



HAL
open science

Survey on p-adic meromorphic functions sharing 5 small ones on a work by Ta Thi Hoai An and Nguyen Viet Phuong, with some additional properties

Alain Escassut

► **To cite this version:**

Alain Escassut. Survey on p-adic meromorphic functions sharing 5 small ones on a work by Ta Thi Hoai An and Nguyen Viet Phuong, with some additional properties. *Mathematics Open*, 2022, 01, 10.1142/S2811007222500055 . hal-04200441

HAL Id: hal-04200441

<https://uca.hal.science/hal-04200441v1>

Submitted on 8 Sep 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Copyright

Survey on p -adic meromorphic functions sharing 5 small ones on a work by Ta Thi Hoai An and Nguyen Viet Phuong, with some additional properties

Alain Escassut

Abstract

Let \mathbb{K} be a complete ultrametric algebraically closed field of characteristic 0, let D be the open disk $\{x \in \mathbb{K} \mid |x| < R\}$ and let $E = \mathbb{K} \setminus D$. Let f, g be two meromorphic functions in \mathbb{K} (resp. two unbounded meromorphic functions in D , resp. two meromorphic functions in E) having infinitely many zeros or poles in E sharing 5 small meromorphic functions in the same set (ignoring multiplicity). Then $f = g$. Moreover, if f and g have finitely many poles in \mathbb{K} (resp. in D , resp. in E), and share 3 small functions, (ignoring multiplicity), then $f = g$. We define archi-branched small functions and show that a meromorphic function f (in \mathbb{K} , D , or E) can't have 5 archi-branched small functions.

1 Introduction and definitions.

Small functions with respect to a meromorphic functions are well known in the general theory of complex functions. Particularly, one knows the Nevanlinna theorem on 3 small functions and its generalization due to K. Yamanoi which shows that two meromorphic functions in \mathbb{C} sharing 5 small functions, ignoring multiplicity, are equal.

Now, we denote by \mathbb{K} a complete ultrametric algebraically closed field of characteristic 0. Let $a \in \mathbb{K}$ and take $r > 0$. We denote by $d(a, r^-)$ the open-disk $\{x \in \mathbb{K} \mid |x - a| < r\}$, by $d(a, r)$ the closed-disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$ and by E the set $\mathbb{K} \setminus d(0, R^-)$. Considering two meromorphic functions f, g in \mathbb{K} or in $d(a, R^-)$, or in E , sharing a few small meromorphic functions, we can ask how many small functions are necessary to be able to prove that f and g are identic. A first work was done in [7], showing that if f and g share 7 small meromorphic functions, then $f = g$. Obtaining a better result is not obvious since Yamanoi's Theorem [14] has no equivalent for ultrametric functions. However, in [13], Ta Thi Hoai An and Nguyen Viet Phuong have found a method to show that 5 small functions are enough to prove that $f = g$. This is the work that we mean to develop with some more explanations. The main difficulty in [13] appears in Lemma 3, that we translate here as Lemma 5.1 in a more simple context and we try to explain the proof with more details.

Now, we denote by D the set $d(0, R^-)$ and by E the set $\mathbb{K} \setminus d(0, R^-) = \{x \in \mathbb{K} \mid |x - a| \geq R\}$. We denote by $\mathcal{A}(\mathbb{K})$ (resp. $\mathcal{A}(D)$), the algebra of analytic functions in \mathbb{K} (resp. in D the \mathbb{K} -

⁰2000 Mathematics Subject Classification: 12J25; 30D35; 30G06.

⁰Keywords: p -adic meromorphic functions, p -adic Nevanlinna theory, small functions.

algebra of power series converging in D). Next, we denote by $\mathcal{A}(E)$ the \mathbb{K} -algebra of Laurent series converging in E [8].

Next, we denote by $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(D)$) the field of fractions of $\mathcal{A}(\mathbb{K})$ (resp. $\mathcal{A}(D)$). We also denote by $\mathcal{A}_u(D)$ the \mathbb{K} -algebra of unbounded analytic functions in D and by $\mathcal{M}_u(D)$ the set of meromorphic functions in D that are not a quotient of two bounded analytic functions in D . Finally, we denote by $\mathcal{M}(E)$ the field of fractions of $\mathcal{A}(E)$.

Throughout the next paragraphs, we will denote by I the interval $]0, +\infty[$, by J an interval of the form $]0, R[$ with $t > 0$ and by G the interval $[R, +\infty[$.

We have to introduce the counting function of zeros and poles of f , counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Definitions: Let $f \in \mathcal{M}(d(0, r))$ and for every $a \in d(0, r)$, let $\theta_a(f)$ be the multiplicity order of a if a is a zero of f and let $\theta_a(f) = 0$, else.

We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way.

Let (a_n) , $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n .

We set $Z(r, f) = \theta_0(f) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|)$ and so, $Z(r, f)$ is called *the counting function of zeros of f in $d(0, r)$, counting multiplicity*.

In order to define the counting function of zeros of f ignoring multiplicity, we put $\bar{\theta}_0(f) = 0$ if $\theta_0(f) = 0$ and $\bar{\theta}_0(f) = 1$ if $\theta_0(f) \geq 1$. Now, we denote by $\bar{Z}(r, f)$ *the counting function of zeros of f ignoring multiplicity*: $\bar{Z}(r, f) = \bar{\theta}_0(f) \log r + \sum_{n=1}^{\sigma(r)} (\log r - \log |a_n|)$ and so, $\bar{Z}(r, f)$ is called *the counting function of zeros of f in $d(0, r)$ ignoring multiplicity*.

In order to define the counting function of poles of f , if 0 is a pole of f , we denote by $\gamma_0(f)$ the order of multiplicity of the pole and we put $\gamma_0(f) = 0$, else. Now, considering the finite sequence (b_n) , $1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n , we put $N(r, f) = \gamma_0(f) \log r + \sum_{n=1}^{\tau(r)} t_n (\log r - \log |b_n|)$ and then $N(r, f)$ is called *the counting function of the poles of f , counting multiplicity*.

Next, in order to define the counting function of poles of f ignoring multiplicity, we put $\bar{\gamma}_0(f) = 1$ if $\gamma_0(f) > 0$ and $\bar{\gamma}_0(f) = 0$ else and we set $\bar{N}(r, f) = \bar{\gamma}_0(f) \log r + \sum_{n=1}^{\tau(r)} (\log r - \log |b_n|)$ and then $\bar{N}(r, f)$ is called *the counting function of the poles of f , ignoring multiplicity*.

Now, we can define the characteristic function of f as $T(r, f) = \max(Z(r, f), N(r, f))$. Thus this definition applies to functions $f \in \mathcal{M}(d(0, R^-))$ as well as functions $f \in \mathcal{M}(\mathbb{K})$.

Remark: If we change the origin, the functions Z , N , T are not changed, up to an additive constant.

Consider now a function $f \in \mathcal{A}(E)$. By the definition, the restriction of f to any annulus $R \leq |x| \leq S$ is an analytic element in that annulus and hence has finitely many zeros in that

annulus [4], [5], [8], [10]. Similarly, a meromorphic function $f \in \mathcal{M}(E)$ has finitely many zeros and finitely many poles in the annulus $R \leq |x| \leq S$. That is summarized in Proposition 1.1:

Proposition 1.1 [1] [4], [5], [10], [11] *Let $f \in \mathcal{M}(E)$. If f has infinitely many zeros in E (resp. infinitely many poles in E), the set of zeros (resp. the set of poles) is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ such that*

$\lim_{n \rightarrow +\infty} |\alpha_n| = +\infty$. If f has no zero in E , then it is of the form $\sum_{-\infty}^{+\infty} a_n x^n$ with $a_n = 0 \forall n > q$ and $|a_q| r^q > |a_n| r^n \forall n \in \mathbb{Z}, n \neq q, \forall r \geq R$.

