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ALGEBRAS OF "SMALL FUNCTIONS" IN \mathbb{C} AND IN \mathbb{C}_p

ALAIN ESCASSUT

ABSTRACT. Small functions were defined in complex analysis and next in ultrametric analysis. Order of growth and type of growth were also defined in complex analysis and have a similar definition in ultrametric analysis. We compare these two notions in the same way, in \mathbb{C} and on a complete ultrametric algebraically closed field \mathbb{K} of characteristic 0 such as \mathbb{C}_p . The set of small functions with regards to an entire function f is a ring. The set of entire functions with an order of growth strictly inferior to a number t is also a ring. If t is the order of f , it is included in the previous one when f is regular, but not always when f is not. If an entire function h is small with regards to an entire function f , that does not imply that its order of growth is inferior to this of f . All these statements are the same on \mathbb{C} and on \mathbb{K} . Moreover, in \mathbb{K} , we have a specific result: if f is clean and the cotype of f is strictly superior to the cotype of h , while the type of f is less than the type of h , then h is a small function with regards to f .

Introduction

In complex analysis, a notion of small functions with regards to a "big" holomorphic function was introduced and particularly applied to the Nevanlinna theory [2], [5]. On the other hand, a notion of order of growth was also examined and led to another kind of small function. These studies suggest symmetric studies in ultrametric analysis: in the same way, we can define small functions with regards to a big function, and order of growth of an entire function [1], [3], [4], [6].

Here we mean to compare the two notions for entire functions, in complex analysis and on an ultrametric field.

Notations and definitions

We denote by Log the Neperian logarithm. We denote by \mathbb{K} a complete algebraically closed field of characteristic 0 such as \mathbb{C}_p and by $\mathcal{A}(\mathbb{C})$ the \mathbb{C} -algebra of entire functions in \mathbb{C} and similarly, by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} .

Let $f \in \mathcal{A}(\mathbb{C})$. (resp. let $f \in \mathcal{A}(\mathbb{K})$). According to classical notations [2], given $f \in \mathcal{A}(\mathbb{C})$ and $r > 0$, we define $M(f, r) = \sup\{|f(x)| \mid |x| = r\}$ and given $f \in \mathcal{A}(\mathbb{K})$ we put $|f|(r) = \sup\{|f(x)| \mid |x| = r\}$ [2].

Next, given $f \in \mathcal{A}(\mathbb{C})$, as in [5], [6], we define $\rho(f) = \limsup_{r \rightarrow +\infty} \frac{\text{Log}(\text{Log}(M(f, r)))}{\text{Log}(r)}$,

$$\tilde{\rho}(f) = \liminf_{r \rightarrow +\infty} \frac{\text{Log}(\text{Log}(M(f, r)))}{\text{Log}(r)}$$

and if $0 < \rho(f) < +\infty$, we put

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$$\begin{aligned}\sigma(f, r) &= \frac{\text{Log}(|f|(r))}{r^{\rho(f)}}, \\ \sigma(f) &= \limsup_{r \rightarrow +\infty} \sigma(f, r), \\ \tilde{\sigma}(f) &= \liminf_{r \rightarrow +\infty} \sigma(f, r),\end{aligned}$$

$\rho(f)$ is called *the order of growth*, $\tilde{\rho}(f)$ is called *the lower order of growth*, $\sigma(f)$ is called *the type of growth*, $\tilde{\sigma}(f)$ is called *the lower type of growth*.

Given $t > 0$, we denote by $\mathcal{A}_t(\mathbb{C})$ the set of functions $h \in \mathcal{A}(\mathbb{C})$ such that $\rho(h) < t$.

A function $f \in \mathcal{A}(\mathbb{C})$ (resp. $f \in \mathcal{A}(\mathbb{K})$). is said to be *regular* if $\tilde{\rho}(f) = \rho(f)$ and it is said to be *clean* if $\tilde{\sigma}(f) = \sigma(f)$.

Given $f, h \in \mathcal{A}(\mathbb{C})$, h is called *a small function with regards to f* if

$\lim_{r \rightarrow +\infty} \frac{\text{Log}(M(h, r))}{\text{Log}(M(f, r))} = 0$ and we denote by $\mathcal{A}_f(\mathbb{C})$ the set of functions $h \in \mathcal{A}(\mathbb{C})$ that are small functions with regards to f .

