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The p -adic Hayman conjecture A survey with some generalizations

by Alain Escassut

In memory of Professor Wolfgang Tutschke

Abstract

Let \mathbb{K} be a complete ultrametric algebraically closed field of characteristic 0. According to the p -adic Hayman conjecture applied to transcendental meromorphic function f in \mathbb{K} or an "unbounded" meromorphic function inside an open disk, for each $n \in \mathbb{N}^*$, $f^n f'$ takes every value $b \neq 0$ infinitely many times. That was proved for $n \geq 3$ by J. Ojeda and next, by herself and the author for meromorphic function in \mathbb{K} , for $n = 2$. Here we recall these proofs and generalize them to meromorphic functions out of a hole whenever $n \geq 3$. We also recall the proof of this theorem: given a meromorphic function f , there exists at most one small function w such that $f - w$ have finitely many zeros.

1 Introduction and main results

Let \mathbb{K} be a complete ultrametric algebraically closed field of characteristic 0. Given $a \in \mathbb{K}$ and $r > 0$, we denote by $d(a, r)$ the disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$ and by $d(a, r^-)$ the disk $\{x \in \mathbb{K} \mid |x - a| < r\}$. Given $a \in \mathbb{K}$, $r > 0$, $s > r$, we denote by $\Delta(a, r, s)$ the set $\{x \in \mathbb{K} \mid r \leq |x - a| \leq s\}$.

We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} , i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$. We denote by $\mathcal{A}(d(a, r^-))$ the \mathbb{K} -algebra of analytic functions in $d(a, r^-)$, by $\mathcal{A}_b(d(a, r^-))$ the \mathbb{K} -algebra of bounded analytic functions in $d(a, r^-)$ and we put $\mathcal{A}_u(d(a, r^-)) = \mathcal{A}(d(a, r^-)) \setminus \mathcal{A}_b(d(a, r^-))$. We denote by $\mathcal{M}(d(a, r^-))$ the field of fractions of $\mathcal{A}(d(a, r^-))$, by $\mathcal{M}_b(d(a, r^-))$ the field of fractions of $\mathcal{A}_b(d(a, r^-))$, and we put $\mathcal{M}_u(d(a, r^-)) = \mathcal{M}(d(a, r^-)) \setminus \mathcal{M}_b(d(a, r^-))$.

Finally, given $R > 0$, we denote by S the disk $d(0, R^-)$, by D the set $\mathbb{K} \setminus d(0, R^-)$, by $\mathcal{A}(D)$ the \mathbb{K} -algebra of analytic functions in D i.e. the set of Laurent series converging in D , by $\mathcal{M}(D)$ the field of fractions of $\mathcal{A}(D)$ and by $\mathcal{M}^c(D)$ the set of $f \in \mathcal{M}(D)$ having infinitely many zeros or poles in D .

Let f be a transcendental meromorphic function in \mathbb{K} , i.e. $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. let $R > 0$ and let $f \in \mathcal{M}_u(d(0, R^-))$, resp. let $f \in \mathcal{M}^c(D)$) and let $b \in \mathbb{K}$. Similarly to classical definitions in complex analysis [18], b is called *an exceptional value for f* or *a Picard value for f* if $f - b$ has no zero in \mathbb{K} (resp. in $d(0, R^-)$, resp. in D) and b is called *a quasi-exceptional value for f* if $f - b$ has finitely many zeros in \mathbb{K} (resp. in $d(0, R^-)$, resp. in D) [5], [6], [7], [13]. By classical results [11], [12], [16], [19], [25], we know that f has at most one quasi-exceptional value and if $f \in \mathcal{A}(\mathbb{K})$, (resp. if $f \in \mathcal{A}_u(d(a, R^-))$, resp. if $f \in \mathcal{A}^c(D)$) then f has no quasi-exceptional value.

