi,1,p}|\mathcal{D}_{\psi,1,p}|^{-1}$. We affirmatively answer this question for a large class of groups including all amenable discrete groups and free groups by showing the following result in the spirit of [AKM] although the approach seems to have no links with [AKM].

Theorem 10.1 (see Theorem 10.39, Theorem 10.41 and Remark 10.43) *Suppose* $1 < p < \infty$. *Let* G *be a weakly amenable discrete group such that the crossed product* $L^\infty(\Omega) \rtimes_\alpha G$ *is* QWEP[4]. *The Hodge-Dirac operator* $\mathcal{D}_{\psi,1,p}$ *is bisectorial on* $L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)$

3. In particular, for any $h \in H$ the random variable $\mathcal{W}(h)$ is a centred real Gaussian.
4. See Subsection 10.1 for this approximation property of a von Neumann algebra

and admits a bounded $\mathrm{H}^\infty(\Sigma_\omega^\pm)$ *functional calculus on a bisector* $\Sigma_\omega^\pm$*. Moreover, the norm of the functional calculus is bounded by a constant* K_ω *which depends neither on* G *nor on the semigroup*[5]*.*

Our result can be seen as a strengthening of the dimension free estimates (10.6) of Riesz transforms of the above authors since it is almost immediate that this bounded functional calculus implies the equivalence (10.6), see Remark 10.40. For a survey of semigroups $(T_t)_{t\geqslant 0}$ to which Theorem 10.1 applies, we refer to [HDR8, Section 1]: Fourier multipliers on abelian groups, Coxeter groups, free groups, cyclic groups and the discrete Heisenberg group.

We also obtain in this section an analogue of the equivalences (10.6) for markovian semigroups $(T_t)_{t\geqslant 0}$ of Schur multipliers acting on Schatten spaces $S_I^p \overset{\text{def}}{=} S^p(\ell_I^2)$ for $1 < p < \infty$ where I is an index set. In this case, by [Arh1, Proposition 5.4], the Schur multiplier symbol $[a_{ij}]$ of the negative generator A of $(T_t)_{t\geqslant 0}$ is given by $a_{ij} = \|\alpha_i - \alpha_j\|_H^2$ for some family $\alpha = (\alpha_i)_{i\in I}$ of vectors of a real Hilbert space H. We define a gradient operator $\partial_{\alpha,1,p}$ as the closure of the unbounded linear operator $\mathrm{M}_{I,\mathrm{fin}} \to L^p(\Omega, S_I^p)$, $e_{ij} \mapsto \mathcal{W}(\alpha_i - \alpha_j) \otimes e_{ij}$ where $\mathcal{W}\colon H \to L^0(\Omega)$ is an H-isonormal Gaussian process, Ω is the associated probability space and where $\mathrm{M}_{I,\mathrm{fin}}$ is the subspace of S_I^p of matrices with a finite number of non null entries. Then the result reads as follows.

Theorem 10.2 (see Theorem 10.28 and (10.65)) *Let* I *be an index set and* A *be the generator of a markovian semigroup* $(T_t)_{t\geqslant 0}$ *of Schur multipliers on* $B(\ell_I^2)$*. Suppose* $1 < p < \infty$*. For any* $x \in \mathrm{M}_{I,\mathrm{fin}}$*, we have*

$$\big\|A_p^{\frac{1}{2}}(x)\big\|_{S_I^p} \approx_p \|\partial_{\alpha,1,p}(x)\|_{L^p(\Omega,S_I^p)}\,. \tag{10.10}$$

We also obtain an analogue of Theorem 10.1.

Moreover, we also relate the equivalences (10.6) and (10.10) with the ones of Meyer's formulation (10.5). To achieve this, we define and study in the spirit of (10.4) a carré du champ Γ (see (10.31) and (10.40)) and its closed extension in the form sense, and we connect this notion to some approximation properties of groups. It leads us to obtain alternative formulations of (10.6) and (10.10). Note that some carré du champ were studied in the papers [CiS, Section 9], [Cip], [JM], [Sau1] and [JuZ] mainly in the σ-finite case and for L^2-spaces (see [DaL] for related things) but unfortunately their approach does not suffice for our work on L^p spaces. By the way, it is rather surprising that even in the *commutative* setting, no one has examined the carré du champ on L^p spaces with $p \neq 2$. The following is an example of result that we have achieved, and which can be compared with (10.5).

Theorem 10.3 (see Theorem 10.20) *Suppose* $2 \leqslant p < \infty$*. Let* A *be the generator of a markovian semigroup of Schur multipliers on* $B(\ell_I^2)$*. For any* $x \in \operatorname{dom} A_p^{\frac{1}{2}}$*, we have*

$$\big\|A_p^{\frac{1}{2}}(x)\big\|_{S_I^p} \approx_p \max\Big\{\big\|\Gamma(x,x)^{\frac{1}{2}}\big\|_{S_I^p}, \big\|\Gamma(x^*,x^*)^{\frac{1}{2}}\big\|_{S_I^p}\Big\}\,. \tag{10.11}$$

The maximum is natural in noncommutative analysis due to the use of noncommutative Khintchine inequalities.

It is well-known that Gaussian variables are not bounded, i.e. do not belong to $L^\infty(\Omega)$. It will become apparent in the follow-up Section 11 on noncommutative geometry that this is

5. In particular it is independent of the dimension of the Hilbert space H associated to the 1-cocycle through Proposition 10.7.

problematic under technical aspects for our so-called spectral triples. Fortunately, the noncommutative setting is very flexible and allows us to introduce a continuum of gradients $\partial_{\psi,q,p}$ and $\partial_{\alpha,q,p}$ indexed by a new parameter $-1 \leqslant q \leqslant 1$ replacing Gaussian variables by *bounded* noncommutative q-deformed Gaussian variables and $L^\infty(\Omega)$ by the von Neumann algebra $\Gamma_q(H)$ of [BoS] and [BKS]. Note that $\Gamma_{-1}(H)$ is the fermion algebra and that $L^\infty(\Omega)$ can be identified with the boson algebra $\Gamma_1(H)$. Our main theorems on Riesz transforms and Hodge-Dirac operators admit extensions in these cases, see (10.51), (10.53), Theorem 10.31, (10.65), Theorem 10.39 and Remark 10.44. We expect some differences of behaviour when q varies.

Note that we present this section a little differently compared to the preceeding ones. Indeed, as often the case in noncommutative harmonic analysis, the preliminary background and technicalities are much more involved than for the Sections 5 – 9 on commutative L^p spaces. We have thus gathered them in a long preliminary Subsection 10.1. Also a particular contribution of this section compared to [JMP1, JMP2] is a careful examination of the formulation such as the domains of A_p, of the gradients $\partial_{\psi,q,p}$ and $\partial_{\alpha,q,p}$ and of Γ, and the spaces $L^p_c(\mathbb{E}), L^p_r(\mathbb{E})$ (see Subsection 10.1). Then, we have decided not to present the proofs, which, depending on the result, can be rather easy, or technically involved or however demanding a lot of machinery and results from different areas of analysis in Banach spaces. In the last case, for main results (i.e. the theorems), we indicate it shortly, and occasionally give a sketch. In any case, for a detailed expository we refer the interested reader to [HDR8] containing the results of this section.

10.1 Preliminaries

The subject of Sections 10 and 11 requires a lot more preliminaries from harmonic analysis on noncommutative L^p spaces. We have regrouped them in the following paragraphs of this subsection.

Noncommutative L^p spaces and operators on it We recall basic background on von Neumann algebras, their L^p spaces and operators acting on them. A von Neumann algebra is an involutive, weak operator topology closed sub-algebra of some $B(H)$ containing the unit, where H is a Hilbert space. In this section, the only appearing examples of von Neumann algebras, besides $B(\ell^2_I)$ itself for some index set I, will be the q-Gaussian algebra and the crossed product algebra, both explained below in this subsection, and the spatial product and the group von Neumann algebra. We turn to the latter two. First note that if M, N are two von Neumann algebras sitting in $B(H), B(K)$, then $M \otimes N$ can be regarded as a subalgebra of $B(H \otimes_2 K)$, where $H \otimes_2 K$ is the Hilbertian tensor product of H and K. Then the spatial product $M \overline{\otimes} N$ is by definition the von Neumann algebra generated by $M \otimes N$. Second if G is a discrete group, we can consider its regular representation λ over the Hilbert space ℓ^2_G given by left translations $\lambda_s \colon \delta_r \mapsto \delta_{sr}$ where $r, s \in G$. Then the group von Neumann algebra $\mathrm{VN}(G)$ is the von Neumann subalgebra of $B(\ell^2_G)$ generated by these left translations. We also recall that $\mathrm{C}^*_r(G)$ stands for the reduced group C^*-algebra sitting inside $\mathrm{VN}(G)$ and generated by these λ_s (i.e. the smallest norm closed involutive subalgebra containing all λ_s).

All our von Neumann algebras M are equipped with a normal semifinite faithful trace $\tau : M_+ \to [0, \infty]$. Then we let $S_M = \mathrm{span}\{x \in M_+ : \tau(x) < \infty\}$, such that $\tau : S_M \to \mathbb{C}$ is a well-defined linear functional. If $x \in S_M$, then for $1 \leqslant p < \infty$, $|x|^p$ defined via the selfadjoint functional calculus belongs again to S_M, and consequently, $\|x\|_p := (\tau(|x|^p))^{\frac{1}{p}}$ is a well-defined expression. It is moreover a norm on S_M.

Definition 10.4 *Let $1 \leqslant p < \infty$. A noncommutative L^p space $L^p(M) = L^p(M, \tau)$ is the completion of S_M with respect to $\|x\|_p$, where M is a von Neumann algebra equipped with*

a normal (for any bounded increasing net $(x_\alpha)_\alpha$ in M_+, we have $\sup_\alpha \tau(x_\alpha) = \tau(\sup_\alpha x_\alpha)$), semifinite (for each non-zero $x \in M_+$ there exists a non-zero $y \in M_+$ such that $0 \leqslant y \leqslant x$ and $\tau(y) < \infty$), faithful (the only element $x \in M_+$ such that $\tau(x) = 0$ is $x = 0$) trace τ ($\tau(xy) = \tau(yx)$ for any $x, y \in S_M$).

It turns out that a noncommutative L^p space $L^p(M)$ can be described by a space consisting of closed densely defined operators over H, affiliated with M (i.e. commuting with all unitaries that commute themselves with all of M). Thus one can consider their intersection for different values of p. It is convenient also to set $L^\infty(M) = M$. Note that the Hölder inequality, complex interpolation, and duality hold literally as in the commutative case. One says that the von Neumann algebra M is finite if it has a normal faithful trace such that $\tau(1)$ is finite. We refer to [JMX, Section 2.A], [PiX] and the references therein for more information.

The space M is not only normed, but its tensor product with the space of $n \times n$ matrices, $\mathrm{M}_n \otimes M$ also carries a canonical norm given by seeing $\mathrm{M}_n \otimes M \subseteq B(\ell_n^2(H))$. In fact, any closed subspace of some $B(K)$ where K is Hilbert (that is, an operator space), carries such tensor amplification norms, and the above $L^p(M)$ are such operator spaces (with however K abstract, different from H in general).

For a linear $T\colon L^p(M_1) \to L^p(M_2)$ (more generally, an operator acting between operator spaces), we say that T is completely bounded, if $\|T\|_{cb} := \sup_{n\in\mathbb{N}} \|\mathrm{Id}_{\mathrm{M}_n} \otimes T\|_{\mathrm{M}_n \otimes L^p(M_1) \to \mathrm{M}_n \otimes L^p(M_2)} < \infty$. Note that $\mathrm{M}_n \otimes L^p(M)$ carries a cone of positivity given by $\{x^*x : x \in \mathrm{M}_n \otimes L^{2p}(M)\}$. Then we say that T is completely positive, if $\mathrm{Id}_{\mathrm{M}_n} \otimes T$ preserves these positive cones for all $n \in \mathbb{N}$. We refer to [Pis5] for more information on operator spaces.

A particular type of completely bounded, completely positive operator appearing below is the conditional expectation. Let N be a von Neumann algebra and $M \subseteq N$ a von Neumann subalgebra.

If $\mathbb{E} : N \to M$ is a positivity preserving projection of norm one, then $\mathbb{E}$ is called conditional expectation (it has other properties then) [Str, page 116]. If moreover N is equipped with a normal semifinite faithful trace τ such that its restriction to M is again semifinite, and $\mathbb{E}$ is normal (i.e. weak* to weak* continuous) and preserves the trace ($\tau(\mathbb{E}(x)) = \tau(x)$ for $x \in S_N$), then $\mathbb{E}$ extends to a contraction $\mathbb{E} : L^p(N) \to L^p(M)$ for all $1 \leqslant p < \infty$ [JMX, page 92]. We refer to [Str] for more information.

Note that if the von Neumann algebra M is commutative then there exists a localisable measure space (Ω, μ) and a bijective $*$-homomorphism $\pi\colon M \to L^\infty(\Omega, \mu)$ preserving the trace (to the μ-integral), such that $\pi|_{L^p(M)\cap M}$ extends to an isometry as well.

Finally, note that if $\pi\colon M \to N$ is a normal (i.e. weak$*$ to weak$*$ continuous) unital $*$-homomorphism between von Neumann algebras equipped with normal semifinite faithful traces, and π preserves the traces, then $\pi|_{L^p(M)\cap M}$ extends to an isometry $L^p(M) \to L^p(N)$ for any $1 \leqslant p < \infty$ [JMX, page 92].

Markovian semigroups of Fourier and Schur multipliers In this paragraph, we recall the basic theory of markovian semigroups of Fourier multipliers. The following definition and properties of a markovian semigroup are fundamental for us. Thus the assumptions and notations which follow these lines are standing for all the section. We need the following extension of the definition of a markovian semigroup acting on noncommutative L^p spaces.

Definition 10.5 *Let M be a von Neumann algebra equipped with a normal semifinite faithful trace. We say that a weak* continuous semigroup $(T_t)_{t\geqslant 0}$ of operators on M is a markovian semigroup if each T_t is a weak* continuous selfadjoint completely positive unital contraction.*

For any $1 \leqslant p < \infty$, such a semigroup induces a strongly continuous semigroup of operators on each $L^p(M)$ satisfying

1. each T_t is a contraction on $L^p(M)$,
2. each T_t is selfadjoint on $L^2(M)$,
3. each T_t is completely positive on $L^p(M)$.
4. $T_t(1) = 1$

Note that according to [JMX, Proposition 5.4], for any $1 < p < \infty$, the negative generator A_p of $(T_t)_{t\geqslant 0}$ on $L^p(M)$ is an ω-sectorial operator with $\omega > \pi \left|\frac{1}{p} - \frac{1}{2}\right|$. Thus we can consider its square root which is again a closed and sectorial operator, and we write $\operatorname{dom} A_p^{\frac{1}{2}}$ for its domain. Consider a discrete group G together with its group von Neumann algebra $\mathrm{VN}(G)$. Let us write

$$\mathcal{P}_G \overset{\text{def}}{=} \operatorname{span}\{\lambda_s : s \in G\} \tag{10.12}$$

for the space of "trigonometric polynomials". The von Neumann algebra $\mathrm{VN}(G)$ is equipped with the tracial normal faithful state $\tau(\lambda_s) \overset{\text{def}}{=} \delta_{s=e} = \langle \lambda_s \delta_e, \delta_e \rangle$. Now, we introduce the main class of multipliers which interest us.

Definition 10.6 *Let G be a discrete group. A Fourier multiplier on $\mathrm{VN}(G)$ is a weak* continuous linear map $T\colon \mathrm{VN}(G) \to \mathrm{VN}(G)$ such that there exists a (unique) complex function $\phi\colon G \to \mathbb{C}$ such that for any $s \in G$ we have $T(\lambda_s) = \phi_s \lambda_s$. In this case, we let $M_\phi = T$ and we say that ϕ is the symbol of T.*

We have the following folklore characterization of markovian semigroups of Fourier multipliers. Proposition 10.7 is central for the remainder of this section and the mappings π and $b_\psi = b$ will be used throughout tacitly. Recall that a function $\psi\colon G \to \mathbb{C}$ is conditionally negative definite if $\psi(s) = \overline{\psi(s^{-1})}$ for any $s \in G$, $\psi(e) \geqslant 0$ and if the condition $\sum_{i=1}^n c_i = 0$ implies $\sum_{i,j=1}^n \overline{c_i} c_j \psi(s_j^{-1} s_i) \leqslant 0$. If H is a real Hilbert space, $O(H)$ stands for the orthogonal group.

Proposition 10.7 (Schoenberg) *Let G be a discrete group and $(T_t)_{t\geqslant 0}$ be a family of operators on $\mathrm{VN}(G)$. Then the following are equivalent.*

1. *$(T_t)_{t\geqslant 0}$ is a markovian semigroup of Fourier multipliers.*
2. *There exists a (unique) real-valued conditionally negative definite function $\psi\colon G \to \mathbb{R}$ satisfying $\psi(e) = 0$ such that $T_t(\lambda_s) = \exp(-t\psi(s))\lambda_s$ for any $t \geqslant 0$ and any $s \in G$.*
3. *There exists a real Hilbert space H together with a mapping $b\colon G \to \mathbb{C}$ and a homomorphism $\pi\colon G \to O(H)$ such that the 1-cocycle law holds*

   $$\pi_s(b_\psi(t)) = b_\psi(st) - b_\psi(s), \quad \textit{i.e.} \quad b_\psi(st) = b_\psi(s) + \pi_s(b_\psi(t)) \tag{10.13}$$

   for any $s, t \in G$ and such that

   $$\psi(s) = \|b_\psi(s)\|_H^2, \quad s \in G. \tag{10.14}$$

Under these conditions, we say that ψ is a conditionally negative length, and (b_ψ, H) is a cocycle.

Let I be any non-empty index set. Let $A = [a_{ij}]_{i,j\in I}$ be a matrix of $\mathbb{M}_I$. By definition, the Schur multiplier on $B(\ell^2_I)$ associated with this matrix is the unbounded linear operator M_A whose domain $\operatorname{dom} M_A$ is the space of all $B = [b_{ij}]_{i,j\in I}$ of $B(\ell^2_I)$ such that $[a_{ij}b_{ij}]_{i,j\in I}$ belongs to $B(\ell^2_I)$, and whose action on $B = [b_{ij}]_{i,j\in I}$ is given by $M_A(B) \stackrel{\text{def}}{=} [a_{ij}b_{ij}]_{i,j\in I}$. Often, we write A for M_A.

Definition 10.8 *Let I be a non-empty index set. Here and below, we write $\mathrm{M}_{I,\mathrm{fin}}$ for the space of $I \times I$ matrices with only a finite number of non-zero coefficients.*

Note the following property. Here and below, $S^p_I = L^p(B(\ell^2_I), \mathrm{Tr})$ is the Schatten space, where Tr is the usual trace on $B(\ell^2_I)$ taking entire values on orthogonal projections.

Definition 10.9 *Let $[a_{ij}]_{i,j\in I}$ be a matrix. For $1 \leqslant p \leqslant \infty$, we define the associated Schur multiplier A_p as a closed operator on S^p_I defined by the closure of $A\colon \mathrm{M}_{I,\mathrm{fin}} \to S^p_I$, $Ax = A[x_{ij}]_{ij} = [a_{ij}x_{ij}]_{ij}$. This closure is given by $\operatorname{dom} A_p = \{x \in S^p_I : [a_{ij}x_{ij}]_{ij} \in S^p_I\}$ and $A_p(x) = [a_{ij}x_{ij}]_{ij}$ for $x \in \operatorname{dom} A_p$.*

Schur multipliers are in some sense the most commutative operations among noncommutative ones, since $A(x) = X(a)$ for two infinite matrices a and x and associated Schur multipliers A and X. Yet they provide a surprisingly rich class of mappings and they have a longstanding usage in various fields of analysis such as complex function theory [ShSh], Banach spaces [KwPe], operator theory [ShWa], multivariate analysis [Sty], theory of absolutely summing operators [Ben, Piet], and S^p boundedness and functional calculus [AlP1, AlP2, CaS, CMPST1, CMPST2, Dou, DoG, JMX, NeRi].

The following description [Arh1, Proposition 5.4] of a markovian semigroup consisting of Schur multipliers is central for this section. See also [Arh2] for a generalization.

Proposition 10.10 (Schoenberg / Arhancet) *Let I be some non-empty index set and $(T_t)_{t\geqslant 0}$ be a family of operators on $B(\ell^2_I)$. Then the following are equivalent.*

1. *$(T_t)_{t\geqslant 0}$ is a markovian semigroup of Schur multipliers.*
2. *There exists a real Hilbert space H and a family*

$$\alpha = (\alpha_i)_{i\in I} \tag{10.15}$$

of elements of H such that each $T_t\colon B(\ell^2_I) \to B(\ell^2_I)$ is the Schur multiplier associated with the matrix

$$\left[e^{-t\|\alpha_i - \alpha_j\|^2_H}\right]_{i,j\in I}. \tag{10.16}$$

In this case, the weak (negative) infinitesimal generator A acts by $A(e_{ij}) = \|\alpha_i - \alpha_j\|^2_H\, e_{ij}$.*

q-Gaussian algebras We recall the basic definitions of q-Gaussian algebras that we need in this section. We refer to [JMX, Chapter 9], [HDR8, Subsection 2.2] for more details. Let H be a real Hilbert space with complexification $H_{\mathbb{C}}$. If $-1 \leqslant q < 1$ the q-Fock space over H is

$$\mathcal{F}_q(H) = \mathbb{C}\Omega \oplus \bigoplus_{n\geqslant 1} H^{\otimes_n}_{\mathbb{C}}$$

where Ω is a unit vector, called the vacuum. Here the scalar product on $H_{\mathbb{C}}^{\otimes_n}$ is given by a certain symmetrisation, involving the parameter q, of the usual Hilbert space tensor power scalar product. The creation operator $\ell(e)$ for $e \in H$ is given by

$$\begin{array}{rccc} \ell(e)\colon & \mathcal{F}_q(H) & \longrightarrow & \mathcal{F}_q(H) \\ & h_1 \otimes \cdots \otimes h_n & \longmapsto & e \otimes h_1 \otimes \cdots \otimes h_n. \end{array}$$

We denote by $s_q(e)\colon \mathcal{F}_q(H) \to \mathcal{F}_q(H)$ the selfadjoint operator $\ell(e) + \ell(e)^*$. This operator is called q-Gaussian. The q-von Neumann algebra $\Gamma_q(H)$ is the von Neumann algebra over $\mathcal{F}_q(H)$ generated by the operators $s_q(e)$ where $e \in H$. It is a finite von Neumann algebra with the trace τ defined by $\tau(x) = \langle \Omega, x.\Omega \rangle_{\mathcal{F}_q(H)}$ where $x \in \Gamma_q(H)$.

Let H and K be real Hilbert spaces and $T\colon H \to K$ be a contraction with complexification $T_{\mathbb{C}}\colon H_{\mathbb{C}} \to K_{\mathbb{C}}$. We define the following linear map

$$\begin{array}{rccc} \mathcal{F}_q(T)\colon & \mathcal{F}_q(H) & \longrightarrow & \mathcal{F}_q(K) \\ & h_1 \otimes \cdots \otimes h_n & \longmapsto & T_{\mathbb{C}}(h_1) \otimes \cdots \otimes T_{\mathbb{C}}(h_n). \end{array}$$

Then there exists a unique map $\Gamma_q(T)\colon \Gamma_q(H) \to \Gamma_q(K)$ such that for every $x \in \Gamma_q(H)$ we have

$$\big(\Gamma_q(T)(x)\big)\Omega = \mathcal{F}_q(T)(x\Omega). \tag{10.17}$$

This map is normal, unital, completely positive and trace preserving. If $T\colon H \to K$ is an isometry, $\Gamma_q(T)$ is an injective $*$-homomorphism. If $1 \leqslant p < \infty$, it extends to a contraction $\Gamma_q^p(T)\colon L^p(\Gamma_q(H)) \to L^p(\Gamma_q(K))$.

If $q = 1$, one can proceed in the same way as above, but the $s_1(e)$ will be unbounded operators. One obtains a commutative von Neumann algebra $\Gamma_1(H)$ and the $s_1(e)$ belong to all $L^p(\Gamma_1(H))$, $1 \leqslant p < \infty$. They identify to an H-isonormal process on a probability space (Ω, μ) [Nua, Definition 1.1.1], [Ne, Definition 6.5], that is a linear mapping $\mathcal{W}\colon H \to L^0(\Omega)$ with the following properties:

(10.18)
for any $h \in H$ the random variable $\mathcal{W}(h)$ is a centred real Gaussian,

(10.19)
for any $h_1, h_2 \in H$ we have $\mathbb{E}\big(\mathcal{W}(h_1)\,\mathcal{W}(h_2)\big) = \langle h_1, h_2 \rangle_H$.

(10.20)
The linear span of the products $\mathcal{W}(h_1)\,\mathcal{W}(h_2)\cdots\mathcal{W}(h_m)$, with $m \geqslant 0$ and $h_1, \ldots, h_m$ in H, is dense in the real Hilbert space $L^2_{\mathbb{R}}(\Omega)$.

This explains the name q-Gaussian by generalisation to the above context.

Approximation properties of C^* algebras and groups We recall that QWEP stands for "quotient of a space with the weak expectation property". A C^* algebra $A \subseteq B(H)$ has the weak expectation property provided that there exists a unital completely positive map $\phi\colon B(H) \to A^{**}$ such that $\phi|_A = \mathrm{Id}_A$. Then a C^* algebra B has by definition QWEP if it is the quotient of a C^*-algebra A having the weak expectation property. It is an open question whether or not every C^* algebra possesses QWEP, we refer to the survey [Oza, Section 3] for more information on this notion.

We shall consider in this section discrete groups G that have certain approximation properties.

1. A discrete group G is weakly amenable if there exists a net (φ_j) of finitely supported functions on G such that $\varphi_j \to 1$ pointwise and $\sup_j \|M_{\varphi_j}\|_{\mathrm{cb},\mathrm{VN}(G)\to\mathrm{VN}(G)} < \infty$. Here, M_{φ_j} is a Fourier multiplier acting on $\mathrm{VN}(G)$ (see above). Note that amenable groups, free groups, $\mathrm{SL}(2,\mathbb{Z})$ and hyperbolic groups are weakly amenable.

2. Recall that a discrete group has AP, if there exists a net (φ_j) of finitely supported functions on G such that $M_{\varphi_j} \otimes \mathrm{Id}_{B(H)} \to \mathrm{Id}_{\mathrm{C}^*_r(G)\otimes B(H)}$ in the point-norm topology for any Hilbert space H by [HK, Theorem 1.9]. Note that a weakly amenable discrete group has AP by [HK, page 677].

3. For the link of the following definition with the usual one in the literature, we refer to [HDR8, Lemma 2.22]. Let G be a discrete group such that $\mathrm{VN}(G)$ has QWEP. Suppose $1 \leqslant p < \infty$. Then we say that the operator space $L^p(\mathrm{VN}(G))$ has CCAP (completely contractive approximation property) if and only if there exists a net (φ_j) of functions $\varphi_j \colon G \to \mathbb{C}$ with finite support which converge pointwise to 1 such that the net (M_{φ_j}) converges to $\mathrm{Id}_{L^p(\mathrm{VN}(G))}$ in the point-norm topology with

$$\sup_j \|M_{\varphi_j}\|_{\mathrm{cb},L^p(\mathrm{VN}(G))\to L^p(\mathrm{VN}(G))} = 1.$$

   Note that by [JR1, Theorem 1.2], if G is a discrete group with AP and such that $\mathrm{VN}(G)$ has QWEP, then $L^p(\mathrm{VN}(G))$ has the completely contractive approximation property CCAP for any $1 < p < \infty$. For more information on this property and the question of L^p spaces without CCAP, we refer to [LaS].

Crossed product von Neumann algebras In the statement of our Riesz transforms associated with markovian semigroups of Fourier multipliers of Subsection 10.5 and later, we will need a sort of common extension von Neumann algebra of the q-Gaussian von Neumann algebra as introduced above, and the group von Neumann algebra $\mathrm{VN}(G)$, respecting hereby the markovian semigroup objects from Proposition 10.7. In order to do this, the appropriate framework is that of a crossed product von Neumann algebra $M \rtimes_\alpha G$, associated with the three objects M a (semi)finite von Neumann algebra, G a discrete group and $\alpha \colon G \to \mathrm{Aut}(M)$ a representation. We refer to [Haa2], [Haa3], [Str1], [Sun] and [Tak2].

Let M be a von Neumann algebra acting on a Hilbert space H. Let G be a discrete group. Let $\alpha \colon G \to \mathrm{Aut}(M)$ be a trace preserving representation of G on M. For any $x \in M$, we define the operators $\pi(x) \colon L^2(G,H) \to L^2(G,H)$ [Str1, (2) page 263] by

$$\big(\pi(x)\xi\big)(s) \stackrel{\mathrm{def}}{=} \alpha_s^{-1}(x)\xi(s), \qquad \xi \in L^2(G,H), s \in G. \tag{10.21}$$

These operators satisfy the following commutation relation [Str1, (2) page 292]:

$$(\lambda_s \otimes \mathrm{Id}_H)\pi(x)(\lambda_s \otimes \mathrm{Id}_H)^* = \pi(\alpha_s(x)), \quad x \in M, s \in G. \tag{10.22}$$

Recall that the crossed product of M and G with respect to α is the von Neumann algebra $M \rtimes_\alpha G$ acting on the Hilbert space $L^2(G,H)$ and generated by the operators $\pi(x)$ and $\lambda_s \otimes \mathrm{Id}_H$ where $x \in M$ and $s \in G$. By [Str1, page 263] or [Dae, Proposition 2.5], π is a normal injective $*$-homomorphism from M into $M \rtimes_\alpha G$ (hence σ-strong* continuous). For any $s \in G$ and any $x \in M$, we let $x \rtimes \lambda_s \stackrel{\mathrm{def}}{=} \pi(x)(\lambda_s \otimes \mathrm{Id}_H)$. We recall the rules of product and adjoint:

$$(x \rtimes \lambda_s)^* = \alpha_{s^{-1}}(x^*) \rtimes \lambda_{s^{-1}} \tag{10.23}$$

and

$$(x \rtimes \lambda_s)(y \rtimes \lambda_t) = x\alpha_s(y) \rtimes \lambda_{st}, \quad s,t \in G, \xi \in H. \tag{10.24}$$

These relations will be used frequently, as well as the following definition. Namely, for $1 \leqslant p \leqslant \infty$ and M a semifinite von Neumann algebra, we let

$$\mathcal{P}_{p,\rtimes,G} = \text{span}\,\{x \rtimes \lambda_s : x \in L^p(M),\ s \in G\} \subseteq L^p(M \rtimes_\alpha G), \tag{10.25}$$

and write in short $\mathcal{P}_{\rtimes,G} = \mathcal{P}_{\infty,\rtimes,G}$. If M is equipped with a normal finite faithful G-invariant trace τ_M, then $M \rtimes_\alpha G$ is equipped with the normal finite faithful trace defined by $\tau_{M\rtimes_\alpha G}(x \rtimes \lambda_s) = \tau_M(x)\delta_{s=e}$. Finally, again if M is equipped with a finite trace, then there is a normal conditional expectation $\mathbb{E}\colon M \rtimes_\alpha G \to \text{VN}(G)$ sending $x \rtimes \lambda_s \mapsto \tau_M(x)\lambda_s$. It thus extends contractively to $\mathbb{E}\colon L^p(M \rtimes_\alpha G) \to L^p(\text{VN}(G))$, see the paragraph on noncommutative L^p spaces above. In the following, when considering a conditional expectation on $M \rtimes_\alpha G$, it will always be this one.

Lemma 10.11 *Assume M to be a finite von Neumann algebra. Let $1 < p < \infty$. Then $\mathcal{P}_{\rtimes,G}$ and $\mathcal{P}_{p,\rtimes,G}$ are dense subspaces of $L^p(M \rtimes_\alpha G)$. In particular, $\mathcal{P}_G$ is dense in $L^p(\text{VN}(G))$.*

Proof : We refer to [HDR9, Lemma 2.17]. ■

The main result of this paragraph is the following transference result. The assumptions of Proposition 10.12 are satisfied in the case where M has QWEP and where the action $\alpha\colon G \to \text{Aut}(M)$ is amenable, see [Oza, Proposition 4.1 (vi)]. See also [Arh1, Proposition 4.8]. We shall apply Proposition 10.12 exclusively for $M = \Gamma_q(H)$, $-1 \leqslant q \leqslant 1$, which does have QWEP [Nou].