Corollary 1.1. [1], [4], [5], [11] : *Let $f \in \mathcal{M}(E)$ have no zero and no pole in E . There exists a unique integer $q \in \mathbb{Z}$ such that $x^{-q} f(x)$ has a limit $b \in \mathbb{K}^*$.*

Definitions: Let $f \in \mathcal{M}(E)$ have no zero and no pole in E . The integer $q \in \mathbb{Z}$ such that $x^{-q} f(x)$ has a limit $b \in \mathbb{K}^*$ is called *the Motzkin index of f* and f is called *a Motzkin factor* if $\lim_{|x| \rightarrow +\infty} x^{-q} f(x) = 1$ [1], [5], [11].

Proposition 1.2 [1], [4], [5], [8], [11] : *Let $f \in \mathcal{M}(E)$. Then f factorizes in a unique way in the form $f^S f^0$ where f^S is a Motzkin factor and $f^0 \in \mathcal{M}(\mathbb{K})$ has no zero in D .*

Notations: We will denote by $\mathcal{A}^c(E)$ the set of $f \in \mathcal{A}(E)$ having infinitely many zeros in E . Similarly, we will denote by $\mathcal{M}^c(E)$ the set of functions $f \in \mathcal{M}(E)$ which have infinitely many zeros or poles in E .

Thus we can define counting functions for zeros and poles in that way: let $f \in \mathcal{M}(E)$ and, for every $r > R$, let $a_1, \dots, a_{\sigma(r)}$ be the sequence of zeros of f in the annulus $R \leq |x| \leq r$, with $|a_j| \leq |a_{j+1}|$, $1 \leq j \leq \sigma(r)$, and let s_j be the order of a_j . Then we put $Z_R(r, f) = \sum_{j=1}^{\sigma(r)} s_j (\log(r) - \log(|a_j|))$ and $Z_R(r, f)$ is called *the counting function of zeros for f in $\mathcal{M}(E)$, counting multiplicity*. And we define $\bar{Z}_R(r, f) = \sum_{j=1}^{\sigma(r)} (\log(r) - \log(|a_j|))$ which is called *the counting function of zeros for f in $\mathcal{M}(E)$, ignoring multiplicity*.

Similarly, let $b_1, \dots, b_{\tau(r)}$ be the sequence of poles of f in the annulus $R \leq |x| \leq r$, with $|b_j| \leq |b_{j+1}|$, $1 \leq j \leq \tau(r)$ and let t_j be the order of b_j . Then we put $N_R(r, f) = \sum_{j=1}^{\tau(r)} t_j (\log(r) - \log(|b_j|))$ which is called *the counting function of poles for f in $\mathcal{M}(E)$, counting multiplicity* and we put $\bar{N}_R(r, f) = \sum_{j=1}^{\tau(r)} (\log(r) - \log(|b_j|))$ which is called *the counting function of poles for f in $\mathcal{M}(E)$, ignoring multiplicity*.

Now, we put $T_R(r, f) = \max(Z_R(r, f), N_R(r, f))$ and the function $T_R(r, f)$ is called *the characteristic function of f in $\mathcal{M}(E)$* .

2 Small functions

Recall that given three functions ϕ, ψ, ζ defined in an interval $J =]R, +\infty[$ (resp. $J =]a, R[$), with values in $[0, +\infty[$, we shall write $\phi(r) \leq \psi(r) + O(\zeta(r))$ if there exists a constant $b \in \mathbb{R}$ such that $\phi(r) \leq \psi(r) + b\zeta(r)$. We shall write $\phi(r) = \psi(r) + O(\zeta(r))$ if $|\psi(r) - \phi(r)|$ is bounded by a function of the form $b\zeta(r)$.

Similarly, we shall write $\phi(r) \leq \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]a, +\infty[$ (resp. from $J =]a, R[$) to \mathbb{R} such that $\lim_{r \rightarrow +\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) \leq$

$\psi(r) + h(r)$. And we shall write $\phi(r) = \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]a, +\infty[$ (resp. from $J =]a, R[$) to \mathbb{R} such that $\lim_{r \rightarrow +\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) = \psi(r) + h(r)$.

Definitions and notations: For each $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(D)$, resp. $f \in \mathcal{M}(E)$) we denote by $\mathcal{M}_f(\mathbb{K})$, (resp. $\mathcal{M}_f(D)$, resp. $\mathcal{M}_f(E)$) then the set of functions $h \in \mathcal{M}(\mathbb{K})$, (resp. $h \in \mathcal{M}(D)$, resp. $\mathcal{M}(E)$) such that $T(r, h) = o(T(r, f))$ when r tends to $+\infty$ (resp. $T(r, h) = o(T(r, f))$ when r tends to R , resp. $T_R(r, h) = o(T_R(r, f))$ when r tends to $+\infty$). Now, if a function Φ from I to \mathbb{R} (resp. from $]0, R[$ to \mathbb{R} , resp. from $[R, +\infty[$ to \mathbb{R}) satisfies $\Phi(r) \leq o(T(r, f))$, or $\phi(r) = o(T(r, f))$, we write $\Phi(r) \leq S(r, f)$, or $\Phi(r) = S(r, f)$ respectively.

Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(D)$, resp. $f \in \mathcal{A}(E)$) we shall denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(D)$, resp. $\mathcal{A}_f(E)$) the set $\mathcal{M}_f(\mathbb{K}) \cap \mathcal{A}(\mathbb{K})$, (resp. $\mathcal{M}_f(D) \cap \mathcal{A}(D)$, resp. $\mathcal{M}_f(E) \cap \mathcal{A}(E)$).

The elements of $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(D)$, resp. $\mathcal{M}_f(E)$) are called *small meromorphic functions with respect to f* , or *small functions* in brief. Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(D)$, resp. $f \in \mathcal{A}(E)$) the elements of $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(D)$, resp. $\mathcal{A}_f(E)$) are called *small analytic functions with respect to f* or *small functions* in brief.

In [7], the following theorem is proved:

Theorem A: *Let $f \in \mathcal{M}(\mathbb{K})$, (resp. let $f \in \mathcal{M}_u(D)$) be nonconstant and let h_1, \dots, h_q ($q \geq 5$) be q distinct small functions with respect to f . We have*

$$\frac{q}{3}T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f - h_i) + S(r, f).$$

Moreover, if f has finitely many poles, and if h_1, \dots, h_q ($q \geq 3$) are q distinct small functions with respect to f . We have

$$\frac{q}{2}T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f - h_i) + S(r, f).$$

From [13], Theorem A is improved in the following way:

Theorem 2.1: *Let $f \in \mathcal{M}(\mathbb{K})$, (resp. let $f \in \mathcal{M}_u(D)$, resp. let $f \in \mathcal{M}^c(E)$) be nonconstant and let h_1, \dots, h_q ($q \geq 5$) be q distinct small functions with respect to f . We have*

$$\frac{2q}{5}T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f - h_i) + S(r, f)$$

(resp.

$$\frac{2q}{5}T(r, f) \leq \sum_{s=1}^5 \bar{Z}(r, f - h_{i_s}) + S(r, f),$$

resp.

$$\frac{2q}{5}T_R(r, f) \leq \sum_{s=1}^5 \bar{Z}_R(r, f - h_{i_s}) + S(r, f).$$

Definition: Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}_u(D)$), resp. $f, g \in \mathcal{M}^c(E)$). Then f and g will be said to share a small function $w \in \mathcal{M}(\mathbb{K})$ (resp. $w \in \mathcal{M}(D)$, resp. $w \in \mathcal{M}^c(E)$), ignoring multiplicity, if $f(x) = w(x)$ implies $g(x) = w(x)$ and if $g(x) = w(x)$ implies $f(x) = w(x)$.