On \mathbb{C} , considering a transcendental meromorphic function f in the whole field, W. Hayman showed that for every $n \geq 3$, the function $f^n f'$ has no quasi-exceptional value different from 0

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⁰Keywords: p -adic meromorphic functions, value distribution, exceptional values

and he conjectured that the statement remains true for $n = 2$ and $n = 1$ [18]. That was proved for $n = 2$ by E.Mues [21] and for $n = 1$ by W. Bergweiler and Eremenko [1] and separately by H. Chen and M. Fang [10].

On the field \mathbb{K} , the same question makes sense too and similarly, J. Ojeda proved that for every $n \geq 3$, $f^n f'$ has no quasi-exceptional value different from 0 [23]. For $n = 2$ and $n = 1$, several particular solutions were given concerning subclasses of meromorphic functions [2], [7], [13], [18] and the general solution was given in [14] for meromorphic functions in \mathbb{K} . However, here we give more precisions on the proof of this very delicate problem in the case $n = 2$. The principal method used in the complex study was the classical Nevanlinna Theory [22] and similarly, here in the p -adic Nevanlinna Theory [8], [9].

We mean to recall results already obtained. The same problem is posed both in $\mathcal{M}(\mathbb{K})$ and in a $\mathcal{M}_u(d(a, R^-))$ ($a \in \mathbb{K}$, $R > 0$) and also in $\mathcal{M}^c(D)$ where it was never published yet.

Small functions are defined among complex meromorphic functions and find the same meaning among meromorphic functions in an ultrametric field, the notion (recalled above) being defined with help of the Nevanlinna characteristic functions. Let us recall that given a meromorphic function $f \in \mathcal{M}(K)$ or $f \in \mathcal{M}_u(d(a, R^-))$, or $f \in \mathcal{M}^c(D)$, there is at most one small function w such that $f - w$ admit finitely many zeros (the proof here is new concerning the case $f \in \mathcal{M}^c(D)$)

Thus, we can now state our main results:

Theorem A: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental. Then for every $n \in \mathbb{N}$, $n \geq 2$, $f' f^n - b$ has infinitely many zeros.*

Theorem B: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. For each $b \in \mathbb{K}^*$, $f' + bf$ has infinitely many zeros that are not zeros of f .*

Theorem C: *Let $a \in \mathbb{K}$, $R > 0$ and let $f \in \mathcal{M}_u(d(a, R^-))$ (resp. let $f \in \mathcal{M}^c(D)$). For every $n \in \mathbb{N}$, $n \geq 3$, for every $b \in \mathbb{K}^*$, $f' f^n - b$ has infinitely many zeros.*

The definition of small functions being recalled below, we have the following Theorem D:

Theorem D: *Let $a \in \mathbb{K}$ and $r > 0$. Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$, (resp. $f \in \mathcal{M}_u(d(a, R^-))$), resp. $f \in \mathcal{M}^c(D)$. There exists at most one function $w \in \mathcal{M}_f(\mathbb{K})$, (resp. $w \in \mathcal{M}_f(d(a, R^-))$), resp. $w \in \mathcal{M}_f(D)$ such that $f - w$ have finitely many zeros. Moreover, if f belongs to $\mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. to $\mathcal{A}_u(d(a, R^-))$), resp. to $\mathcal{A}^c(\mathbb{K})$) then there exists no function $w \in \mathcal{M}_f(\mathbb{K}) \setminus \mathbb{K}(x)$, (resp. $w \in \mathcal{M}_f(d(a, R^-))$), resp. $w \in \mathcal{M}_f(D)$) such that $f - w$ have finitely many zeros.*

2 Generalities

The proofs of these theorems require many technical results and the 2nd Main Theorem of the Nevanlinna Theory in the three situations that we will examine.

Lemma 1 is immediate.