Proposition 10.12 *[HDR8, Proposition 2.20] Suppose $1 \leqslant p \leqslant \infty$. Let $\phi\colon G \to \mathbb{C}$ a function which induces a completely bounded Fourier multiplier $M_\phi\colon L^p(\text{VN}(G)) \to L^p(\text{VN}(G))$. If $1 \leqslant p < \infty$, assume in addition that $M \rtimes_\alpha G$ has* QWEP*. Then $x \rtimes \lambda_s \mapsto \phi(s)x \rtimes \lambda_s$ induces a completely bounded map $\text{Id}_{L^p(M)} \rtimes M_\phi\colon L^p(M \rtimes_\alpha G) \to L^p(M \rtimes_\alpha G)$ and*

$$\left\|\text{Id}_{L^p(M)} \rtimes M_\phi\right\|_{\text{cb},L^p(M\rtimes_\alpha G)\to L^p(M\rtimes_\alpha G)} \leqslant \|M_\phi\|_{\text{cb},L^p(\text{VN}(G))\to L^p(\text{VN}(G))}. \tag{10.26}$$

Proof : See [HDR8, Proposition 2.20]. ■

Hilbertian valued L^p spaces We will need the following vector valued noncommutative L^p spaces. Suppose $1 \leqslant p < \infty$. Let H be a Hilbert space. For any elements $\sum_{k=1}^n x_k \otimes a_k, \sum_{j=1}^m y_j \otimes b_j$ of $L^p(M) \otimes H$, we define the $L^{\frac{p}{2}}(M)$-valued inner product

$$\left\langle \sum_{k=1}^n x_k \otimes a_k, \sum_{j=1}^m y_j \otimes b_j \right\rangle_{L^p(M,H_{c,p})} \overset{\text{def}}{=} \sum_{k,j=1}^{n,m} \langle a_k, b_j\rangle_H\, x_k^* y_j. \tag{10.27}$$

For any element $\sum_{k=1}^n x_k \otimes a_k$ of $L^p(M) \otimes H$, we set [JMX, (2.9)]

$$\left\|\sum_{k=1}^n x_k \otimes a_k\right\|_{L^p(M,H_{c,p})} \overset{\text{def}}{=} \left\|\left(\left\langle \sum_{k=1}^n x_k \otimes a_k, \sum_{j=1}^n x_j \otimes a_j \right\rangle_{L^p(M,H_{c,p})}\right)^{\frac{1}{2}}\right\|_{L^p(M)} \tag{10.28}$$

$$= \left\| \left(\sum_{k,j=1}^{n} \langle a_k, a_j \rangle_H \, x_k^* x_j \right)^{\frac{1}{2}} \right\|_{L^p(M)} .$$

The space $L^p(M, H_{c,p})$ is the completion of $L^p(M) \otimes H$ for this norm [JMX, page 10]. If $(e_1, \ldots, e_n)$ is an orthonormal family of H and if $x_1, \ldots, x_n$ belong to $L^p(M)$, it follows from (10.28) (see [JMX, (2.10)]) that

$$\left\| \sum_{k=1}^{n} x_k \otimes e_k \right\|_{L^p(M, H_{c,p})} = \left\| \left(\sum_{k=1}^{n} |x_k|^2 \right)^{\frac{1}{2}} \right\|_{L^p(M)} = \left\| \sum_{k=1}^{n} e_{k1} \otimes x_k \right\|_{S^p(L^p(M))} .$$

We define similarly $L^p(M, H_{r,p})$ by exchanging the two factors in the inner product (10.27) and in (10.28). Then the spaces $L^p(M, H_{c,p})$ and $L^p(M, H_{r,p})$ are compatible in the sense of interpolation, and we let $L^p(M, H_{\mathrm{rad},p}) \overset{\mathrm{def}}{=} L^p(M, H_{r,p}) \cap L^p(M, H_{c,p})$ if $p \geqslant 2$ and $L^p(M, H_{\mathrm{rad},p}) \overset{\mathrm{def}}{=} L^p(M, H_{r,p}) + L^p(M, H_{c,p})$ if $1 \leqslant p \leqslant 2$. Recall that [JMX, (2.25)]

$$L^p(M, H_{\mathrm{rad},p})^* = L^{p^*}(M, \overline{H}_{\mathrm{rad},p^*}). \tag{10.29}$$

We need some semifinite variant of spaces introduced in [Jun] in the σ-finite case. Let $\mathbb{E} \colon \mathcal{N} \to M$ be a trace preserving normal faithful conditional expectation between von Neumann algebras equipped with normal semifinite faithful traces. If $2 \leqslant p \leqslant \infty$, for any $f, g \in L^p(\mathcal{N})$, using the boundedness[6] of the conditional expectation $\mathbb{E} \colon L^{\frac{p}{2}}(\mathcal{N}) \to L^{\frac{p}{2}}(M)$ we let $\langle f, g \rangle_{L_c^p(\mathbb{E})} \overset{\mathrm{def}}{=} \mathbb{E}(f^* g)$. We denote by $L_c^p(\mathbb{E})$ the completion of $L^p(\mathcal{N})$ with respect to the norm

$$\|f\|_{L_c^p(\mathbb{E})} \overset{\mathrm{def}}{=} \left\| \mathbb{E}(f^* f) \right\|_{L^{\frac{p}{2}}(M)}^{\frac{1}{2}}. \tag{10.30}$$

We still denote by $\langle f, g \rangle_{L_c^p(\mathbb{E})}$ the extension of the bracket on this space. Similarly, we define $\|f\|_{L_r^p(\mathbb{E})} \overset{\mathrm{def}}{=} \left\| \mathbb{E}(f f^*) \right\|_{L^{\frac{p}{2}}(M)}^{\frac{1}{2}}$, which gives rise to a Banach space $L_r^p(\mathbb{E})$. Consider the particular case of $\mathcal{N} = N \rtimes_\alpha G$ a crossed product as considered in the preceding paragraph, and $\mathbb{E} : N \rtimes_\alpha G \to \mathrm{VN}(G)$ the canonical conditional expectation sending $x \rtimes \lambda_s \mapsto \tau_N(x) \lambda_s$. Then it is shown in [HDR8, Subsection 2.5] that $L_c^p(\mathbb{E})$ and $L_r^p(\mathbb{E})$ are compatible Banach spaces, where the compatibility respects $\mathcal{P}_{\rtimes, G}$. We then let $L_{cr}^p(\mathbb{E}) = L_c^p(\mathbb{E}) \cap L_r^p(\mathbb{E})$ in case $2 \leqslant p < \infty$, and $L_{cr}^p(\mathbb{E}) = L_c^p(\mathbb{E}) + L_r^p(\mathbb{E})$ in case $1 < p < 2$.

10.2 The noncommutative gradient and the carré du champ for Fourier multipliers

Let G be a discrete group. We fix in this subsection a markovian semigroup $(T_t)_{t \geqslant 0}$ of Fourier multipliers on $\mathrm{VN}(G)$ as given in Definition 10.6 and Proposition 10.7, with generator A. We equally catch the notations b_ψ, π and H from the cocycle. We shall introduce the objects of the carré du champ Γ and of the noncommutative gradient $\partial_{\psi,q}$ and interrelate them together with $A^{\frac{1}{2}}$, the square root of the generator. This will be pursued more profoundly in Subsection 10.5. For any $1 \leqslant p \leqslant \infty$, note that we have that $\mathcal{P}_G$ from (10.12) is a (dense) subset of $L^p(\mathrm{VN}(G))$. Then for $x, y \in \mathcal{P}_G$, we define the element

$$\Gamma(x, y) \overset{\mathrm{def}}{=} \frac{1}{2} \left[A(x^*) y + x^* A(y) - A(x^* y) \right] \tag{10.31}$$

6. In the case $0 < p < 2$, note that the conditional expectation $\mathbb{E} \colon L^{\frac{p}{2}}(\mathcal{N}) \to L^{\frac{p}{2}}(M)$ is not bounded in general.

of $\mathcal{P}_G$. Suppose $-1 \leqslant q \leqslant 1$. For any $s \in G$, we will use the second quantization from (10.17) by letting

$$\alpha_s \overset{\text{def}}{=} \Gamma_q^\infty(\pi_s) \colon \Gamma_q(H) \to \Gamma_q(H) \tag{10.32}$$

which is trace preserving. We obtain an action $\alpha \colon G \to \mathrm{Aut}(\Gamma_q(H))$. So we can consider the crossed product $\Gamma_q(H) \rtimes_\alpha G$ as studied in Subsection 10.1, which comes equipped with its canonical normal finite faithful trace $\tau_\rtimes$.

Suppose $1 \leqslant p < \infty$. We introduce the map $\partial_{\psi,q} \colon \mathcal{P}_G \to L^p(\Gamma_q(H) \rtimes_\alpha G)$ defined by

$$\partial_{\psi,q}(\lambda_s) = s_q(b_\psi(s)) \rtimes \lambda_s. \tag{10.33}$$

which is a slight generalization of the map of [JMP2, page 535].

Note that $L^p(\Gamma_q(H) \rtimes_\alpha G)$ is a $\mathrm{VN}(G)$-bimodule with left and right actions induced by

$$\lambda_s(z \rtimes \lambda_t) \overset{\text{def}}{=} \alpha_s(z) \rtimes \lambda_{st} \quad \text{and} \quad (z \rtimes \lambda_t)\lambda_s \overset{\text{def}}{=} z \rtimes \lambda_{ts}, \qquad z \in \Gamma_q(H), s, t \in G. \tag{10.34}$$

The following is stated in the particular case $q = 1$ in [JMP2, page 544].

Lemma 10.13 *Suppose $-1 \leqslant q \leqslant 1$. Let G be a discrete group. For any $x, y \in \mathcal{P}_G$, we have*

$$\partial_{\psi,q}(xy) = x\partial_{\psi,q}(y) + \partial_{\psi,q}(x)y. \tag{10.35}$$

Proof : See [HDR8, Lemma 2.34]. ∎

The following is a slight generalization of [JMP2, Remark 1.3]. For the easy proof, we refer to [HDR8, Proposition 2.35]. Here $\mathbb{E} \colon \Gamma_q(H) \rtimes_\alpha G \to \mathrm{VN}(G)$ denotes the canonical conditional expectation.

Proposition 10.14 *Suppose $-1 \leqslant q \leqslant 1$. For any $x, y \in \mathcal{P}_G$, we have*

$$\Gamma(x, y) = \mathbb{E}\big[\big(\partial_{\psi,q}(x)\big)^* \partial_{\psi,q}(y)\big]. \tag{10.36}$$

Suppose $1 \leqslant p < \infty$. The following equalities are in [JRZ, pages 930-931] for $q = 1$. For any $x, y \in \mathcal{P}_G$, we have

$$\Gamma(x, y) = \big\langle \partial_{\psi,q}(x), \partial_{\psi,q}(y) \big\rangle_{L^p(\mathrm{VN}(G), L^2(\Gamma_q(H))_{c,p})}, \tag{10.37}$$

and

$$\big\| \Gamma(x, x)^{\frac{1}{2}} \big\|_{L^p(\mathrm{VN}(G))} = \big\| \partial_{\psi,q}(x) \big\|_{L^p(\mathrm{VN}(G), L^2(\Gamma_q(H))_{c,p})}. \tag{10.38}$$

We shall also need the following Riesz transform norm equivalence for markovian semigroups of Fourier multipliers from [JMP2, Theorem A2, Remark 1.3].

Theorem 10.15 (Junge-Mei-Parcet) *Let G be a discrete group and $(T_t)_{t \geqslant 0}$ a markovian semigroup of Fourier multipliers with symbol ψ of the negative generator A. Then for $2 \leqslant p < \infty$, we have the norm equivalence*

$$\big\| A^{\frac{1}{2}}(x) \big\|_{L^p(\mathrm{VN}(G))} \cong_p \max\Big\{ \big\| \Gamma(x, x)^{\frac{1}{2}} \big\|_{L^p(\mathrm{VN}(G))}, \big\| \Gamma(x^*, x^*)^{\frac{1}{2}} \big\|_{L^p(\mathrm{VN}(G))} \Big\}, \quad x \in \mathcal{P}_G. \tag{10.39}$$

Remark 10.16 *Note that in case $p \geqslant 2$, we can extend the domain of $\Gamma \colon \mathcal{P}_G \times \mathcal{P}_G \to L^{\frac{p}{2}}(\mathrm{VN}(G))$ to $\mathrm{dom}\, A_p^{\frac{1}{2}} \times \mathrm{dom}\, A_p^{\frac{1}{2}}$ by means of a closed form method [HDR8, Lemma 3.16]. Then we obtained that Proposition 10.14 holds literally for $x, y \in \mathrm{dom}\, A_p^{\frac{1}{2}}$ if $p \geqslant 2$, and in case that $\mathrm{VN}(G)$ is QWEP and $L^p(\mathrm{VN}(G))$ has the CCAP, then Theorem 10.15 equally holds literally for $x \in \mathrm{dom}\, A_p^{\frac{1}{2}}$. We refer to [HDR8, Lemma 3.20, Theorem 3.21] for the details.*

10.3 The noncommutative gradient and the carré du champ for Schur multipliers

The first part of this subsection on markovian semigroups of Schur multipliers is the companion of Subsection 10.2 on Fourier multipliers. We suppose that we are given a markovian semigroup of Schur multipliers $(T_t)_{t\geqslant 0}$ as in Definition 10.10, and fix the associated Hilbert space H and the family $(\alpha_i)_{i\in I}$ from (10.15). We shall link, more profoundly in Subsection 10.6, the (square root of the) negative generator A_p, the carré du champ Γ from (10.40) and some noncommutative gradient from (10.41). For any $x, y \in \mathrm{M}_{I,\mathrm{fin}}$, we define the element

$$\Gamma(x,y) \overset{\mathrm{def}}{=} \frac{1}{2}\big[A(x^*)y + x^*A(y) - A(x^*y)\big]. \tag{10.40}$$

of $\mathrm{M}_{I,\mathrm{fin}}$. We recall that $\mathrm{M}_{I,\mathrm{fin}}$ was given in (10.8) as the subspace of $B(\ell_I^2)$ given by the matrices with only a finite number of non-null entries.

Suppose $1 \leqslant p < \infty$ and $-1 \leqslant q \leqslant 1$. We recall $\Gamma_q(H)$ the q-Gaussian functor of Subsection 10.1 associated with the real Hilbert space H stemming from Definition 10.10. Now we can consider the linear map $\partial_{\alpha,q}\colon \mathrm{M}_{I,\mathrm{fin}} \to \Gamma_q(H)\otimes \mathrm{M}_{I,\mathrm{fin}}$ (resp. $\partial_{\alpha,1}\colon \mathrm{M}_{I,\mathrm{fin}} \to L^0(\Omega)\otimes \mathrm{M}_{I,\mathrm{fin}}$ if $q=1$) defined by

$$\partial_{\alpha,q}(e_{ij}) \overset{\mathrm{def}}{=} s_q(\alpha_i - \alpha_j)\otimes e_{ij}, \qquad i,j\in I. \tag{10.41}$$

We have the following Leibniz rule. Note that the space $L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))$ is equipped with a canonical structure of S_I^∞-bimodule whose operations are defined by $(f\otimes x)y \overset{\mathrm{def}}{=} f\otimes xy$ and $y(f\otimes x) \overset{\mathrm{def}}{=} f\otimes yx$ where $x\in S_I^p$, $y\in S_I^\infty$ and $f\in L^p(\Gamma_q(H))$.

Lemma 10.17 *Suppose $-1\leqslant q\leqslant 1$. For any $x,y\in \mathrm{M}_{I,\mathrm{fin}}$, we have*

$$\partial_{\alpha,q}(xy) = x\partial_{\alpha,q}(y) + \partial_{\alpha,q}(x)y. \tag{10.42}$$

Proof : See [HDR8, Lemma 2.38]. ∎

Now, we describe a connection between the carré du champ and the map $\partial_{\alpha,q}$ which is analogous to the equality of [Sau1, Section 1.4] (see also [Sau2]). For that, we introduce the canonical trace preserving normal faithful conditional expectation $\mathbb{E}\colon \Gamma_q(H)\overline{\otimes}B(\ell_I^2) \to B(\ell_I^2)$, $x\otimes y \mapsto \tau_{\Gamma_q(H)}(x)y$.

Proposition 10.18 *Suppose $-1\leqslant q\leqslant 1$ and $1\leqslant p<\infty$.*

1. *For any $x,y\in \mathrm{M}_{I,\mathrm{fin}}$, we have*

$$\Gamma(x,y) = \mathbb{E}\big(\big(\partial_{\alpha,q}(x)\big)^*\partial_{\alpha,q}(y)\big) = \big\langle \partial_{\alpha,q}(x), \partial_{\alpha,q}(y)\big\rangle_{S_I^p(L^2(\Gamma_q(H))_{c,p})}. \tag{10.43}$$

2. *For any $x\in \mathrm{M}_{I,\mathrm{fin}}$, we have*

$$\big\|\Gamma(x,x)^{\frac{1}{2}}\big\|_{S_I^p} = \big\|\partial_{\alpha,q}(x)\big\|_{S_I^p(L^2(\Gamma_q(H))_{c,p})}, \tag{10.44}$$

Proof : See [HDR8, Proposition 2.39]. ∎

Remark 10.19 *Note that in case $p\geqslant 2$, we can extend the domain of $\Gamma\colon \mathrm{M}_{I,\mathrm{fin}}\times\mathrm{M}_{I,\mathrm{fin}} \to S_I^{\frac{p}{2}}$ to $\operatorname{dom} A_p^{\frac{1}{2}} \times \operatorname{dom} A_p^{\frac{1}{2}}$ by means of the closed form method [HDR8, Lemma 3.44]. Then Proposition 10.18 holds literally for $x,y\in \operatorname{dom} A_p^{\frac{1}{2}}$ in case $p\geqslant 2$. We refer to [HDR8, Lemma 3.46] for the details.*

Theorem 10.20 *[HDR8, Theorem 3.47] Suppose $2 \leqslant p < \infty$. For any $x \in \operatorname{dom} A_p^{\frac{1}{2}}$, we have*

$$\left\| A_p^{\frac{1}{2}}(x) \right\|_{S_I^p} \approx_p \max \left\{ \left\| \Gamma(x,x)^{\frac{1}{2}} \right\|_{S_I^p}, \left\| \Gamma(x^*,x^*)^{\frac{1}{2}} \right\|_{S_I^p} \right\}. \tag{10.45}$$

Proof : See [HDR8, Theorem 3.47] for the proof which needs Proposition 10.18 and the result from Theorem 10.28 below. ∎

At the end of this subsection, we shall see that directional Riesz transforms associated with markovian semigroups of Schur multipliers decompose the S_I^p norm. If $h \in H$, we define the h-directional Riesz transform $R_{\alpha,h}$ defined on $\mathrm{M}_{I,\mathrm{fin}}$ by

$$R_{\alpha,h}(e_{ij}) \overset{\mathrm{def}}{=} \frac{\langle \alpha_i - \alpha_j, h \rangle_H}{\|\alpha_i - \alpha_j\|_H} e_{ij} \text{ if } i,j \text{ satisfy } \alpha_i \neq \alpha_j \tag{10.46}$$

and $R_{\alpha,h}(e_{ij}) = 0$ if it is not the case. If $(e_k)_{k \in K}$ is an orthonormal basis of the Hilbert space H, we let

$$R_{\alpha,k} \overset{\mathrm{def}}{=} R_{\alpha,e_k}. \tag{10.47}$$

For a related result on markovian semigroups of Fourier multipliers of the following, we refer to [JMP2, Theorem A1].

Proposition 10.21 *Suppose $1 < p < \infty$.*

1. *If $1 < p \leqslant 2$ and if $x \in \mathrm{M}_{I,\mathrm{fin}} \cap \operatorname{Ran} A$, we have*

$$\|x\|_{S_I^p} \approx_p \inf_{R_{\alpha,k}(x) = a_k + b_k} \left\| \left(\sum_{k \in K} |a_k|^2 \right)^{\frac{1}{2}} \right\|_{S_I^p} + \left\| \left(\sum_{k \in K} |b_k^*|^2 \right)^{\frac{1}{2}} \right\|_{S_I^p}$$

   where the infimum is taken over all $(a_k), (b_k) \in S_I^p(\ell^2_{K,c})$.

2. *If $2 \leqslant p < \infty$ and if $x \in \mathrm{M}_{I,\mathrm{fin}} \cap \operatorname{Ran} A$, we have*

$$\|x\|_{S_I^p} \approx_p \max \left\{ \left\| \left(\sum_{k \in K} |R_{\alpha,k}(x)|^2 \right)^{\frac{1}{2}} \right\|_{S_I^p}, \left\| \left(\sum_{k \in K} |(R_{\alpha,k}(x))^*|^2 \right)^{\frac{1}{2}} \right\|_{S_I^p} \right\}.$$

Proof : See [HDR8, Theorem 3.52]. ∎

10.4 Khintchine inequalities for q-Gaussians in crossed products

In this subsection, we consider a markovian semigroup $(T_t)_{t \geqslant 0}$ of Fourier multipliers on $\mathrm{VN}(G)$, where G is a discrete group, from Definition 10.6, and with cocycle objects b_ψ, π, H from Proposition 10.7. Moreover, we have its second quantization $\alpha : G \to \mathrm{Aut}(\Gamma_q(H))$ from (10.32). Our Theorem 10.22 below generalises [JMP2, Theorem 1.1]. In the following, the conditional expectation is again $\mathbb{E} \colon \Gamma_q(H) \rtimes_\alpha G \to \mathrm{VN}(G)$, $x \rtimes \lambda_s \mapsto \tau_{\Gamma_q(H)}(x)\lambda_s$. We let

$$\mathrm{Gauss}_{q,p,\rtimes}(L^p(\mathrm{VN}(G))) \overset{\mathrm{def}}{=} \overline{\operatorname{span}\left\{ s_q(h) \rtimes x : h \in H, x \in L^p(\mathrm{VN}(G)) \right\}} \tag{10.48}$$

where the closure is taken in $L^p(\Gamma_q(H) \rtimes_\alpha G)$ (for the weak* topology if $p = \infty$ and $-1 \leqslant q < 1$).

Now our first main result in this Section 10 is the noncommutative Khintchine inequality for q-Gaussians on crossed product spaces, that can be rewritten under the following form.

Theorem 10.22 *[HDR8, Theorem 3.4] Consider $-1 \leqslant q \leqslant 1$ and $1 < p < \infty$. Let G be a discrete group.*

1. *Suppose $1 < p < 2$. For any $f = \sum_{s,h} f_{s,h} s_q(h) \rtimes \lambda_s \in \operatorname{span}\{s_q(h) : h \in H\} \rtimes \mathcal{P}_G$, we have*

(10.49)
$$\|f\|_{\mathrm{Gauss}_{q,p,\rtimes}(L^p(\mathrm{VN}(G)))} \approx_p \inf_{f=g+h} \left\{ \left\| \left(\mathbb{E}(g^*g)\right)^{\frac{1}{2}} \right\|_{L^p(\mathrm{VN}(G))}, \left\| \left(\mathbb{E}(hh^*)\right)^{\frac{1}{2}} \right\|_{L^p(\mathrm{VN}(G))} \right\}.$$

Here the infimum runs over all $g \in L^p_c(\mathbb{E})$ and $h \in L^p_r(\mathbb{E})$ such that $f = g + h$.

In case that $\mathrm{VN}(G)$ is QWEP *and $L^p(\mathrm{VN}(G))$ has the* CCAP*, the infimum can be taken over all $g, h \in \operatorname{span}\{s_q(e) : e \in H\} \rtimes \mathcal{P}_G$.*

2. *Suppose $2 \leqslant p < \infty$. For any $f = \sum_{s,h} f_{s,h} s_q(h) \rtimes \lambda_s \in \mathrm{Gauss}_{q,p,\rtimes}(L^p(\mathrm{VN}(G)))$ with $f_{s,h} \in \mathbb{C}$, we have*

(10.50)
$$\begin{aligned} \max\left\{ \left\| \left(\mathbb{E}(f^*f)\right)^{\frac{1}{2}} \right\|_{L^p(\mathrm{VN}(G))}, \left\| \left(\mathbb{E}(ff^*)\right)^{\frac{1}{2}} \right\|_{L^p(\mathrm{VN}(G))} \right\} &\leqslant \|f\|_{\mathrm{Gauss}_{q,p,\rtimes}(L^p(\mathrm{VN}(G)))} \\ &\lesssim \sqrt{p} \underbrace{\max\left\{ \left\| \left(\mathbb{E}(f^*f)\right)^{\frac{1}{2}} \right\|_{L^p(\mathrm{VN}(G))}, \left\| \left(\mathbb{E}(ff^*)\right)^{\frac{1}{2}} \right\|_{L^p(\mathrm{VN}(G))} \right\}}_{\|f\|_{L^p_{cr}(\mathbb{E})}}. \end{aligned}$$

Proof : The proof is highly involved. See [HDR8, Theorem 3.4]. ∎

10.5 Kato's square root problem for semigroups of Fourier multipliers

Throughout this subsection, we consider a discrete group G, and fix a markovian semigroup of Fourier multipliers $(T_t)_{t\geqslant 0}$ acting on $\mathrm{VN}(G)$ with negative generator A_p on $L^p(\mathrm{VN}(G))$ and representing objects $b_\psi \colon G \to H$, $\pi \colon G \to O(H)$, $\alpha \colon G \to \mathrm{Aut}(\Gamma_q(H))$ (see Definition 10.6, Proposition 10.7 and (10.32)). We also have the noncommutative gradient $\partial_{\psi,q} \colon \mathcal{P}_G \subseteq L^p(\mathrm{VN}(G)) \to L^p(\Gamma_q(H) \rtimes_\alpha G)$ from (10.33). The aim of this subsection is to compare $A_p^{\frac{1}{2}}(x)$ and $\partial_{\psi,q}(x)$ in L^p norm, known as Kato's square root problem. We shall also extend $\partial_{\psi,q}$ to a closed operator and identify the domain of that closure. Suppose $1 < p < \infty$, $-1 \leqslant q \leqslant 1$. In [HDR8, (3.29)], we obtained the following generalisation of the result from [JMP2]:

$$\left\| A_p^{\frac{1}{2}}(x) \right\|_{L^p(\mathrm{VN}(G))} \approx_p \left\| \partial_{\psi,q}(x) \right\|_{L^p(\Gamma_q(H)\rtimes_\alpha G)}, \quad x \in \mathcal{P}_G. \tag{10.51}$$

Its proof uses our Theorem 10.22 on the Khintchine equivalence in a decisive way. We shall extend this equivalence in (10.53) below. For later use in Subsection 10.7, we define the densely defined unbounded operator $\partial^*_{\psi,q} \colon \mathcal{P}_{\rtimes,G} \subseteq L^p(\Gamma_q(H) \rtimes_\alpha G) \to L^p(\Gamma_q(H))$ by

$$\partial^*_{\psi,q}(f \rtimes \lambda_s) = \left\langle s_q(b_\psi(s)), f \right\rangle_{L^{p^*}(\Gamma_q(H)), L^p(\Gamma_q(H))} \lambda_s, \quad s \in G, f \in \Gamma_q(H). \tag{10.52}$$

The following lemma is left to the reader.

Lemma 10.23 *The operators $\partial_{\psi,q}$ and $\partial^*_{\psi,q}$ are formal adjoints.*

Proposition 10.24 *Let G be a discrete group. Suppose $1 < p < \infty$ and $-1 \leqslant q \leqslant 1$.*

1. *The operator* $\partial_{\psi,q}\colon \mathcal{P}_G \subseteq L^p(\mathrm{VN}(G)) \to L^p(\Gamma_q(H) \rtimes_\alpha G)$ *is closable as a densely defined operator on* $L^p(\mathrm{VN}(G))$ *into* $L^p(\Gamma_q(H) \rtimes_\alpha G)$. *We denote by* $\partial_{\psi,q,p}$ *its closure. So* $\mathcal{P}_G$ *is a core of* $\partial_{\psi,q,p}$.

2. *We have* $\operatorname{dom}\partial_{\psi,q,p} = \operatorname{dom} A_p^{\frac{1}{2}}$. *Moreover, for any* $x \in \operatorname{dom} A_p^{\frac{1}{2}}$, *we have*

$$\big\|A_p^{\frac{1}{2}}(x)\big\|_{L^p(\mathrm{VN}(G))} \approx_p \big\|\partial_{\psi,q,p}(x)\big\|_{L^p(\Gamma_q(H)\rtimes_\alpha G)}. \tag{10.53}$$

3. *If* $x \in \operatorname{dom}\partial_{\psi,q,p}$, *we have* $x^* \in \operatorname{dom}\partial_{\psi,q,p}$ *and*

$$(\partial_{\psi,q,p}(x))^* = -\partial_{\psi,q,p}(x^*). \tag{10.54}$$

Proof : See [HDR8, Proposition 3.12] for a proof which uses the formal adjoint $\partial^*_{\psi,q}$ from (10.52). ■

Proposition 10.25 *Assume* $-1 \leqslant q < 1$. *Let* G *be a discrete group with* AP. *The operator* $\partial_{\psi,q}\colon \mathcal{P}_G \subseteq \mathrm{VN}(G) \to \Gamma_q(H) \rtimes_\alpha G$ *is weak* closable*[7]. *We denote by* $\partial_{\psi,q,\infty}$ *its weak* closure.*

Proof : See [HDR8, Proposition 3.13]. ■

We observe that the estimates in (10.53) come with a constant independent of the group G and the cocycle (b_ψ, H). We start with the case $q = 1$ which is essentially the referee's proof [JMP2, pages 574-575] of the paper [JMP2]. Note that we are unfortunately unable[8] to check the original proof given in [JMP2, page 544].

Lemma 10.26 (Junge-Mei-Parcet) *Suppose* $1 < p < \infty$. *For any* $x \in \operatorname{dom}\partial_{\psi,1,p}$, *we have*

$$\frac{1}{K\max(p,p^*)}\big\|A_p^{\frac{1}{2}}(x)\big\|_{L^p(\mathrm{VN}(G))} \leqslant \big\|\partial_{\psi,1,p}(x)\big\|_{L^p(L^\infty(\Omega)\rtimes_\alpha G)} \leqslant K\max(p,p^*)^{\frac{3}{2}}\big\|A_p^{\frac{1}{2}}(x)\big\|_{L^p(\mathrm{VN}(G))} \tag{10.55}$$

with an absolute constant K *not depending on* G *nor the cocycle* (b_ψ, H). *Here,* $L^\infty(\Omega) = \Gamma_1(H)$ *is the Gaussian space from Subsection 10.1.*

Proof : See [HDR8, Lemma 3.9] ■

The next theorem extends Lemma 10.26 to the case of q-Gaussians.

7. That is, if (x_n) is a sequence in $\mathcal{P}_G$ such that $x_n \to 0$ and $\partial_{\psi,q}(x_n) \to y$ for some $y \in \Gamma_q(H) \rtimes_\alpha G$ both for the weak* topology, then $y = 0$.

8. More precisely, with the notations of [JMP2] we are unable to check that "$H \rtimes \mathrm{Id}_G$ extends to a bounded operator on L_p". A part of the very concise explanation given in [JMP2, page 544] is "H is G-equivariant". But it seems to be strange. Indeed we have an action $\alpha\colon G \to \mathrm{Aut}(L_\infty(\mathbb{R}^n_{\mathrm{bohr}}))$, $f \mapsto \big[x \mapsto \alpha_g(f)(x) = f(\pi_g(x))\big]$ for some map $\pi_g\colon \mathbb{R}^n_{\mathrm{bohr}} \to \mathbb{R}^n_{\mathrm{bohr}}$ where $g \in G$ and an induced action α from G on $L_\infty(\mathbb{R}^n_{\mathrm{bohr}} \times \mathbb{R}^n, \nu \times \gamma)$. Now, note that

$$(H(\alpha_g f))(x,y) = \Big(\text{p. v.}\int_{\mathbb{R}} \beta_t \alpha_g f \frac{\mathrm{d}t}{t}\Big)(x,y) = \Big(\text{p. v.}\int_{\mathbb{R}} \beta_t (f\circ\pi_g)\frac{\mathrm{d}t}{t}\Big)(x,y) = \text{p. v.}\int_{\mathbb{R}} f(\pi_g(x+ty))\frac{\mathrm{d}t}{t}$$

and

$$(\alpha_g(H(f))(x,y) = \alpha_g\Big(\text{p. v.}\int_{\mathbb{R}} \beta_t f \frac{\mathrm{d}t}{t}\Big)(x,y) = \text{p. v.}\int_{\mathbb{R}} f(\pi_g(x)+ty)\frac{\mathrm{d}t}{t}$$

which could be different if π is not trivial.

Theorem 10.27 *[HDR8, Proposition 3.11] Suppose $1 < p < \infty$ and $-1 \leqslant q \leqslant 1$. For any $x \in \operatorname{dom} \partial_{\psi,q,p}$, we have*

(10.56)
$$\frac{1}{K \max(p,p^*)^{\frac{3}{2}}} \big\| A_p^{\frac{1}{2}}(x) \big\|_{L^p(\mathrm{VN}(G))} \leqslant \big\| \partial_{\psi,q,p}(x) \big\|_{L^p(\Gamma_q(H) \rtimes_\alpha G)} \leqslant K \max(p,p^*)^2 \big\| A_p^{\frac{1}{2}}(x) \big\|_{L^p(\mathrm{VN}(G))}$$

with an absolute constant K not depending on G nor the cocycle (b_ψ, H).