The same notations and definitions apply in $\mathcal{A}(\mathbb{K})$ just by replacing $M(f, r)$ by $|f|(r)$.

Results

Theorem 1 *Let $f(x) \in \mathcal{A}(\mathbb{C})$ (resp. $f(x) \in \mathcal{A}(\mathbb{K})$) be such that $\tilde{\sigma}(f) > 0$. Then f is regular.*

Proof. Suppose first $f(x) \in \mathcal{A}(\mathbb{C})$. Let $a = \tilde{\sigma}(f)$, hence $a = \liminf_{r \rightarrow +\infty} \frac{\text{Log}(M(f, r))}{r^{\rho(f)}}$, then there exists $R > 0$ such that $\frac{\text{Log}(M(f, r))}{r^{\rho(f)}} \geq \frac{a}{2} \forall r > R$ hence $\text{Log}(M(f, r)) > \frac{a}{2} r^{\rho(f)} \forall r > R$ and hence

$$\frac{\text{Log}(\text{Log}(M(f, r)))}{\text{Log}(r)} \geq \frac{\text{Log}(a) - \text{Log}(2)}{\text{Log}(r)} + \rho(f) \forall r \geq R.$$

Thus, $\tilde{\rho}(f) \geq \rho(f)$, which ends the proof when $f(x) \in \mathcal{A}(\mathbb{C})$. Suppose now $f(x) \in \mathcal{A}(\mathbb{K})$: we just replace $M(f, r)$ by $|f|(r)$ and have the same proof. \square

Theorem 2 *Let $f(x) \in \mathcal{A}(\mathbb{C})$ (resp. $f(x) \in \mathcal{A}(\mathbb{K})$). Then $\mathcal{A}_f(\mathbb{C})$ is a \mathbb{C} -subalgebra of $\mathcal{A}(\mathbb{C})$ (resp. $\mathcal{A}_f(\mathbb{K})$ is a \mathbb{K} -subalgebra of $\mathcal{A}(\mathbb{K})$).*

Proof. Suppose first $f \in \mathcal{A}(\mathbb{C})$ and let $g, h \in \mathcal{A}_f(\mathbb{C})$. Consider

$$\text{Log}(M(g + h, r)) \leq \text{Log}(M(g, r) + M(h, r)) \leq \text{Log}(M(g, r)) + \text{Log}(M(h, r)).$$

But by hypotheses,

$$\lim_{r \rightarrow +\infty} \frac{\text{Log}(M(g, r))}{\text{Log}(M(f, r))} = \lim_{r \rightarrow +\infty} \frac{\text{Log}(M(h, r))}{\text{Log}(M(f, r))} = 0$$

hence $g + h \in \mathcal{A}_f(\mathbb{C})$.

$$\text{Next, } \text{Log}(M(gh, r)) \leq \text{Log}(M(g, r)M(h, r)) = \text{Log}(M(g, r)) + \text{Log}(M(h, r))$$

hence

$$\lim_{r \rightarrow +\infty} \frac{\text{Log}(M(gh, r))}{\text{Log}(M(f, r))} = 0$$

which shows that $gh \in \mathcal{A}_f(\mathbb{C})$. Consequently, $\mathcal{A}_f(\mathbb{C})$ is a subalgebra of $\mathcal{A}(\mathbb{C})$.

If we set in $\mathcal{A}_f(\mathbb{K})$, the proof is identical with $|f|(r)$, $|g|(r)$, $|h|(r)$ instead of $M(f, r)$, $M(g, r)$, $M(h, r)$, respectively and hence $\mathcal{A}_f(\mathbb{K})$ is a subalgebra of $\mathcal{A}(\mathbb{K})$. \square

Theorem 3 *Let $t \in \mathbb{R}, t > 0$. Then $\mathcal{A}_t(\mathbb{C})$ is a \mathbb{C} -subalgebra of $\mathcal{A}(\mathbb{C})$ (resp. $\mathcal{A}_t(\mathbb{K})$ is a \mathbb{K} -subalgebra of $\mathcal{A}(\mathbb{K})$).*

Proof. Let $g, h \in \mathcal{A}_t(\mathbb{C})$. For all $r > 0$, we put $S(r) = \max \left(\text{Log}(M(g, r)), \text{Log}(M(h, r)) \right)$ and $\limsup_{r \rightarrow +\infty} t - \frac{\text{Log}(S(r))}{\text{Log}(r)} = 2b$.