In [7] the following theorem is proved:

Theorem B: Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(D)$) be distinct and share 7 distinct small functions (other than the constant ∞) ignoring multiplicity, $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ ($j = 1, \dots, 7$) (resp. $w_j \in \mathcal{M}_f(D) \cap \mathcal{M}_g(D)$ ($j = 1, \dots, 7$),). Then $f = g$.
Moreover, if f and g have finitely many poles and share 3 small functions, then $f = g$.

Definition: A small function $h \in \mathcal{M}(\mathbb{K})$ (resp. $h \in \mathcal{M}_u(D)$, resp. $h \in \mathcal{M}^c(E)$) will be said to be archi-branched with respect to a meromorphic function $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(D)$, resp. $f \in \mathcal{M}^c(E)$) if all zeros of $f - h$ except finitely many are of order at least 3.

Theorem 2.2: Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f \in \mathcal{M}_u(D)$, resp. $f \in \mathcal{M}^c(E)$). Then f admits at most 4 archi-branched small functions. Moreover, if f has finitely many poles, then f admits at most one archi-branched small function.

Remark: Thanks to Yamanoy's theorem [14], it is easily seen that a meromorphic function f in \mathbb{C} can't admit more than 4 totally branched small functions, i.e. small function h such that all zeros of $f - h$ has order at least 2, except finitely many [2]. Similarly, a meromorphic function f in \mathbb{C} can't admit more than 3 archi-branched small functions.

From [13], Theorem B is improved in the following way:

Theorem 2.3: Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(D)$, resp. $f, g \in \mathcal{M}^c(E)$) be distinct and share 5 distinct small meromorphic functions ignoring multiplicity, $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ ($j = 1, \dots, 5$) (resp. $w_j \in \mathcal{M}_f(D) \cap \mathcal{M}_g(D)$ ($j = 1, \dots, 5$), resp. $w_j \in \mathcal{M}_f(E) \cap \mathcal{M}_g(E)$ ($j = 1, \dots, 5$),). Then $f = g$.

Moreover, if f and g have finitely many poles in \mathbb{K} (resp. in D , resp in E) and share 3 small meromorphic functions, ignoring multiplicity, then $f = g$.

Remarks: 1) When we consider meromorphic functions on \mathbb{C} , thanks to the main Nevanlinna Theorem on n small functions due to Yamanoi [14], it is easily seen that two meromorphic functions sharing 5 small ones, ignoring multiplicity, are equal. The absence of such a theorem with p-adic meromorphic functions makes much more difficult the study.

2) If a meromorphic function $f \in \mathcal{M}^c(E)$ has finitely many poles, then it must have infinitely many zeros.

3 Nevanlinna Theorems

The Nevanlinna Theory is well known in \mathbb{C} [12]. It was constructed in a field like \mathbb{K} in the eighties and next, in a disk and out of a hole [3], [9], [5], [6], [8]. We have to recall the two main Theorems, applied to each domain of definition of meromorphic functions.

Theorem 3.1 (First Main Theorem in a disk and in \mathbb{K}) [3], [5], [9]: Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}(D)$). Then $T(r, f+b) = T(r, f) + O(1)$. Let $P(X) \in \mathbb{K}[X]$. Then $T(r, P(f)) = \deg(P)T(r, f) + O(1)$ and $T(r, f'P(f)) \geq T(r, P(f))$.

Suppose now $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \mathcal{A}(D)$). Then $Z(r, fg) = Z(r, f) + Z(r, g)$, $T(r, f) = Z(r, f)$, $T(r, fg) = T(r, f) + T(r, g) + O(1)$ and $T(r, f+g) \leq \max(T(r, f), T(r, g))$. Moreover, if $\lim_{r \rightarrow +\infty} T(r, f) - T(r, g) = +\infty$ then $T(r, f+g) = T(r, f)$ when r is big enough.

Theorem 3.2 (First Main Theorem out of a hole): [6] Let $f, g \in \mathcal{M}(E)$. Then for every $b \in \mathbb{K}$, we have $T_R(r, f+b) = T_R(r, f) + O(\log(r))$, ($r \in I$) $T_R(r, f \cdot g) \leq T_R(r, f) + T_R(r, g) + O(\log(r))$ ($r \in I$), $T_R(r, \frac{1}{f}) = T_R(r, f)$, $T_R(r, f+g) \leq T_R(r, f) + T_R(r, g) + O(\log(r))$ ($r \geq R$) and $T_R(r, f^n) = nT_R(r, f)$. Given a polynomial $P(X) \in \mathbb{K}[X]$ of degree q , then $T_R(r, P \circ f) = qT_R(r, f) + O(\log(r))$.

Moreover, if both f and g belong to $\mathcal{A}(E)$, then

$$T_R(r, f+g) \leq \max(T_R(r, f), T_R(r, g)) + O(\log(r)) \quad (r \geq R)$$

and $T_R(r, fg) = T_R(r, f) + T_R(r, g)$, ($r \geq R$). Particularly, if $f \in \mathcal{A}(E)$, then $T_R(r, f+b) = T_R(r, f) + O(1)$ ($r \geq R$).

Theorem 3.3 (Second Main Theorem in \mathbb{K} and in a disk) [3], [5], [9]: Let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$ be distinct, with $q \geq 2$, let $S = \{\alpha_1, \dots, \alpha_q\}$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then

$$\sum_{j=1}^q (Z(r, f - \alpha_j) - \bar{Z}(r, f - \alpha_j)) \leq T(r, f) + \bar{N}(r, f) - \log r + O(1) \quad \forall r > 0 \quad (\text{resp. } \forall r < R).$$

Theorem 3.4 (Second Main Theorem out of a hole) [6], [8] : Let $f \in \mathcal{M}(E)$ and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct with $q \geq 2$. Then

$$(q-1)T_R(r, f) \leq \sum_{j=1}^q Z_R(r, f - a_j) + O(\log(r)) \quad (r \geq R).$$

Theorem 3.5 (Second Main Theorem on three small functions) [6], [8], [9] : Let $f \in \mathcal{M}(\mathbb{K})$, resp. $f \in \mathcal{M}_u(D)$, resp. $f \in \mathcal{M}^c(E)$ and let $h_1, h_2, h_3 \in \mathcal{M}(\mathbb{K})$, resp. $h_1, h_2, h_3 \in \mathcal{M}(D)$, resp. $h_1, h_2, h_3 \in \mathcal{M}(E)$ be small functions with respect to f . Then

$$\begin{aligned} T(r, f) &\leq \sum_{j=1}^3 \bar{Z}(r, f - h_j) + o(T(r, f)) \\ (\text{resp. } T(r, f) &\leq \sum_{j=1}^3 \bar{Z}(r, f - h_j) + o(T(r, f)), \\ \text{resp. } T_R(r, f) &\leq \sum_{j=1}^3 \bar{Z}(r, f - h_j) + o(T(r, f))). \end{aligned}$$

Considering a small function h_3 whose zeros are the poles of f_1 and f_2 , we can derive Corollary 3.6:

Corollary 3.6: Let $f \in \mathcal{M}(\mathbb{K})$, resp. $f \in \mathcal{M}_u(D)$, resp. $f \in \mathcal{M}^c(E)$ having finitely many poles and let $h_1, h_2 \in \mathcal{M}(\mathbb{K})$, resp. $h_1, h_2 \in \mathcal{M}(D)$, resp. $h_1, h_2 \in \mathcal{M}(E)$ be small functions with respect to f . Then

$$\begin{aligned}
T(r, f) &\leq \sum_{j=1}^2 \overline{Z}(r, f - h_j) + o(T(r, f)) \\
(\text{resp. } T(r, f) &\leq \sum_{j=1}^2 \overline{Z}(r, f - h_j) + o(T(r, f)), \\
\text{resp. } T_R(r, f) &\leq \sum_{j=1}^2 \overline{Z}(r, f - h_j) + o(T(r, f))).
\end{aligned}$$

Now we have to recall the function $m(r, f)$ defined for a function $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}(D)$ by $m(r, f) = T(r, f) - N(r, f)$ and for a function $f \in \mathcal{M}(E)$ by $m_R(r, f) = T_R(r, f) - N_R(r, f)$. This function satisfies the Logarithmic derivative lemma [9]:

Lemma 3.7: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then given $k \in \mathbb{N}^*$, $m(r, \frac{f^k}{f}) = S(r, f)$. Let $f \in \mathcal{M}(E)$. Then given $k \in \mathbb{N}^*$, $m_R(r, \frac{f^k}{f}) = S(r, f)$.*

4 Proofs of Theorems 2.1 and 2.2.

Notation: Let f be a meromorphic function in an annulus which is the set of $x \in \mathbb{K}$ such that $R < |x| < S$ and let us fix $r \in]R, S[$. We know that $|f(x)|$ admits a limit when $|x|$ tends to r , but is not equal to r [4], [5]. This limit is usually denoted $|f|(r)$.