Proof : See [HDR8, Proposition 3.11]. It uses Theorem 10.22 in a decisive way. ∎

10.6 Kato's square root problem for semigroups of Schur multipliers

In this subsection, we shall consider the Kato square root problem for markovian semigroups of Schur multipliers. Thus, we fix for the whole subsection such a markovian semigroup $(T_t)_{t \geqslant 0}$ from Definition 10.10, with its generator A_p on S_I^p and also the gradient type operator $\partial_{\alpha,q}$ from (10.41), together with the family $(\alpha_i)_{i \in I}$ in the Hilbert space H. Kato's square root problem is then the question whether $A_p^{\frac{1}{2}}(x)$ and $\partial_{\alpha,q}(x)$ are comparable in L^p norm. We shall answer affirmatively to this question in this subsection. Also the problem of exact description of the domain of the closure of $\partial_{\alpha,q}$ is addressed.

Suppose $1 \leqslant p < \infty$. We denote by $A_p^{\frac{1}{2}} \colon \operatorname{dom} A_p^{\frac{1}{2}} \subseteq S_I^p \to S_I^p$ the square root of the sectorial operator $A_p \colon \operatorname{dom} A_p \subseteq S_I^p \to S_I^p$. It is again a Schur multiplier, associated with symbol $\|\alpha_i - \alpha_j\|$.

Theorem 10.28 *[HDR8, Theorem 3.28] Suppose $1 < p < \infty$. For any $x \in \mathrm{M}_{I,\mathrm{fin}}$, we have*

$$\big\| A_p^{\frac{1}{2}}(x) \big\|_{S_I^p} \approx_p \| \partial_{\alpha,1}(x) \|_{L^p(\Omega, S_I^p)} . \tag{10.57}$$

Proof : The proof is highly involved and uses the Hilbert transform on $L^p(\mathrm{VN}(H_{\mathrm{disc}}))$, transference of bounded c_0-groups of operators [BGM], and an intertwining formula. Ideas for this stem from work by Junge, Mei, Parcet [JMP2] and Pisier's approach to a certain Meyer formula [Pis1]. See [HDR8, Theorem 3.28]. ∎

We will extend (10.57) in (10.65) below.

Remark 10.29 *Keeping track of the constants in the two-sided estimate of Theorem 10.28, we obtain*

$$\frac{1}{K \max(p,p^*)^{\frac{3}{2}}} \big\| A_p^{\frac{1}{2}}(x) \big\|_{L^p(\Omega, S_I^p)} \leqslant \big\| \partial_{\alpha,1}(x) \big\|_{S_I^p} \leqslant K \max(p,p^*)^{\frac{3}{2}} \big\| A_p^{\frac{1}{2}}(x) \big\|_{L^p(\Omega, S_I^p)}, \tag{10.58}$$

where K is an absolute constant.

Proof : See [HDR8, Remark 3.29]. ∎

The q-Gaussian derivations equally satisfy the equivalence with $A_p^{\frac{1}{2}}$. To this end, the following Khintchine type lemma 10.30 (which is probably folklore) will be useful. We denote here the canonical conditional expectation $\mathbb{E} \colon \Gamma_q(H) \overline{\otimes} B(\ell_I^2) \to B(\ell_I^2)$. We let

$$\mathrm{Gauss}_{q,p}(S_I^p) \overset{\mathrm{def}}{=} \overline{\operatorname{span} \big\{ s_q(h) \otimes x : h \in H, x \in S_I^p \big\}} \tag{10.59}$$

where the closure is taken in $L^p(\Gamma_q(H) \overline{\otimes} B(\ell_I^2))$ (for the weak* topology if $p = \infty$ and $-1 \leqslant q < 1$).

Lemma 10.30 *Consider $-1 \leqslant q \leqslant 1$.*

1. *Suppose $1 < p < 2$. For any $f = \sum_{i,j,h} f_{i,j,h} s_q(h) \otimes e_{ij} \in \operatorname{span}\{s_q(h) : h \in H\} \otimes \mathrm{M}_{I,\mathrm{fin}}$, we have*

(10.60)
$$\|f\|_{\mathrm{Gauss}_{q,p}(S_I^p)} \approx_p \|f\|_{S_I^p(L^2(\Gamma_q(H))_{\mathrm{rad},p})} \approx_p \inf_{f=g+h} \left\{ \left\| \left(\mathbb{E}(g^*g)\right)^{\frac{1}{2}} \right\|_{S_I^p}, \left\| \left(\mathbb{E}(hh^*)\right)^{\frac{1}{2}} \right\|_{S_I^p} \right\}$$

   where the infimum can equally be taken over all $g, h \in \operatorname{span}\{s_q(e) : e \in H\} \otimes \mathrm{M}_{I,\mathrm{fin}}$.

2. *Suppose $2 \leqslant p < \infty$. For any $f = \sum_{i,j,h} f_{i,j,h} s_q(h) \otimes e_{ij} \in \mathrm{Gauss}_{q,p}(S_I^p)$ with $f_{i,j,h} \in \mathbb{C}$*

(10.61)
$$\begin{aligned} \max\left\{ \left\| \left(\mathbb{E}(f^*f)\right)^{\frac{1}{2}} \right\|_{S_I^p}, \left\| \left(\mathbb{E}(ff^*)\right)^{\frac{1}{2}} \right\|_{S_I^p} \right\} &\leqslant \|f\|_{\mathrm{Gauss}_{q,p}(S_I^p)} \\ &\lesssim \sqrt{p} \underbrace{\max\left\{ \left\| \left(\mathbb{E}(f^*f)\right)^{\frac{1}{2}} \right\|_{S_I^p}, \left\| \left(\mathbb{E}(ff^*)\right)^{\frac{1}{2}} \right\|_{S_I^p} \right\}}_{\|f\|_{L^p_{cr}(\mathbb{E})}}. \end{aligned}$$

Proof : We refer to [HDR8, Lemma 3.32] for a (non-original) proof. ∎

Now we can state the Kato square root problem for the case of the derivation taking values in a q-deformed algebra.

Theorem 10.31 *[HDR8, Proposition 3.36] Suppose $-1 \leqslant q \leqslant 1$ and $1 < p < \infty$. For any $x \in \mathrm{M}_{I,\mathrm{fin}}$, we have*

(10.62)
$$\left\|A_p^{\frac{1}{2}}(x)\right\|_{S_I^p} \approx_p \left\|\partial_{\alpha,q}(x)\right\|_{L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))}.$$

Proof : See [HDR8, Proposition 3.36] for a proof which uses Lemma 10.30 above. ∎

Remark 10.32 *Assume $-1 \leqslant q \leqslant 1$ and $1 < p < \infty$. In Theorem 10.31 above, we obtain again constants depending on p as in (10.58) of a slightly different form, that is, for some absolute constant $K > 0$, we have for all $x \in \mathrm{M}_{I,\mathrm{fin}}$,*

(10.63)
$$\frac{1}{K\max(p,p^*)^2}\left\|\partial_{\alpha,q}(x)\right\|_{L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))} \leqslant \left\|A_p^{\frac{1}{2}}(x)\right\|_{S_I^p} \leqslant K\max(p,p^*)^2\left\|\partial_{\alpha,q}(x)\right\|_{L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))}$$

Proof : See [HDR8, Remark 3.37]. ∎

At the end of this subsection, we turn to an extension of $\partial_{\alpha,q}$ to a closed operator. This contains also an extension of the Riesz transform equivalences to the domain of $A_p^{\frac{1}{2}}$. For later use, we define the densely defined unbounded operator

$$\partial_{\alpha,q}^* \colon L^p(\Gamma_q(H)) \otimes \mathrm{M}_{I,\mathrm{fin}} \subseteq L^p(\Gamma_q(H))\overline{\otimes}B(\ell_I^2)) \to S_I^p$$

by

(10.64)
$$\partial_{\alpha,q}^*(f \otimes e_{ij}) = \left\langle s_q(\alpha_i - \alpha_j), f\right\rangle_{L^{p^*},L^p} e_{ij} = \tau(s_q(\alpha_i - \alpha_j)f)e_{ij}, \quad i,j \in I, f \in L^p(\Gamma_q(H)).$$

Lemma 10.33 *The operators $\partial_{\alpha,q}$ and $\partial_{\alpha,q}^*$ are formal adjoints (with respect to duality brackets $\langle x, y\rangle = \tau(x^*y)$).*

Proof : See [HDR8, Lemma 3.39] for the easy proof. ∎

Proposition 10.34 *Suppose* $1 < p < \infty$ *and* $-1 \leqslant q \leqslant 1$.

1. *The operator* $\partial_{\alpha,q} \colon \mathrm{M}_{I,\mathrm{fin}} \subseteq S_I^p \to L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))$ *is closable as a densely defined operator on* S_I^p *into* $L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))$. *We denote by* $\partial_{\alpha,q,p}$ *its closure. So* $\mathrm{M}_{I,\mathrm{fin}}$ *is a core of* $\partial_{\alpha,q,p}$.

2. *We have* $\operatorname{dom}\partial_{\alpha,q,p} = \operatorname{dom} A_p^{\frac{1}{2}}$. *Moreover, for any* $x \in \operatorname{dom} A_p^{\frac{1}{2}}$, *we have*

$$\left\|A_p^{\frac{1}{2}}(x)\right\|_{S_I^p} \approx_p \left\|\partial_{\alpha,q,p}(x)\right\|_{\mathrm{Gauss}_{q,p}(S_I^p)}. \tag{10.65}$$

3. *If* $x \in \operatorname{dom}\partial_{\alpha,q,p}$, *we have* $x^* \in \operatorname{dom}\partial_{\alpha,q,p}$ *and*

$$(\partial_{\alpha,q,p}(x))^* = -\partial_{\alpha,q,p}(x^*). \tag{10.66}$$

4. *Suppose that* $-1 \leqslant q < 1$. *Then the operator* $\partial_{\alpha,q} \colon \mathrm{M}_{I,\mathrm{fin}} \subseteq B(\ell_I^2) \to \Gamma_q(H)\overline{\otimes}B(\ell_I^2)$ *is weak* closable. We denote by* $\partial_{\alpha,q,\infty}$ *its weak* closure.*

5. *The Riesz transform norm equivalences* (10.57), (10.58), (10.62), (10.63) *hold literally for* $x \in \operatorname{dom} A_p^{\frac{1}{2}}$, *and* $\partial_{\alpha,q}$ *replaced by its closed extension* $\partial_{\alpha,q,p}$.

Proof : See [HDR8, Proposition 3.40] for the proof which uses the formal adjoint $\partial_{\alpha,q}^*$ from (10.64). ∎

10.7 Boundedness of H^∞ calculus of Hodge-Dirac operators for multipliers

In this subsection, we let G be a discrete group and we consider a semigroup of Markov Fourier multipliers $(T_t)_{t\geqslant 0}$ from Definition 10.6 and Proposition 10.7. If $1 \leqslant p < \infty$, we denote by A_p the (negative) infinitesimal generator on $L^p(\mathrm{VN}(G))$. Also we fix the cocycle objects b_ψ, π, H from Proposition 10.7. If $-1 \leqslant q \leqslant 1$, recall that by Proposition 10.24, we have a closed operator

$$\partial_{\psi,q,p} \colon \operatorname{dom}\partial_{\psi,q,p} \subseteq L^p(\mathrm{VN}(G)) \to L^p(\Gamma_q(H) \rtimes_\alpha G), \quad \lambda_s \mapsto s_q(b_\psi(s)) \rtimes \lambda_s$$

and a closed operator $(\partial_{\psi,q,p^*})^* \colon \operatorname{dom}(\partial_{\psi,q,p^*})^* \subseteq L^p(\Gamma_q(H) \rtimes G) \to L^p(\mathrm{VN}(G))$. The noncommutative gradient $\partial_{\psi,q,p}$ comes with graph norm equivalent to that of the square root of the generator $A_p^{\frac{1}{2}}$ according to our Kato square root result, see (10.53). We will now define a Hodge-Dirac operator $D_{\psi,q,p}$ in (10.69), out of $\partial_{\psi,q,p}$ and its adjoint, on some larger space. Then the main result of this subsection is that $D_{\psi,q,p}$ is R-bisectorial (Theorem 10.37) and has a bisectorial H^∞ calculus on $L^p(\mathrm{VN}(G)) \oplus \overline{\operatorname{Ran}\partial_{\psi,q,p}}$ (Theorem 10.39).

We have the following useful formula.

Proposition 10.35 *Suppose* $1 < p < \infty$ *and* $-1 \leqslant q \leqslant 1$. *As unbounded operators, we have*

$$A_p = (\partial_{\psi,q,p^*})^* \partial_{\psi,q,p}. \tag{10.67}$$

Proof : See [HDR8, Proposition 4.1]. ∎

We let $B_p \overset{\mathrm{def}}{=} (\mathrm{Id}_{L^p(\Gamma_q(H))} \rtimes A_p)|\overline{\operatorname{Ran}\partial_{\psi,q,p}}$.

Proposition 10.36 *Let $1 < p < \infty$ and $-1 \leqslant q \leqslant 1$. Assume that $\Gamma_q(H) \rtimes_\alpha G$ has QWEP. We have $\operatorname{dom} B_p = \operatorname{dom}(\partial_{\psi,q,p}(\partial_{\psi,q,p^*})^* | \overline{\operatorname{Ran}\, \partial_{\psi,q,p}})$ and for any y belonging to this space,*

$$\partial_{\psi,q,p}(\partial_{\psi,q,p^*})^*(y) = B_p(y). \tag{10.68}$$

Proof : See [HDR8, Proposition 4.10]. ∎

Suppose $1 < p < \infty$. We introduce the unbounded operator

$$D_{\psi,q,p} \overset{\text{def}}{=} \begin{bmatrix} 0 & (\partial_{\psi,q,p^*})^* \\ \partial_{\psi,q,p} & 0 \end{bmatrix} \tag{10.69}$$

on the Banach space $L^p(\mathrm{VN}(G)) \oplus_p \overline{\operatorname{Ran}\, \partial_{\psi,q,p}}$ defined by

$$D_{\psi,q,p}(x,y) \overset{\text{def}}{=} \big((\partial_{\psi,q,p^*})^*(y), \partial_{\psi,q,p}(x)\big), \quad x \in \operatorname{dom} \partial_{\psi,q,p},\ y \in \operatorname{dom}(\partial_{\psi,q,p^*})^* \cap \overline{\operatorname{Ran}\, \partial_{\psi,q,p}}. \tag{10.70}$$

We call it the Hodge-Dirac operator of the semigroup. This operator is a closed operator and can be seen as a differential square root of the generator of the semigroup $(T_{t,p})_{t \geqslant 0}$ since we have Proposition 10.38. For the definition of R-bisectoriality we refer to the end of Subsection 4.3.

Theorem 10.37 *[HDR8, Theorem 4.13] Suppose $1 < p < \infty$ and $-1 \leqslant q \leqslant 1$. Assume that $\Gamma_q(H) \rtimes_\alpha G$ has QWEP. The Hodge-Dirac operator $D_{\psi,q,p}$ is R-bisectorial on $L^p(\mathrm{VN}(G)) \oplus_p \overline{\operatorname{Ran}\, \partial_{\psi,q,p}}$.*

Proof : See [HDR8, Theorem 4.13, Section 4.1]. The main ingredient of the proof is the (dimension free) estimate of the Riesz transform from Theorem 10.27, and the extension of Fourier multipliers to crossed products from Proposition 10.12. ∎

Proposition 10.38 *Suppose $1 < p < \infty$ and $-1 \leqslant q \leqslant 1$. Assume that $\Gamma_q(H) \rtimes_\alpha G$ has QWEP. As densely defined closed operators on $L^p(\mathrm{VN}(G)) \oplus_p \overline{\operatorname{Ran}\, \partial_{\psi,q,p}}$, we have*

$$D_{\psi,q,p}^2 = \begin{bmatrix} A_p & 0 \\ 0 & (\mathrm{Id}_{L^p(\Gamma_q(H))} \rtimes A_p) | \overline{\operatorname{Ran}\, \partial_{\psi,q,p}} \end{bmatrix}. \tag{10.71}$$

Proof : According to (10.67) and (10.68), we have

$$\begin{aligned} D_{\psi,q,p}^2 &= \begin{bmatrix} 0 & (\partial_{\psi,q,p^*})^* \\ \partial_{\psi,q,p} & 0 \end{bmatrix}^2 \\ &= \begin{bmatrix} (\partial_{\psi,p^*})^* \partial_{\psi,p} & 0 \\ 0 & \partial_{\psi,p}(\partial_{\psi,p^*})^* \end{bmatrix} \\ &= \begin{bmatrix} A_p & 0 \\ 0 & B_p \end{bmatrix}. \end{aligned}$$

See [HDR8, Proposition 4.14] for the details. ∎

Theorem 10.39 *[HDR8, Theorem 4.15] Suppose $1 < p < \infty$ and $-1 \leqslant q \leqslant 1$. Suppose that the von Neumann algebra $\Gamma_q(H) \rtimes_\alpha G$ has* QWEP *(e.g. G is amenable or G is a free group and $q = \pm 1$). Then the Hodge-Dirac operator $D_{\psi,q,p}$ is R-bisectorial on $L^p(\mathrm{VN}(G)) \oplus_p \overline{\operatorname{Ran}\, \partial_{\psi,q,p}}$ and admits a bounded $\mathrm{H}^\infty(\Sigma_\omega^\pm)$ functional calculus on a bisector.*

Proof : This time, we give a sketch of proof. Note that A_p has an $\mathrm{H}^\infty(\Sigma_\theta)$ calculus for any $\theta > \frac{\pi}{2}$ being the generator of a markovian semigroup of Fourier multipliers [Arh4, Theorem 4.1]. By selfadjointness of A_2, one can reduce the angle to some $\theta_p < \frac{\pi}{2}$. Then according to the extension of Fourier multipliers to the crossed product $L^p(\Gamma_q(H) \rtimes_\alpha G)$ from Proposition 10.12, $\mathrm{Id}_{L^p(\Gamma_q(H))} \rtimes A_p$ also has an $\mathrm{H}^\infty(\Sigma_{\theta_p})$ calculus, thus also its restriction to the subspace $\overline{\operatorname{Ran} \partial_{\psi,q,p}}$. From the preceding Proposition 10.38, we see then that $D^2_{\psi,q,p}$ has a bounded $\mathrm{H}^\infty(\Sigma_{\theta_p})$ calculus. Then according to Theorem 4.13, $D_{\psi,q,p}$ has an $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ calculus for some $\omega < \frac{\pi}{4}$. ∎

Remark 10.40 *The boundedness of the* H^∞ *functional calculus of the operator* $D_{\psi,q,p}$ *implies the boundedness of the Riesz transforms and this result may be thought of as a strengthening of the equivalence* (10.53)*. Indeed, consider the function* $\operatorname{sgn} \in \mathrm{H}^\infty(\Sigma^\pm_\omega)$ *defined by* $\operatorname{sgn}(z) \overset{\text{def}}{=} 1_{\Sigma^+_\omega}(z) - 1_{\Sigma^-_\omega}(z)$*. By Theorem 10.39, the operator* $D_{\psi,q,p}$ *has a bounded* $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ *functional calculus on* $L^p(\mathrm{VN}(G)) \oplus_p \overline{\operatorname{Ran} \partial_{\psi,q,p}}$*. Hence the operator* $\operatorname{sgn}(D_{\psi,q,p})$ *is bounded. This implies that*

$$\big(D^2_{\psi,q,p}\big)^{\frac{1}{2}} = \operatorname{sgn}(D_{\psi,q,p}) D_{\psi,q,p} \quad \text{and} \quad D_{\psi,q,p} = \operatorname{sgn}(D_{\psi,q,p}) \big(D^2_{\psi,q,p}\big)^{\frac{1}{2}}. \tag{10.72}$$

For any $x \in \operatorname{dom} D_{\psi,q,p} = \operatorname{dom} \big(D^2_{\psi,q,p}\big)^{\frac{1}{2}}$*, we deduce that*

$$\begin{aligned} \big\| D_{\psi,q,p}(x) \big\|_{L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)} &\overset{(10.72)}{=} \big\| \operatorname{sgn}(D_{\psi,q,p}) (D^2_{\psi,q,p})^{\frac{1}{2}}(x) \big\|_{L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)} \\ &\lesssim_p \big\| (D^2_{\psi,q,p})^{\frac{1}{2}}(x) \big\|_{L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)} \end{aligned}$$

and

$$\begin{aligned} \big\| (D^2_{\psi,q,p})^{\frac{1}{2}}(x) \big\|_{L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)} &\overset{(10.72)}{=} \big\| \operatorname{sgn}(D_{\psi,q,p}) D_{\psi,q,p}(x) \big\|_{L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)} \\ &\lesssim_p \big\| D_{\psi,q,p}(x) \big\|_{L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)}. \end{aligned}$$

Recall that on $L^p(\mathrm{VN}(G)) \oplus_p \overline{\operatorname{Ran} \partial_{\psi,q,p}}$*, we have*

$$(D^2_{\psi,q,p})^{\frac{1}{2}} \overset{(10.71)}{=} \begin{bmatrix} A_p^{\frac{1}{2}} & 0 \\ 0 & \mathrm{Id}_{L^p(\Gamma_q(H))} \rtimes A_p^{\frac{1}{2}} \end{bmatrix}.$$

By restricting to elements of the form $(y, 0)$ *with* $y \in \operatorname{dom} A_p^{\frac{1}{2}}$*, we obtain the desired result.*

We remark that in case that G is weakly amenable, we obtained in [HDR8, Section 4.2] an extension of the operator $D_{\psi,q,p}$ to a densely defined bisectorial operator $\mathcal{D}_{\psi,q,p}$ on $L^p(\mathrm{VN}(G)) \oplus L^p(\Gamma_q(H) \rtimes_\alpha G)$ which is also bisectorial and has an $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ calculus on a bisector. We let

$$\mathcal{D}_{\psi,q,p} = \begin{bmatrix} 0 & (\partial_{\psi,q,p^*})^* \\ \partial_{\psi,q,p} & 0 \end{bmatrix} \tag{10.73}$$

along the decomposition $L^p(\mathrm{VN}(G)) \oplus L^p(\Gamma_q(H) \rtimes_\alpha G)$, with natural domains for $\partial_{\psi,q,p}$ and $(\partial_{\psi,q,p^*})^*$.

Theorem 10.41 *[HDR8, Theorem 4.23] Let* $-1 \leqslant q \leqslant 1$ *and* $1 < p < \infty$*. Let* G *be a weakly amenable discrete group such that* $\Gamma_q(H) \rtimes_\alpha G$ *is* QWEP *(e.g.* G *amenable, or* G *is a free group and* $q = \pm 1$*). Consider the operator* $\mathcal{D}_{\psi,q,p}$ *from* (10.73)*. Then* $\mathcal{D}_{\psi,q,p}$ *is bisectorial and has a bounded* $\mathrm{H}^\infty(\Sigma^\pm_\omega)$ *functional calculus.*

Proof : See [HDR8, Theorem 4.23]. ∎

As a corollary of the bounded $\mathrm{H}^\infty(\Sigma_\omega^\pm)$ calculus from Theorem 10.41 above we obtain the following important result. In fact, in [HDR8, Section 4.2] we first had to prove Theorem 10.42 to obtain then the other way around Theorem 10.41 from this.

Theorem 10.42 (Hodge decomposition) *[HDR8, Theorem 4.24] Suppose $1 < p < \infty$ and $-1 \leqslant q \leqslant 1$. Let G be a weakly amenable discrete group such that $\Gamma_q(H) \rtimes_\alpha G$ is* QWEP *(e.g. G amenable, or G is a free group and $q = \pm 1$). If we identify $\overline{\operatorname{Ran} \partial_{\psi,q,p}}$ and $\overline{\operatorname{Ran}(\partial_{\psi,q,p^*})^*}$ as the closed subspaces $\{0\} \oplus \overline{\operatorname{Ran} \partial_{\psi,q,p}}$ and $\overline{\operatorname{Ran}(\partial_{\psi,q,p^*})^*} \oplus \{0\}$ of $L^p(\mathrm{VN}(G)) \oplus L^p(\Gamma_q(H) \rtimes_\alpha G)$, we have*

$$L^p(\mathrm{VN}(G)) \oplus L^p(\Gamma_q(H) \rtimes_\alpha G) = \overline{\operatorname{Ran} \partial_{\psi,q,p}} \oplus \overline{\operatorname{Ran}(\partial_{\psi,q,p^*})^*} \oplus \operatorname{Ker} \mathcal{D}_{\psi,q,p}. \tag{10.74}$$

Proof : See [HDR8, Subsections 4.1, 4.2, Theorem 4.24] for the involved proof. ∎

Now we can give an answer to a variant of [JMP2, Problem C.5].

Remark 10.43 *An inspection in all the steps of the proof of Theorem 10.41 shows that the angle of the $\mathrm{H}^\infty(\Sigma_\omega^\pm)$ calculus can be chosen $\omega > \frac{\pi}{2}|\frac{1}{p} - \frac{1}{2}|$ and that the norm of the calculus is bounded by a constant K_ω not depending on G nor the cocycle (b_ψ, H), in particular it is independent of the dimension of H.*

Proof : See [HDR8, Remark 4.25] for the proof which uses results from quite different areas such as H^∞ functional calculus and R-boundedness, transference of markovian semigroups of Fourier multipliers, UMD constants of noncommutative L^p spaces and the dimension free estimates of the Riesz transforms from Theorem 10.27. ∎

Remark 10.44 *With the same method as above in this subsection, we could also show that a Hodge-Dirac operator associated to a markovian semigroup of Schur multipliers is R-bisectorial and has a bounded bisectorial H^∞ calculus with dimension free bounds. Here the situation is slightly simpler as we do not have any approximation hypotheses such as (weak) amenability or the QWEP property any more.*

We pass to the details. Let $-1 \leqslant q \leqslant 1$ and $1 < p < \infty$. We let

$$\mathcal{D}_{\alpha,q,p} = \begin{bmatrix} 0 & \partial_{\alpha,q,p^*}^* \\ \partial_{\alpha,q,p} & 0 \end{bmatrix} \tag{10.75}$$

along the decomposition $S_I^p \oplus L^p(\Gamma_q(H) \overline{\otimes} B(\ell_I^2))$, with natural domains for $\partial_{\alpha,q,p}$ and $(\partial_{\alpha,q,p^*})^*$. Then $\mathcal{D}_{\alpha,q,p}$ is bisectorial and has a bounded $\mathrm{H}^\infty(\Sigma_\omega^\pm)$ calculus. Moreover, if we identify $\overline{\operatorname{Ran} \partial_{\alpha,q,p}}$ and $\overline{\operatorname{Ran}(\partial_{\alpha,q,p^*})^*}$ as the closed subspaces $\{0\} \oplus \overline{\operatorname{Ran} \partial_{\alpha,q,p}}$ and $\overline{\operatorname{Ran}(\partial_{\alpha,q,p^*})^*} \oplus \{0\}$ of $S_I^p \oplus L^p(\Gamma_q(H) \overline{\otimes} B(\ell_I^2))$, we have

$$S_I^p \oplus L^p(\Gamma_q(H) \overline{\otimes} B(\ell_I^2)) = \overline{\operatorname{Ran} \partial_{\alpha,q,p}} \oplus \overline{\operatorname{Ran}(\partial_{\alpha,q,p^*})^*} \oplus \operatorname{Ker} \mathcal{D}_{\alpha,q,p}. \tag{10.76}$$

Finally the $\mathrm{H}^\infty(\Sigma_\omega^\pm)$ calculus norm of $\mathcal{D}_{\alpha,q,p}$ is controlled independently of H and α, that is, $\|m(D_{\alpha,q,p})\| \leqslant C_{q,p} \|m\|_{\infty,\omega}$. Here, ω can be chosen $> \frac{\pi}{2} \left|\frac{1}{p} - \frac{1}{2}\right|$.

Proof : See [HDR8, Subsections 4.5, 4.6]. ∎

11 Locally compact quantum metric spaces and spectral triples

The results of this section are in [HDR8, Section 5]. It is a continuation of Section 10 and we shall keep its definitions and notations. In particular, we need the notions of markovian semigroup of noncommutative Fourier and Schur multipliers from Definitions 10.6 and 10.10, the noncommutative gradients $\partial_{\psi,q,p}$ and $\partial_{\alpha,q,p}$ from Propositions 10.24 and 10.34, the carré du champ Γ from (10.31) and (10.40), as well as the Hodge-Dirac operators $\mathcal{D}_{\psi,q,p}$ and $\mathcal{D}_{\alpha,q,p}$ from (10.73) and (10.75). The results of Theorems 10.15 and 10.3 on boundedness of Riesz transforms and of Theorem 10.1 and Remark 10.44 on bounded H^∞ calculus for the Hodge-Dirac operators will be used in this section.

It is remarkable that the point of view of Hodge-Dirac operators fits perfectly in the setting of noncommutative geometry if $p = 2$. If G is a discrete group, the Hilbert space $K \overset{\text{def}}{=} L^2(\mathrm{VN}(G)) \oplus_2 L^2(\Gamma_q(H) \rtimes_\alpha G)$, the $*$-algebra $\mathcal{P}_G = \mathrm{span}\,\{\lambda_s : s \in G\}$ of trigonometric polynomials and the Hodge-Dirac operator $\mathcal{D}_{\psi,1,2}$ on $L^2(\mathrm{VN}(G)) \oplus_2 L^2(\Gamma_q(H) \rtimes_\alpha G)$ define a triple $(\mathcal{P}_G, K, \mathcal{D}_{\psi,1,2})$ in the spirit of noncommutative geometry [Con1], [GVF], [Var] (see also Subsection 11.4) . Recall that the notion of spectral triple (A, K, D) (= noncommutative manifold) à la Connes covers a huge variety of different geometries such as Riemannian manifolds, fractals, quantum groups or even non-Hausdorff spaces. From here, it is apparent that we can see Markov semigroups of Fourier multipliers as geometric objects. The same observation is true for Markov semigroups of Schur multipliers. Nevertheless, the Hilbert space setting of the noncommutative geometry is too narrow to encompass our setting on L^p spaces. So, we develop in Subsection 11.4 a natural Banach space variant (A, X, D) of a spectral triple where the selfadjoint operator D acting on the Hilbert space H is replaced by a bisectorial operator D acting on a (reflexive) Banach space X, allowing us to use (noncommutative) L^p spaces $(1 < p < \infty)$. We then obtain for the setting of Markov semigroups of Fourier multipliers that $(\mathrm{C}^*_r(G), L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G), \mathcal{D}_{\psi,q,p})$ form – under suitable additional hypotheses – a compact Banach spectral triple (see Theorem 11.15). Second we obtain for the setting of Markov semigroups of Schur multipliers that $(\mathrm{M}_{I,\mathrm{fin}}, S^p_I \oplus_p L^p(\Gamma_q(H)\overline{\otimes}B(\ell^2_I)), \mathcal{D}_{\alpha,q,p})$ form – under additional hypotheses – a locally compact Banach spectral triple (see Theorem 11.18).

We will be equally interested by the metric aspect [Lat6] of noncommutative geometry, see Subsection 11.1 for background on quantum (locally) compact metric spaces. We introduce new quantum (locally) compact metric spaces in the sense of [Lat1], [Lat6], [Rie1] associated with these spectral triples. It relies on L^p variants of the seminorms of [JM, Section 1.2]. Here, we check carefully the axioms taking into account all problems of domains required by this theory. Note that it is not clear how to do the same analysis at the level $p = \infty$ considered in [JM, Section 1.2] since we cannot hope the boundedness of Riesz transform on L^∞ which is an important tool in this part. So, on this point, L^p seminorms seem to be more natural.

We observe significant differences between the case of Fourier multipliers (Theorem 11.11) and the one of Schur multipliers (Theorem 11.12). For example, we need to use C^*-*algebras* for semigroups of Fourier multipliers and which produces quantum *compact* metric spaces contrarily to the case of semigroups of Schur multipliers which requires *order-unit spaces* and that leads to quantum *locally* compact metric spaces if I is infinite. Furthermore, our analysis with quantum compact metric spaces relying on noncommutative L^p spaces makes appear a new phenomenon when the value of the parameter p changes, see Theorem 11.11. Also in the part on spectral triples: those in Theorem 11.15 associated with Fourier multipliers will be *compact*, contrary to those in Theorem 11.18 associated with Schur multipliers which are *locally compact*.

Finally, note that the combination of our spectral triples and our quantum (locally) compact

metric spaces is in the spirit of the papers [Lat8] and [BMR] (see also [Con2]) but it is more subtle here since the link between the norms of the commutators and the seminorms of our quantum metric spaces is not as direct as the ones of [Lat8] and [BMR].