Now we have $\text{Log}(\text{Log}(M(g+h, r))) \leq \text{Log}(\text{Log}(M(g, r) + M(h, r)))$
 $\leq \text{Log}(\text{Log}(M(g, r) + \text{Log}(M(h, r)))) \leq \text{Log}(\text{Log}(M(g, r))) + (\text{Log}(\text{Log}(M(h, r))))$
 $\leq \text{Log}(2S(r))$. Therefore

$$\frac{\text{Log}(\text{Log}(M(g+h, r)))}{\text{Log}(r)} \leq \frac{\text{Log}(S(r) + \text{Log}(2))}{\text{Log}(r)} \quad \forall r \geq R$$

and hence $\rho(g+h) \leq t - b$, which shows that $g+h \in \mathcal{A}_t(\mathbb{C})$.

Next, $\text{Log}(\text{Log}(M(gh, r))) \leq \text{Log}(\text{Log}(M(g, r)M(h, r))) \leq \text{Log}(\text{Log}(M(g, r) + \text{Log}(M(h, r))))$ hence

$$\frac{\text{Log}(\text{Log}(M(gh, r)))}{\text{Log}(r)} \leq \frac{\text{Log}(2S(r))}{\text{Log}(r)} = \frac{\text{Log}(S(r))}{\text{Log}(r)} + \frac{\text{Log}(2)}{\text{Log}(r)},$$

and hence there exists $R' \geq R$ such that

$$\frac{\text{Log}(S(r))}{\text{Log}(r)} + \frac{\text{Log}(2)}{\text{Log}(r)} \leq \frac{\text{Log}(S(r))}{\text{Log}(r)} + \frac{\text{Log}(2)}{\text{Log}(r)} \leq t - b \quad \forall r \geq R'.$$

Consequently, $gh \in \mathcal{A}_t(\mathbb{C})$.

The proof in \mathbb{K} is identical by replacing $M(f, r)$ by $|f|(r)$ and so on. \square

Theorem 4 *Let $f \in \mathcal{A}(\mathbb{C})$, the \mathbb{C} -algebra $\mathcal{A}_{\tilde{\rho}(f)}(\mathbb{C})$ is included in $\mathcal{A}_f(\mathbb{C})$, (resp. let $f \in \mathcal{A}(\mathbb{K})$, the \mathbb{K} -algebra $\mathcal{A}_{\tilde{\rho}(f)}(\mathbb{K})$ is included in $\mathcal{A}_f(\mathbb{K})$).*

Proof. There exists $R > 0$ such that $\frac{\text{Log}(\text{Log}(M(f, r)))}{\text{Log}(r)} \geq \tilde{\rho}(f) - b \quad \forall r \geq R$.

Consequently when $r \geq R$, we have

$$\frac{\text{Log}(\text{Log}(M(f, r)))}{\text{Log}(r)} \geq \tilde{\rho}(f) - b \geq \frac{\text{Log}(\text{Log}(M(g, r)))}{\text{Log}(r)} + b,$$

hence $\text{Log}(\text{Log}(M(f, r))) \geq \text{Log}(\text{Log}(M(g, r))) + b\text{Log}(r)$ therefore $\text{Log}(M(f, r)) \geq (\text{Log}(M(g, r)))r^b$ which leads to

$$\frac{\text{Log}(M(g, r))}{\text{Log}(M(f, r))} \leq r^{-b}.$$

This proves that $g \in \mathcal{A}_f(\mathbb{C})$.

Now, if we are in \mathbb{K} instead of \mathbb{C} , we have exactly the same proof just by replacing $M(f, r)$ by $|f|(r)$, $M(g, r)$ by $|g|(r)$ etc...