Lemma 4.1: *Let $f \in \mathcal{M}(\mathbb{K})$, (resp. let $f \in \mathcal{M}_u(D)$, resp. let $f \in \mathcal{M}^c(E)$) be nonconstant and let h_1, \dots, h_5 be distinct small functions with respect to f . We have*

$$2T(r, f) \leq \sum_{i=1}^5 \overline{Z}(r, f - h_i) + S(r, f).$$

Proof: We first suppose $f \in \mathcal{M}(\mathbb{K})$, or $f \in \mathcal{M}_u(D)$. Let $g = \frac{(f - h_2)(h_3 - h_1)}{(f - h_1)(h_3 - h_2)}$. Then it is easily seen that:

$$\begin{aligned}
\overline{Z}(r, g) &= \overline{Z}(r, f - h_2) + S(r, f), \\
\overline{N}(r, g) &= \overline{Z}(r, f - h_1) + S(r, f) \\
\overline{Z}(r, g - 1) &= \overline{Z}(r, f - h_3) + S(r, f).
\end{aligned}$$

Consequently, in order to prove Lemma 4.1, it is sufficient to prove the following inequality:

$$(4.1) \quad 2T(r, f) \leq \overline{N}(r, f) + \overline{Z}(r, f) + \overline{Z}(r, f - 1) + \overline{Z}(r, f - h_4) + \overline{Z}(r, f - h_5) + S(r, f)$$

where $h_1 = \infty$, $h_2 = 0$, $h_3 = 1$ and h_4 and h_5 are two small functions with respect to f different from $0, 1, \infty$ and $h_4 \neq h_5$.

If one of the functions h_4, h_5 is a constant, then (4.1) is immediate by the second main theorem for constants 3.3. Consequently, now we can assume that both h_4, h_5 are nonconstant small functions.

Let

$$H = \det \begin{pmatrix} ff' & f' & f(f-1) \\ h_4 h_4' & h_4' & h_4(h_4-1) \\ h_5 h_5' & h_5' & h_5(h_5-1) \end{pmatrix}.$$

By a simple computation, we get:

$$(4.2) \quad H = f(f-1)h_4(h_4-1)h_5(h_5-1) \left(\left(\frac{h_4'}{h_4} - \frac{h_5'}{h_5} \right) \left(\frac{f'}{f-1} - \frac{h_5'}{h_5-1} \right) - \left(\frac{h_4'}{h_4-1} - \frac{h_5'}{h_5-1} \right) \left(\frac{f'}{f} - \frac{h_5'}{h_5} \right) \right).$$

Suppose first that H is identically zero. Since f is not a constant and since h_4, h_5 are not identically 0 or 1, it follows from the definition of H that

$$(4.3) \quad \begin{aligned} & \left(\frac{h_4'}{h_4} - \frac{h_5'}{h_5} \right) \frac{f'}{f-1} - \left(\frac{h_4'}{h_4-1} - \frac{h_5'}{h_5-1} \right) \frac{f'}{f} \\ & \equiv \left(\frac{h_4'}{h_4} - \frac{h_5'}{h_5} \right) \frac{fh_5'}{h_5-1} - \left(\frac{h_4'}{h_4-1} - \frac{h_5'}{h_5-1} \right) \frac{h_5'}{h_5}. \end{aligned}$$

We must now distinguish 4 cases.

Case 1. $\frac{h_4'}{h_4} = \frac{h_5'}{h_5}$. It follows from (4.3) that either $\frac{h_4'}{h_4-1} = \frac{h_5'}{h_5-1}$ or $\frac{f'}{f} = \frac{h_5'}{h_5}$. If $\frac{h_4'}{h_4-1} = \frac{h_5'}{h_5-1}$, then h_4 and h_5 are constants, which is excluded by hypothesis. Hence $\frac{f'}{f} = \frac{h_5'}{h_5}$ and then f is of the form ch_5 , where c is a constant, which contradicts our hypothesis: h_5 is a small function with respect to f .

Case 2. $\frac{h_4'}{h_4-1} = \frac{h_5'}{h_5-1}$. Similarly to Case 1, we have a contradiction.

Case 3. $\frac{h_4'}{h_4} - \frac{h_5'}{h_5} = \frac{h_4'}{h_4-1} - \frac{h_5'}{h_5-1} \neq 0$. It follows from (4.3) that

$$\frac{f'}{f-1} - \frac{f'}{f} = \frac{h_5'}{h_5-1} - \frac{h_5'}{h_5},$$

which implies

$$\frac{f-1}{f} = \frac{C(h_5-1)}{h_5}$$

where C is a constant. Thus we obtain

$$\frac{1}{f} = 1 - \frac{C(h_5-1)}{h_5}$$

and hence $T(r, \frac{1}{f}) = S(r, f)$, which is absurd because $T(r, \frac{1}{f}) = T(r, f) + S(r, f)$.

Case 4.

$$\frac{h_4'}{h_4} \neq \frac{h_5'}{h_5}, \quad \frac{h_4'}{h_4-1} \neq \frac{h_5'}{h_5-1} \quad \text{and} \quad \frac{h_4'}{h_4} - \frac{h_5'}{h_5} \neq \frac{h_4'}{h_4-1} - \frac{h_5'}{h_5-1}.$$

Then it follows from (4.3) that the zeros of $f-1$ can only occur when h_j takes value 0 or 1, for $j = 4, 5$ or when $\frac{h_4'}{h_4} - \frac{h_5'}{h_5}$ has a zero. In the same way, the zeros of f can only occur when h_j

takes value 0 or 1, for $j = 4, 5$ and when $\frac{h'_4}{h_4 - 1} - \frac{h'_5}{h_5 - 1}$ has a zero. Moreover, by (4.3) we can see that the poles of f can only occur when h_j takes value 1 or when h_j has a pole, for $j = 4, 5$ or when $\frac{h'_4}{h_4} - \frac{h'_5}{h_5} - \frac{h'_4}{h_4 - 1} + \frac{h'_5}{h_5 - 1}$ has a zero. Therefore we have

$$(4.4) \quad \overline{N}(r, f) + \overline{Z}(r, f) + \overline{Z}(r, f - 1) = S(r, f).$$

Applying the second Main Theorem 3.3 to f we can derive

$$T(r, f) \leq \overline{Z}(r, f) + \overline{Z}(r, f - 1) + \overline{N}(r, f) - \log(r) = S(r, f),$$

which is absurd and finishes proving that $H \neq 0$.