11.1 Background on quantum compact metric spaces

We recall definitions and characterizations of the notions that we need in the first half of this section. The main notion is that of quantum compact metric space. This concept has its origins in Connes' paper [Con2] of 1989 (see also [Con1, Chapter 6] and [Var, Chapter 3]), in which he shows that we can recover the geodesic distance d of a compact riemannian spin manifold M using the Dirac operator D by the formula

$$d(p,q) = \sup_{f\in \mathrm{C}(M), \|[D,f]\|\leqslant 1} |f(p) - f(q)|, \quad p,q \in M$$

where the commutator $[D,f] = Df - fD$ extends to a bounded operator[9]. Indeed, it is known that the commutator $[D,f]$ induces a bounded operator if and only if f is a Lipschitz function and in this case the Lipschitz norm of f is equal to $\|[D,f]\|$. Moreover, this space of functions is norm dense in $\mathrm{C}(M)$. If we identify the points p,q as pure states ω_p and ω_q on the algebra $\mathrm{C}(M)$, this formula can be seen as

$$d(\omega_p,\omega_q) = \sup_{f\in \mathrm{C}(M), \|[D,f]\|\leqslant 1} |\omega_p(f) - \omega_q(f)|, \quad p,q \in M.$$

Afterwards, Rieffel [Rie1] and Latrémolière [Lat1] axiomatised this formula replacing $\mathrm{C}(M)$ by a unital C*-algebra A (or even by an order-unit space $\mathcal{A}$), $f \mapsto \|[D,f]\|$ by a seminorm $\|\cdot\|$ defined on a subspace of A and ω_p, ω_q by arbitrary states obtaining essentially the formula (11.3) below and giving rise to a theory of *quantum* compact metric spaces. With this notion, Rieffel was able to define a quantum analogue of Gromov-Hausdorff distance and to give a meaning to many approximation found in the physics literature. We refer to the surveys [Lat6] and [Rie1] and references therein for more information.

Recall that an order-unit space [Alf, page 69], [AlS, Definition 1.8] is an ordered $\mathbb{R}$-vector space $\mathcal{A}$ with a closed positive cone and an element $1_{\mathcal{A}}$, satisfying $\|a\|_{\mathcal{A}} = \inf\{\lambda > 0 : -\lambda 1_{\mathcal{A}} \leqslant a \leqslant \lambda 1_{\mathcal{A}}\}$. The element $1_{\mathcal{A}}$ is called the distinguished order unit. The definition of an order-unit space is due to Kadison [Kad]. Important examples of order-unit spaces are given by real linear subspaces of selfadjoint elements containing the unit element in a unital C*-algebra.

The following is a slight generalization of [Lat6, Definition 2.3] and [Lat3, Definition 2.2].

Definition 11.1 *A unital Lipschitz pair $(\mathcal{A}, \|\cdot\|)$ is a pair where $\mathcal{A}$ is an order-unit space and where $\|\cdot\|$ is a seminorm defined on a dense subspace $\operatorname{dom}\|\cdot\|$ of $\mathcal{A}$ such that*

$$\big\{a \in \operatorname{dom}\|\cdot\| : \|a\| = 0\big\} = \mathbb{R}1_{\mathcal{A}}. \tag{11.1}$$

Remark 11.2 *Note that if a seminorm $\|\cdot\|$ is defined on some subspace $\operatorname{dom}\|\cdot\|$ of a unital C*-algebra A such that $A_{\mathrm{sa}} \cap \operatorname{dom}\|\cdot\|$ is dense in A_{sa} (here we write A_{sa} for the real subspace of A formed by the selfadjoint elements) and such that $\big\{a \in \operatorname{dom}\|\cdot\| : \|a\| = 0\big\} = \mathbb{C}1_A$, then its restriction on $A_{\mathrm{sa}} \cap \operatorname{dom}\|\cdot\|$ defines a unital Lipschitz pair. In this case, we also say that $(A, \|\cdot\|)$ is a unital Lipschitz pair (or a compact quantum metric space if Definition 11.3 is satisfied).*

9. Recall that D is an unbounded operator acting on the Hilbert space of L^2-spinors and that the functions of $\mathrm{C}(M)$ act on the same Hilbert space by multiplication operators.

If (X, dist) is a compact metric space, a fundamental example for Definition 11.1 is given by $(\mathrm{C}(X)_{\mathrm{sa}}, \mathrm{Lip})$ where $\mathrm{C}(X)$ is the commutative C*-algebra of continuous functions and where Lip is the Lipschitz seminorm, defined for any Lipschitz function $f\colon X \to \mathbb{C}$ by

$$\mathrm{Lip}(f) \stackrel{\mathrm{def}}{=} \sup\left\{ \frac{|f(x) - f(y)|}{\mathrm{dist}(x,y)} : x, y \in X, x \neq y \right\}. \tag{11.2}$$

See [Lat6, Example 2.6], [Lat4, Example 2.9]. It is immediate that a function f has zero Lipschitz constant if and only if it is constant on X. Moreover, the set of real Lipschitz functions is norm-dense in $\mathrm{C}(X)_{\mathrm{sa}}$ by the Stone-Weierstrass theorem.

Now, following [Rie2, Definition 2.2] (see also [Lat6, Theorem 2.42], [Lat5, Definition 1.2] or [Lat3, Definition 2.6]), we introduce a notion of quantum compact metric space. Recall that a linear functional φ on an order-unit space $\mathcal{A}$ is a state [Alf, page 72] if $\|\varphi\| = \varphi(1_{\mathcal{A}}) = 1$.

Definition 11.3 *A quantum compact metric space $(\mathcal{A}, \|\cdot\|)$ is a unital Lipschitz pair whose associated Monge-Kantorovich metric*

$$\mathrm{dist}_{\mathrm{mk}}(\varphi, \psi) \stackrel{\mathrm{def}}{=} \sup\left\{ |\varphi(a) - \psi(a)| : a \in \mathcal{A}, \|a\| \leqslant 1 \right\}, \quad \varphi, \psi \in \mathrm{S}(\mathcal{A}) \tag{11.3}$$

metrises the weak topology restricted to the state space $\mathrm{S}(\mathcal{A})$ of $\mathcal{A}$. When a Lipschitz pair $(\mathcal{A}, \|\cdot\|)$ is a quantum compact metric space, the seminorm $\|\cdot\|$ is referred to as a Lip-norm.*

The Lipschitz seminorm Lip associated to a compact metric space (X, dist) enjoys a natural property with respect to the multiplication of functions in $\mathrm{C}(X)$, called the Leibniz property. Namely, for any Lipschitz functions $f, g\colon X \to \mathbb{C}$,

$$\mathrm{Lip}(fg) \leqslant \|f\|_{\mathrm{C}(X)} \mathrm{Lip}(g) + \mathrm{Lip}(f) \|g\|_{\mathrm{C}(X)}. \tag{11.4}$$

Moreover, the Lipschitz seminorm is lower-semicontinuous with respect to the C*-norm of $\mathrm{C}(X)$, i.e. the uniform convergence norm on X. These two additional properties were not assumed in the above Definition 11.3, yet they are quite natural. However, as research in noncommutative metric geometry progressed, the need for a noncommutative analogue of these properties for some developments became evident. So, sometimes, some additional conditions are often added to Definition 11.3 which brings us to the following definition.

The following is a slight generalization of [Lat6, Definition 2.21] for order-unit spaces embedding in unital C*-algebras (see also [Lat7, Definition 1.3]) and [Lat6, Definition 2.45], [Lat4, Definition 2.19], [Lat3, Definition 2.19] and [Lat2, Definition 2.2.2]. Here $a \circ b \stackrel{\mathrm{def}}{=} \frac{1}{2}(ab + ba)$ and $\{a, b\} \stackrel{\mathrm{def}}{=} \frac{1}{2i}(ab - ba)$.

Definition 11.4 *1. A unital Leibniz pair $(\mathcal{A}, \|\cdot\|)$ is a unital Lipschitz pair where $\mathcal{A}$ is a real linear subspace of selfadjoint elements containing the unit element in a unital C*-algebra A such that:*

(a) the domain of $\|\cdot\|$ is a Jordan-Lie subalgebra of A_{sa},

(b) for any $a, b \in \mathrm{dom}\,\|\cdot\|$, we have:

$$\|a \circ b\| \leqslant \|a\|_A \|b\| + \|a\| \|b\|_A \quad \text{and} \quad \|\{a, b\}\| \leqslant \|a\|_A \|b\| + \|a\| \|b\|_A.$$

2. A unital Leibniz pair $(\mathcal{A}, \|\cdot\|)$ is a Leibniz quantum compact metric space when $\|\cdot\|$ is a lower semicontinuous Lip-norm.

We also need a notion of quantum locally compact metric spaces. The paper [Lat1] gives such a definition in the case of a Lipschitz pair $(\mathcal{A}, \|\cdot\|)$ where $\mathcal{A}$ is a C*-algebra. However, we need a version for order-unit spaces not covered by [Lat1].

The following is a variant of [Lat1, Definition 2.3] and [Lat1, Definition 2.27]. Here uA is the unitisation of A.

Definition 11.5 *A Lipschitz pair $(\mathcal{A}, \|\cdot\|)$ is a closed subspace $\mathcal{A}$ of selfadjoint elements of a non-unital C*-algebra A and a seminorm $\|\cdot\|$ defined on a dense subspace of $\mathcal{A} \oplus \mathbb{R}1_{uA}$ such that $\{x \in \mathcal{A} \oplus \mathbb{R}1_{uA} : \|x\| = 0\} = \mathbb{R}1_{uA}$.*

A Lipschitz triple $(\mathcal{A}, \|\cdot\|, \mathfrak{M})$ is a Lipschitz pair $(\mathcal{A}, \|\cdot\|)$ and an abelian C-algebra $\mathfrak{M}$ of A such that $\mathfrak{M}$ contains an approximate unit of A.*

The following is [Lat1, Definition 2.23].

Definition 11.6 *Let A be a non-unital C*-algebra and $\mathfrak{M}$ be an abelian C*-subalgebra of A containing an approximate unit of A. Let $\mu\colon A \to \mathbb{C}$ be a state of A. We call μ a local state (of $(A, \mathfrak{M})$) provided that there exists a projection e in $\mathfrak{M}$ of compact support[10] (in the Gelfand spectrum) such that $\mu(e) = 1$.*

Inspired by [Lat1, Theorem 3.10], [Lat6, Theorem 2.73], we introduce the following definition. Of course, we recognise that this definition is a bit artificial. A better choice could be to generalise the results and definitions of [Lat1] to a larger context. Here μ is extended on the unitisation as in [Lat1, Notation 2.2].

Definition 11.7 *We say that a Lipschitz triple $(\mathcal{A}, \|\cdot\|, \mathfrak{M})$ is a quantum locally compact metric space if for a local state μ of $(A, \mathfrak{M})$ and any compactly supported $a, b \in \mathfrak{M}$, the set*

$$a\big\{x \in \mathcal{A} \oplus \mathbb{R}1_{uA} : \|x\| \leqslant 1, \mu(x) = 0\big\}b$$

is totally bounded in the norm topology of uA.

Now, we give a variant of Definition 11.4.

Definition 11.8 *Let $(\mathcal{A}, \|\cdot\|, \mathfrak{M})$ be a quantum locally compact metric space. We call it a Leibniz quantum locally compact metric space if $(\mathcal{A} \oplus \mathbb{R}1_{uA}, \|\cdot\|)$ is a unital Leibniz pair in the sense of Definition 11.4 and if the seminorm $\|\cdot\|$ is lower semicontinuous with respect to the C*-norm $\|\cdot\|_{uA}$.*

Finally, following [Lat1, Condition 4.3] (see also [BMR] for a related discussion), we introduce the following definition.

Definition 11.9 *We say that a locally compact metric space $(\mathcal{A}, \|\cdot\|, \mathfrak{M})$ in the sense of Definition 11.7 is bounded if the Lipschitz ball $\big\{a \in \mathcal{A} : \|a\| \leqslant 1\big\}$ is norm bounded.*

10. It is not clear if the support must be in addition open in [Lat1] since the indicator function of a subset A is continuous if and only if A is both open and closed.

11.2 Quantum compact metric spaces associated to semigroups of Fourier multipliers

In this subsection, we consider a markovian semigroup $(T_t)_{t\geqslant 0}$ of Fourier multipliers on a discrete group von Neumann algebra $\mathrm{VN}(G)$ as in Definition 10.6. We introduce new compact quantum metric spaces in the spirit of the ones of [JM]. We also add to the picture the lower semicontinuity and a careful examination of the domains. Recall the carré du champ Γ from Subsection 10.2. Note that according to Remark 10.16, the following definition is correct.

Definition 11.10 *Suppose $2 \leqslant p < \infty$. Let G be a discrete group. Let A_p denote the L^p realization of the (negative) generator of $(T_t)_{t\geqslant 0}$. For any $x \in \mathrm{dom}\, A_p^{\frac{1}{2}}$ we let*

$$\|x\|_{\Gamma,p} \overset{\mathrm{def}}{=} \max\Big\{ \big\|\Gamma(x,x)^{\frac{1}{2}}\big\|_{L^p(\mathrm{VN}(G))}, \big\|\Gamma(x^*,x^*)^{\frac{1}{2}}\big\|_{L^p(\mathrm{VN}(G))} \Big\}. \tag{11.5}$$

In the next theorem, recall that if A is the generator of a markovian semigroup of Fourier multipliers, then there exists a real Hilbert space H together with a mapping $b_\psi : G \to H$ such that the symbol $\psi : G \to \mathbb{C}$ of A satisfies $\psi(s) = \|b_\psi(s)\|_H^2$. Following [JMP1, page 1962], we define

$$\mathrm{Gap}_\psi \overset{\mathrm{def}}{=} \inf_{b_\psi(s)\neq b_\psi(t)} \|b_\psi(s) - b_\psi(t)\|_H^2 \,. \tag{11.6}$$

By [BHV, Proposition 2.10.2], note that Gap_ψ is independent of b_ψ, that is, if $b_\psi : G \to H$ and $b'_\psi : G \to H'$ define the same markovian semigroup of Fourier multipliers, then $\mathrm{Gap}_\psi = \mathrm{Gap}_{\psi'}$. We refer to [HDR8, Subsection 5.6] for this quantity in examples.

Theorem 11.11 *[HDR8, Theorem 5.16] Let $2 \leqslant p < \infty$. Let G be a discrete group. Let $b_\psi : G \to H$ be an injective cocycle with values in a finite-dimensional Hilbert space of dimension $n < p$. Assume that $\mathrm{Gap}_\psi > 0$.*

1. *If $\mathrm{VN}(G)$ is QWEP and $L^p(\mathrm{VN}(G))$ has the CCAP, then $(\mathrm{C}_r^*(G), \|\cdot\|_{\Gamma,p})$ is a quantum compact metric space.*
2. *If in addition G is weakly amenable and second countable then $(\mathrm{C}_r^*(G), \|\cdot\|_{\Gamma,p})$ is a Leibniz quantum compact metric space.*

Proof : 1. The injectivity of the cocycle b_ψ and the density of $\mathcal{P}_G \subseteq \mathrm{dom}\, A_p^{\frac{1}{2}}$ in $\mathrm{C}_r^*(G)$ imply that $(\mathrm{C}_r^*(G), \|\cdot\|_{\Gamma,p})$ is a unital Lipschitz pair. Note that according to Theorem 10.15, the seminorm $\|x\|_{\Gamma,p}$ is equivalent to $\big\|A_p^{\frac{1}{2}}(x)\big\|_{L^p(\mathrm{VN}(G))}$. The compacity in 1. follows then from the compacity of $A^{-\frac{1}{2}} : L_0^p(\mathrm{VN}(G)) \to \mathrm{VN}(G)$ (the index 0 stands for the subspace of $L^p(\mathrm{VN}(G))$ of elements with trace 0). This in turn uses the finite dimensionality of H, the gap condition $\mathrm{Gap}_\psi > 0$ and again the injectivity of b_ψ.

2. The Leibniz property and the lower semicontinuity from Definition 11.4 can be shown with the formula (10.37) which connects the carré du champ with the noncommutative gradient living in the Hilbertian valued noncommutative L^p space $L^p(\mathrm{VN}(G), L^2(\Gamma_q(H))_{c,p})$, and the Leibniz rule from Lemma 10.13. See [HDR8, Theorem 5.16] for the details. ∎

11.3 Quantum locally compact metric spaces associated to semigroups of Schur multipliers

In this subsection, we let $(T_t)_{t\geqslant 0}$ be a markovian semigroup of Schur multipliers on $B(\ell^2_I)$ from Definition 10.10. Recall from Subsection 10.3 that we have a generator A_p and also a carré du champ Γ. Note also that A_∞, the generator of the strongly continuous semigroup $(T_t)_{t\geqslant 0}$ on S^∞_I, is sectorial, and $\overline{\operatorname{Ran} A_\infty}$ is a closed subspace of S^∞_I. We shall introduce the following family of seminorms. Suppose $2 \leqslant p < \infty$. For any $x \in \operatorname{dom} A_p^{\frac{1}{2}}$ we let

$$\|x\|_{\Gamma,p} \overset{\text{def}}{=} \max\left\{ \left\|\Gamma(x,x)^{\frac{1}{2}}\right\|_{S^p_I}, \left\|\Gamma(x^*,x^*)^{\frac{1}{2}}\right\|_{S^p_I} \right\}. \tag{11.7}$$

Hereby, we recall the definition of Γ from (10.40) extended to $\operatorname{dom} A_p^{\frac{1}{2}}$ according to Remark 10.19 (closed form method).

In the following theorem, we consider the abelian C*-subalgebra $\mathfrak{M}$ of S^∞_I consisting of its diagonal operators. The restriction to $p \geqslant 2$ seems to be natural in view of the same restriction in Theorem 10.20. We define the gap of α by

$$\operatorname{Gap}_\alpha \overset{\text{def}}{=} \inf_{\alpha_i-\alpha_j \neq \alpha_k-\alpha_l} \|(\alpha_i-\alpha_j)-(\alpha_k-\alpha_l)\|^2_H \,. \tag{11.8}$$

Note that by the proof of [Arh1, Proposition 5.4] and [BHV, Theorem C.2.3], $\operatorname{Gap}_\alpha$ is independent of α, that is, if $\alpha\colon I \to H$ and $\alpha'\colon I \to H'$ define the same markovian semigroup of Schur multipliers, then $\operatorname{Gap}_\alpha = \operatorname{Gap}_{\alpha'}$. We refer to [HDR8, Subsection 5.6] for this quantity in examples.

Theorem 11.12 *[HDR8, Theorem 5.30] Assume that the Hilbert space H is of finite dimension and that $\operatorname{Gap}_\alpha > 0$.*

1. *Suppose $2 \leqslant p \leqslant \infty$ and that I is finite. Then $\left(\left(\overline{\operatorname{Ran} A_\infty}\right)_{\mathrm{sa}} \oplus \mathbb{R}\mathrm{Id}_{\ell^2_I}, \|\cdot\|_{\Gamma,\alpha,p}\right)$ is a Leibniz quantum compact metric space (Definition 11.4).*
2. *Suppose $2 \leqslant p < \infty$ and that I is infinite. We consider the abelian C*-subalgebra $\mathfrak{M}$ of diagonal operators of S^∞_I. Then $\left(\left(\overline{\operatorname{Ran} A_\infty}\right)_{\mathrm{sa}} \oplus \mathbb{R}\mathrm{Id}_{\ell^2_I}, \|\cdot\|_{\Gamma,\alpha,p}, \mathfrak{M}\right)$ is a bounded Leibniz quantum locally compact metric space (see Definitions 11.7, 11.8 and 11.9).*

Proof : The strategy of proof is partially the same as that of the Fourier counterpart, Theorem 11.11. However the compacity of $A^{-\frac{1}{2}}\colon \overline{\operatorname{Ran} A_p}^{S^p_I} \to S^\infty_I$ fails if I is infinite. Thus we obtain in 2. a locally compact statement. We refer to [HDR8, Theorem 5.30] for the details. ∎

11.4 Banach spectral triples

In the remainder of this Section 11, we will establish that our Hodge-Dirac operators associated with markovian semigroups of Fourier multipliers (10.73) resp. of Schur multipliers (10.75) give rise to so called compact resp. locally compact Banach spectral triples (see Theorem 11.15 resp. Proposition 11.17, Theorem 11.18).

We refer to [CGRS], [GVF] and [CGIS] for more information on spectral triples. Let us recall this notion. A (possibly kernel-degenerate, compact) spectral triple (A, H, D) consists of a unital C*-algebra A, a Hilbert space H, a (densely defined, unbounded) selfadjoint operator D and a representation $\pi\colon A \to B(H)$ which satisfy the following properties.

1. D^{-1} is compact on $\overline{\operatorname{Ran} D}$.

2. The set

$$\mathrm{Lip}_D(A) \stackrel{\text{def}}{=} \big\{a \in A : \pi(a) \cdot \mathrm{dom}\, D \subseteq \mathrm{dom}\, D \text{ and the unbounded operator } [D, \pi(a)] \colon \mathrm{dom}\, D \subseteq H \to H \text{ extends to an element of } B(H)\big\}$$

is dense in A.

In the next subsections, we give new examples of spectral triples. Our examples can be generalised to the context of L^p spaces instead of Hilbert spaces. So, it is natural to state the following definition.

Definition 11.13 *A (compact) Banach spectral triple (A, X, D) consists of the following data: a reflexive Banach space X, a bisectorial operator D on X with dense domain $\mathrm{dom}\, D \subseteq X$, a Banach algebra A and a homomorphism $\pi \colon A \to B(X)$ such that for all $a \in A$ we have:*

1. *D admits a bounded H^∞ functional calculus on a bisector $\Sigma_\omega^\pm$.*
2. *$|D|^{-1}$ is a compact operator on $\overline{\mathrm{Ran}\, D}$.*
3. *The set*

$$\mathrm{Lip}_D(A) \stackrel{\text{def}}{=} \big\{a \in A : \pi(a) \cdot \mathrm{dom}\, D \subseteq \mathrm{dom}\, D \text{ and the unbounded operator } [D, \pi(a)] \colon \mathrm{dom}\, D \subseteq X \to X \text{ extends to an element of } B(X)\big\} \tag{11.9}$$

is dense in A.

Here, $|D|^{-1} = m(D)$ with the spectral multiplier $m(\lambda) = 1/\sqrt{\lambda^2}$, which is an (unbounded) holomorphic function on any bisector $\Sigma_\omega^\pm$ with $\omega < \frac{\pi}{2}$. Note that for such functions m, $m(D)$ is always well-defined closed densely defined. Next we define the following notion of a locally compact spectral triple. When X is a Hilbert space, then compare to the one in [SuZ], [GGISV, page 588].

Definition 11.14 *Let X be a reflexive Banach space, D a densely defined closed operator on X and A a subalgebra of some Banach algebra. Let $\pi \colon A \to B(X)$ be a homomorphism. We call (A, X, D) a locally compact (Banach) spectral triple, provided that*

1. *D is bisectorial on X and has a bounded H^∞ functional calculus on some bisector.*
2. *For any $a \in A$, we have $\pi(a) \cdot \mathrm{dom}\, D \subseteq \mathrm{dom}\, D$ and $[D, \pi(a)] \colon \mathrm{dom}\, D \to X$ which is densely defined, extends to a bounded operator on X.*
3. *For any $a \in A$, $\pi(a)(i\mathrm{Id} + D)^{-1}|\overline{\mathrm{Ran}\, D}$ is a compact operator $\overline{\mathrm{Ran}\, D} \to X$.*

11.5 Spectral triples associated to semigroups of Fourier multipliers

In this subsection, we consider a markovian semigroup of Fourier multipliers as in Definition 10.6 and Proposition 10.7, together with the noncommutative gradient $\partial_{\psi,q,p}$ from Proposition 10.24 and its adjoint. Now, we generalise the construction of [JMP2, pages 587-589] which corresponds to the case $q = 1$ and $p = 2$ below. So we obtain a scale of L^p Banach spectral triples associated to Fourier multipliers.

Suppose $1 < p < \infty$ and $-1 \leqslant q < 1$. Recall that the Hodge-Dirac operator is defined by

$$\mathcal{D}_{\psi,q,p} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & (\partial_{\psi,q,p^*})^* \\ \partial_{\psi,q,p} & 0 \end{bmatrix} \tag{11.10}$$

on the subspace $\operatorname{dom}\mathcal{D}_{\psi,q,p} = \operatorname{dom}\partial_{\psi,q,p} \oplus \operatorname{dom}(\partial_{\psi,q,p^*})^*$ of $L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)$. If $a \in \mathrm{C}^*_r(G)$, we define the bounded operator $\pi(a)\colon L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G) \to L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)$ by

$$\pi(a) \overset{\mathrm{def}}{=} \begin{bmatrix} L_a & 0 \\ 0 & \tilde{L}_a \end{bmatrix}, \quad a \in \mathrm{C}^*_r(G) \tag{11.11}$$

where $L_a\colon L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$, $x \mapsto ax$ is the left multiplication operator and where $\tilde{L}_a\colon L^p(\Gamma_q(H) \rtimes_\alpha G) \to L^p(\Gamma_q(H) \rtimes_\alpha G)$, $x \mapsto (1 \rtimes a)x$ is the left action of the bimodule.

In the following theorem, recall the weak*-closed operator $\partial_{\psi,q,\infty}\colon \operatorname{dom}\partial_{\psi,q,\infty} \subseteq \Gamma_q(H) \rtimes_\alpha G \to \Gamma_q(H) \rtimes_\alpha G$ from Proposition 10.25. The latter is valid if G has AP (see Subsection 10.1 for AP) and $q \neq 1$. We also recall the definition of Gap_ψ from (11.6).

Theorem 11.15 *[HDR8, Theorem 5.49] Suppose $1 < p < \infty$ and $-1 \leqslant q < 1$. Consider the triple $(\mathrm{C}^*_r(G), L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G), \mathcal{D}_{\psi,q,p})$. It satisfies the following properties. In particular, it is a Banach spectral triple in the sense of Definition 11.13, in case that $b_\psi\colon G \to H$ is injective, $\mathrm{Gap}_\psi > 0$, H is finite dimensional and $\Gamma_q(H) \rtimes_\alpha G$ has* QWEP *(e.g. G is amenable, or G is a free group and $q = -1$).*

1. *We have $(\mathcal{D}_{\psi,q,p})^* = \mathcal{D}_{\psi,q,p^*}$. In particular, the operator $\mathcal{D}_{\psi,q,2}$ is selfadjoint.*
2. *We have*
$$\mathcal{P}_G \subseteq \mathrm{Lip}_{\mathcal{D}_{\psi,q,p}}(\mathrm{C}^*_r(G)). \tag{11.12}$$
3. *For any $a \in \mathcal{P}_G$, we have*
(11.13)
$$\big\|\big[\mathcal{D}_{\psi,q,p}, \pi(a)\big]\big\|_{L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G) \to L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)} \leqslant \|\partial_{\psi,q}(a)\|_{\Gamma_q(H) \rtimes_\alpha G}.$$
4. *Suppose that G has* AP. *We have*
$$\mathrm{C}^*_r(G) \cap \operatorname{dom}\partial_{\psi,q,\infty} \subseteq \mathrm{Lip}_{\mathcal{D}_{\psi,q,p}}(\mathrm{C}^*_r(G)). \tag{11.14}$$
5. *Suppose that G has* AP. *For any $a \in \mathrm{C}^*_r(G) \cap \operatorname{dom}\partial_{\psi,q,\infty}$, we have*
$$\big\|\big[\mathcal{D}_{\psi,q,p}, \pi(a)\big]\big\|_{L^p \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G) \to L^p \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G)} \leqslant \big\|\partial_{\psi,q,\infty}(a)\big\|_{\Gamma_q(H) \rtimes_\alpha G}. \tag{11.15}$$
6. *Assume that $\Gamma_q(H) \rtimes_\alpha G$ has* QWEP. *If $b_\psi\colon G \to H$ is injective, $\mathrm{Gap}_\psi > 0$ and if H is finite-dimensional, then the operator*
$$|\mathcal{D}_{\psi,q,p}|^{-1}\colon \overline{\mathrm{Ran}(\partial_{\psi,q,p^*})^*} \oplus \overline{\mathrm{Ran}\,\partial_{\psi,q,p}} \to \overline{\mathrm{Ran}(\partial_{\psi,q,p^*})^*} \oplus \overline{\mathrm{Ran}\,\partial_{\psi,q,p}}$$
is compact.

Proof : See [HDR8, Theorem 5.49]. ∎

Question 11.16 *Spectral triples (A, H, D) carry the notion of a dimension $r \in [0, \infty]$ with $r = \inf\{q \geqslant 0 : \mathrm{Tr}\,|D|^{-q} < \infty\}$, see [EcI, page 4], [GVF, page 450] for the Hilbert space case. It would be interesting to find conditions on the cocycle associated with the Markov semigroup of Fourier multipliers such that the triple $(\mathrm{C}^*_r(G), L^p(\mathrm{VN}(G)) \oplus_p L^p(\Gamma_q(H) \rtimes_\alpha G), \mathcal{D}_{\psi,q,p})$ is finite dimensional. There are also other notions of dimension / summability for spectral triples.*

11.6 Spectral triples associated to semigroups of Schur multipliers

In this subsection, we consider a markovian semigroup of Schur multipliers on $B(\ell_I^2)$ from Definition 10.10 with associated gradient $\partial_{\alpha,q,p}$ from Proposition 10.34. Suppose $1 < p < \infty$ and $-1 \leqslant q < 1$. Recall that the (full) Hodge-Dirac operator $\mathcal{D}_{\alpha,q,p}$ with domain $\operatorname{dom} \mathcal{D}_{\alpha,q,p} = \operatorname{dom} \partial_{\alpha,q,p} \oplus \operatorname{dom}(\partial_{\alpha,q,p^*})^*$ is defined in (10.75) by the formula

$$\mathcal{D}_{\alpha,q,p} = \begin{bmatrix} 0 & (\partial_{\alpha,q,p^*})^* \\ \partial_{\alpha,q,p} & 0 \end{bmatrix}. \tag{11.16}$$

We will see in the main results of this section (Proposition 11.17, Theorem 11.18) how this Hodge-Dirac operator gives rise to a Banach spectral triple. Note that the compactness criterion considers particular attention and supplementary hypotheses. Thus the Banach spectral triple of this subsection will be locally compact.

Let us turn to the description of the homomorphism π. For any $a \in B(\ell_I^2)$, we denote by $L_a \colon S_I^p \to S_I^p$, $x \mapsto ax$ the left multiplication operator. Moreover, we write $\tilde{L}_a \overset{\text{def}}{=} \mathrm{Id}_{L^p(\Gamma_q(H))} \otimes L_a \colon L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2)) \to L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))$, $f \otimes e_{ij} \mapsto f \otimes a e_{ij}$ for the left action. If $a \in B(\ell_I^2)$, we define the bounded operator $\pi(a) \colon S_I^p \oplus_p L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2)) \to S_I^p \oplus_p L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))$ by

$$\pi(a) \overset{\text{def}}{=} \begin{bmatrix} L_a & 0 \\ 0 & \tilde{L}_a \end{bmatrix}, \quad a \in B(\ell_I^2). \tag{11.17}$$

It is then easy to check that $\pi(a)^* = \pi(a^*)$ in case that $p = 2$.

Proposition 11.17 *Let $1 < p < \infty$ and $-1 \leqslant q < 1$. Then the triple*

$$(S_I^\infty, S_I^p \oplus_p L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2)), \mathcal{D}_{\alpha,q,p})$$

satisfies the following properties.

1. *We have $(\mathcal{D}_{\alpha,q,p})^* = \mathcal{D}_{\alpha,q,p^*}$. In particular, the operator $\mathcal{D}_{\alpha,q,2}$ is selfadjoint.*
2. *We have*
$$S_I^\infty \cap \operatorname{dom} \partial_{\alpha,q,\infty} \subseteq \mathrm{Lip}_{\mathcal{D}_{\alpha,q,p}}(S_I^\infty). \tag{11.18}$$
3. *For any $a \in S_I^\infty \cap \operatorname{dom} \partial_{\alpha,q,\infty}$, we have*
$$\big\| [\mathcal{D}_{\alpha,q,p}, \pi(a)] \big\|_{S_I^p \oplus_p L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2)) \to S_I^p \oplus_p L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))} \leqslant \big\| \partial_{\alpha,q,\infty}(a) \big\|_{\Gamma_q(H)\overline{\otimes}B(\ell_I^2)}. \tag{11.19}$$

Proof : See [HDR8, Proposition 5.57]. ∎

In the following theorem, we recall that Gap_α was defined in (11.8).