□

Corollary 1: Let $f(x) \in \mathcal{A}(\mathbb{C})$ (resp. $f(x) \in \mathcal{A}(\mathbb{K})$) be regular, then the \mathbb{C} -algebra $\mathcal{A}_{\rho(f)}(\mathbb{C})$ is included in $\mathcal{A}_f(\mathbb{C})$, (resp. let $f \in \mathcal{A}(\mathbb{K})$, then the \mathbb{K} -algebra $\mathcal{A}_{\rho(f)}(\mathbb{K})$ is included in $\mathcal{A}_f(\mathbb{K})$).

Corollary 2: Let $f(x) \in \mathcal{A}(\mathbb{C})$ (resp. $f(x) \in \mathcal{A}(\mathbb{K})$) be such that $\tilde{\sigma}(f) > 0$. Then $\mathcal{A}_{\rho(f)}(\mathbb{C})$ is a \mathbb{C} -subalgebra of $\mathcal{A}_f(\mathbb{C})$ (resp. $\mathcal{A}_{\rho(f)}(\mathbb{K})$ is a \mathbb{K} -subalgebra of $\mathcal{A}_f(\mathbb{K})$).

Corollary 3 Let $f(x) \in \mathcal{A}(\mathbb{C})$ (resp. $f(x) \in \mathcal{A}(\mathbb{K})$) be clean and such that $\sigma(f) > 0$. Then $\mathcal{A}_{\rho(f)}(\mathbb{C})$ is a \mathbb{C} -subalgebra of $\mathcal{A}_f(\mathbb{C})$ (resp. $\mathcal{A}_{\rho(f)}(\mathbb{K})$ is a \mathbb{K} -subalgebra of $\mathcal{A}_f(\mathbb{K})$).

Remark 1: Let $f \in \mathcal{A}_{\rho(f)}(\mathbb{C})$ (resp. $f \in \mathcal{A}_{\rho(f)}(\mathbb{K})$) be not regular. Then f does not lie always in $\mathcal{A}_f(\mathbb{C})$ (resp. $\mathcal{A}_f(\mathbb{K})$). Indeed, suppose that $\rho(f) - \tilde{\rho}(f) = 2b$, with $b > 0$. We can construct a function $h \in \mathcal{A}(\mathbb{C})$ such that $\rho(h) \leq \rho(f) - b$ and such that there exists a sequence $(r_n)_{n \in \mathbb{N}}$ where $\lim_{n \rightarrow +\infty} r_n = +\infty$, satisfying

$$\frac{\text{Log}(\text{Log}(M(f, r_n)))}{\text{Log}(r_n)} \leq \frac{\text{Log}(\text{Log}(M(h, r_n)))}{\text{Log}(r_n)} \leq \frac{\text{Log}(\text{Log}(M(f, r_n)))}{\text{Log}(r_n)} - b$$

So, we have

$$\text{Log}(\text{Log}(M(f, r_n))) \leq \text{Log}(\text{Log}(M(h, r_n))) \leq \text{Log}(\text{Log}(M(f, r_n))) - b \text{Log}(r_n)$$

hence $\text{Log}(M(f, r_n)) \leq \text{Log}(M(h, r_n))$ and hence $1 \leq \frac{\text{Log}(M(h, r_n))}{\text{Log}(M(f, r_n))}$ which proves that $h \notin \mathcal{A}_f(\mathbb{C})$, although $h \in \mathcal{A}_{\rho(f)}(\mathbb{C})$.

Remark 2: Let $f \in \mathcal{A}(\mathbb{C})$ and $g \in \mathcal{A}_f(\mathbb{C})$. Can we tell that $g \in \mathcal{A}_{\rho(f)}(\mathbb{C})$? Suppose that $\rho(f) = 1$ and consider a function $h \in \mathcal{A}(\mathbb{C})$ such that $\text{Log}(M(h, r))$ is of the form $\omega(r)\text{Log}(M(f, r))$ with $\omega(r) \in \left[\frac{a}{\text{Log}(\text{Log}(M(f, r)))}, \frac{b}{\text{Log}(\text{Log}(M(f, r)))} \right]$ and $0 < a < b$. Then

$$\frac{\text{Log}(\text{Log}(M(h, r)))}{\text{Log}(\text{Log}(M(f, r)))} = \frac{\text{Log}(\omega(r)) + \text{Log}(\text{Log}(M(f, r)))}{\text{Log}(\text{Log}(M(f, r)))} = \frac{\text{Log}(\omega(r))}{\text{Log}(\text{Log}(M(f, r)))} + 1$$