Now, given $r > 0$, we put

$$\delta(r) = \min(1, |h_4|(r), |h_5|(r), |h_4 - 1|(r), |h_5 - 1|(r), |h_4 - h_5|(r)).$$

Then we have

$$\begin{aligned} \log^+\left(\frac{1}{\delta(r)}\right) &\leq \log^+\left(\max\left(1, \frac{1}{|h_4|(r)}, \frac{1}{|h_5|(r)}, \frac{1}{|h_4 - 1|(r)}, \frac{1}{|h_5 - 1|(r)}, \frac{1}{|h_4 - h_5|(r)}\right)\right) \\ &\leq \log^+\left(\frac{1}{|h_4|(r)}\right) + \log^+\left(\frac{1}{|h_5|(r)}\right) + \log^+\left(\frac{1}{|h_4 - 1|(r)}\right) + \log^+\left(\frac{1}{|h_5 - 1|(r)}\right) + \log^+\left(\frac{1}{|h_4 - h_5|(r)}\right) \\ &\quad + \log(6) = m\left(r, \frac{1}{h_4}\right) + m\left(r, \frac{1}{h_5}\right) + m\left(r, \frac{1}{h_4 - 1}\right) + m\left(r, \frac{1}{h_5 - 1}\right) + m\left(r, \frac{1}{h_4 - h_5}\right) + \log(6). \end{aligned}$$

Consequently, we can see that

$$(4.5) \quad \log^+\left(\frac{1}{\delta(r)}\right) = S(r, f).$$

First, we consider the case when $|f - h_j|(r) > \frac{\delta(r)}{2}$ for $2 \leq j \leq 5$. Then, we have

$$(4.6) \quad m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f - 1}\right) + m\left(r, \frac{1}{f - h_4}\right) + m\left(r, \frac{1}{f - h_5}\right) \leq 5 \log^+\left(\frac{1}{\delta(r)}\right) + O(1) = S(f, r).$$

Now, let i be an index such that $2 \leq i \leq 5$ such that $|f - h_i|(r) \leq \frac{\delta(r)}{2}$. Then for every $j \neq i$, with $2 \leq j \leq 5$, we have

$$\delta(r) \leq |h_i - h_j|(r) \leq |f - h_i|(r) + |f - h_j|(r) \leq \frac{\delta(r)}{2} + |f - h_j|(r)$$

hence

$$\frac{\delta(r)}{2} \leq |f - h_j|(r).$$

Therefore, for $i \neq j$, we have

$$\sum_{\substack{j=2 \\ j \neq i}}^5 m\left(r, \frac{1}{f - h_j}\right) = \sum_{\substack{j=2 \\ j \neq i}}^5 \log^+\left(\frac{1}{|f - h_j|(r)}\right) \leq 3 \log^+\left(\frac{1}{\delta(r)}\right).$$

Combining (4.5) and the last inequality, we have

$$(4.7) \quad \sum_{\substack{j=2 \\ j \neq i}}^5 m(r, \frac{1}{f-h_j}) = S(r, f).$$

On the other hand, for $2 \leq i \leq 5$, we can write

$$\begin{aligned} ff' &= (f-h_i)(f'-h'_i) + h'_i(f-h_i) + h_i(f'-h'_i) + h_i h'_i \\ f' &= (f'-h'_i) + h'_i, \\ f(f-1) &= f^2 - f = (f-h_i)^2 + (2h_i-1)(f-h_i) + h_i^2 - h_i \end{aligned}$$

And now we put $g_i = (f-h_i)(f'-h'_i) + h'_i(f-h_i) + h_i(f'-h'_i)$ and $l_i = (f-h_i)^2 + (2h_i-1)(f-h_i)$ and then, thanks to properties of determinants, we obtain

$$(4.8) \quad H = \det \begin{pmatrix} g'_i & f'-h'_i & l_i \\ h_4 h'_4 & h'_4 & h_4(h_4-1) \\ h_5 h'_5 & h'_5 & h_5(h_5-1) \end{pmatrix}.$$

Now, we have $\log^+(\delta(r)) \leq \log^+(1+|h_i|(r)) \forall i = 2, \dots, 5$ because it is obvious from the definition for $i = 4, 5$ and also for $i = 2$ and $i = 3$ because $h_2 = 0$ and $h_3 = 1$. Consequently, for every $i = 2, \dots, 5$, we have $\log^+(\delta(r)) \leq \log^+(|h_i|(r)) + \log(2) = m(r, h_i) + \log(2) = S(r, f)$. Then by (4.8) and using Lemma 3.7, we obtain for every $i = 2, \dots, 5$:

$$\begin{aligned} \log^+ \left| \frac{H}{f-h_i} \right|(r) &\leq \log^+ \left(\left| \frac{f'-h'_i}{f-h_i} \right|(r) \right) + \log^+(|f-h_i|(r)) \\ &+ O\left(\log^+(|h_i|(r)) + \log^+(|h'_i|(r)) + \log^+(|h_4(r)|) + \log^+(|h'_4|(r)) + \log^+(|h_5(r)|) + \log^+(|h'_5|(r)) \right) \\ &\leq m\left(r, \frac{f'-h'_i}{f-h_i}\right) + \log^+(\delta(r)) + S(r, f) = S(r, f) \end{aligned}$$

Hence, for every $i = 2, \dots, 5$ we obtain

$$(4.9) \quad \begin{aligned} m\left(\frac{1}{f-h_i}\right) &= \log^+ \left(\frac{1}{|f-h_i|(r)} \right) \leq \log^+ \left| \frac{H}{f-h_i} \right|(r) + \log^+ \left| \frac{1}{H} \right|(r) \\ &\leq m\left(r, \frac{1}{H}\right) + S(r, f). \end{aligned}$$

Then by (4.6), (4.7), (4.8) and (4.9) we can check that in all cases we have

$$(4.10) \quad m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f-1}\right) + m\left(r, \frac{1}{f-h_4}\right) + m\left(r, \frac{1}{f-h_5}\right) \leq m\left(r, \frac{1}{H}\right) + S(r, f).$$

Now, by Theorem 3.1, we can write

$$\begin{aligned} 4T(r, f) &= m\left(r, \frac{1}{f}\right) + m\left(r, \frac{1}{f-1}\right) + m\left(r, \frac{1}{f-h_4}\right) + m\left(r, \frac{1}{f-h_5}\right) \\ &+ N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-1}\right) + N\left(r, \frac{1}{f-h_4}\right) + N\left(r, \frac{1}{f-h_5}\right) + S(r, f) \end{aligned}$$

hence by (4.10):

$$\begin{aligned} 4T(r, f) &\leq m(r, \frac{1}{H}) + N(r, \frac{1}{f}) + N(r, \frac{1}{f-1}) + N(r, \frac{1}{f-h_4}) + N(r, \frac{1}{f-h_5}) + S(r, f). \\ (4.11) \quad &\leq T(r, H) - Z(r, H) + Z(r, f) + Z(r, f-1) + Z(r, f-h_4) + Z(r, f-h_5) + S(r, f). \end{aligned}$$

Now, given a zero α of order s of any function $f - h_i$ ($2 \leq i \leq 5$), then α is also a zero of H of order at least $s - 1$. Then from (4.11) we can see that we have:

$$(4.12) \quad 4T(r, f) \leq \bar{Z}(r, f) + \bar{Z}(r, f-1) + \bar{Z}(r, f-h_4) + \bar{Z}(r, f-h_5) + T(r, H) + S(r, f).$$

Next, by (4.2) we can check that

$$\begin{aligned} m(r, H) &\leq 2m(r, f) + S(r, f) \\ N(r, H) &\leq 2N(r, f) + \bar{N}(r, f) + S(r, f) \end{aligned}$$

Consequently

$$(4.13) \quad T(r, H) \leq 2T(r, f) + \bar{N}(r, f) + S(r, f).$$

Then by (4.12) and (4.13), we obtain

$$2T(r, f) \leq \bar{N}(r, f) + \bar{Z}(r, f) + \bar{Z}(r, f-1) + \bar{Z}(r, f-h_4) + \bar{Z}(r, f-h_5) + S(r, f)$$

which finishes proving (4.1) and hence ends the proof of Lemma 4.1 when f belongs to $\mathcal{M}(\mathbb{K})$ or $\mathcal{M}_u(D)$. Next, when f belongs to $\mathcal{M}^c(E)$ we have a similar proof, writing corresponding counting functions $T_R(r, \cdot)$ instead of $T(r, \cdot)$, $Z_R(r, \cdot)$ instead of $Z(r, \cdot)$, $N_R(r, \cdot)$ instead of $N(r, \cdot)$, etc...