Theorem 11.18 *[HDR8, Proposition 5.61] Let $1 < p < \infty$ and $-1 \leqslant q < 1$. Assume that H is finite dimensional, that $\alpha \colon I \to H$ is injective and that $\mathrm{Gap}_\alpha > 0$. Then $(\mathrm{M}_{I,\mathrm{fin}}, S_I^p \oplus L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2)), \mathcal{D}_{\alpha,q,p})$ is a locally compact Banach spectral triple. In other words, we have the following properties.*

1. *$\mathcal{D}_{\alpha,q,p}$ is densely defined and has a bisectorial H^∞ calculus.*
2. *For any $a \in \mathrm{M}_{I,\mathrm{fin}}$, we have $a \in \mathrm{Lip}_{\mathcal{D}_{\alpha,q,p}}(S_I^\infty)$.*
3. *For any $a \in \mathrm{M}_{I,\mathrm{fin}}$, $\pi(a)(i\mathrm{Id} + \mathcal{D}_{\alpha,q,p})^{-1}$ and $\pi(a)|\mathcal{D}_{\alpha,q,p}|^{-1}$ are compact operators between the spaces $\overline{\operatorname{Ran} \mathcal{D}_{\alpha,q,p}} \to S_I^p \oplus L^p(\Gamma_q(H)\overline{\otimes}B(\ell_I^2))$.*

Proof : See [HDR8, Proposition 5.61]. ∎

12 Projections, multipliers and decomposable maps on noncommutative L^p spaces

This section contains some of the results from [HDR6]. The absolute value $|T|$ and the regular norm $\|T\|_{\mathrm{reg}}$ of a regular operator T already appear in the seminal work of Kantorovich [Kan] on operators on linear ordered spaces. These constructions essentially rely on the structure of (Dedekind complete) Banach lattices. These notions are of central importance in the theory of linear operators between Banach lattices, including classical L^p spaces, since the absolute value is a positive operator. Indeed it is well-known that positive contractions are well-behaved operators. Actually, contractively regular operators on L^p spaces share in general the same nice properties as contractions on Hilbert spaces. We refer to the books [AbA], [MeN] and [Sch] and to the papers [Pis6] and [Pis1] for more information.

Due to the lack of local unconditional structure, on a Schatten space and more generally on a noncommutative L^p space, the canonical order on the space of selfadjoint elements does not induce a structure of a Banach lattice, see [DJT, Chapter 17] and [PiX, page 1478]. Nevertheless, there exists a purely Banach space characterization of regular operators on classical L^p spaces [HvNVW1, Theorem 2.7.2] which says that a linear operator $T\colon L^p(\Omega) \to L^p(\Omega')$ is regular if and only if for any Banach space X the map $T \otimes \mathrm{Id}_X$ induces a bounded operator between the Bochner spaces $L^p(\Omega, X)$ and $L^p(\Omega', X)$. In this case, the regular norm is given by

$$\|T\|_{\mathrm{reg}, L^p(\Omega)\to L^p(\Omega')} = \sup_X \|T \otimes \mathrm{Id}_X\|_{L^p(\Omega,X)\to L^p(\Omega',X)}\,, \tag{12.1}$$

where the supremum runs over all Banach spaces X. Using this property, a natural extension of this notion for noncommutative L^p spaces is introduced in [Pis2]. A linear map $T\colon L^p(M) \to L^p(N)$ between noncommutative L^p spaces, associated with approximately finite-dimensional von Neumann algebras M and N, is said regular if for any noncommutative Banach space E (that is, an operator space), the map $T \otimes \mathrm{Id}_E$ induces a bounded operator between the vector-valued noncommutative L^p spaces $L^p(M, E)$ and $L^p(N, E)$ (see Subsection 12.1 for their definition). Nevertheless, the paper [Pis2] does not give a definition of the absolute value of a regular operator and the definition of a regular operator is only usable for *approximately finite-dimensional* von Neumann algebras.

In this section, we define a noncommutative analogue of the absolute value of a regular operator acting on an arbitrary noncommutative L^p space for any $1 \leqslant p \leqslant \infty$. This gives rise to the notion of decomposable operators. We shall present some important results of the works [HDR6, HDR3]. After introducing the main notions from noncommutative (vector valued) L^p spaces, and the definitions of regular and decomposable operators in Subsection 12.1, we show in Subsection 12.2 that these two notions coincide isometrically on approximately finite-dimensional noncommutative L^p spaces. We shall also show that decomposable operators generalise the absolute value $|T|$ in Proposition 12.5.

Henceforth, in view of its more handy flexibility, we will prefer the notion of decomposable operators to regular operators. Then our interest will be focussed on the decomposability of important classes of noncommutative operations. Namely, we will consider Fourier and Schur multipliers that we have already encountered in Sections 10 and 11. We will also be able to treat a certain combination of the two, see Definition 12.6. For our Fourier multipliers, the group "sits on the frequency side". Note that convolution operators on locally compact amenable groups (i.e. the group "sits on the spatial side") are decomposable iff the convolution measure is finite [Are, Proposition 3.3], and non-decomposability of the Hilbert transform in various settings is discussed in [ArV], see also [HDR6, Section 7.3] for a generalisation. In our setting, an important type of result will be a projection onto the space of Fourier-Schur multipliers, that

is, a mapping

$$P^p_{I,G}\colon \mathrm{CB}(L^p(B(\ell^2_I)\overline{\otimes}\mathrm{VN}(G))) \to \mathfrak{M}^{p,\mathrm{cb}}_I(G), \tag{12.2}$$

where CB stands for the space of completely bounded operators (see Subsection 10.1) and $\mathfrak{M}^{p,\mathrm{cb}}_I(G)$ for the subspace consisting of Fourier-Schur multipliers. We will show in Theorem 12.7 that in case G is a discrete group, this projection preserves completely positive operators (see Subsection 12.1 for a definition of this notion). In view of the fact that decomposable operators are exactly the linear combinations of the completely positive ones (see Proposition 12.1), we will be able afterwards to describe the decomposable norm of Fourier multipliers and that of Schur multipliers (see Proposition 12.9 and 12.10).

In the sequel, we turn our attention to Fourier multipliers acting on locally compact group von Neumann algebras and their L^p spaces. We will recall this notion in Subsection 12.4. Note that the method of Theorem 12.7 on projection is naturally restricted to the case of discrete groups. In order to extend the result (12.2) to the case of more general locally compact groups, we introduce the notion of ALSS groups. This means that the group G contains a sequence of lattices such that one can choose the fundamental domains being included in small neighborhoods of the identity e_G (see Subsection 12.4 for a precise definition). For these groups and the fundamental domains satisfying a certain condition on the limit of their measures (12.13), we extend in Subsection 12.5 the projection result from (12.2), see Theorem 12.14. In a simpler variant, we are also able to give this result for pro-discrete groups, see Theorem 12.17. Then in Propositions 12.18 and 12.19, we obtain as corollaries the description of decomposable Fourier multipliers acting on $L^p(\mathrm{VN}(G))$ with such ALSS and pro-discrete groups G. At the end of Subsection 12.5, we illustrate the projection result from Theorem 12.14 by giving several examples of ALSS groups, paying a particular attention to the measure convergence criterion (12.13).

12.1 Preliminaries on noncommutative vector valued L^p spaces

The readers are referred to [ER], [Pau] and [Pis5] for details on operator spaces and completely bounded maps. See also Subsection 10.1.

If $T\colon E \to F$ is a completely bounded map between two operators spaces E and F, we denote by $\|T\|_{\mathrm{cb},E\to F}$ its completely bounded norm. We also let $\mathrm{CB}(E,F)$ denote the space of completely bounded operators $E \to F$. Note that if E is an operator space, then E^{op} is the operator space defined by the matricial norms $\|[x_{ij}]_{ij}\|_{M_n\otimes E} = \|[x_{ji}]_{ji}\|_{M_n\otimes E}$ (exchanging row and column index). If M is a von Neumann algebra, then M^{op} is the von Neumann algebra obtained from M by inverting the product $x\odot y \overset{\mathrm{def}}{=} y\cdot x$. Then if M is equipped with a normal semifinite faithful trace (which is the case for all our von Neumann algebras in this memoir), $L^p(M^{\mathrm{op}}) = L^p(M)^{\mathrm{op}}$ for $1 \leqslant p \leqslant \infty$.

A von Neumann algebra M is called approximately finite-dimensional if $M = \overline{\bigcup_\alpha M_\alpha}^{w^*}$ (weak* closure), where the M_α are a net of finite dimensional $*$-subalgebras directed by inclusion [Pis4, Theorem 3.4]. The theory of vector-valued noncommutative L^p spaces was initiated by Pisier [Pis4] for the case where the underlying von Neumann algebra is approximately finite-dimensional and equipped with a normal semifinite faithful trace. Suppose $1 \leqslant p \leqslant \infty$. Under these assumptions, for any operator space E, we can define by complex interpolation

$$L^p(M,E) \overset{\mathrm{def}}{=} \left(M\otimes_{\min} E, L^1(M^{\mathrm{op}})\widehat{\otimes}E\right)_{\frac{1}{p}} \tag{12.3}$$

where $\otimes_{\min}$ and $\widehat{\otimes}$ denote the injective and the projective tensor product of operator spaces. When $E=\mathbb{C}$, we get the noncommutative L^p space $L^p(M)$.

A linear map $T\colon L^p(M) \to L^p(N)$ between noncommutative L^p spaces, associated with approximately finite-dimensional von Neumann algebras M and N equipped with normal semifinite faithful traces, is said regular if for any operator space E, the map $T\otimes \mathrm{Id}_E$ induces a bounded operator between the vector-valued noncommutative L^p spaces $L^p(M,E)$ and $L^p(N,E)$. Then the regular norm is defined by

$$\|T\|_{\mathrm{reg},L^p(M)\to L^p(N)} = \sup_E \|T\otimes \mathrm{Id}_E\|_{L^p(M,E)\to L^p(N,E)}\,, \tag{12.4}$$

where the supremum runs over all operator spaces E. In fact, it is shown in [Pis2, (2.1)] that it suffices to take the supremum over all finite dimensional matrix spaces $E = \mathrm{M}_d$, $d \in \mathbb{N}$. We denote by $\mathrm{Reg}\big(L^p(M), L^p(N)\big)$ the space(!) of regular operators, normed by $\|\cdot\|_{\mathrm{reg},L^p(M)\to L^p(N)}$. Note also that if $E = L^p(M_0)$ is itself a noncommutative L^p space associated with a normal semifinite faithful trace, then $L^p(M, L^p(M_0)) = L^p(M\overline{\otimes}M_0)$ (completely) isometrically [Pis4, (3.6)]. This fact will be tacitly used in the sequel; in particular, $S^p_I(L^p(M_0))$ is the noncommutative L^p space $L^p(B(\ell^2_I)\overline{\otimes}M_0)$. According to [Pis2, Corollary 3.3] and [Pis2, Theorem 3.7] (see also [Pis1, (6) page 264]), we have the isometric interpolation identity[11]

$$\mathrm{Reg}\big(L^p(M), L^p(N)\big) = \big(\mathrm{CB}_{\mathrm{w}^*}(M,N), \mathrm{CB}(L^1(M), L^1(N))\big)^{\frac{1}{p}} \tag{12.5}$$

where we use the Caldéron's second method or upper method [BeL, page 88] and where the subscript w* means "weak* continuous".

Recall that a linear map $T\colon L^p(M) \to L^p(N)$ is decomposable [Haa1, JR2] if there exist linear maps $v_1, v_2\colon L^p(M) \to L^p(N)$ such that the linear map

$$\Phi = \begin{bmatrix} v_1 & T \\ T^\circ & v_2 \end{bmatrix} : S^p_2(L^p(M)) \to S^p_2(L^p(N)), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} v_1(a) & T(b) \\ T^\circ(c) & v_2(d) \end{bmatrix} \tag{12.6}$$

is completely positive, where $T^\circ(c) \overset{\mathrm{def}}{=} T(c^*)^*$ and where $S^p_2(L^p(M))$ and $S^p_2(L^p(N))$ are vector-valued Schatten spaces. Note that a mapping $S\colon L^p(M) \to L^p(N)$ between noncommutative L^p spaces is called completely positive if $S \otimes \mathrm{Id}_{\mathrm{M}_d}\colon \mathrm{M}_d(L^p(M)) \to \mathrm{M}_d(L^p(N))$ is positivity preserving for all $d \in \mathbb{N}$, where $\mathrm{M}_d(L^p(M))$ carries naturally a positive cone formed by elements x^*x with $x \in \mathrm{M}_d(L^{2p}(M))$ (cf. Subsection 10.1). If T is decomposable, then v_1 and v_2 are completely positive and the decomposable norm of T is defined by

$$\|T\|_{\mathrm{dec},L^p(M)\to L^p(N)} = \inf\big\{\max\{\|v_1\|, \|v_2\|\}\big\}, \tag{12.7}$$

where the infimum is taken over all possible maps v_1 and v_2. See the books [BLM], [ER] and [Pis5] for more information on this classical notion in the case $p = \infty$. The space $\mathrm{Dec}(L^p(M), L^p(N))$ of decomposable operators is a normed space [HDR6, Proposition 3.8].

12.2 Decomposable equals regular on approximately finite dimensional von Neumann algebras

We have the following first easy result, we refer e.g. to [HDR6, Proposition 3.12].

Proposition 12.1 *Let M and N be von Neumann algebras equipped with normal semifinite faithful traces. Suppose $1 \leqslant p \leqslant \infty$. Let $T\colon L^p(M) \to L^p(N)$ be a linear map. Then the following are equivalent.*

11. The compatibility means, roughly speaking, that the elements of $\mathrm{CB}(M,N) \cap \mathrm{CB}(L^1(M), L^1(N))$ are the maps simultaneous bounded from M into N and from $L^1(M)$ into $L^1(N)$.

1. *The map T is decomposable.*

2. *The map T belongs to the span of the completely positive maps from $L^p(M)$ into $L^p(N)$.*

3. *There exist some completely positive maps $T_1, T_2, T_3, T_4 \colon L^p(M) \to L^p(N)$ such that*

$$T = T_1 - T_2 + i(T_3 - T_4).$$

If the latter case is satisfied, we have $\|T\|_{\mathrm{dec}, L^p(M) \to L^p(N)} \leqslant \|T_1 + T_2 + T_3 + T_4\|_{L^p(M) \to L^p(N)}$.

If $1 < p < \infty$ and if M and N are approximately finite-dimensional, it is alluded in the introduction of [JR2] that decomposable maps coincide with the regular maps. We greatly strengthen this statement by showing that the regular norm $\|T\|_{\mathrm{reg}, L^p(M) \to L^p(N)}$ and the decomposable norm $\|T\|_{\mathrm{dec}, L^p(M) \to L^p(N)}$ are identical for a regular map T.

Theorem 12.2 *[HDR6, Theorem 3.24] Let M and N be approximately finite-dimensional von Neumann algebras which are equipped with normal semifinite faithful traces. Suppose $1 \leqslant p \leqslant \infty$. Let $T \colon L^p(M) \to L^p(N)$ be a linear mapping. Then T is regular if and only if T is decomposable. In this case, we have*

$$\|T\|_{\mathrm{dec}, L^p(M) \to L^p(N)} = \|T\|_{\mathrm{reg}, L^p(M) \to L^p(N)}.$$

We give the strategy of proof which is outlined in detail in [HDR6, Subsections 3.4-3.6].
Proof :

1st step of the proof of Theorem 12.2: Reduction to selfadjoint operators We have the following intermediate step.

Proposition 12.3 *Let M and N be approximately finite-dimensional von Neumann algebras equipped with normal semifinite faithful traces. Suppose $1 \leqslant p \leqslant \infty$ and that $T \colon L^p(M) \to L^p(N)$ is a linear mapping. Define $\tilde{T} \colon S_2^p(L^p(M)) \to S_2^p(L^p(N))$ by*

$$\tilde{T}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & T(b) \\ T^\circ(c) & 0 \end{bmatrix}.$$

Then $\tilde{T}$ is selfadjoint in the sense that $\tilde{T}(x^*) = \big(\tilde{T}(x)\big)^*$. Moreover, T is regular if and only if the map $\tilde{T} \colon S_2^p(L^p(M)) \to S_2^p(L^p(N))$ is regular and in this case, we have $\|T\|_{\mathrm{reg}, L^p(M) \to L^p(N)} = \|\tilde{T}\|_{\mathrm{reg}, S_2^p(L^p(M)) \to S_2^p(L^p(N))}$.

Proof : See [HDR6, Proposition 3.17]. ∎

Proposition 12.4 *Let M and N be von Neumann algebras equipped with normal semifinite faithful traces. Suppose $1 \leqslant p \leqslant \infty$. Let $T \colon L^p(M) \to L^p(N)$ be a linear map. Then T is decomposable if and only if the map $\tilde{T} \colon S_2^p(L^p(M)) \to S_2^p(L^p(N))$ from Proposition 12.3 is decomposable, and in this case, we have $\|T\|_{\mathrm{dec}, L^p(M) \to L^p(N)} = \|\tilde{T}\|_{\mathrm{dec}, S_2^p(L^p(M)) \to S_2^p(L^p(N))}$.*

Proof : See [HDR6, Proposition 3.18]. ∎

2nd step: Proof for the case $L^p(M) = S^p_m$, $L^p(N) = S^p_n$ We give some ideas of the proof. According to the 1st step and the fact that $S^p_2(S^p_m) = S^p_{2m}$, $S^p_2(S^p_n) = S^p_{2n}$ are again Schatten spaces, it suffices to consider the case that T is selfadjoint.

First we show $\|T\|_{\mathrm{reg}} \leqslant \|T\|_{\mathrm{dec}}$. In [HDR6, Proposition 3.19], we show that since T is selfadjoint, for every $\varepsilon > 0$ there exist $T_1, T_2 \colon S^p_m \to S^p_n$ completely positive such that $T = T_1 - T_2$ and $\|T_1 + T_2\| \leqslant \|T\|_{\mathrm{dec}} + \varepsilon$. According to Choi's characterization of completely positive maps acting on matrix spaces [Ch1, Theorem 1], there exist $a_1, \ldots, a_l, b_1, \ldots, b_l \in \mathrm{M}_{m,n}$ such that $T_1(x) = \sum_{k=1}^l a_k^* x a_k$ and $T_2(x) = \sum_{k=1}^l b_k^* x b_k$. Using the calculation from [Pis2, Lemma 2.3] used in a related context in this source, we obtain for any $d \in \mathbb{N}$ that $\|T \otimes \mathrm{Id}_{\mathrm{M}_d}\|_{S^p_m(\mathrm{M}_d) \to S^p_n(\mathrm{M}_d)} \leqslant \|T_1 + T_2\|_{S^p_m \to S^p_n} \leqslant \|T\|_{\mathrm{dec}} + \varepsilon$. Passing to the supremum over all d and letting $\varepsilon \to 0$ yields $\|T\|_{\mathrm{reg}} \leqslant \|T\|_{\mathrm{dec}}$.

Then we show $\|T\|_{\mathrm{dec}} \leqslant \|T\|_{\mathrm{reg}}$. Assume that $\|T\|_{\mathrm{reg}} \leqslant 1$. According to [Pis5, Theorem 5.12], note that we have isometrically

$$\mathrm{CB}(S^\infty_n) = \mathrm{M}_n \otimes_h \mathrm{M}_n \tag{12.8}$$

where $\otimes_h$ denotes the Haagerup tensor product. Moreover, using the properties of this tensor product [Pis4, pages 95-97], we obtain

$$\begin{aligned} \mathrm{M}_n^{\mathrm{op}} \otimes_h \mathrm{M}_n^{\mathrm{op}} &= (\mathrm{C}_n \otimes_h \mathrm{R}_n)^{\mathrm{op}} \otimes_h (\mathrm{C}_n \otimes_h \mathrm{R}_n)^{\mathrm{op}} = \mathrm{R}_n^{\mathrm{op}} \otimes_h \mathrm{C}_n^{\mathrm{op}} \otimes_h \mathrm{R}_n^{\mathrm{op}} \otimes_h \mathrm{C}_n^{\mathrm{op}} \\ &= \mathrm{C}_n \otimes_h \mathrm{R}_n \otimes_h \mathrm{C}_n \otimes_h \mathrm{R}_n = \mathrm{C}_n \otimes_h S^1_n \otimes_h \mathrm{R}_n = \mathrm{M}_n(S^1_n) = \mathrm{M}_n \otimes_{\min} S^1_n = \mathrm{CB}(S^1_n). \end{aligned}$$

Here C_n resp. R_n stand for the column resp. row space (i.e. the operator space formed by the first column resp. row matrices in M_n). We have $\gamma_\theta(T) \leqslant 1$ with $\theta = \frac{1}{p}$ and γ_θ defined in [Pis3, Theorem 8.5], according to [Pis2, Corollary 3.3]. Then the selfadjointness of T together with [Pis3, Corollary 8.7] yields that $\|T\|_{\mathrm{dec}} \leqslant \|T_1 + T_2\|_{S^p_m \to S^p_n} \leqslant 1$ where $T = T_1 - T_2$ and T_1, T_2 are completely positive mappings $S^p_m \to S^p_n$ given there.

3rd step: The general case of approximately finite-dimensional algebras So we assume that $T \colon L^p(M) \to L^p(N)$ is a bounded operator, and want to show that its regular norm (if finite) coincides with the decomposable norm (if finite). We divide into substeps.

3.1 If M and N are finite dimensional, then by [Tak1, Theorem 11.2] and [Dix, proof of Proposition 7 page 109, Theorem 5 page 105, Corollary page 103], there exist $m_1, \ldots, m_K, n_1, \ldots, n_L \in \mathbb{N}$ and $\lambda_1, \ldots, \lambda_K, \mu_1, \ldots, \mu_L \in (0, \infty)$ such that $(M, \tau) = (\mathrm{M}_{m_1} \oplus \cdots \oplus \mathrm{M}_{m_K}, \lambda_1 \operatorname{Tr}_{m_1} \oplus \cdots \oplus \lambda_K \operatorname{Tr}_{m_K})$ and $(N, \sigma) = (\mathrm{M}_{n_1} \oplus \cdots \oplus \mathrm{M}_{n_L}, \mu_1 \operatorname{Tr}_{n_1} \oplus \cdots \oplus \mu_L \operatorname{Tr}_{n_L})$. Considering subsequently the subcases $\lambda_l, \mu_l \in \mathbb{N} \,/\, \in \mathbb{Q}_+ \,/\, \in \mathbb{R}_+$, we could reduce in [HDR6, proof of Theorem 3.24] this case to that of Schatten spaces $L^p(M) \rightsquigarrow S^p_m$, $L^p(N) \rightsquigarrow S^p_n$. Then we can appeal to the 2nd step.

3.2 If M and N are approximately finite-dimensional, but their traces are finite, then [Bla, page 291], $M = \overline{\bigcup_\alpha M_\alpha}^{\mathrm{w}^*}$ and $N = \overline{\bigcup_\beta N_\beta}^{\mathrm{w}^*}$ where (M_α) and (N_β) are nets directed by inclusion of finite dimensional unital $*$-subalgebras (as in case 3.1). Using operators between $L^p(M)$ and the approximations $L^p(M_\alpha)$ (resp. $L^p(N)$ and $L^p(N_\beta)$), which are moreover regular and decomposable of norm one, we were able to reduce this case to the above case 3.1.

3.3 M and N are general approximately finite-dimensional semifinite von Neumann algebras. By [Sun, page 57], there exist an increasing net of projections (e_i) which is strongly convergent to 1 with $\tau(e_i) < \infty$ for any i. We set $M_i \overset{\mathrm{def}}{=} e_i M e_i$. The trace $\tau|_{M_i}$ is obviously finite. Moreover,

it is well-known[12] that M_i is approximately finite-dimensional. We conclude that M_i is a von Neumann algebra satisfying the properties of case 3.2. Then again we find mappings between $L^p(M)$ and $L^p(M_i)$ (resp. $L^p(N)$ and $L^p(N_j)$) which are both regular and decomposable of norm one, and can thus reduce to the case 3.2 above. ∎

For any regular operator $T\colon L^p(\Omega) \to L^p(\Omega')$ on classical L^p spaces, it is well-known that there exists some (minimal in some sense) positivity preserving linear operator $|T|\colon L^p(\Omega) \to L^p(\Omega')$ such that $|Tf(x)| \leqslant |T|f(x)$ for all $f \in L^p(\Omega)$ and a.e. $x \in \Omega'$, and moreover $\big\||T|\big\|_{L^p(\Omega)\to L^p(\Omega')} = \|T\|_{\mathrm{reg},L^p(\Omega)\to L^p(\Omega')}$, see e.g. [MeN, Proposition 1.3.6].

Proposition 12.5 *Let Ω and Ω' be (localizable) measure spaces. Suppose $1 \leqslant p \leqslant \infty$. Let $T\colon L^p(\Omega) \to L^p(\Omega')$ be a (weak* continuous if $p = \infty$) regular operator. Then the map $\Phi = \begin{bmatrix} |T| & T \\ T^\circ & |T| \end{bmatrix} : S^p_2(L^p(\Omega)) \to S^p_2(L^p(\Omega'))$ is completely positive, i.e. the infimum of (12.6) in the definition of decomposability is attained with $v_1 = v_2 = |T|$.*

Proof : See [HDR6, Theorem 3.27]. ∎

12.3 Complementation on discrete groups

We note that Fourier multipliers on discrete group von Neumann algebras, and Schur multipliers acting on S^p_I have been introduced in Definitions 10.6 and 10.9. The former extends verbatim to a notion of L^p Fourier multiplier M_φ, provided that the operator in question is bounded on $L^p(\mathrm{VN}(G))$. We shall also need the following combination of the two notions of Fourier and Schur multiplier.

Definition 12.6 *Let G be a discrete group and I an index set. We denote by $\mathfrak{M}^p_I(G)$ the space of all operators $T\colon L^p(B(\ell^2_I)\overline{\otimes}\mathrm{VN}(G)) \to L^p(B(\ell^2_I)\overline{\otimes}\mathrm{VN}(G))$ such that for any $i,j \in I$, there is a function $\varphi_{ij}\colon G \to \mathbb{C}$ such that $T(e_{ij}\otimes f) = e_{ij}\otimes M_{\varphi_{ij}}(f)$ for any $f \in L^p(\mathrm{VN}(G))$. We then write $T = [M_{\varphi_{ij}}]$. We also denote by $\mathfrak{M}^{p,\mathrm{cb}}_I(G)$ the subspace of $\mathfrak{M}^p_I(G)$ consisting of completely bounded operators, and by $\mathfrak{M}^p(G)$ resp. $\mathfrak{M}^{p,\mathrm{cb}}(G)$ the space of (resp. completely) bounded Fourier multipliers $L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$.*

The following theorem generalises an average trick of Haagerup [Haa3, proof of Lemma 2.5][13]. The important point of the proof (for $1 \leqslant p \leqslant \infty$) is the fact that the map Δ below is trace preserving. The trace is not preserved for any non-discrete locally compact group G.

Theorem 12.7 *[HDR6, Theorem 4.2] Let I be an index set equipped with the counting measure. Let G be a discrete group. Suppose $1 \leqslant p < \infty$. Assume that $\mathrm{VN}(G)$ has QWEP if $p < \infty$. Let $T\colon S^p_I(L^p(\mathrm{VN}(G))) \to S^p_I(L^p(\mathrm{VN}(G)))$ be a completely bounded operator. For any $i,j \in I$, we define the complex function $\varphi_{ij}\colon G \to \mathbb{C}$ by*

$$\varphi_{ij}(s) = (\mathrm{Tr}\otimes\tau_G)\big(T(e_{ij}\otimes\lambda_s)(e_{ij}\otimes\lambda_s)^*\big), \qquad (s \in G)$$

Then the map

$$\begin{array}{rccc} P^p_{I,G}\colon & \mathrm{CB}(S^p_I(L^p(\mathrm{VN}(G)))) & \longrightarrow & \mathrm{CB}(S^p_I(L^p(\mathrm{VN}(G)))) \\ & T & \longmapsto & [M_{\varphi_{ij}}] \end{array}$$

is a well-defined contractive projection onto $\mathfrak{M}^{p,\mathrm{cb}}_I(G)$. There are the following additional properties of $P^p_{I,G}$.

12. This observation relies on the equivalence between "injective" and "approximately finite-dimensional".
13. We warn the reader that the assumption "normal" is lacking in [Haa3, Lemma 2.5] for maps defined on $\mathfrak{M}(\Gamma)$.

1. *For $p=\infty$, the same assertions are true by replacing* $\mathrm{CB}(S^p_I(L^p(\mathrm{VN}(G))))$ *by the space* $\mathrm{CB}_{\mathrm{w}^*}(B(\ell^2_I)\overline{\otimes}\mathrm{VN}(G))$.

2. *If T is completely positive then the map $P^p_{I,G}(T)$ is completely positive.*

3. *For any values $p,q\in[1,\infty]$ and any $T\in \mathrm{CB}(S^p_I(L^p(\mathrm{VN}(G))))\cap \mathrm{CB}(S^q_I(L^q(\mathrm{VN}(G))))$ we have $(P^p_{I,G}(T))([x_{ij}]) = (P^q_{I,G}(T))([x_{ij}])$ for any element $[x_{ij}]$ of $S^p_I(L^p(\mathrm{VN}(G)))\cap S^q_I(L^q(\mathrm{VN}(G)))$. So the mappings $P^p_{I,G}$, $1\leqslant p\leqslant\infty$, are compatible.*

Proof : Note that the following mapping is a unital normal $*$-monomorphism:

$$\begin{array}{ccc}\mathrm{VN}(G) & \longrightarrow & \mathrm{VN}(G)\overline{\otimes}\mathrm{VN}(G)\\ \lambda_s & \longmapsto & \lambda_s\otimes\lambda_s\end{array}\tag{12.9}$$

Then we can define a well-defined unital normal $*$-isomorphism

$$\Delta\colon \mathrm{M}_I(\mathrm{VN}(G))\to \mathrm{M}_I(\mathrm{VN}(G))\overline{\otimes}\mathrm{M}_I(\mathrm{VN}(G))$$

onto the sub von Neumann algebra $\Delta\big(\mathrm{M}_I(\mathrm{VN}(G))\big)$ of $\mathrm{M}_I(\mathrm{VN}(G))\overline{\otimes}\mathrm{M}_I(\mathrm{VN}(G))$ such that

$$\Delta(e_{ij}\otimes\lambda_s) = e_{ij}\otimes\lambda_s\otimes e_{ij}\otimes\lambda_s,\qquad (s\in G).$$

It is not difficult to check that the operator Δ preserves the traces. Consequently Δ admits a canonical extension $\Delta_p\colon S^p_I(L^p(\mathrm{VN}(G)))\to L^p(B(\ell^2_I)\overline{\otimes}\mathrm{VN}(G)\overline{\otimes}B(\ell^2_I)\overline{\otimes}\mathrm{VN}(G))$ which is completely contractive and completely positive (and normal if $p=\infty$).

Suppose that $T\colon S^p_I(L^p(\mathrm{VN}(G)))\to S^p_I(L^p(\mathrm{VN}(G)))$ is a completely bounded operator. The operator

$$P^p_{I,G}(T) = (\Delta^*)_p\big(T\otimes\mathrm{Id}_{S^p_I(L^p(\mathrm{VN}(G)))}\big)\Delta_p$$

is a completely bounded map on the space $S^p_I(L^p(\mathrm{VN}(G)))$. Note that if T is completely positive then $P^p_{I,G}(T)$ is also a completely positive map. Moreover, we have

$$\begin{aligned}\big\|P^p_{I,G}(T)\big\|_{\mathrm{cb},S^p_I(L^p(\mathrm{VN}(G)))\to S^p_I(L^p(\mathrm{VN}(G)))} &\leqslant \big\|(\Delta^*)_p\big(T\otimes\mathrm{Id}_{S^p_I(L^p(\mathrm{VN}(G)))}\big)\Delta_p\big\|_{\mathrm{cb}}\\ &\leqslant \|T\|_{\mathrm{cb},S^p_I(L^p(\mathrm{VN}(G)))\to S^p_I(L^p(\mathrm{VN}(G)))}.\end{aligned}$$

Thus $P^p_{I,G}$ is contractive. For any $i,j,k,l\in I$ and any $s,s'\in G$, we have

$$\begin{aligned}&(\mathrm{Tr}\otimes\tau_G)\Big(\big((\Delta^*)_p\big(T\otimes\mathrm{Id}_{S^p_I(L^p(\mathrm{VN}(G)))}\big)\Delta_p(e_{ij}\otimes\lambda_s)\big)(e_{kl}\otimes\lambda_{s'})^*\Big)\\
&\quad= (\mathrm{Tr}\otimes\tau_G)\big((\Delta^*)_p\big(T\otimes\mathrm{Id}_{S^p_I(L^p(\mathrm{VN}(G)))}\big)(e_{ij}\otimes\lambda_s\otimes e_{ij}\otimes\lambda_s)(e^*_{kl}\otimes\lambda^*_{s'})\big)\\
&\quad= (\mathrm{Tr}\otimes\tau_G)\Big(\big((\Delta^*)_p\big(T(e_{ij}\otimes\lambda_s)\otimes e_{ij}\otimes\lambda_s\big)\big)\big(e_{lk}\otimes\lambda_{s'^{-1}}\big)\Big)\\
&\quad= (\mathrm{Tr}\otimes\tau_G\otimes\mathrm{Tr}\otimes\tau_G)\Big(\big(T(e_{ij}\otimes\lambda_s)\otimes e_{ij}\otimes\lambda_s\big)\Delta_{p^*}\big(e_{lk}\otimes\lambda_{s'^{-1}}\big)\Big)\\
&\quad= (\mathrm{Tr}\otimes\tau_G\otimes\mathrm{Tr}\otimes\tau_G)\big((T(e_{ij}\otimes\lambda_s)\otimes e_{ij}\otimes\lambda_s)(e_{lk}\otimes\lambda_{s'^{-1}}\otimes e_{lk}\otimes\lambda_{s'^{-1}})\big)\\
&\quad= (\mathrm{Tr}\otimes\tau_G)\big(T(e_{ij}\otimes\lambda_s)(e_{lk}\otimes\lambda_{s'^{-1}})\big)(\mathrm{Tr}\otimes\tau_G)\big(e_{ij}e_{lk}\otimes\lambda_s\lambda_{s'^{-1}}\big)\\
&\quad= (\mathrm{Tr}\otimes\tau_G)\big(T(e_{ij}\otimes\lambda_s)(e_{lk}\otimes\lambda_{s'^{-1}})\big)\delta_{i,k}\delta_{j,l}\delta_{s,s'}\\
&\quad= (\mathrm{Tr}\otimes\tau_G)\big(T(e_{ij}\otimes\lambda_s)(e_{lk}\otimes\lambda_{s'^{-1}})\big)\delta_{i,k}\delta_{j,l}\delta_{s,s'}\\
&\quad= (\mathrm{Tr}\otimes\tau_G)\big(T(e_{ij}\otimes\lambda_s)(e_{kl}\otimes\lambda_{s'})^*\big)\delta_{i,k}\delta_{j,l}\delta_{s,s'}.\end{aligned}$$

From here, it is not difficult to see that $P^p_{I,G}(T)$ is the operator $[M_{\varphi_{ij}}]$. If $T = [M_{\psi_{ij}}]$ right from the beginning, for some symbols $\psi_{ij}\colon G \to \mathbb{C}$, then for $s \in G$

$$\begin{aligned}\varphi_{ij}(s) &= (\mathrm{Tr}\otimes\tau_G)\big(T(e_{ij}\otimes\lambda_s)(e_{ij}\otimes\lambda_s)^*\big)\\ &= \psi_{ij}(s)(\mathrm{Tr}\otimes\tau_G)\big((e_{ij}\otimes\lambda_s)(e_{ij}\otimes\lambda_s)^*\big)\\ &= \psi_{ij}(s)\tau_G(\lambda_s\lambda_s^*) = \psi_{ij}(s)\tau_G(\lambda_s\lambda_{s^{-1}})\\ &= \psi_{ij}(s)\delta_{s,s} = \psi_{ij}(s).\end{aligned}$$

Thus, in this case $P^p_{I,G}(T) = T$, so that $P^p_{I,G}$ is indeed a projection onto $\mathfrak{M}^{p,\mathrm{cb}}_I(G)$.