and that belongs to

$$\left[\frac{\text{Log}(a) - \text{Log}(\text{Log}(\text{Log}(M(f, r))))}{\text{Log}(\text{Log}(M(f, r)))} + 1, \frac{\text{Log}(b) - \text{Log}(\text{Log}(\text{Log}(M(f, r))))}{\text{Log}(\text{Log}(M(f, r)))} + 1 \right],$$

therefore

$$\lim_{r \rightarrow +\infty} \frac{\text{Log}(\text{Log}(M(h, r)))}{\text{Log}(\text{Log}(M(f, r)))} = 1,$$

and hence $\rho(h) = \rho(f)$.

On the other hand, by definition,

$$\frac{\text{Log}(M(h, r))}{\text{Log}(M(f, r))} = \omega(r) \in \left[\frac{a}{\text{Log}(\text{Log}(M(f, r)))}, \frac{b}{\text{Log}(\text{Log}(M(f, r)))} \right],$$

hence $\lim_{r \rightarrow +\infty} \frac{\text{Log}(M(h, r))}{\text{Log}(M(f, r))} = 0$ hence h belongs to $\mathcal{A}_f(\mathbb{C})$.

The same reasoning works in \mathbb{K} and is easier because we can more easily construct a function $h \in \mathcal{A}_f(\mathbb{K})$ such that $\text{Log}(|h|(r)) = \omega(r)|f|(r)$, just by choosing the zeros of h among the zeros of f .

In order to prove the following Theorem 5, we must recall some notations and this Theorem A which only concern analytic functions in \mathbb{K} [3], [4].

Notations: Let $f \in \mathcal{A}(\mathbb{K})$. and let $d(0, r)$ be the disk of \mathbb{K} $\{x \in \mathbb{K} \mid |x| \leq r\}$. We denote by $s(r, f)$ the number of zeros of f in $d(0, r)$ each counted with its multiplicity.

Assuming that $0 < \rho(f) < +\infty$, here we put $\psi(f, r) = \frac{s(r, f)}{r^{\rho(f)}}$, $\psi(f) = \limsup_{r \rightarrow +\infty} \frac{s(r, f)}{r^{\rho(f)}}$,

which is called *the cotype of f* and we put $\tilde{\psi}(f) = \liminf_{r \rightarrow +\infty} \frac{s(r, f)}{r^{\rho(f)}}$.

Theorem A: Let $f \in \mathcal{A}(\mathbb{K})$ be such that $0 < \rho(f) < +\infty$. Then $\sigma(f) < +\infty$ if and only if $\psi(f) < +\infty$. Suppose that these hypotheses are satisfied. Then

$$\rho(f)\sigma(f) \leq \psi(f) \leq \rho(f)\left(e\sigma(f) - \tilde{\sigma}(f)\right)$$

and

$$\rho(f)\left(\tilde{\sigma}(f) - \frac{\sigma(f)}{e}\right) \leq \tilde{\psi}(f) \leq \rho(f)\tilde{\sigma}(f).$$

Further, the hypotheses $\sigma(f) = \tilde{\sigma}(f)$ and $\psi(f) = \tilde{\psi}(f)$ are equivalent and if they are satisfied, then $\psi(f) = \rho(f)\sigma(f)$.

Theorem 5 Let $f \in \mathcal{A}(\mathbb{K})$ be clean and let $h \in \mathcal{A}(\mathbb{K})$ be such that $\psi(h) < \psi(f)$ and $\sigma(h) \geq \sigma(f)$. Then f is regular and $h \in \mathcal{A}_f(\mathbb{K})$.

Proof. Since f is clean, by Theorem A, we have $\psi(f) = \rho(f)\sigma(f) > \psi(h) \geq \rho(h)\sigma(h)$, hence $\rho(f) > \rho(h)$. Moreover, we notice that $\rho(f)\sigma(f) > 0$, hence $\sigma(f) > 0$. But since f is clean, $\sigma(f) = \tilde{\sigma}(f)$ and hence by Theorem 1, f is regular. Finally, from the hypothesis, we obtain $\rho(h) < \rho(f)$, therefore. by Corollary 3, $h \in \mathcal{A}_f(\mathbb{K})$. \square

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