Proof of Theorem 2.1: By Lemma 4.1, for every subset $\{i_1, \dots, i_5\}$ of $\{1, \dots, q\}$ such that $1 \leq i_1 < \dots < i_5 \leq q$, we have

$$2T(r, f) \leq \sum_{s=1}^5 \bar{Z}(r, f - h_{i_s}) + S(r, f),$$

(resp.

$$2T(r, f) \leq \sum_{s=1}^5 \bar{Z}(r, f - h_{i_s}) + S(r, f),$$

resp.

$$2T_R(r, f) \leq \sum_{s=1}^5 \bar{Z}_R(r, f - h_{i_s}) + S(r, f).$$

The number of such inequalities is C_q^5 . Summing up over all subsets $\{i_1, \dots, i_5\}$ of $\{1, \dots, q\}$, we can get

$$\begin{aligned} &2C_q^5 T(r, f) \\ &\leq \sum_{\substack{\{i_1, \dots, i_5\} \subset \{1, \dots, q\} \\ 1 \leq i_1 < \dots < i_5 \leq q}} \left(\bar{Z}(r, f - h_{i_1}) + \bar{Z}(r, h_{i_2}) + \bar{Z}(r, h_{i_3}) + \bar{Z}(r, h_{i_4}) + \bar{Z}(r, h_{i_5}) \right) + S(r, f) \end{aligned}$$

(resp.

$$2C_q^5 T(r, f)$$

$$\leq \sum_{\substack{\{i_1, \dots, i_5\} \subset \{1, \dots, q\} \\ 1 \leq i_1 < \dots < i_5 \leq q}} \left(\overline{Z}(r, f - h_{i_1}) + \overline{Z}(r, h_{i_2}) + \overline{Z}(r, h_{i_3}) + \overline{Z}(r, h_{i_4}) + \overline{Z}(r, h_{i_5}) \right) + S(r, f),$$

resp.

$$2C_q^5 T_R(r, f)$$

$$\leq \sum_{\substack{\{i_1, \dots, i_5\} \subset \{1, \dots, q\} \\ 1 \leq i_1 < \dots < i_5 \leq q}} \left(\overline{Z}_R(r, f - h_{i_1}) + \overline{Z}_R(r, h_{i_2}) + \overline{Z}_R(r, h_{i_3}) + \overline{Z}_R(r, h_{i_4}) + \overline{Z}_R(r, h_{i_5}) \right) + S(r, f).$$

In each one of the last inequalities, we can check that for each index i_k , the term $\overline{Z}(r, f - h_{i_k})$ (resp. $\overline{Z}_R(r, f - h_{i_k})$, resp. $\overline{Z}_R(r, f - h_{i_k})$) intervenes C_{q-1}^4 times. Consequently, we can derive that

$$2C_q^5 T(r, f) \leq C_{q-1}^4 \sum_{i=1}^q \overline{Z}(r, f - h_i) + S(r, f),$$

(resp.

$$2C_q^5 T(r, f) \leq C_{q-1}^4 \sum_{i=1}^q \overline{Z}(r, f - h_i) + S(r, f),$$

resp.

$$2C_q^5 T_R(r, f) \leq C_{q-1}^4 \sum_{i=1}^q \overline{Z}_R(r, f - h_i) + S(r, f).$$

Consequently, it follows that

$$\frac{2q}{5} T(r, f) \leq \sum_{i=1}^q \overline{Z}(r, f - h_i) + S(r, f),$$

(resp.

$$\frac{2q}{5} T(r, f) \leq \sum_{i=1}^q \overline{Z}(r, f - h_i) + S(r, f),$$

resp.

$$\frac{2q}{5} T_R(r, f) \leq \sum_{i=1}^q \overline{Z}_R(r, f - h_i) + S(r, f).$$

This completes the proof of Theorem 2.1.

Proof of Theorem 2.2: Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and let f_j , $1 \leq j \leq 5$ be archi-branched small functions with respect to f . Let us apply Theorem 2.1:

$$(1) \quad 5T(r, f) \leq \frac{5}{2} \left(\sum_{j=1}^5 \overline{Z}(r, f - h_j) \right) + o(T(r, f)).$$

But for each $j = 1, \dots, 5$, we have

$$\overline{Z}(r, f - h_j) \leq \frac{Z(r, f - h_j)}{3} + o(T(r, f)) \leq \frac{T(r, f)}{3} + o(T(r, f))$$

hence $\sum_{j=1}^5 \overline{Z}(r, f - h_j) \leq T(r, f) + o(T(r, f))$, therefore by (1):

$$5T(r, f) \leq \left(\frac{5}{2}\right) \frac{5T(r, f)}{3} + o(T(r, f)),$$

which is absurd and finishes the proof in the general case when $f \in \mathcal{M}(\mathbb{K})$. The proof is similar when $f \in \mathcal{M}_u(D)$ and when $f \in \mathcal{M}^c(E)$.

Suppose now f, h_1, h_2 have finitely many poles. By Corollary 3.6 we have

$$T(r, f) \leq \overline{Z}(r, f - h_1) + \overline{Z}(r, f - h_2) + o(T(r, f))$$

hence

$$T(r, f) \leq \frac{Z(r, f - h_1) + Z(r, f - h_2)}{3} + o(T(r, f)) \leq \frac{2T(r, f)}{3} + o(T(r, f))$$

which is absurd. Similar proofs when $f \in \mathcal{M}_u(D)$ and when $f \in \mathcal{M}^c(E)$.

5 Proof of Theorem 2.3.

We first need to establish the following Lemma 5.1.

Lemma 5.1: *Let f, g be distinct and belong to $\mathcal{M}(\mathbb{K})$, (resp. to $\mathcal{M}_u(D)$, resp. to $\mathcal{M}^c(E)$) sharing q distinct small functions ignoring multiplicity: h_1, \dots, h_q with $q \geq 5$. Then, for every subset $\{i_1, i_2, i_3, i_4\}$ of $\{1, \dots, q\}$, we have*

$$\sum_{j \in \{1, \dots, q\} \setminus \{i_1, \dots, i_4\}} \overline{Z}(r, f - h_j) + \overline{Z}(r, g - h_j) \leq S(r, f) + S(r, g).$$

(resp.

$$\sum_{j \in \{1, \dots, q\} \setminus \{i_1, \dots, i_4\}} \overline{Z}(r, f - h_j) + \overline{Z}(r, g - h_j) \leq S(r, f) + S(r, g),$$

resp.

$$\sum_{j \in \{1, \dots, q\} \setminus \{i_1, \dots, i_4\}} \overline{Z}_R(r, f - h_j) + \overline{Z}_R(r, g - h_j) \leq S(r, f) + S(r, g).$$

Proof: Let us first suppose $f, g \in \mathcal{M}(\mathbb{K})$, or $f, g \in \mathcal{M}_u(D)$. Clearly, we only have to prove that

$$\sum_{j=5}^q \overline{Z}(r, f - h_j) \leq S(r, f) + S(r, g).$$

Thus, we assume that

$$\sum_{j=5}^q \overline{Z}(r, f - h_j) \neq S(r, f) + S(r, g).$$

Given any meromorphic function $w \in \mathcal{M}(\mathbb{K})$, (resp. in $\mathcal{M}_u(D)$) we put

$$L(w) = \frac{(w - h_1)(h_3 - h_2)}{(w - h_2)(h_3 - h_4)}$$

and $F = L(f)$, $G = L(g)$.