The statement about the compatibility of $P^p_{I,G}$ for different values of $p \in [1,\infty]$ follows directly from the defining formula of $P^p_{I,G}$ and the fact that $(\Delta^*)_p, \Delta_p$ and $\mathrm{Id}_{S^p_I(L^p(\mathrm{VN}(G)))}$ are all compatible for two different values of p.

■

We now give some consequences of Theorem 12.7 above.

Proposition 12.8 *Let G be a discrete group. Suppose $1 \leqslant p \leqslant \infty$. Suppose that $\mathrm{VN}(G)$ has* QWEP *if $p < \infty$. Then a function $\phi\colon G \to \mathbb{C}$ induces a decomposable Fourier multiplier on $L^p(\mathrm{VN}(G))$ if and only if it induces a decomposable Fourier multiplier on $\mathrm{VN}(G)$.*

Proof : $\Rightarrow$: Let $M_\phi\colon L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ be a decomposable Fourier multiplier. By Proposition 12.1, we can write $M_\phi = T_1 - T_2 + i(T_3 - T_4)$ where each T_j is a completely positive map on $L^p(\mathrm{VN}(G))$. Using the projection $P^p_G = P^p_{\{0\},G}$ of Theorem 12.7 with $I = \{0\}$ a singleton, we obtain that

$$M_\phi = P^p_G(M_\phi) = P^p_G\big(T_1 - T_2 + i(T_3 - T_4)\big) = P^p_G(T_1) - P^p_G(T_2) + i\big(P^p_G(T_3) - P^p_G(T_4)\big)$$

and that each $P^p_G(T_j) = M_{\phi_j}$ is a completely positive Fourier multiplier on $L^p(\mathrm{VN}(G))$. By the proof of [DCH, Proposition 4.2], we see that the (continuous) function ϕ_j is[14] positive definite. Hence it induces a completely positive Fourier multiplier on $\mathrm{VN}(G)$ again by [DCH, Proposition 4.2]. We conclude that ϕ induces a decomposable Fourier multiplier on $\mathrm{VN}(G)$.

$\Leftarrow$: Let $M_\phi\colon \mathrm{VN}(G) \to \mathrm{VN}(G)$ be a decomposable Fourier multiplier. Similarly, with Theorem 12.7, we can write $M_\phi = M_{\phi_1} - M_{\phi_2} + i(M_{\phi_3} - M_{\phi_4})$ where each $M_{\phi_j}\colon \mathrm{VN}(G) \to \mathrm{VN}(G)$ is completely positive. By [Har, page 216], each Fourier multiplier ϕ_j induces a completely positive multiplier on $L^p(\mathrm{VN}(G))$. We conclude that ϕ induces a decomposable Fourier multiplier on $L^p(\mathrm{VN}(G))$. ■

Let G be a discrete group. Recall that the group von Neumann algebra $\mathrm{VN}(G)$ is approximately finite-dimensional if and only if G is amenable, see [SiSm, Theorem 3.8.2]. Using Theorem 12.7, we obtain the following result.

Proposition 12.9 *Let G be an amenable discrete group. Suppose $1 \leqslant p \leqslant \infty$. Then a function $\phi\colon G \to \mathbb{C}$ induces a decomposable Fourier multiplier $M_\phi\colon L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ if and only if it induces a (completely) bounded Fourier multiplier $M_\phi\colon \mathrm{VN}(G) \to \mathrm{VN}(G)$. In this case, we have the isometric identity*

$$\|M_\phi\|_{\mathrm{dec},L^p(\mathrm{VN}(G))\to L^p(\mathrm{VN}(G))} = \|M_\phi\|_{\mathrm{cb},\mathrm{VN}(G)\to\mathrm{VN}(G)} = \|M_\phi\|_{\mathrm{VN}(G)\to\mathrm{VN}(G)}\,.$$

14. Here we use the inclusion $\mathrm{VN}(G) \subseteq L^p(\mathrm{VN}(G))$ and the realization of $L^p(\mathrm{VN}(G))$ as a space of measurable operators.

Proof : By [DCH, Corollary 1.8], since G is amenable, we have $\mathfrak{M}^\infty(G) = \mathfrak{M}^{\infty,\mathrm{cb}}(G)$ isometrically. The first part is Proposition 12.8 using [Haa1, Theorem 2.1] (which says that the decomposable norm and the completely bounded norm coincide for operators on approximately finite-dimensional von Neumann algebras). By [Har], we have $\mathfrak{M}^\infty(G) = \mathfrak{M}^1(G)$ isometrically. Now, we use the intersection interpolation theorem [Tri, Section 1.17.1 Theorem 1] with the interpolation couple (12.5) and with intersection with $\mathfrak{M}^\infty(G)$. We also use the projection $P_G^p = P_{\{0\},G}^p$ from Theorem 12.7, so that P_G^∞ projects onto $\mathfrak{M}^\infty(G)$. Here it is important that Theorem 12.7 3. gives compatibility of the P_G^p for different values of p. Note that we have isometrically

$$\big(\mathrm{CB}_{\mathrm{w}^*}(\mathrm{VN}(G)) \cap \mathfrak{M}^\infty(G), \mathrm{CB}(L^1(\mathrm{VN}(G))) \cap \mathfrak{M}^\infty(G)\big)^{\frac{1}{p}} = \big(\mathfrak{M}^\infty(G), \mathfrak{M}^\infty(G)\big)^{\frac{1}{p}} = \mathfrak{M}^\infty(G).$$

We infer that the space $\mathrm{Reg}(L^p(\mathrm{VN}(G))) \cap \mathfrak{M}^\infty(G) = (\mathrm{CB}_{\mathrm{w}^*}(\mathrm{VN}(G)), \mathrm{CB}(L^1(\mathrm{VN}(G))))^{\frac{1}{p}} \cap \mathfrak{M}^\infty(G)$ equipped with the regular norm $\|\cdot\|_{\mathrm{reg},L^p(\mathrm{VN}(G))\to L^p(\mathrm{VN}(G))}$ is isometric to the space $\mathfrak{M}^\infty(G) = \mathfrak{M}^{\infty,\mathrm{cb}}(G)$. We finally employ Theorem 12.2 to pass isometrically from regular operators to decomposable operators. ∎

Similarly, we obtain the following description of decomposable Schur multipliers with the projection of Theorem 12.7. In this result, we write M_ϕ for the Schur multiplier associated with $\phi : I \times I \to \mathbb{C}$.

Proposition 12.10 *Suppose $1 \leqslant p \leqslant \infty$. Then a function $\phi\colon I\times I \to \mathbb{C}$ induces a decomposable Schur multiplier on S_I^p if and only if it induces a (completely) bounded Schur multiplier on $B(\ell_I^2)$. In this case, we have the isometric identity*

$$\|M_\phi\|_{\mathrm{dec},S_I^p\to S_I^p} = \|M_\phi\|_{\mathrm{reg},S_I^p\to S_I^p} = \|M_\phi\|_{\mathrm{cb},B(\ell_I^2)\to B(\ell_I^2)} = \|M_\phi\|_{B(\ell_I^2)\to B(\ell_I^2)}\,.$$

Proof : See [HDR6, Theorem 4.10]. ∎

At the end of this subsection, we spell out a variant of Theorem 12.7 valid on the most classical group $\mathbb{R}^d$ and (vector valued) Fourier multipliers in the classical sense.

Proposition 12.11 *[HDR3, Theorem 3] Let X be a Banach space. Suppose $1 \leqslant p < \infty$. Denote by $\mathfrak{M}^p(\mathbb{R}^d, X)$ the subspace of $B(L^p(\mathbb{R}^d, X))$ consisting of Fourier multipliers, and by $\mathfrak{M}^p_{\mathrm{rad}}(\mathbb{R}^d, X)$ in turn the subspace with radial symbols $\phi(x) = \phi_0(|x|)$, $x \in \mathbb{R}^d$.*

Then there exists a contractive projection $P_{p,X} : \mathfrak{M}^p(\mathbb{R}^d, X) \to \mathfrak{M}^p_{\mathrm{rad}}(\mathbb{R}^d, X)$, preserving positive operators.

12.4 Preliminaries on locally compact groups, Fourier multipliers and groups approximable by lattices

Now we extend the notion of Fourier multiplier to the context of group von Neumann algebras of unimodular locally compact groups.

Group von Neumann algebras of locally compact groups Let G be a locally compact group equipped with a fixed left invariant Haar measure μ_G. For a complex function $g\colon G \to \mathbb{C}$, we write $\lambda(g)$ for the left convolution operator (in general unbounded) by g on $L^2(G)$. In case that this operator is bounded, we say that g is left bounded. Note that if $g \in L^1(G)$, then g is indeed left bounded. Let $\mathrm{VN}(G) \subseteq B(L^2(G))$ be the von Neumann algebra generated by the set $\big\{\lambda(g) : g \in L^1(G)\big\}$. It is called the group von Neumann algebra of G. Recall that for any $g \in L^1(G)$ we have $\lambda(g) = \int_G g(s)\lambda_s \,\mathrm{d}\mu_G(s)$ where the latter integral is understood in the weak

operator sense[15], and λ_s is the left translation $\lambda_s f(t) = f(s^{-1}t)$. Then the Plancherel weight $\tau_G \colon \mathrm{VN}(G)^+ \to [0,\infty]$ is[16] defined by the formula

$$\tau_G(x) = \begin{cases} \|g\|^2_{L^2(G)} & \text{if } x^{\frac{1}{2}} = \lambda(g) \text{ for some left bounded function } g \in L^2(G) \\ +\infty & \text{otherwise} \end{cases}.$$

By [Haa2, page 125] or [Ped, Proposition 7.2.8], the Plancherel weight τ_G on $\mathrm{VN}(G)$ is tracial if and only if G is unimodular, which means that the left Haar measure of G and the right Haar measure of G coincide. Now, in the sequel, we suppose that the locally compact group G is unimodular, and μ_G always denotes its Haar measure. As the trace τ_G is normal semifinite faithful, we then have noncommutative $L^p(\mathrm{VN}(G))$ spaces.

Fourier multipliers on noncommutative L^p spaces Note that if $\phi \in L^2_{\mathrm{loc}}(G)$ is a 2-locally integrable function and if $f \in C_c(G)$ then the product ϕf belongs to $L^1(G)$ and consequently induces a bounded operator $\lambda(\phi f) \colon L^2(G) \to L^2(G)$. Recall that this operator is equal to the weak integral $\int_G \phi(s) f(s) \lambda_s \, \mathrm{d}\mu_G(s)$. Finally, recall that $\lambda(\operatorname{span} C_c(G) * C_c(G))$ is dense in $L^p(\mathrm{VN}(G))$ for $1 \leqslant p < \infty$ and that $\lambda(C_c(G))$ is weak* dense in $\mathrm{VN}(G)$.

Definition 12.12 *Let G be a unimodular locally compact group. Suppose $1 \leqslant p \leqslant \infty$. Then we say that a (weak* continuous if $p = \infty$) bounded operator $T \colon L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ is a (L^p) Fourier multiplier if there exists a locally 2-integrable function $\phi \in L^2_{\mathrm{loc}}(G)$ such that for any $f \in C_c(G) * C_c(G)$ ($f \in C_c(G)$ if $p = \infty$) the element $\int_G \phi(s) f(s) \lambda_s \, \mathrm{d}\mu_G(s)$ belongs to $L^p(\mathrm{VN}(G))$ and*

$$T\left(\int_G f(s) \lambda_s \, \mathrm{d}\mu_G(s) \right) = \int_G \phi(s) f(s) \lambda_s \, \mathrm{d}\mu_G(s), \quad \textit{i.e.} \quad T(\lambda(f)) = \lambda(\phi f). \tag{12.10}$$

In this case, we let $T = M_\phi$. Then in fact $\phi \in L^\infty(G)$ a posteriori.

In accordance with Definitions 10.6 and 12.6 above, we define $\mathfrak{M}^p(G)$ to be the space of all bounded L^p Fourier multipliers and $\mathfrak{M}^{p,\mathrm{cb}}(G)$ to be the subspace consisting of completely bounded L^p Fourier multipliers.

Groups approximable by discrete subgroups In the next Subsection 12.5, we shall be interested in groups that are approximable by (finer and finer) discrete subgroups. A lattice Γ in a locally compact group G is a discrete subgroup for which G/Γ has a bounded G-invariant Borel measure [BHV, Definition B.2.1 page 332]. A locally compact group G that admits a lattice is necessarily unimodular [BHV, Proposition B.2.2 page 332].

Let Γ be a discrete subgroup of a locally compact group G. A fundamental domain X relative to Γ is a Borel measurable subset of G satisfying the following two properties:

$$\mathrm{X}\Gamma = G, \tag{12.11}$$

$$\mathrm{X}\gamma \cap \mathrm{X}\gamma' = \emptyset \text{ for any distinct elements } \gamma, \gamma' \text{ of } \Gamma. \tag{12.12}$$

15. That means (see e.g. [Gaa, Theorem 5 page 289]) that $\lambda(g) \colon L^2(G) \to L^2(G)$ is the unique bounded operator such that

$$\langle \lambda(g) f, h \rangle_{L^2(G)} = \int_G g(s) \langle \lambda_s f, h \rangle_{L^2(G)} \, \mathrm{d}\mu_G(s), \quad f, h \in L^2(G).$$

16. This is the natural weight associated with the left Hilbert algebra $C_c(G)$.

These properties say that every element $x \in G$ is covered by one and only one of the sets $\mathrm{X}\gamma$, $(\gamma \in \Gamma)$. These conditions are equivalent to the following statement: X is a Borel measurable subset of G such that the restriction of the canonical mapping $G \to G/\Gamma$ of G onto left cosets, restricted to X, becomes a bijection onto G/Γ. If G is σ-compact the result [BHV, Proposition B.2.4 page 333] gives the existence of a fundamental domain for any discrete subgroup Γ and if in addition Γ is a lattice in G then every fundamental domain for Γ has finite Haar measure [BHV, Proposition B.2.4 page 333].

Definition 12.13 *Let G be a second countable locally compact group. The group G is said to be approximable by lattice subgroups by shrinking (*ALSS*) if there exists a sequence (Γ_j) of lattice subgroups with associated fundamental domains (X_j) such that for any neighborhood V of the identity e_G, there exists some integer j_0 such that $\mathrm{X}_j \subseteq V$ for any $j \geqslant j_0$.*

We refer to [HDR6, Section 5] for a discussion of the notion ALSS and to [HDR6, Theorem 5.13] for a characterization in terms of other approximation properties.

12.5 Complementation on groups approximable by lattice subgroups

The following theorem gives a variant of Theorem 12.7 for a particular class of unimodular groups. Recall that the symmetric difference of $X_1, X_2 \subseteq G$ is defined by $X_1 \Delta X_2 = (X_1 \backslash X_2) \cup (X_2 \backslash X_1)$.

Theorem 12.14 *[HDR6, Theorem 6.16] Let G be a second countable unimodular locally compact group which satisfies* ALSS *with respect to a sequence of lattices $(\Gamma_j)_{j\geqslant 1}$ and associated fundamental domains $(\mathrm{X}_j)_{j\geqslant 1}$. Suppose $1 \leqslant p \leqslant \infty$. We assume that G is amenable if $1 < p < \infty$. Suppose that for some constant $c > 0$ and any compact subset K of G we have*

$$\lim_{j\to\infty} \sup_{\gamma\in\Gamma_j\cap K} \left| \frac{1}{\mu_G(\mathrm{X}_j)} \int_G \frac{\mu_G^2(\mathrm{X}_j \cap \gamma \mathrm{X}_j s)}{\mu_G^2(\mathrm{X}_j)} \, \mathrm{d}\mu_G(s) - c \right| = 0 \tag{12.13}$$

where μ_G is a Haar measure of G. Then for $1 \leqslant p \leqslant \infty$, there exists a linear mapping

$$P_G^p \colon \mathrm{CB}(L^p(\mathrm{VN}(G))) \to \mathfrak{M}^{p,\mathrm{cb}}(G)$$

of norm at most $\frac{1}{c}$ with the properties:

1. *If $T \colon L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ is completely positive, then $P_G^p(T)$ is completely positive.*
2. *If $T = M_\psi$ is a Fourier multiplier on $L^p(\mathrm{VN}(G))$ with bounded continuous symbol $\psi \colon G \to \mathbb{C}$, then $P_G^p(M_\psi) = M_\psi$. Moreover, if we have $\gamma \mathrm{X}_j = \mathrm{X}_j \gamma$ for any $j \in \mathbb{N}$ and any $\gamma \in \Gamma_j$, or alternatively, if X_j is symmetric in the sense that $\mu(\mathrm{X}_j \Delta \mathrm{X}_j^{-1}) = 0$ for any $j \in \mathbb{N}$, then $P_G^p(M_\psi) = M_\psi$ for any bounded measurable symbol such that $M_\psi \in \mathfrak{M}^{p,\mathrm{cb}}(G)$.*

For an element T belonging to $\mathrm{CB}(L^p(\mathrm{VN}(G)))$ and to $\mathrm{CB}(L^q(\mathrm{VN}(G)))$ for two values $p, q \in [1, \infty]$, we have $P_G^p(T)x = P_G^q(T)x$ for $x \in L^p(\mathrm{VN}(G)) \cap L^q(\mathrm{VN}(G))$.

In the above, if $p = \infty$, then we have to take $\mathrm{CB}_{\mathrm{w}^*}(\mathrm{VN}(G))$ as the domain space of P_G^∞.

Proof : See [HDR6, Theorem 6.16] for the highly involved proof, building in part on Caspers, Parcet, Perrin and Ricard's work [CPPR] on noncommutative de Leeuw theorems. ∎

Remark 12.15 *Suppose $1 < p < \infty$. The amenability assumption has only been used once in the proof because of the use of Jodeit's Theorem [CPPR, Theorem B.1] (in fact, an extension of it from [HDR6, Theorem 6.15]). It would be interesting to find a non-amenable version of Jodeit's Theorem.*

Remark 12.16 *We ignore if the condition* (12.13) *can be removed.*

There is the following variant of Theorem 12.14 for pro-discrete groups. Recall that a locally compact group is pro-discrete if and only if it admits a basis of neighborhoods of the identity e_G consisting of open compact normal subgroups. We refer to [Wil], [Sau, Lemma 1.3] for more information on this class of groups.

Theorem 12.17 *[HDR6, Theorem 6.39] Let $G = \varprojlim G_j$ be a second countable pro-discrete locally compact group with respect to an inverse system indexed by $\mathbb{N}$. Suppose $1 \leqslant p \leqslant \infty$. Assume that G is amenable if $1 < p < \infty$. Then there exists a contractive map*

$$P_G^p \colon \mathrm{CB}(L^p(\mathrm{VN}(G))) \to \mathfrak{M}^{p,\mathrm{cb}}(G)$$

with the properties:

1. *If T is completely positive, then $P_G^p(T)$ is also completely positive.*
2. *If $T = M_\psi$ is a Fourier multiplier on $L^p(\mathrm{VN}(G))$ with bounded measurable symbol $\psi \colon G \to \mathbb{C}$ then $P_G^p(M_\psi) = M_\psi$.*

Moreover, P_G^p has the following compatibility: if $T \in \mathrm{CB}(L^p(\mathrm{VN}(G))) \cap \mathrm{CB}(L^q(\mathrm{VN}(G)))$ for some $1 \leqslant p, q \leqslant \infty$, then $P_G^p(T)$ being twice defined as an element of $\mathfrak{M}^{p,\mathrm{cb}}(G)$ and $\mathfrak{M}^{q,\mathrm{cb}}(G)$ coincides on $L^p(\mathrm{VN}(G)) \cap L^q(\mathrm{VN}(G))$. Note that in the case $p = \infty$, one has to take $\mathrm{CB}_{\mathrm{w}^*}(\mathrm{VN}(G))$ as the domain space of P_G^∞.

Proof : See [HDR6, Theorem 6.39] for the involved proof. ∎

We turn to some consequences of Theorems 12.14 and 12.17. The following is a variant of Proposition 12.9.

Proposition 12.18 *Let G be an amenable second countable unimodular locally compact group which is* ALSS *satisfying the assumption* (12.13). *Suppose $1 \leqslant p \leqslant \infty$. Then a measurable function $\phi \colon G \to \mathbb{C}$ induces a decomposable Fourier multiplier on $L^p(\mathrm{VN}(G))$ if and only if it induces a (completely) bounded Fourier multiplier on $\mathrm{VN}(G)$. In this case, we have*

$$c\,\|M_\phi\|_{\mathrm{VN}(G)\to\mathrm{VN}(G)} \leqslant \|M_\phi\|_{\mathrm{dec},L^p(\mathrm{VN}(G))\to L^p(\mathrm{VN}(G))} \leqslant \|M_\phi\|_{\mathrm{VN}(G)\to\mathrm{VN}(G)}\,. \tag{12.14}$$

Proof : See [HDR6, Theorem 6.47] for the involved proof. ∎

Similarly, we obtain the following result, see [HDR6, Theorem 6.49]:

Proposition 12.19 *Let G be a second countable amenable pro-discrete locally compact group. Suppose $1 \leqslant p \leqslant \infty$. Then a function $\phi \colon G \to \mathbb{C}$ induces a decomposable Fourier multiplier $M_\phi \colon L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ if and only if it induces a (completely) bounded Fourier multiplier on $M_\phi \colon \mathrm{VN}(G) \to \mathrm{VN}(G)$. In this case, we have*

$$\|M_\phi\|_{\mathrm{dec},L^p(\mathrm{VN}(G))\to L^p(\mathrm{VN}(G))} = \|M_\phi\|_{\mathrm{cb},\mathrm{VN}(G)\to\mathrm{VN}(G)} = \|M_\phi\|_{\mathrm{VN}(G)\to\mathrm{VN}(G)}\,.$$

Remark 12.20 *In both situations of Propositions 12.18 and 12.19, a function $\phi \colon G \to \mathbb{C}$ which induces a decomposable Fourier multiplier $M_\phi \colon L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))$ is equal to a continuous function almost everywhere, see e.g. [Haa3, Corollary 3.3].*

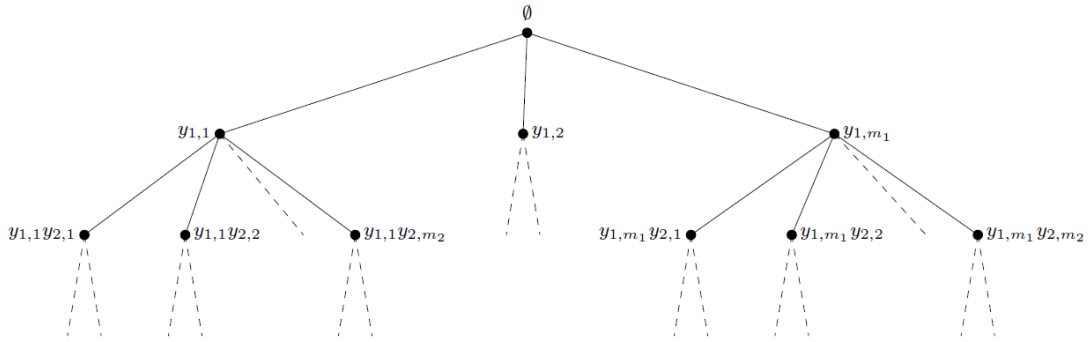
We will describe a concrete non-abelian group in which Theorem 12.14 applies.

Groups acting on locally finite trees We give now some examples of compact non-discrete ALSS groups acting on locally finite trees for which Theorem 12.14 yields a bounded map $P_G^p \colon \mathrm{CB}(L^p(\mathrm{VN}(G))) \to \mathfrak{M}^{p,\mathrm{cb}}(G)$ with sharp norm, i.e. with a norm equal to one.

Let $(m_j)_{j\geqslant 1}$ be a sequence of integers with $m_j \geqslant 2$. Let $\overline{Y} = (Y_j)_{j\geqslant 1}$ be a sequence of alphabets with $|Y_j| = m_j$ and $Y_j = \{y_{j,1}, \dots, y_{j,m_j}\}$. If $n \geqslant 0$, a word of length n over $\overline{Y}$ is a sequence of letters of the form $w = w_1 w_2 \dots w_n$ with $w_j \in Y_j$ for all j. The unique word of length 0, the empty word, is denoted by $\emptyset$. The set of words of length n is called the nth level.

Now we introduce the prefix relation $\leqslant$ on the set of all words over $\overline{Y}$. Namely, we let $w \leqslant z$ if w is an initial segment of the sequence z, i.e. if $w = w_1 \dots w_n$, $z = z_1 \dots z_k$ with $n \leqslant k$ and $w_j = z_j$ for all $j \in \{1, \dots, n\}$. This relation is a partial order and the partially ordered set $\mathcal{T}$ of words over $\overline{Y}$ is called the spherically homogeneous tree over $\overline{Y}$. We refer to [BGS] and [Gri1] for more information.

Let us give now the graph-theoretical interpretation of $\mathcal{T}$. Every word over $\overline{Y}$ represents a vertex in a rooted tree. Namely, the empty word $\emptyset$ represents the root, the m_1 one-letter words $y_{1,1}, \dots, y_{1,m_1}$ represent the m_1 children of the root, the m_2 two-letter words $y_{1,1}y_{2,1}, \dots, y_{1,1}y_{2,m_2}$ represent the m_2 children of the vertex $y_{1,1}$, etc.



An automorphism of $\mathcal{T}$ is a bijection of $\mathcal{T}$ which preserves the prefix relation. From the graph-theoretical point of view, an automorphism is a bijection which preserves edge incidence and the distinguished root vertex $\emptyset$. We denote by $\mathrm{Aut}(\mathcal{T})$ the group of automorphisms of $\mathcal{T}$ and if $j \geqslant 0$ by $\mathrm{Aut}_{[j]}(\mathcal{T})$ the subgroup of automorphisms whose vertex permutations at level j and below[17] are trivial.

We equip $\mathcal{T}$ with the discrete topology and $\mathrm{Aut}(\mathcal{T})$ with the topology of pointwise convergence. By [Gri1, page 133], the sequence $(\mathrm{Aut}_{[j]}(\mathcal{T}))_{j\geqslant 0}$ of finite groups and the canonical inclusions $\psi_{ij} \colon \mathrm{Aut}_{[j]}(\mathcal{T}) \to \mathrm{Aut}_{[i]}(\mathcal{T})$ where $j \geqslant i \geqslant 0$ define an inverse system and we have an isomorphism

$$\mathrm{Aut}(\mathcal{T}) = \varprojlim \mathrm{Aut}_{[j]}(\mathcal{T}). \tag{12.15}$$

In particular, $\mathrm{Aut}(\mathcal{T})$ is a profinite group, hence compact and totally disconnected by [Wil, Corollary 1.2.4].

If $j \geqslant 0$, we denote by $\mathrm{St}(j)$ the jth level stabilizer consisting of automorphisms of $\mathcal{T}$ which fix all the vertices on the level j (and of course on the levels $0, 1, \dots, j-1$). Then $\mathrm{St}(j)$ is a normal subgroup of $\mathrm{Aut}(\mathcal{T})$ which is open if $j \geqslant 1$. By [BGS, page 20], for any $j \geqslant 0$, we have an isomorphism

$$\mathrm{Aut}(\mathcal{T}) = \mathrm{St}(j) \rtimes \mathrm{Aut}_{[j]}(\mathcal{T}). \tag{12.16}$$

17. The action is trivial on the levels $j, j+1, j+2, \dots$.

Proposition 12.21 *The compact group* $\mathrm{Aut}(\mathcal{T})$ *is second countable and* ALSS *with respect to the sequence* $(\mathrm{Aut}_{[j]}(\mathcal{T}))_{j\geqslant 1}$ *of finite lattice subgroups and to the sequence* $(\mathrm{St}(j))_{j\geqslant 1}$ *of symmetric fundamental domains. Moreover,* (12.13) *holds with* $c = 1$*. More precisely, for any integer* $j \in \mathbb{N}$ *and any* $\gamma \in \mathrm{Aut}_{[j]}(\mathcal{T})$*, we have*

$$\frac{1}{\mu_G(\mathrm{St}(j))} \int_{\mathrm{Aut}(\mathcal{T})} \frac{\mu_G(\mathrm{St}(j) \cap \gamma \mathrm{St}(j)s)^2}{\mu_G(\mathrm{St}(j))^2} \, \mathrm{d}\mu(s) = 1. \tag{12.17}$$

Consequently, Theorem 12.14 applies.

Proof : See [HDR6, Proposition 6.28]. ∎

We refer to [HDR6, Subsection 6.5] for other examples of groups to which Theorem 12.14 applies: direct products of ALSS groups, under additional hypotheses also semidirect products, locally compact abelian groups of Lie type and Heisenberg groups.

At the end of this section, we mention that we have continued the investigation of decomposable operators in [HDR6]. In place of Fourier multipliers on $L^p(\mathrm{VN}(G))$ (group / symbol on frequency side), one can also consider convolutors on $L^p(G)$ (translation invariant operators, and group on spatial side). We equally obtained a complementation as in Theorem 12.7 and 12.14 building on preceding work by Arendt-Voigt [ArV] and Derighetti [Der].