Without loss of generality, we may assume that $h_1 = \infty$, $h_2 = 0$, $h_3 = 1$ and we put $h = h_4$. Now, we set

$$B = \frac{f'(h'g - hg')(f - g)}{f(f - 1)(g - h)} - \frac{g'(h'f - hf')(f - g)}{g(g - 1)(f - h)}.$$

Now, let

$$\begin{aligned} Q &= f'(h'g - hg')(f - h)(g - 1) - g'(h'f - hf')(g - h)(f - 1) \\ &= h'ff'g^2 - h'f'g - h(h - 1)ff'g' - hh'f'g^2 + hh'f'g \\ &\quad - h'f^2gg' + h'f'gg' + h(h - 1)f'gg' + hh'f^2g' - hh'f'g'. \end{aligned}$$

Then

$$B = \frac{(f - g)Q}{f(f - 1)g(g - 1)}.$$

Suppose first that B is identically zero. Then we have

$$(5.1) \quad \frac{f'(h'g - hg')(f - g)}{f(f - 1)(g - h)} = \frac{g'(h'f - hf')(f - g)}{g(g - 1)(f - h)}.$$

If h is a constant, then $f = g$ which contradicts our hypothesis. Hence h is not a constant and then, by (5.1) we have

$$\frac{(f - 1)(g - h)}{g - 1)(f - h)} - 1 = \frac{f'(h'g - hg')}{g'(h'f - hf')} - 1$$

therefore

$$\frac{(f - g)(g - h)}{(g - 1)(f - h)} = \frac{h'(f' - g')g - (f - g)g'}{g'(h'f - hf')}$$

and hence

$$(5.2) \quad \frac{f' - g'}{f - g} = \frac{(1 - h)g'(h'f - hf')}{h'g(g - 1)(f - h)} + \frac{g'}{g}.$$

Let us fix $j \geq 5$, $j \leq q$. By (5.1), there exists a common zero $\alpha \in \mathbb{K}$ (resp. $\alpha \in D$, resp. $\alpha \in E$) of $f - h_j$ and $g - h_j$ which is not a zero or a pole of $h, h', h_j - 1, h_j - h$. Then, α must be a pole of the left hand side of (5.2) but cannot be a pole of the right hand side of (5.2). This is a contradiction showing that B cannot be identically zero.

Now, let us fix $j \geq 5$ such that $j \leq q$ and suppose that β is a common zero of $f - h_j$ and $g - h_j$ but is not a zero or a pole of $h, h_j, h_j - 1, h_j - h$. Then it is a zero of $f - g$ and it is not a pole of

$$\frac{Q}{f(f - 1)(f - h)g(g - 1)(g - h)},$$

and hence it is a zero of B .

Now, since f and g share each h_j ignoring multiplicity, we have

$$\sum_{j=1}^5 \bar{Z}(r, g - h_j) = \sum_{j=1}^5 \bar{Z}(r, f - h_j) \leq Z(r, B) + S(r, f) + S(r, g)$$

$$(5.3) \quad \leq T(r, B) + S(r, f) + S(r, g) = m(r, B) + N(r, B) + S(r, f) + S(r, g).$$

We will estimate $m(r, B)$. By computation, B can be written

$$\begin{aligned} B &= \frac{f'(h'g - hg')}{(f-1)g(g-h)} - \left(\frac{f'}{f-1} - \frac{f'}{f}\right) \left(\frac{h'g - hg'}{g-h}\right) \\ &\quad + \frac{g'(h'f - hf')}{(g-1)f(f-h)} - \left(\frac{g'}{g-1} - \frac{g'}{g}\right) \left(\frac{h'f - hf'}{f-h}\right) \\ &= \left(\frac{f'}{f-1}\right) \left(\frac{g'}{g} - \frac{g' - h'}{g-h}\right) - \left(\frac{f'}{f-1} - \frac{f'}{f}\right) \left(h' - \frac{h(g' - h')}{g-h}\right) \\ &\quad + \left(\frac{g'}{g-1}\right) \left(\frac{f'}{f} - \frac{f' - h'}{f-h}\right) - \left(\frac{g'}{g-1} - \frac{g'}{g}\right) \left(h' - \frac{h(f' - h')}{f-h}\right) \end{aligned}$$

In this last relation we have a sum of logarithmic derivative and hence, by Lemma 3.7, we can see that

$$(5.4) \quad m(r, B) = S(r, f) + S(r, g).$$

Next, we will estimate $N(r, B)$. Clearly, the poles of B can only occur at the zeros of $f, g, f-1, g-1, f-h, g-h$ and at the poles of f, g, h .

Let A be the set of all zeros, 1-points and poles of h . Then we will first estimate the counting function of poles of B when we are in A . Recall that f and g share the constants 0, 1 and ∞ .

Let γ be a zero of h . Then we can check that B has no pole at γ .

Let γ be a 1-value of h . If B had a pole, then $f-1$ would have a zero of order s , $g-1$ would have a zero of order t , $f-g$ would have a zero and Q would have a zero of order $s+t$, hence B would have a zero, not a pole.

Now, let γ be a pole of h of order n .

If $f(\gamma) = 1$, then $g(\gamma) = 1$ because f and g share h_3 , hence B has no pole at γ .

If f has a pole of order s at γ then g also has a pole of order t because f and g share h_1 . We can assume $s \geq t$, then B has a pole of order at most $n+2$, hence the counting function of poles of B when γ is a pole of f, g and h and hence is a pole of B of order $n+2$, is bounded by thrice the counting function of the poles of h and hence is of the form $S(r, f) + S(r, g)$.

Next, suppose that $f(\gamma) \neq 1, \infty$, hence $g(\gamma) \neq 1, \infty$. If f and g have no zero at γ , it is clear that B has at most a pole of order n , hence the counting function of poles of B when $f(\gamma) \neq 0, 1, \infty$, (hence $g(\gamma) \neq 0, 1, \infty$) is of the form $S(r, f) + S(r, g)$. Finally, if $f(\gamma) = 0$, then $g(\gamma) = 0$ because f and g share h_2 and then we can check that B may admits a pole at γ of order at most $2n+2$. Consequently the counting function of poles of B when γ lies in A and is a pole of h is bounded by thrice the counting function of the poles of h and hence is of the form $S(r, f) + S(r, g)$.

Thus, the counting function of poles of B when γ lies in A is of the form $S(r, f) + S(r, g)$.

Consider now the poles of B when $\gamma \notin A$.

Let $\gamma \notin A$ be a common pole of f and g of order s_1 and t_1 respectively. Then, γ is a pole of Q of order at most $2s_1 + 2t_1 + 1$. Next, γ is a pole of $f-g$ of order $\max(s_1, t_1)$. Hence from the definition of B , γ is a pole of the numerator of B of order at most $2s_1 + 2t_1 + 1 + \max(s_1, t_1)$ and on the other hand, it is a pole of the denominator of B of order $3s_1 + 3t_1$. But since

$$2s_1 + 2t_1 + 1 + \max(s_1, t_1) - 3(s_1 + t_1) = 1 + \max(s_1, t_1) - s_1 - t_1 \leq 0,$$

it follows that γ is not a pole of B .

Let $\gamma \notin A$ be a common zero of f and g of order s_2 and t_2 respectively. We can check that γ is a zero of Q of order at least $s_2 + t_2 + 1$ and that γ is a zero of $f - g$ of order at least $\min(s_2, t_2)$. On the other hand, we can check that the denominator of B has a zero at γ of order at most $s_2 + t_2$. Consequently, B has no pole at γ .

Suppose now that $\gamma \notin A$ is a common zero of $f - h$ and $g - h$. Then γ is a zero of $f - g$. On the other hand, γ is a pole of order 1 of $\frac{f'}{f - h}$ and $\frac{g'}{g - h}$. Consequently, by the definition of B we can check that γ is not a pole of B . Similarly, when γ is a common zero of $f - 1$ and $g - 1$, γ is not a pole of B .