Note that the Hilbert transform on $\ell^p_{\mathbb{Z}}$ and on $L^p(\mathbb{R})$ provides an example of a non-decomposable operator. In fact it is not even approximable by decomposable operators in $B(L^p)$ norm [ArV], what we called strongly non-decomposable in [HDR6]. Then we generalised this and found strongly non-decomposable Fourier multipliers on arbitrary infinite locally compact abelian groups. We also provide an example of a strongly non-decomposable Schur multiplier and a strongly non-decomposable operator on an arbitrary infinite dimensional approximately finite-dimensional von Neumann algebra, when $p \neq 2$. These existence results rely on (previously known) structure theorems of locally compact abelian groups and of $L^p(M)$ spaces where M is approximately finite-dimensional. Finally, we were able to connect the decomposability of Fourier multipliers M_ϕ with the validity of the so-called noncommutative Matsaev inequality: for any complex polynomial P and S the right shift on ℓ^p,

$$\|P(M_\phi)\|_{\mathrm{cb}, L^p(\mathrm{VN}(G)) \to L^p(\mathrm{VN}(G))} \leqslant \|P(S)\|_{\mathrm{cb}, \ell^p \to \ell^p} .$$

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Bibliography

[HDR1] C. Kriegler. Hörmander Functional Calculus for Poisson Estimates, Int. Equ. Op. Theory, 80(3):379–413, 2014. 14, 22, 23, 24, 25, 26, 27

[HDR2] L. Deléaval and C. Kriegler. Dunkl spectral multipliers with values in UMD lattices, Journal of Functional Analysis, 272(5):2132–2175, 2017. 14, 22, 28, 29, 30, 32

[HDR3] C. Arhancet and C. Kriegler. Complementation of the subspace of radial multipliers in the space of Fourier multipliers on $\mathbb{R}^n$, Archiv der Mathematik, 112(1):93–100, 2018. 14, 106, 114

[HDR4] L. Deléaval and C. Kriegler. Dimension free bounds for the vector-valued Hardy-Littlewood maximal operator, Revista Matemática Iberoamericana, 35(1):101–123, 2019. 14, 32, 33, 34, 36, 38, 39

[HDR5] L. Deléaval, M. Kemppainen and C. Kriegler. Hörmander functional calculus on UMD lattice valued L^p spaces under generalised Gaussian estimates, Journal d'Analyse Mathématique, to appear. 14, 19, 32, 40, 41, 43, 44, 46, 47, 48, 49

[HDR6] C. Arhancet and C. Kriegler. Decomposable Schur multipliers and Fourier multipliers on noncommutative L^p-spaces. Submitted. Preprint on arXiv.org. 13, 14, 106, 108, 109, 110, 111, 114, 116, 117, 119

[HDR7] K. Domelevo, C. Kriegler and S. Petermichl. Diffusion semigroups on weighted L^2 spaces. Submitted. Preprint on arXiv.org. 14, 65, 67, 68, 69, 70, 71, 72, 73

[HDR8] C. Arhancet and C. Kriegler. Riesz transforms, Hodge-Dirac operators and functional calculus for multipliers I. Submitted. Preprint on arXiv.org. 14, 74, 77, 78, 81, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 101, 102, 104, 105

[HDR9] L. Deléaval and C. Kriegler. Maximal and q-variational Hörmander Functional Calculus. Submitted. Preprint here. 14, 19, 54, 55, 56, 57, 58, 59, 60, 61, 62, 64, 84

[AbA] Y. A. Abramovich and C. D. Aliprantis. An invitation to operator theory. Graduate Studies in Mathematics, 50. American Mathematical Society, Providence, RI, 2002. 106

[ACMM] J. Ahrens, M. G. Cowling, A. Martini and D. Müller. Quaternionic spherical harmonics and a sharp multiplier theorem on quaternionic spheres. Math. Z. 294 (2020), no. 3-4, 1659–1686. 9

[AcS] M. A. Akcoglu and L. Sucheston. Dilations of positive contractions on L^p spaces. Canad. Math. Bull. 20 (1977), no. 3, 285–292. 13

[Ald] D. Aldous. Unconditional bases and martingales in $L_p(F)$. Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 1, 117–123. 21

[AlP1] A. B. Alexandrov and V. Peller. Hankel and Toeplitz-Schur multipliers. Math. Ann. 324 (2002), no. 2, 277–327. 81

[AlP2] A. B. Alexandrov and V. Peller. Schur multipliers of SchattenâĂŞvon Neumann classes S_p. J. Funct. Anal. 279 (2020), no. 8, 108683. 81

[Ale] G. Alexopoulos. Spectral multipliers on Lie groups of polynomial growth. Proc. Amer. Math. Soc. 120 (1994), no. 3, 973–979. 9, 22

[Alf] E. M. Alfsen. Compact convex sets and boundary integrals. Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 57. Springer-Verlag, New York-Heidelberg, 1971. 98, 99

[AlS] E. M. Alfsen and F. W. Shultz. Geometry of state spaces of operator algebras. Mathematics: Theory & Applications. Birkhauser Boston, Inc., Boston, MA, 2003. 98

[ALV] A. Amenta, E. Lorist and M. Veraar. Rescaled extrapolation for vector-valued functions. Publ. Mat. 63 (2019), no. 1, 155–182. 49, 50

[ABSDH] J-P. Anker, N. Ben Salem, J. Dziubański and N. Hamda. The Hardy space H^1 in the rational Dunkl setting. Constr. Approx. 42 (2015), no. 1, 93–128. 28

[Are] W. Arendt. On the o-spectrum of regular operators and the spectrum of measures. Math. Z. 178 (1981), no. 2, 271–287. 106

[ArB1] W. Arendt and S. Bu. The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. Math. Z. 240 (2002), no. 2, 311–343. 43

[ArB2] W. Arendt and S. Bu. Operator-valued Fourier multipliers on periodic Besov spaces and applications. Proc. Edinb. Math. Soc. (2) 47 (2004), no. 1, 15–33. 43

[ArV] W. Arendt and J. Voigt. Approximation of multipliers by regular operators. Indag. Math. (N.S.) 2 (1991), no. 2, 159–170. 106, 119

[Arh1] C. Arhancet. On Matsaev's conjecture for contractions on noncommutative L^p-spaces. J. Operator Theory 69 (2013), no. 2, 387–421 77, 81, 84, 102

[Arh2] C. Arhancet. On a conjecture of Pisier on the analyticity of semigroups. Semigroup Forum 91 (2015), no. 2, 450–462. 81

[Arh3] C. Arhancet. Dilations of semigroups on von Neumann algebras and noncommutative L^p-spaces. J. Funct. Anal. 276 (2019), no. 7, 2279–2314. 10, 12

[Arh4] C. Arhancet. Dilations of markovian semigroups of Fourier multipliers on locally compact groups. Proc. Amer. Math. Soc. 148 (2020), no. 6, 2551–2563. 10, 12, 95

[Arh5] C. Arhancet. Dilations of markovian semigroups of measurable Schur multipliers. Preprint, arxiv:1910.14434 10, 12

[AFLM] C. Arhancet, S. Fackler and C. Le Merdy. Isometric dilations and H^∞ calculus for bounded analytic semigroups and Ritt operators. Trans. Amer. Math. Soc. 369 (2017), no. 10, 6899–6933. 10, 65, 66

[ArhM] C. Arhancet and C. Le Merdy. Dilation of Ritt operators on L^p-spaces. Israel J. Math. 201 (2014), no. 1, 373–414. 10

[ArnM] L. Arnold and C. Le Merdy. New counterexamples on Ritt operators, sectorial operators and R-boundedness. Bull. Aust. Math. Soc. 100 (2019), no. 3, 498–506. 10

[ArrM] O. Arrigoni and C. Le Merdy. H^∞-functional calculus for commuting families of Ritt operators and sectorial operators. Oper. Matrices 13 (2019), no. 4, 1055–1090. 10

[AHLLMT] P. Auscher, S. Hofmann, M. Lacey, J. Lewis, A. McIntosh and P. Tchamitchian. The solution of Kato's conjectures. C. R. Acad. Sci. Paris Sér. I Math. 332 (2001), no. 7, 601–606. 10, 65

[AHLMT] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and P. Tchamitchian. The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$. Ann. of Math. (2) 156 (2002), no. 2, 633–654. 10, 75

[AuSt1] P. Auscher and S. Stahlhut. Remarks on functional calculus for perturbed first-order Dirac operators. Operator theory in harmonic and non-commutative analysis, 31–43, Oper. Theory Adv. Appl., 240, Birkhäuser/Springer, Cham, 2014. 10

[AuSt2] P. Auscher and S. Stahlhut. Functional calculus for first order systems of Dirac type and boundary value problems. Mém. Soc. Math. Fr. (N.S.) No. 144 (2016), vii+164 pp. 10

[AuTc] P. Auscher and P. Tchamitchian. Square root problem for divergence operators and related topics. Astérisque No. 249 (1998), viii+172 pp. 10, 65

[AKM] A. Axelsson, S. Keith and A. McIntosh. Quadratic estimates and functional calculi of perturbed Dirac operators. Invent. Math. 163 (2006), no. 3, 455–497. 10, 75, 76

[BFG] H. Bahouri, C. Fermanian-Kammerer, I. Gallagher. Dispersive estimates for the Schrödinger operator on step-2 stratified Lie groups. Anal. PDE 9 (2016), no. 3, 545–574. 53

[BaW] R. Banuelos and G. Wang. Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms. Duke Math. J. 80 (1995), no. 3, 57–600. 75

[BGS] L. Bartholdi, R. I. Grigorchuk and Z. Sunik. Branch groups. Handbook of algebra, Vol. 3, 989–1112, Handb. Algebr., 3, Elsevier/North-Holland, Amsterdam, 2003. 118

[BHM] C. Batty, M. Haase and J. Mubeen. The holomorphic functional calculus approach to operator semigroups. Acta Sci. Math. (Szeged) 79 (2013), no. 1-2, 289–323. 10

[BMV] C. Batty, J. Mubeen and I. Vörös. Bounded H^∞-calculus for strip-type operators. Integral Equations Operator Theory 72 (2012), no. 2, 159–178. 10

[BeGo] M. Beceanu and M. Goldberg. Strichartz estimates and maximal operators for the wave equation in $\mathbb{R}^3$. J. Funct. Anal. 266 (2014), no. 3, 1476–1510. 64

[BHV] B. Bekka, P. de la Harpe, Pierre and A. Valette. Kazhdan's property (T). New Mathematical Monographs, 11. Cambridge University Press, Cambridge, 2008. 101, 102, 115, 116

[BMR] J. V. Bellissard, M. Marcolli and K. Reihani. Dynamical Systems on Spectral Metric Spaces. Preprint, arXiv:1008.4617. 98, 100

[Ben] G. Bennett. Schur multipliers. Duke Math. J. 44 (1977), no. 3, 603–639. 81

[BeL] J. Bergh, J. Löfström. Interpolation spaces. An Introduction. Springer-Verlag, Berlin, Heidelberg, New York, 1976. 108

[BGM] E. Berkson, T. A. Gillespie and P. S. Muhly. Generalized analyticity in UMD spaces. Ark. Mat. 27 (1989), no. 1, 1–14. 91

[BCRM] J. Betancor, A. Castro and L. Rodríguez-Mesa. Square functions and spectral multipliers for Bessel operators in UMD spaces. Banach J. Math. Anal. 10 (2016), no. 2, 338–384. 51, 54

[BSS] O. Bilyi, E. Schrohe and J. Seiler. H_∞-calculus for hypoelliptic pseudodifferential operators. Proc. Amer. Math. Soc. 138 (2010), no. 5, 1645–1656. 9

[Bla] B. Blackadar. Operator algebras. Theory of C^*-algebras and von Neumann algebras. Encyclopaedia of Mathematical Sciences, 122. Operator Algebras and Noncommutative Geometry, III. Springer-Verlag, Berlin, 2006. 110

[BLM] D. Blecher and C. Le Merdy. Operator algebras and their modules-an operator space approach. London Mathematical Society Monographs. New Series, 30. Oxford Science Publications. The Clarendon Press, Oxford University Press, Oxford, 2004. 108

[BlDo] G. Blower and I. Doust. A maximal theorem for holomorphic semigroups. Q. J. Math. 56 (2005), no. 1, 21–30. 9

[BlDT] G. Blower, I. Doust and R. J. Taggart. A maximal theorem for holomorphic semigroups on vector-valued spaces. The AMSI-ANU Workshop on Spectral Theory and Harmonic Analysis, 105–114, Proc. Centre Math. Appl. Austral. Nat. Univ., 44, Austral. Nat. Univ., Canberra, 2010. 9, 12

[Bl] S. Blunck. A Hörmander type spectral multiplier theorem for operators without heat kernel. Ann. Scuo. Norm. Sup. Pisa (5) Vol.II (2003), 449–459. 22, 49, 53

[Bl07] S. Blunck. Generalized Gaussian estimates and Riesz means of Schrödinger groups. J. Aust. Math. Soc. 82 (2007), no. 2, 149–162. 47

[BK02] S. Blunck and P. Kunstmann. Weighted norm estimates and maximal regularity. Adv. Differential Equations 7 (2002), no. 12, 1513–1532. 47

[BK03] S. Blunck and P. Kunstmann. Calderón-Zygmund theory for non-integral operators and the H^∞ functional calculus. Rev. Mat. Iberoamericana 19 (2003), no. 3, 919–942. 65

[BK05] S. Blunck and P. Kunstmann. Generalized Gaussian estimates and the Legendre transform. J. Operator Theory 53 (2005), no. 2, 351–365. 47

[BoCl] A. Bonami and J.-L. Clerc. Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques. Trans. Amer. Math. Soc. 183 (1973), 223–263. 28

[Bou83] J. Bourgain. Some remarks on Banach spaces in which martingale difference sequences are unconditional. Ark. Mat. (1983), no. 2, 163–168. 21

[Bou84] J. Bourgain. Extension of a result of Benedek, Calderón and Panzone. Ark. Mat. 22 (1984), no. 1, 91–95. 40

[Bou85] J. Bourgain. On dimension free maximal inequalities for convex symmetric bodies in $\mathbb{R}^n$. Geometrical aspects of functional analysis (1985/86), 168–176, Lecture Notes in Math. 1267, Springer, Berlin, 1987. 33

[Bou] J. Bourgain. Vector valued singular integrals and the H^1-BMO duality. Probability theory and harmonic analysis (Cleveland, Ohio, 1983), 1–19, Monogr. Textbooks Pure Appl. Math., 98 (1986) Dekker, New York. 43

[Bou86] J. Bourgain. On high dimensional maximal functions associated to convex bodies. Amer. J. Math. 108 (1986), no. 6, 1467–1476. 33

[Bou86bis] J. Bourgain. Averages in the plane over convex curves and maximal operators. J. Analyse Math. 47 (1986), 69–85. 34

[Bou86ter] J. Bourgain. On the L^p-bounds for maximal functions associated to convex bodies in $\mathbb{R}^n$. Israel J. Math. 54 (1986), no. 3, 257–265. 33

[Bou14] J. Bourgain. On the Hardy-Littlewood maximal function for the cube. Israel J. Math. 203, (2014), no. 1, 275–293. 33

[BMSW1] J. Bourgain, M. Mirek, E. M. Stein and B. Wróbel. On dimension-free variational inequalities for averaging operators in $\mathbb{R}^d$. Geom. Funct. Anal. 28 (2018), no. 1, 58–99. 33, 39, 60, 62

[BMSW2] J. Bourgain, M. Mirek, E. M. Stein and B. Wróbel. Dimension-free estimates for discrete Hardy-Littlewood averaging operators over the cubes in $\mathbb{Z}^d$. Amer. J. Math. 141 (2019), no. 4, 857–905. 33, 39

[BKS] M. Bożejko, B. Kümmerer and R. Speicher. q-Gaussian processes: non-commutative and classical aspects. Comm. Math. Phys. 185 (1997), no. 1, 129–154. 78

[BoS] M. Bożejko and R. Speicher. Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. Math. Ann. 300 (1994), no. 1, 97–120. 78

[Bur81] D. L. Burkholder. A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. Ann. Probab. 9 (1981), no. 6, 997–1011. 21, 41

[Bur83] D. L. Burkholder. A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions. Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981), 270–286, Wadsworth Math. Ser., Wadsworth, Belmont, CA, 1983. 21

[Bur01] D. L. Burkholder. Martingales and singular integrals in Banach spaces. Handbook of the geometry of Banach spaces, Vol. I, 233–269, North-Holland, Amsterdam, 2001. 21, 41

[CaZ] A. P. Calderon and A. Zygmund. On the existence of certain singular integrals. Acta Math. 88 (1952), 85–139. 75

[Car] A. Carbery. An almost-orthogonality principle with applications to maximal functions associated to convex bodies. Bull. Amer. Math. Soc. (N.S.) 14 (1986), no. 2, 269–273. 33

[CaDr1] A. Carbonaro and O. Dragičević. Functional calculus for generators of symmetric contraction semigroups. Duke Math. J. 166 (2017), no. 5, 937–974. 9, 11, 66, 69, 71

[CaDr2] A. Carbonaro and O. Dragičević. Bounded holomorphic functional calculus for nonsymmetric Ornstein-Uhlenbeck operators. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 19 (2019), no. 4, 1497–1533. 11

[CMM] A. Carbonaro, G. Mauceri and S. Meda. Spectral multipliers for Laplacians associated with some Dirichlet forms. Proc. Edinb. Math. Soc. (2) 51 (2008), no. 3, 581–607. 10

[CGRS] A. L. Carey, V. Gayral, A. Rennie and F. A. Sukochev. Index theory for locally compact noncommutative geometries. Mem. Amer. Math. Soc. 231 (2014), no. 1085. 102

[CaCoOu] G. Carron, T. Coulhon and E.-M. Ouhabaz. Gaussian estimates and L^p-boundedness of Riesz means. J. Evol. Equ. 2 (2002) 299–317. 23, 45, 46

[CCM] V. Casarino, P. Ciatti and A. Martini. From refined estimates for spherical harmonics to a sharp multiplier theorem on the Grushin sphere. Adv. Math. 350 (2019), 816–859. 10

[CCMS] V. Casarino, M. Cowling, A. Martini and A. Sikora. Spectral multipliers for the Kohn Laplacian on forms on the sphere in $\mathbb{C}^n$. 22

[CaS] M. Caspers and M. de la Salle. Schur and Fourier multipliers of an amenable group acting on non-commutative L^p-spaces. Trans. Amer. Math. Soc. 367 (2015), no. 10, 6997–7013. 81

[CPPR] M. Caspers, J. Parcet, M. Perrin and É Ricard. Noncommutative de Leeuw theorems. Forum Math. Sigma 3 (2015), e21, 59 pp. 116

[CDLWY] P. Chen, X. T. Duong, J. Li, L. A. Ward and L. Yan. Marcinkiewicz-type spectral multipliers on Hardy and Lebesgue spaces on product spaces of homogeneous type. J. Fourier Anal. Appl. 23 (2017), no. 1, 21–64. 10

[CDY] P. Chen, X. T. Duong and L. Yan. L^p-bounds for Stein's square functions associated to operators and applications to spectral multipliers. J. Math. Soc. Japan 65 (2013), no. 2, 389–409. 22

[CO] P. Chen and E. M. Ouhabaz. Weighted restriction type estimates for Grushin operators and application to spectral multipliers and Bochner-Riesz summability. Math. Z. 282 (2016), no. 3-4, 663–678. 22

[COSY] P. Chen, E. M. Ouhabaz, A. Sikora and L. Yan. Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner-Riesz means. J. Anal. Math. 129 (2016), 219–283. 10, 22, 49

[ChHa] X. Chen and A. Hassell. Resolvent and spectral measure on non-trapping asymptotically hyperbolic manifolds II: Spectral measure, restriction theorem, spectral multipliers. Ann. Inst. Fourier (Grenoble) 68 (2018), no. 3, 1011–1075. 9

[Ch1] M. D. Choi. Completely positive linear maps on complex matrices. Linear Algebra Appl. 10 (1975), 285–290. 110

[Choi] W. Choi. Maximal functions for multipliers on stratified groups. Math. Nachr. 288 (2015), no. 10, 1098–1121. 12, 55, 60, 65

[Chr] M. Christ. L^p bounds for spectral multipliers on nilpotent groups. Trans. Amer. Math. Soc. 328 (1991), no. 1, 73–81. 22

[CGHS] M. Christ, L. Grafakos, P. Honzík and A. Seeger. Maximal functions associated with Fourier multipliers of Mikhlin-Hörmander type. Math. Z. 249 (2005), no. 1, 223–240. 12, 54, 55, 60, 65

[Cip] F. Cipriani. Noncommutative potential theory: a survey. J. Geom. Phys. 105 (2016), 25–59. 76, 77

[CGIS] F. Cipriani, D. Guido, T. Isola and J.-L. Sauvageot. Spectral triples for the Sierpinski gasket. J. Funct. Anal. 266 (2014), no. 8, 4809–4869. 76, 102

[CiS] F. Cipriani and J.-L. Sauvageot. Derivations as square roots of Dirichlet forms. J. Funct. Anal. 201 (2003), no. 1, 78–120. 76, 77

[CoWe1] R. Coifman and G. Weiss. Analyse harmonique non-commutative sur certains espaces homogènes. (French) Étude de certaines intégrales singulières. Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971. v+160 pp. 20

[CoWe2] R. Coifman and G. Weiss. Some examples of transference methods in harmonic analysis. Symposia Mathematica, Vol. XXII (Convegno sull'Analisi Armonica e Spazi di Funzioni su Gruppi Localmente Compatti, INDAM, Rome, 1976), pp. 33–45. Academic Press, London, 1977. 9, 66

[CMPST1] C. Coine, C. Le Merdy, D. Potapov, F. Sukochev and A. Tomskova. Peller's problem concerning Koplienko-Neidhardt trace formulae: the unitary case. J. Funct. Anal. 271 (2016), no. 7, 1747–1763. 81

[CMPST2] C. Coine, C. Le Merdy, D. Potapov, F. Sukochev and A. Tomskova. Resolution of Peller's problem concerning Koplienko-Neidhardt trace formulae. Proc. Lond. Math. Soc. (3) 113 (2016), no. 2, 113–139. 81

[Con1] A. Connes. Noncommutative geometry. Academic Press, Inc., San Diego, CA, 1994. 97, 98

[Con2] A. Connes. Compact metric spaces, Fredholm modules, and hyperfiniteness. Ergodic Theory Dynam. Systems 9 (1989), no. 2, 207–220. 98

[CSS] S. Coriasco, E. Schrohe and J. Seiler. Bounded H_∞-calculus for differential operators on conic manifolds with boundary. Comm. Partial Differential Equations 32 (2007), no. 1-3, 229–255. 9

[CoD] T. Coulhon and X. T. Duong. Riesz transform and related inequalities on noncompact Riemannian manifolds. Comm. Pure Appl. Math. 56 (2003), no. 12, 1728–1751. 75

[CouSi] T. Coulhon and A. Sikora. Gaussian heat kernel upper bounds via the Phragmén–Lindelöf theorem. Proc. Lond. Math. Soc. (3) 96 (2008), no. 2, 507–544. 45

[Cow] M. Cowling. Harmonic analysis on semigroups. Ann. of Math. (2) 117 (1983), no. 2, 267–283. 2, 8, 9, 66

[CDMY] M. Cowling, I. Doust, A. McIntosh and A. Yagi. Banach space operators with a bounded H^∞ functional calculus. J. Austral. Math. Soc. Ser. A 60 (1996), no. 1, 51–89. 10, 17, 24, 65, 66, 69, 72

[Dae] A. van Daele. Continuous crossed products and type III von Neumann algebras. London Mathematical Society Lecture Note Series, 31. Cambridge University Press, Cambridge-New York, 1978. 83

[DaWa] F. Dai and H. Wang. A transference theorem for the Dunkl transform and its applications. J. Funct. Anal. 258 (2010), no. 12, 4052–4074. 28

[DaXu] F. Dai and Y. Xu. Analysis on h-harmonics and Dunkl transforms. Advanced Courses in Mathematics – CRM Barcelona. viii, 118 p. (2015) Birkhäuser/Springer, Basel. 30

[DaL] E. B. Davies and J. M. Lindsay. Noncommutative symmetric Markov semigroups. Math. Z. 210 (1992), no. 3, 379–411. 77

[DCH] J. De Cannière and U. Haagerup. Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups. Amer. J. Math. 107 (1985), no. 2, 455–500. 113, 114

[Del] L. Deleaval. Two results on the Dunkl maximal operator. Studia Math. 203 (2011), no. 1, 47–68. 27

[DGM] L. Deleaval, O. Guédon and B. Maurey. Dimension free bounds for the Hardy-Littlewood maximal operator associated to convex sets. Ann. Fac. Sci. Toulouse Math. (6) 27 (2018), no. 1, 1–198. 11, 33, 37, 39

[DeGu] C. Demeter and S. Guo. Schrödinger maximal function estimates via the pseudo-conformal transformation. Preprint on arxiv:1608.07640 64

[DDHPV] R. Denk, G. Dore, M. Hieber, J. Prüss and A. Venni. New thoughts on old results of R. T. Seeley. Math. Ann. 328 (2004), no. 4, 545–583. 65

[DSS] R. Denk, J. Saal and J. Seiler. Bounded H_∞-calculus for pseudo-differential Douglis-Nirenberg systems of mild regularity. Math. Nachr. 282 (2009), no. 3, 386–407. 9

[Der] A. Derighetti. Relations entre les convoluteurs d'un groupe localement compact et ceux d'un sous-groupe fermé. (French) [Relations between the multipliers of a locally compact group and those of a closed subgroup]. Bull. Sci. Math. (2) 106 (1982), no. 1, 69–84. 119

[DJT] J. Diestel, H. Jarchow and A. Tonge. Absolutely summing operators. Cambridge Studies in Advanced Mathematics, 43. Cambridge University Press, 1995. 106

[Dix] J. Dixmier. Von Neumann algebras. With a preface by E. C. Lance. Translated from the second French edition by F. Jellett North-Holland Mathematical Library, 27. North-Holland Publishing Co., Amsterdam-New York, 1981. 110

[DoPe] K. Domelevo and S. Petermichl. Differential Subordination under change of law. preprint on https://arXiv.org/abs/1604.01606 71

[Dou] I. Doust. Norms of 0-1 matrices in C^p. Geometric analysis and applications (Canberra, 2000), 50–55, Proc. Centre Math. Appl. Austral. Nat. Univ., 39, Austral. Nat. Univ., Canberra, 2001. 81

[DoG] I. Doust and T. A. Gillespie. Schur multiplier projections on the von Neumann-Schatten classes. J. Operator Theory 53 (2005), no. 2, 251–272. 81

[Dun] C. F. Dunkl. Differential-difference operators associated to reflection groups. Trans. Amer. Math. Soc., 311(1):167–183, 1989. 27

[DuXu] C. F. Dunkl and Y. Xu. Orthogonal polynomials of several variables, volume 81 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, Cambridge, 2001. 27

[Duo] X. T. Duong. H^∞ functional calculus of second order elliptic partial differential operators on L^p spaces. Miniconference on Operators in Analysis (Sydney, 1989), 91–102, Proc. Centre Math. Anal. Austral. Nat. Univ., 24, Austral. Nat. Univ., Canberra, 1990. 22, 66

[DuMc] X. T. Duong and A. McIntosh. Singular integral operators with non-smooth kernels on irregular domains. Rev. Mat. Iberoam. 15(2):233–265, 1999. 25

[DuOS] X. T. Duong, E. M. Ouhabaz and A. Sikora. Plancherel-type estimates and sharp spectral multipliers. J. Funct. Anal. 196(2):443–485, 2002. 8, 9, 22, 23, 50, 51, 52, 65

[DuRo] X. T. Duong and D. W. Robinson. Semigroup kernels, Poisson bounds, and holomorphic functional calculus. J. Funct. Anal. 142(1):89–128, 1996. 22, 24, 48

[DSY] X. T. Duong, A. Sikora and L. Yan. Weighted norm inequalities, Gaussian bounds and sharp spectral multipliers. Journal of Functional Analysis 260 (2011), 1106–1131. 65, 66, 68

[DPW] J. Dziubański, M. Preisner and B. Wróbel. Multivariate Hörmander-type multiplier theorem for the Hankel transform. J. Fourier Anal. Appl. 19 (2013), no. 2, 417–437. 9

[EcI] M. Eckstein and B. Iochum. Spectral action in noncommutative geometry. SpringerBriefs in Mathematical Physics, 27. Springer, Cham, 2018. 104

[ER] E. Effros and Z.-J. Ruan. Operator spaces. Oxford University Press, 2000. 107, 108

[EHDT1] M. Egert, R. Haller-Dintelmann and P. Tolksdorf. The Kato square root problem for mixed boundary conditions. J. Funct. Anal. 267 (2014), no. 5, 1419–1461. 10

[EHDT2] M. Egert, R. Haller-Dintelmann and P. Tolksdorf. The Kato square root problem follows from an extrapolation property of the Laplacian. Publ. Mat. 60 (2016), no. 2, 451–483. 10

[EnN] K.-J. Engel and R. Nagel. One-parameter semigroups for linear evolution equations. Graduate Texts in Mathematics, 194. Springer-Verlag, New York, 2000. 15, 16

[ErGr] M. Erdoğan and W. Green. Dispersive estimates for the Schrödinger equation for $C^{\frac{n-3}{2}}$ potentials in odd dimensions. Int. Math. Res. Not. IMRN (2010), no. 13, 2532–2565. 52

[FeSt] C. Fefferman and E. M. Stein. Some maximal inequalities. Amer. J. Math. 93 (1971), 107–115. 11, 33, 40

[Fen1] G. Fendler. Dilations of one parameter semigroups of positive contractions on L^p spaces. Canad. J. Math. 49 (1997), no. 4, 736–748. 10

[Fen2] G. Fendler. On dilations and transference for continuous one-parameter semigroups of positive contractions on L^p-spaces. Ann. Univ. Sarav. Ser. Math. 9 (1998), no. 1, iv+97 pp. 10, 49, 66

[FMP] D. Frey, A. McIntosh and P. Portal. Conical square function estimates and functional calculi for perturbed Hodge-Dirac operators in L^p. J. Anal. Math. 134 (2018), no. 2, 399–453. 10

[FrWe] A. Fröhlich and L. Weis. H^∞ calculus and dilations. Bull. Soc. Math. France 134 (2006), no. 4, 487–508. 10

[Gaa] S. A. Gaal. Linear analysis and representation theory. Die Grundlehren der mathematischen Wissenschaften, Band 198. Springer-Verlag, New York-Heidelberg, 1973. 115

[GMT1] J. García-Cuerva, R. Macías and J. L. Torrea. The Hardy-Littlewood property of Banach lattices. Israel J. Math. 83 (1993), no. 1-2, 177–201. 40

[GMT2] J. García-Cuerva, R. Macías and J. L. Torrea. Maximal operators and B.M.O. for Banach lattices. Proc. Edinburgh Math. Soc. (2) 41 (1998), no. 3, 585–609. 40

[GCMMST] J. García-Cuerva, G. Mauceri, S. Meda, P. Sjögren and J. L. Torrea. Functional calculus for the Ornstein-Uhlenbeck operator. J. Funct. Anal. 183 (2001), no. 2, 413–450. 65, 66

[GCRdF] J. Garciá-Cuerva and J. L Rubio de Francia. Weighted Norm Inequalities and Related Topics. North-Holland Mathematics Studies, 116. North-Holland Publishing Co., Amsterdam, 1985. 33, 68

[GGISV] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker and J. C. Varilly. Moyal planes are spectral triples. Comm. Math. Phys. 246 (2004), no. 3, 569–623. 103

[GHT] M. Geissert, H. Heck and C. Trunk. $\mathcal{H}^\infty$-calculus for a system of Laplace operators with mixed order boundary conditions. Discrete Contin. Dyn. Syst. Ser. S 6 (2013), no. 5, 1259–1275. 9

[Gig] Y. Giga. Domains of fractional powers of the Stokes operator in L_r spaces. Arch. Rational Mech. Anal. 89 (1985), no. 3, 251–265. 65

[GiGr] H. Gimperlein and G. Grubb. Heat kernel estimates for pseudodifferential operators, fractional Laplacians and Dirichlet-to-Neumann operators. J. Evol Equ. (2014), 14:49–83. 27

[GiWe] M. Girardi and L. Weis. Operator-valued Fourier multiplier theorems on $L^p(X)$ and geometry of Banach spaces. J. Funct. Anal. 204 (2003), no. 2, 320–354. 10, 43, 49, 62

[GoS] M. Goldberg and W. Schlag. Dispersive estimates for Schrödinger operators in dimensions one and three. Comm. Math. Phys. 251 (2004), no. 1, 157–178. 52

[GoTo] A. Gomilko and Y. Tomilov. On discrete subordination of power bounded and Ritt operators. Indiana Univ. Math. J. 67 (2018), no. 2, 781–829. 10

[GoY] R. Gong and L. Yan. Littlewood-Paley and spectral multipliers on weighted L^p spaces. J. Geom. Anal. 24 (2014), no. 2, 873–900. 9, 49, 50, 65, 66, 68

[GPJP] A. González-Pérez, M. Junge and J. Parcet. Smooth Fourier multipliers in group algebras via Sobolev dimension. Ann. Sci. Éc. Norm. Supér. (4) 50 (2017), no. 4, 879–925. 10, 12

[GVF] J. M. Gracia-Bondía, J. C. Varilly and H. Figueroa. Elements of noncommutative geometry. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Boston, Inc., Boston, MA, 2001. 97, 102, 104

[Gr] L. Grafakos. Classical Fourier analysis. Second Edition, Graduate Texts in Mathematics, 249. Springer, New York, 2008. 33, 35

[GLY] L. Grafakos, L. Liu and D. Yang. Vector-valued singular integrals and maximal functions on spaces of homogeneous type. Math. Scand. 104 (2009), no. 2, 296–310. 40, 43

[Gri1] R. I. Grigorchuk. Just infinite branch groups. New horizons in pro-p groups, 121–179, Progr. Math., 184, Birkhäuser Boston, Boston, MA, 2000. 118

[Gri] A. Grigor'yan. The heat equation on noncompact Riemannian manifolds. Math. USSR Sbornik 72 (1992), no. 1, 47–77. 51

[GHL] A. Grigor'yan, J. Hu and K. Lau. Heat kernels on metric measure spaces and an application to semilinear elliptic equations. Trans. Amer. Math. Soc. 355 (2003), no. 5, 2065–2095. 52

[GriTel] A. Grigor'yan and A. Telcs. Two-sided estimates of heat kernels on metric measure spaces. Ann. Probab. 40 (2012), no. 3, 1212–1284. 50, 52

[Haa1] U. Haagerup. Injectivity and decomposition of completely bounded maps. Operator algebras and their connections with topology and ergodic theory (Buşteni, 1983), 170–222, Lecture Notes in Math., 1132, Springer, Berlin, 1985. 108, 114

[Haa2] U. Haagerup. On the dual weights for crossed products of von Neumann algebras. II. Application of operator-valued weights. Math. Scand. 43 (1978/79), no. 1, 119–140. 83, 115

[Haa3] U. Haagerup. Group C*-algebras without the completely bounded approximation property. J. Lie Theory 26 (2016), no. 3, 861–887. 83, 111, 117

[HK] U. Haagerup and J. Kraus. Approximation properties for group C^*-algebras and group von Neumann algebras. Trans. Amer. Math. Soc. 344 (1994), no. 2, 667–699. 83

[HaLe] B. Haak and C. Le Merdy. α-admissibility of observation and control operators. Houston J. Math. 31 (2005), no. 4, 1153–1167. 65

[HaOu] B. Haak and E. M. Ouhabaz. Exact observability, square functions and spectral theory. J. Funct. Anal. 262 (2012), no. 6, 2903–2927. 65

[Haas] M. Haase. The functional calculus for sectorial operators. Operator Theory: Advances and Applications, 169. Birkhäuser Verlag, Basel (2006). xiv+392 pp. 10, 65

[Har] A. Harcharras. Fourier analysis, Schur multipliers on S^p and non-commutative $\Lambda(p)$-sets. Studia Math. 137 (1999), no. 3, 203–260. 113, 114

[HHL] D. He, G. Hong and W. Liu. Dimension-free estimates for the vector-valued variational operators. Preprint on https://arxiv.org/abs/1807.10145 62

[HNS] Y. Heo, F. Nazarov and A. Seeger. Radial Fourier multipliers in high dimensions. Acta Math. 206 (2011), no. 1, 55–92. 65

[HiPr] M. Hieber and J. Prüß. Functional calculi for linear operators in vector-valued L^p-spaces via the transference principle. Adv. Differential Equations 3 (1998), no. 6, 847–872. 9, 44, 49, 65, 66

[Hir] F. Hirsch. Opérateurs carré du champ (d'après J. P. Roth). (French). Séminaire Bourbaki, 29e année (1976/77), Exp. No. 501, pp. 167–182, Lecture Notes in Math., 677, Springer, Berlin, 1978. 75

[HLM] S. Hofmann, M. Lacey and A. McIntosh. The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds. Ann. of Math. (2) 156 (2002), no. 2, 623–631. 75

[HoMa1] G. Hong and T. Ma. Vector valued q-variation for differential operators and semigroups I. Math. Z. 286 (2017), no. 1-2, 89–120. 12

[HoMa2] G. Hong and T. Ma. Vector valued q-variation for differential operators and semigroups II. Preprint on https://arxiv.org/abs/1411.1254 12, 60, 61

[Hor] L. Hörmander. Estimates for translation invariant operators in L^p spaces. Acta Math. 104, (1960), 93–140. 43, 63

[Hy04] T. Hytönen. Fourier embeddings and Mihlin-type multiplier theorems. Math. Nachr. 274/275 (2004), 74–103. 10, 43, 49

[Hy06] T. Hytönen. Reduced Mihlin-Lizorkin multiplier theorem in vector-valued L^p spaces. Partial differential equations and functional analysis, 137–151, Oper. Theory Adv. Appl., 168, Birkhäuser, Basel, 2006. 10, 62

[Hy10] T. Hytönen. New thoughts on the vector-valued Mihlin-Hörmander multiplier theorem. Proc. Amer. Math. Soc. 138 (2010), no. 7, 2553–2560. 10, 43

[HyKa] T. Hytönen and A. Kairema. Systems of dyadic cubes in a doubling metric space. Colloq. Math. 126 (2012), no. 1, 1–33. 42

[HMP1] T. Hytönen, A. McIntosh and P. Portal. Kato's square root problem in Banach spaces. J. Funct. Anal. 254 (2008), no. 3, 675–726. 10, 76

[HMP2] T. Hytönen, A. McIntosh and P. Portal. Holomorphic functional calculus of Hodge-Dirac operators in Lp. J. Evol. Equ. 11 (2011), no. 1, 71–105. 10, 76

[HvNP] T. Hytönen, J. van Neerven and P. Portal. Conical square function estimates in UMD Banach spaces and applications to H^∞-functional calculi. J. Anal. Math. 106 (2008), 317–351. 65

[HvNVW1] T. Hytönen, J. van Neerven, M. Veraar and L. Weis. Analysis in Banach spaces, Volume I: Martingales and Littlewood-Paley theory. Springer, 2016. 10, 21, 74, 106

[HvNVW2] T. Hytönen, J. van Neerven, M. Veraar and L. Weis. Analysis in Banach spaces, Volume II: Probabilistic Methods and Operator Theory. Springer, 2018. 10, 15, 17, 19, 74

[IwG] T. Iwaniec and G. Martin. Riesz transforms and related singular integrals. J. Reine Angew. Math. 473 (1996), 25–57. 75

[JSW] R. L. Jones, A. Seeger and J. Wright. Strong variational and jump inequalities in harmonic analysis. Trans. Amer. Math. Soc. 360 (2008), no. 12, 6711–6742. 62

[Jun] M. Junge. Doob's inequality for non-commutative martingales. J. Reine Angew. Math. 549 (2002), 149–190. 85

[JM] M. Junge and T. Mei. Noncommutative Riesz transforms–a probabilistic approach. Amer. J. Math. 132 (2010), no. 3, 611–680. 77, 97, 101

[JMP1] M. Junge, T. Mei and J. Parcet. Smooth Fourier multipliers on group von Neumann algebras. Geom. Funct. Anal. 24 (2014), no. 6, 1913–1980. 10, 12, 78, 101

[JMP2] M. Junge, T. Mei and J. Parcet. Noncommutative Riesz transforms-dimension free bounds and Fourier multipliers. J. Eur. Math. Soc. (JEMS) 20 (2018), no. 3, 529–595. 10, 12, 13, 75, 76, 78, 86, 88, 89, 90, 91, 96, 103

[JMe] M. Junge and C. Le Merdy. Dilations and rigid factorisations on noncommutative L^p-spaces. J. Funct. Anal. 249 (2007), no. 1, 220–252. 13

[JMX] M. Junge, C. Le Merdy and Q. Xu. H^∞ functional calculus and square functions on noncommutative L^p-spaces. Astérisque No. 305 (2006). 10, 12, 79, 80, 81, 84, 85

[JRZ] M. Junge, S. Rezvani and Q. Zeng. Harmonic analysis approach to Gromov-Hausdorff convergence for noncommutative tori. Comm. Math. Phys. 358 (2018), no. 3, 919–994. 86

[JR1] M. Junge and Z.-J. Ruan. Approximation properties for noncommutative L_p-spaces associated with discrete groups. Duke Math. J. 117 (2003), no. 2, 313–341. 83

[JR2] M. Junge and Z.-J. Ruan. Decomposable maps on non-commutative L_p-spaces. Operator algebras, quantization, and noncommutative geometry, 355–381, Contemp. Math., 365, Amer. Math. Soc., Providence, RI, 2004. 108, 109

[JuZ] M. Junge and Q. Zeng. Noncommutative martingale deviation and Poincaré type inequalities with applications. Probab. Theory Related Fields 161 (2015), no. 3-4, 449–507. 77

[Kad] R. V. Kadison. A representation theory for commutative topological algebra. Mem. Amer. Math. Soc., No. 7 (1951), 39 pp. 98

[KaKuWe] N. Kalton, P. Kunstmann and L. Weis. Perturbation and interpolation theorems for the H$^\infty$-calculus with applications to differential operators. Math. Ann. 336 (2006), no. 4, 747–801. 65

[KaWe1] N. Kalton and L. Weis. The H$^\infty$-calculus and sums of closed operators. Math. Ann. 321 (2001), no. 2, 319–345. 65, 66

[KaWe2] N. Kalton and L. Weis. The H$^\infty$-Functional Calculus and Square Function Estimates. Preprint on https://arxiv.org/abs/1411.0472v2 65, 66

[Kan] L. Kantorovitch. Linear operations in semi-ordered spaces. I. Rec. Math. [Mat. Sbornik] N.S. (7) 49 (1940), 209–284. 106

[Kem] M. Kemppainen. On the Rademacher maximal function. Studia Math. 203 (2011), no. 1, 1–31. 41

[Kim] J. Kim. A characterization of maximal operators associated with radial Fourier multipliers. Proc. Amer. Math. Soc. 145 (2017), no. 3, 1077–1085. 65

[Kin1] F. King. Hilbert transforms. Vol. 1. Encyclopedia of Mathematics and its Applications, 124. Cambridge University Press, Cambridge, 2009. 74

[Kin2] F. King. Hilbert transforms. Vol. 2. Encyclopedia of Mathematics and its Applications, 125. Cambridge University Press, Cambridge, 2009. 74

[KPS] V. Kovalenko, M. Perelmuter and Y. Semenov. Schrödinger operators with $L^{\frac{1}{2}}_W(\mathbb{R}^l)$-potentials. J. Math. Phys. 22 (1981), no. 5, 1033–1044. 53

[KrPhD] C. Kriegler. Spectral multipliers, R-bounded homomorphisms, and analytic diffusion semigroups. PhD thesis. Université de Franche-Comté, 2010. Available at https://hal.archives-ouvertes.fr/tel-00461310 18, 44

[KrW2] C. Kriegler and L. Weis. Paley-Littlewood decomposition for sectorial operators and interpolation spaces. Math. Nachr. 289 (2016), no. 11-12, 1488–1525. 10, 19

[KrW3] C. Kriegler and L. Weis. Spectral multiplier theorems via H^∞ calculus and R-bounds. Math. Z. 289 (2018), no. 1-2, 405–444. 10, 18, 19, 22, 44, 55, 73

[Krol] S. Król. Fourier multipliers on weighted L^p spaces. Math. Res. Lett. 21 (2014), no. 4, 807–830. 10

[KuU1] P. C. Kunstmann and M. Uhl. L^p-spectral multipliers for some elliptic systems. Proc. Edinb. Math. Soc. (2) 58 (2015), no. 1, 231–253. 22, 26

[KuU2] P. Kunstmann and M. Uhl. Spectral multiplier theorems of Hörmander type on Hardy and Lebesgue spaces. J. Operator Theory 73 (2015), no. 1, 27–69. 10, 22, 49

[KuUl1] P. Kunstmann and A. Ullmann. $\mathcal{R}_s$-sectorial operators and generalized Triebel-Lizorkin spaces. J. Fourier Anal. Appl. 20 (2014), no. 1, 135–185. 9

[KuUl2] P. Kunstmann and A. Ullmann. $\mathcal{R}_s$-bounded H^∞-calculus for sectorial operators via generalized Gaussian estimates. Math. Nachr. 288 (2015), no. 11-12, 1371–1387. 10, 48, 53

[KW1] P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. pp. 65-311 in Functional analytic methods for evolution equations, Lect. Notes in Math. 1855, Springer, 2004. 8, 9, 10, 15, 17, 31, 43, 65, 66

[KW2] P. C. Kunstmann and L. Weis. New criteria for the H^∞-calculus and the Stokes operator on bounded Lipschitz domains. J. Evol. Equ. 17 (2017), no. 1, 387–409. 9

[KwPe] S. Kwapień and A. Pełczyński. The main triangle projection in matrix spaces and its applications. Studia Math. 34 (1970), 43–68. 81

[LaS] V. Lafforgue and M. de la Salle. Noncommutative L^p-spaces without the completely bounded approximation property. Duke Math. J. 160 (2011), no. 1, 71–116. 83

[LaLaMe] F. Lancien, G. Lancien and C. Le Merdy. A joint functional calculus for sectorial operators with commuting resolvents. Proc. London Math. Soc. (3) 77 (1998), no. 2, 387–414. 65

[LaMe1] G. Lancien and C. Le Merdy. A generalized H^∞ functional calculus for operators on subspaces of L^p and application to maximal regularity. Illinois J. Math. 42 (1998), no. 3, 470–480. 65

[LaMe2] F. Lancien and C. Le Merdy. On functional calculus properties of Ritt operators. Proc. Roy. Soc. Edinburgh Sect. A 145 (2015), no. 6, 1239–1250. 10

[Lat1] F. Latrémolière. Quantum locally compact metric spaces. J. Funct. Anal. 264 (2013), no. 1, 362–402. 97, 98, 100

[Lat2] F. Latrémolière. Convergence of fuzzy tori and quantum tori for the quantum Gromov-Hausdorff propinquity: an explicit approach. Münster J. Math. 8 (2015), no. 1, 57–98. 99

[Lat3] F. Latrémolière. The dual Gromov-Hausdorff propinquity. J. Math. Pures Appl. (9) 103 (2015), no. 2, 303–351. 98, 99

[Lat4] F. Latrémolière. The quantum Gromov-Hausdorff propinquity. Trans. Amer. Math. Soc. 368 (2016), no. 1, 365–411. 99

[Lat5] F. Latrémolière. Equivalence of quantum metrics with a common domain. J. Math. Anal. Appl. 443 (2016), no. 2, 1179–1195. 99

[Lat6] F. Latrémolière. Quantum metric spaces and the Gromov-Hausdorff propinquity. Noncommutative geometry and optimal transport, 47–133, Contemp. Math., 676, Amer. Math. Soc., Providence, RI, 2016. 97, 98, 99, 100

[Lat7] F. Latrémolière. Heisenberg Modules over Quantum 2-tori are metrized quantum vector bundles. Preprint, arXiv:1703.07073. 99

[Lat8] F. Latrémolière. The Gromov-Hausdorff propinquity for metric Spectral Triples. Preprint, arxiv:1811.10843. 98

[LeM1] C. Le Merdy. H$^\infty$-functional calculus and applications to maximal regularity. Publ. Math. UFR Sci. Tech. Besançon 16 (1998), 41–77. 65

[LeM2] C. Le Merdy. The Weiss conjecture for bounded analytic semigroups. J. London Math. Soc. (2) 67 (2003), no. 3, 715–738. 65

[LeM3] C. Le Merdy. H^∞ functional calculus and square function estimates for Ritt operators. Rev. Mat. Iberoam. 30 (2014), no. 4, 1149–1190. 10

[LMX] C. Le Merdy and Q. Xu. Strong q-variation inequalities for analytic semigroups. Ann. Inst. Fourier (Grenoble) 62 (2012), no. 6, 2069–2097 (2013). 60, 61

[Li] H.-Q. Li. Fonctions maximales centrées de Hardy-Littlewood pour les opérateurs de Grushin. Preprint on arxiv:1207.3128. 38

[LY] P. Li and Sh. Yau. On the parabolic kernel of the Schrödinger operator. Acta Math. 156 (1986), no. 3-4, 153–201. 50

[Lin] N. Lindemulder. Parabolic Initial-Boundary Value Problems with Inhomogeneous Data A Maximal Weighted $L^q - L^p$-Regularity Approach. Master thesis, Utrecht University. 21

[LiVe] N. Lindemulder and M. Veraar. The heat equation with rough boundary conditions and holomorphic functional calculus. J. Differential Equations 269 (2020), no. 7, 5832–5899. 10

[LTz] J. Lindenstrauss and L. Tzafriri. Classical Banach spaces. II. Function spaces. Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas], Vol. 97. Springer-Verlag, Berlin-New York, 1979. x+243 pp. 21

[Lor] E. Lorist. Maximal functions, factorization, and the R-boundedness of integral operators. Master thesis, Delft Institute of Applied Mathematics, October 2016, online available. 51

[Lus1] F. Lust-Piquard. Riesz transforms associated with the number operator on the Walsh system and the fermions. J. Funct. Anal. 155 (1998), no. 1, 263–285. 75, 76

[Lus2] F. Lust-Piquard. Dimension free estimates for discrete Riesz transforms on products of abelian groups. Adv. Math. 185 (2004), no. 2, 289–327. 75, 76

[MaN1] J. Maas and J. van Neerven. Boundedness of Riesz transforms for elliptic operators on abstract Wiener spaces. J. Funct. Anal. 257 (2009), no. 8, 2410–2475. 76

[Man] R. Manna. Weighted inequalities for spherical maximal operator. Proc. Japan Acad. Ser. A Math. Sci. 91 (2015), no. 9, 135–140. 34, 36

[MMMM] J. M. Martell, D. Mitrea, I. Mitrea and M. Mitrea. The higher order regularity Dirichlet problem for elliptic systems in the upper-half space. Harmonic Analysis and Partial Differential Equations. Proceedings of the 9th International Conference on Harmonic Analysis and Partial Differential Equations, El Escorial, June 11-15, 2012, Contemporary Mathematics 612:123–141, 2014, Preprint available online at http://www.uam.es/personal_pdi/ciencias/martell/Investigacion/MMMM-proc.pdf 25, 26

[Mar] A. Martini. Joint functional calculi and a sharp multiplier theorem for the Kohn Laplacian on spheres. Math. Z. 286 (2017), no. 3-4, 1539–1574. 22

[MaMe] G. Mauceri and S. Meda. Vector-valued multipliers on stratified groups. Rev. Mat. Iberoamericana 6 (1990), no. 3-4, 141–154. 12, 54, 55, 60

[McC] T. McConnell. On Fourier multiplier transformations of Banach-valued functions. Trans. Amer. Math. Soc. 285 (1984), no. 2, 739–757. 43, 49

[Med] S. Meda. A general multiplier theorem. Proc. Amer. Math. Soc. 110 (1990), no. 3, 639–647. 66

[Mel] A. D. Melas. The best constant for the centered Hardy-Littlewood maximal inequality. Ann. of Math. (2) 157 (2003), no. 2, 647–688. 33

[Mey] P. A. Meyer. Transformations de Riesz pour les lois gaussiennes. (French) [Riesz transforms for Gaussian laws]. Seminar on probability, XVIII, 179–193, Lecture Notes in Math., 1059, Springer, Berlin, 1984. 75

[MeN] P. Meyer-Nieberg. Banach lattices. Universitext. Springer-Verlag, Berlin, 1991. 106, 111

[MoLu] H.-X. Mo and S.-Z. Lu. Vector-valued Singular Integral Operators with Non-smooth Kernels and Related Multilinear Commutators. Pure and Appl. Math. Quart. 3(2):451–480, 2007. 25

[Mul] D. Müller. A geometric bound for maximal functions associated to convex bodies. Pacific J. Math. 142 (1990), no. 2, 297–312. 9, 33

[MuSt] D. Müller and E. M. Stein. On spectral multipliers for Heisenberg and related groups. J. Math. Pures Appl. (9) 73 (1994), no. 4, 413–440. 9

[Naor] A. Naor. Discrete Riesz transforms and sharp metric X_p inequalities. Ann. of Math. (2) 184 (2016), no. 3, 991–1016. 76

[Ne] J. van Neerven. Stochastic Evolution Equations. ISEM Lecture Notes 2007/08. 82

[NeP] J. van Neerven and P. Portal. The Weyl calculus for group generators satisfying the canonical commutation relations. Preprint on https://arxiv.org/abs/1806.00980 65

[NeV] J. v. Neerven and R. Versendaal. L^p-Analysis of the Hodge-Dirac Operator Associated with Witten Laplacians on Complete Riemannian Manifolds. J Geom Anal (2017), 1–30. 76

[NeRi] S. Neuwirth and É. Ricard. Transfer of Fourier multipliers into Schur multipliers and sumsets in a discrete group. Canad. J. Math. 63 (2011), no. 5, 1161–1187. 81

[Nou] A. Nou. Asymptotic matricial models and QWEP property for q–Araki–Woods algebras. J. Funct. Anal. 232 (2006), no. 2, 295–327. 84

[Nua] D. Nualart. The Malliavin calculus and related topics. Second edition. Springer-Verlag, Berlin, 2006. 82

[Ouh95] E.-M. Ouhabaz. Gaussian estimates and holomorphy of semigroups. Proc. Amer. Math. Soc. 123 (1995), no. 5, 1465–1474. 51

[Ouh05] E.-M. Ouhabaz. Analysis of heat equations on domains. London Mathematical Society Monographs Series, 31. Princeton University Press, Princeton, NJ, 2005. 22, 51

[Ouh06] E.-M. Ouhabaz. Sharp Gaussian bounds and L^p-growth of semigroups associated with elliptic and Schrödinger operators. Proc. Amer. Math. Soc. 134 (2006), no. 12, 3567–3575. 51

[OtE] E. M. Ouhabaz and A. F. M. ter Elst. Analysis of the heat kernel of the Dirichlet-to-Neumann operator. J. Funct. Anal. 267 (2014), no. 11, 4066–4109. 26, 27

[Oza] N. Ozawa. About the QWEP conjecture. Internat. J. Math. 15 (2004), no. 5, 501–530. 82, 84

[PRS] J. Parcet, É. Ricard and M. de la Salle. Fourier multipliers in $SL_n(\mathbb{R})$. Preprint on https://arxiv.org/abs/1811.07874 12

[Pau] V. Paulsen. Completely bounded maps and operator algebras. Cambridge Univ. Press, 2002. 107

[Ped] G. K. Pedersen. C^*-algebras and their automorphism groups. London Mathematical Society Monographs, 14. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], London-New York, 1979. 115

[Pier] V. Pierfelice. Dispersive estimates and NLS on product manifolds. Preprint on arxiv:1012.0442 53

[Piet] A. Pietsch. Absolut p-summierende Abbildungen in normierten Räumen. Studia Math. 28 1966/1967 333–353. 81

[Pis1] G. Pisier. Riesz transforms: a simpler analytic proof of P.-A. Meyer's inequality. Séminaire de Probabilités, XXII, 485–501. Lecture Notes in Math., 1321, Springer, Berlin, 1988. 91, 106, 108

[Pis2] G. Pisier. Regular operators between non-commutative L_p-spaces. Bull. Sci. Math. 119 (1995), no. 2, 95–118. 13, 106, 108, 110

[Pis3] G. Pisier. The operator Hilbert space OH, complex interpolation and tensor norms. Mem. Amer. Math. Soc. 122 (1996), no. 585. 110

[Pis4] G. Pisier. Non-commutative vector valued L_p-spaces and completely p-summing maps. Astérisque, 247, 1998. 107, 108, 110

[Pis5] G. Pisier. Introduction to operator space theory. Cambridge University Press, Cambridge, 2003. 79, 107, 108, 110

[Pis6] G. Pisier. Complex interpolation between Hilbert, Banach and operator spaces. Mem. Amer. Math. Soc. 208 (2010), no. 978. 106

[PiX] G. Pisier and Q. Xu. Non-commutative L^p-spaces. In Handbook of the Geometry of Banach Spaces, Vol. II, edited by W.B. Johnson and J. Lindenstrauss, Elsevier, 1459–1517, 2003. 79, 106

[PrSi] J. Prüss and G. Simonett. H^∞-calculus for the sum of non-commuting operators. Trans. Amer. Math. Soc. 359 (2007), no. 8, 3549–3565. 65

[Rie1] M. A. Rieffel. Compact quantum metric spaces. Operator algebras, quantization, and noncommutative geometry, 315–330, Contemp. Math., 365, Amer. Math. Soc., Providence, RI, 2004. 97, 98

[Rie2] M. A. Rieffel. Gromov-Hausdorff distance for quantum metric spaces. Appendix 1 by Hanfeng Li. Gromov-Hausdorff distance for quantum metric spaces. Matrix algebras converge to the sphere for quantum Gromov-Hausdorff distance. Mem. Amer. Math. Soc. 168 (2004), no. 796, 1–65. 99

[Ri] M. Riesz. Sur les fonctions conjuguées. (French). Math. Z. 27 (1928), no. 1, 218–244. 74

[Rog] K. Rogers. Strichartz estimates via the Schrödinger maximal operator. Math. Ann. 343 (2009), no. 3, 603–622. 64

[RV1] K. Rogers and P. Villarroya. Global estimates for the Schrödinger maximal operator. Ann. Acad. Sci. Fenn. Math. 32 (2007), no. 2, 425–435. 63, 64

[RV2] K. Rogers and P. Villarroya. Sharp estimates for maximal operators associated to the wave equation. Ark. Mat. 46 (2008), no. 1, 143–151. 63, 64

[Ros] M. Rösler. Dunkl operators: theory and applications. In Orthogonal polynomials and special functions (Leuven, 2002), volume 1817 of Lecture Notes in Math., pages 93–135. Springer, Berlin, 2003. 27

[Rot] J.-P. Roth. Opérateurs dissipatifs et semi-groupes dans les espaces de fonctions continues. (French. English summary). Ann. Inst. Fourier (Grenoble) 26 (1976), no. 4, ix, 1–97. 75

[RoVe] J. Rozendaal and M. Veraar. Fourier multiplier theorems involving type and cotype. J. Fourier Anal. Appl. 24 (2018), no. 2, 583–619. 10

[RdF1] J. L. Rubio de Francia. Martingale and integral transforms of Banach space valued functions. Probability and Banach spaces (Zaragoza, 1985), 195–222, Lecture Notes in Math., 1221, Springer, Berlin, 1986. 21, 40, 47

[RdF2] J. L. Rubio de Francia. Maximal functions and Fourier transforms. Duke Math. J. 53 (1986), no. 2, 395–404. 12, 34, 54

[Sal] L. Saloff-Coste. A note on Poincaré, Sobolev, and Harnack inequalities. Int. Math. Res. Not. (1992), no. 2. 50, 51

[Sau] J. Sauer. K-theory for proper smooth actions of totally disconnected groups. High-dimensional manifold topology, 427–448, World Sci. Publ., River Edge, NJ, 2003. 117

[Sau1] J.-L. Sauvageot. Tangent bimodule and locality for dissipative operators on C*-algebras. Quantum probability and applications, IV (Rome, 1987), 322–338, Lecture Notes in Math., 1396, Springer, Berlin, 1989. 76, 77, 87

[Sau2] J.-L. Sauvageot. Quantum Dirichlet forms, differential calculus and semigroups. Quantum probability and applications. V (Heidelberg, 1988), 334–346, Lecture Notes in Math., 1442, Springer, Berlin, 1990. 87

[Sch] H. H. Schaefer. Banach lattices and positive operators. Die Grundlehren der mathematischen Wissenschaften, Band 215. Springer-Verlag, New York-Heidelberg, 1974. 106

[SSe] E. Schrohe and J. Seiler. Bounded H_∞-calculus for cone differential operators. J. Evol. Equ. 18 (2018), no. 3, 1395–1425. 9

[Schw] F. L. Schwenninger. Functional calculus estimates for Tadmor-Ritt operators. J. Math. Anal. Appl. 439 (2016), no. 1, 103–124. 10

[See] A. Seeger. On quasiradial Fourier multipliers and their maximal functions. J. Reine Angew. Math. 370 (1986), 61–73. 12

[SeSo] A. Seeger and C. D. Sogge. Bounds for eigenfunctions of differential operators. Indiana Univ. Math. J. 38(3):669–682, 1989. 27

[ShSh] H. Shapiro and A. Shields. On some interpolation problems for analytic functions. Amer. J. Math. 83 (1961), 513–532. 81

[ShWa] A. Shields and L. Wallen. The commutants of certain Hilbert space operators. Indiana Univ. Math. J. 20 (1970/1971), 777–788. 81

[SYY] A. Sikora, L. Yan and X. Yao. Sharp spectral multipliers for operators satisfying generalized Gaussian estimates. J. Funct. Anal. 266 (2014), no. 1, 368–409. 10, 22, 49

[Sim] B. Simon. Schrödinger semigroups. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 3, 447–526. 51

[SiSm] A. Sinclair and R. Smith. Finite von Neumann algebras and masas. London Mathematical Society Lecture Note Series, 351. Cambridge University Press, Cambridge, 2008. 113

[Ste70] E. M. Stein. Topics in harmonic analysis related to the Littlewood-Paley theory. Annals of Mathematics Studies, No. 63 Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo (1970) viii+146 pp. 2, 8, 9, 66

[Ste82] E. M. Stein. The development of square functions in the work of A. Zygmund. Bull. Amer. Math. Soc. (N.S.) 7 (1982), no. 2, 359–376. 32

[Ste83] E. M. Stein. Some results in harmonic analysis in $\mathbb{R}^n$, for $n \to \infty$. Bull. Amer. Math. Soc. (N.S.) 9 (1983), no. 1, 71–73. 33, 75

[StSt] E. M. Stein and J.-O. Strömberg. Behavior of maximal functions in $\mathbf{R}^n$ for large n Ark. Mat. 21 (1983), no. 2, 259–269. 32, 37

[Str] S. Stratila. Modular theory in operator algebras. Translated from the Romanian by the author. Editura Academiei Republicii Socialiste România, Bucharest; Abacus Press, Tunbridge Wells, 1981. 79

[SW] Ž. Štrkalj and L. Weis. On operator-valued Fourier multiplier theorems. Trans. Amer. Math. Soc. 359 (2007), no. 8, 3529–3547. 43

[Str1] D. W. Stroock. Probability theory. An analytic view. Second edition. Cambridge University Press, Cambridge, 2011. 83

[Sty] G. Styan. Hadamard products and multivariate statistical analysis. Linear Algebra and Appl. 6 (1973), 217–240. 81

[SuZ] F. Sukochev and D. Zanin. The Connes character formula for locally compact spectral triples. Preprint, arXiv:1803.01551. 103

[Sun] V. Sunder. An invitation to von Neumann algebras. Universitext. Springer-Verlag, New York, 1987. 83, 110

[Tagg] R. Taggart. Evolution equations and vector-valued L^p spaces: Strichartz estimates and symmetric diffusion semigroups. PhD thesis, online available at http://maths-people.anu.edu.au/˜taggart/thesis.pdf 49

[Tak1] M. Takesaki. Theory of operator algebras. I. Reprint of the first (1979) edition. Encyclopaedia of Mathematical Sciences, 124. Operator Algebras and Non-commutative Geometry, 5. Springer-Verlag, Berlin, 2002. 110

[Tak2] M. Takesaki. Theory of operator algebras. II. Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003. 83

[Tch] P. Tchamitchian. The solution of Kato's conjecture (after Auscher, Hofmann, Lacey, McIntosh and Tchamitchian). Journées "Équations aux Dérivées Partielles" (Plestin-les-Grèves, 2001), Exp. No. XIV, 14 pp., Univ. Nantes, Nantes, 2001. 75

[Than] S. Thangavelu. Summability of Hermite expansions. I, II. Trans. Amer. Math. Soc. 314 (1989), no. 1, 119–142, 143–170. 24

[TJ] N. Tomczak-Jaegermann. Banach-Mazur distances and finite-dimensional operator ideals. Pitman Monographs and Surveys in Pure and Applied Mathematics, 38. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, 1989. xii+395 pp. available online at https://sites.ualberta.ca/˜ntj/bm_book/index.html 21, 47, 51

[Tri] H. Triebel. Interpolation theory, function spaces, differential operators. Second edition. Johann Ambrosius Barth, Heidelberg, 1995. 114

[Var] J. C. Varilly. Dirac operators and spectral geometry. Lecture notes on non-commutative geometry and quantum groups edited by P. M. Hajac. Online at https://www.mimuw.edu.pl/˜pwit/toknotes/ 97, 98

[VSCC] N. Th. Varopoulos, L. Saloff-Coste and T. Coulhon Analysis and geometry on groups. Cambridge Tracts in Mathematics, 100. Cambridge University Press, Cambridge, 1992. 38

[We01] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. Math. Ann. 319 (2001), no. 4, 735–758. 43

[We01bis] L. Weis. A new approach to maximal L_p-regularity. volution equations and their applications in physical and life sciences (Bad Herrenalb, 1998), 195–214, Lecture Notes in Pure and Appl. Math., 215, Dekker, New York, (2001). 9, 65

[Wil] J. S. Wilson. Profinite groups. London Mathematical Society Monographs. New Series, 19. The Clarendon Press, Oxford University Press, New York, 1998. 117, 118

[Wro1] B. Wróbel. Multivariate spectral multipliers for systems of Ornstein-Uhlenbeck operators. Studia Math. 216 (2013), no. 1, 47–67. 9

[Wro2] B. Wróbel. On the consequences of a Mihlin-Hörmander functional calculus: maximal and square function estimates. Math. Z. 287 (2017), no. 1-2, 143–153. 9, 12, 55, 60

[Xu15] Q. Xu. H^∞ functional calculus and maximal inequalities for semigroups of contractions on vector-valued L_p-spaces. Int. Math. Res. Not. IMRN (2015), no. 14, 5715–5732. 9, 12, 54, 58, 59, 66

[Yag] A. Yagi. Coïncidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs. C. R. Acad. Sci. Paris Sér. I Math. 299 (1984), no. 6, 173–176. 65

[Zien] J. Zienkiewicz. Estimates for the Hardy-Littlewood maximal function on the Heisenberg group. Colloq. Math. 103 (2005), no. 2, 199–205. 40

[Zim] F. Zimmermann. On vector-valued Fourier multiplier theorems. Studia Math. 93 (1989), no. 3, 201–222. 43, 49

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