Suppose now that $\gamma \notin A$ is a pole of f but is not a pole of g and is not a zero of g , $g - 1$, and $g - h$. Since f and g share h_0 , that situation does not occur.

Suppose now that $\gamma \notin A$ is a zero of f but is not a pole of g and is not a zero of g . Since f and g share h_0 , that situation does not occur.

Suppose now that $\gamma \notin A$ is a zero of $f - h_i$ with $i = 3, 4$ but is not a pole of g and is not a zero of g , $g - 1$ and $g - h$. Then γ is a pole of B of order at most 1. Similarly, if γ is a zero of $g - h_i$ with $i = 3, 4$ but is not a pole of f and is not a zero of f , $f - 1$ and $f - h$. Then γ is a pole of B of order at most 1. However, since f and g share h_3 and h_4 , such a situation does not occur.

Suppose now that $\gamma \notin A$ is a common zero of $f - h_i$ and $g - h_j$ for some $1 \leq i \leq 4$ and $1 \leq j \leq 4$ with $i \neq j$ and $i \neq 2$, $j \neq 2$. Then γ is a pole of B of order at most 2. However, in such a situation, since f and g share h_i , then γ is a zero of $h_i - h_j$. But since h_i and h_j are small with respect to f and g , the counting function of points γ that are common zeros of $f - h_i$ and $g - h_j$ for some $1 \leq i \leq 4$ and $1 \leq j \leq 4$ with $i \neq j$ and $i \neq 2$, $j \neq 2$, is bounded by $S(r, f) + S(r, g)$.

Finally, suppose that $\gamma \notin A$ is a zero of $f - h_i$ and a pole of g for some $i = 2, 3, 4$ or a zero of $g - h_i$ and a pole of f for some $i = 2, 3, 4$. But since f and g share h_2 , h_3 , h_4 , such a situation is impossible.

In conclusion, we have

$$N(r, B) \leq S(r, f) + S(r, g)$$

and therefore, by (5.3) and (5.4) that completes the proof of Lemma 5.1 when $f, g \in \mathcal{M}(\mathbb{K})$, or $f, g \in \mathcal{M}_u(D)$. Now, when $f, g \in \mathcal{M}^c(E)$, the proof is similar.

Corollary 5.2: *Let f, g be distinct and belong to $\mathcal{M}(\mathbb{K})$, (resp. to $\mathcal{M}_u(D)$), (resp. to $\mathcal{M}^c(E)$) sharing q distinct small functions ignoring multiplicity: h_1, \dots, h_q (with $q \geq 5$). Then, for every subset $\{i_1, i_2, i_3, i_4\}$ of $\{1, \dots, q\}$, we have*

$$\overline{Z}(r, f - h_j) + \overline{Z}(r, g - h_j) \leq S(r, f) + S(r, g) \quad \forall j = 1, \dots, q.$$

(resp.

$$\overline{Z}(r, f - h_j) + \overline{Z}(r, g - h_j) \leq S(r, f) + S(r, g) \quad \forall j = 1, \dots, q$$

resp.

$$\overline{Z}_R(r, f - h_j) + \overline{Z}_R(r, g - h_j) \leq S(r, f) + S(r, g) \quad \forall j = 1, \dots, q.)$$

Proof of Theorem 2.3: Suppose first that $f, g \in \mathcal{M}(\mathbb{K})$ or $f, g \in \mathcal{M}_u(D)$. Suppose that f and g are not identic. Applying Corollary 5.2 with $q = 5$, we have $\overline{Z}(r, f - h_j) = S(r, f) + S(r, g)$ and $\overline{Z}(r, g - h_j) = S(r, f) + S(r, g)$ for every $j \in \{1, \dots, 5\}$. Therefore:

$$(5.5) \quad \sum_{j=1}^5 \bar{Z}(r, f - h_j) + \bar{Z}(r, g - h_j) = S(r, f) + S(r, g).$$

Now, by Theorem 2.1, we have

$$2(T(r, f) + T(r, g)) \leq \sum_{j=1}^5 \bar{Z}(r, f - h_j) + \bar{Z}(r, g - h_j) + S(r, f) + S(r, g)$$

hence, by (5.5): $2(T(r, f) + T(r, g)) \leq S(r, f) + S(r, g)$, which is absurd and proves that $f = g$.

Suppose now that $f, g \in \mathcal{M}^c(E)$. Replacing each symbol $T(r, \cdot)$ by $T_R(r, \cdot)$ and $\bar{Z}(r, \cdot)$ by $\bar{Z}_R(r, \cdot)$ we can make the same reasoning and conclude in the same way $f = g$.

The conclusion concerning meromorphic functions f, g having finitely many poles comes from [7] and from [8], Theorem C.9.18 and Theorem C.9.16.

Remark: After obtaining Relation (5.5), by Theorem A, we could state that

$$\frac{5}{3}(T(r, f) + T(r, g)) \leq \sum_{j=1}^5 \bar{Z}(r, f - h_j) + \bar{Z}(r, g - h_j) + S(r, f) + S(r, g),$$

which could also let us conclude.

Acknowledgement: I am grateful to the anonymous referee who pointed out to me several misprints.

References

- [1] Boussaf, K. *Motzkin factorization in algebras of analytic elements*, Annales Mathématiques Blaise Pascal 2, n1, p.73-91. (1995).
- [2] Boussaf, K., Escassut, A. and Ojeda, J. *Complex and p-adic branched functions and growth of entire functions*, Bulletin of the Belgian Mathematical Society Simon Stevin 22, p.781-796, (2015).
- [3] Boutabaa, A. *Théorie de Nevanlinna p-adique*, Manuscripta Math. 67, p. 251-269 (1990).
- [4] Escassut, A. *Analytic Elements in p-adic Analysis*, World Scientific Publishing Co. Pte. Ltd. Singapore, (1995).
- [5] Escassut, A. *Value Distribution in p-adic Analysis*. World Scientific Publishing Co. Pte. Ltd. Singapore, (2015).
- [6] Escassut, A. and Ta Thi Hoai An *p-adic Nevanlinna theory outside of a hole* Vietnam Journal of Mathematics 45,n.4, p. 681-694 (2017) .
- [7] Escassut, A. and Yang, C.C. *A short note on two p-adic meromorphic functions sharing a few small ones*, Rendiconti del Circolo Matematico di Palermo, 70 (2), p. 623-630 (2021).

- [8] Escassut, A. *p-adic Analytic Functions*. World Scientific Publishing Co. Pte. Ltd. Singapore, (2021).
- [9] Hu, P.C. and Yang, C.C. *Meromorphic Functions over non-Archimedean Fields*, Kluwer Academic Publishers, (2000).
- [10] Krasner, M. *Prolongement analytique uniforme et multiforme dans les corps valués complets. Les tendances géométriques en algèbre et théorie des nombres*, Clermont-Ferrand, p.94-141 (1964). Centre National de la Recherche Scientifique (1966), (Colloques internationaux de C.N.R.S. Paris, 143).
- [11] Motzkin, E. *La décomposition d'un élément analytique en facteurs singuliers*, Ann. Inst. Fourier 27, n 1, pp. 67-82 (1977).
- [12] Nevanlinna, R. *Le théorème de Picard-Borel et la théorie des fonctions méromorphes*. Gauthiers-Villars, Paris (1929).
- [13] Ta Thi Hoai An and Phuong Nguyen Viet *Non-Archimedean Second Main Theorem sharing small functions*. ArXiv: 2011.08917v2
- [14] Yamanoi, K. *The second main theorem for small functions and related problems*, Acta Mathematica 192, p. 225-294 (2004).

Laboratoire de Mathématiques Blaise Pascal, UMR 6620
Université Clermont Auvergne
63 000 Clermont-Ferrand
FRANCE
alain.escassut@uca.fr