Lemma 1: *Let $g \in \mathcal{M}(\mathbb{K})$ (resp. let $g \in \mathcal{M}(d(a, R^-))$), $a \in \mathbb{K}$, $R > 0$, (resp. $g \in \mathcal{M}(D)$), set $f = \frac{1}{g}$ and let $n \in \mathbb{N}^*$. Then $g' g^n$ admits a quasi-exceptional value $b \in \mathbb{K}^*$ if and only if $f' + b f^{n+2}$ has finitely many zeros that are not zeros of f .*

Notation: Given $a \in \mathbb{K}$ and $r > 0$, we denote by $C(a, r)$ the circle $\{x \in \mathbb{K} \mid |x - a| = r\}$, by \log the Neperian logarithm and by e the number such that $\log(e) = 1$. Next we put $\exp(s) = e^s$, $x \in \mathbb{R}$.

Lemma 2 [16]: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$, $R > 0$, resp. $f \in \mathcal{M}(D)$) and let $r \in \mathbb{R}_+$ (resp. let $r \in]0, R[$, resp. let $r > R$). Then $|f(x)|$ is constant in all disks $d(a, r^-)$ of $C(0, r)$ where f has no zero and no pole and is noted $|f|(r)$. Moreover, the function Ψ in μ defined as $\Psi(\mu) = \log(|f|(\exp(\mu)))$ is continuous and piecewise affine in \mathbb{R} (resp. in $] -\infty, \log(R)[$, resp. in $[\log(R), +\infty[$).

Notation: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$, $R > 0$, resp. $f \in \mathcal{M}(D)$) and let $r \in \mathbb{R}_+$ (resp. let $r \in]0, R[$, resp. let $r > R$). We denote by $\nu^+(f, \mu)$ the right-side derivative of the function Ψ at μ and by $\nu^-(f, \mu)$ the left-side derivative of the function Ψ at μ . When $\nu^+(f, \mu) = \nu^-(f, \mu)$ we just write $\nu(f, \mu)$.

Lemma 3: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$, $R > 0$, suppose that f admits infinitely many zeros in \mathbb{K} (resp. in $d(0, R^-)$) and suppose that there exists a sequence of non-empty intervals $[r'_n, r''_n]$ such that $\lim_{n \rightarrow +\infty} r'_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} r'_n = \lim_{n \rightarrow +\infty} r''_n = R$) and such that $|(f' + f^m)|(r) = |f^m|(r) \forall r \in \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$. Let $m \in \mathbb{N}^*$ be $\neq 2$. Then $f' + f^m$ has infinitely many zeros that are not zeros of f .

Proof. Let $J = \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$. By Corollary B.13.6 in [16], we have

$$\nu^+(f' + f^m, \log r) = \nu^+(f^m, \log r), \quad \nu^-(f' + f^m, \log r) = \nu^-(f^m, \log r) \quad \forall r \in J.$$

Consequently, in each disk $d(0, r)$ with $r \in J$, f and $f' + f^m$ have the same difference between the number of zeros and poles (taking multiplicity into account). Now, if $m \geq 3$ the poles of $f' + f^m$ and f^m are the same taking multiplicity into account. And when $m = 1$, each pole of f is a pole of $f' + f$ with a strictly greater order. Consequently, for each $r \in J$, the number of zeros of $f' + f^m$ in $d(0, r)$ is superior or equal to this of f^m (taking multiplicity into account).

Now, for each $n \in \mathbb{N}$, let s_n be the number of distinct zeros of f in $d(0, r''_n)$. Since f has infinitely many zeros, the sequence s_n is increasing and tends to $+\infty$. On the other hand, for each zero α of order u of f , either α is not a zero of $f' + f^m$ (when $u = 1$), or it is a zero of order $u - 1$. Consequently, the difference between the sum of multiplicities of the number of zeros of $f' + f^m$ in $d(0, r''_n)$ and the number of distinct zeros of f in $d(0, r''_n)$ is at least s_n and hence the number of zeros of $f' + f^m$ in $d(0, r''_n)$ which are not zeros of f in $d(0, r''_n)$ is at least s_n . Thus, we have proved that $f' + f^m$ has infinitely many zeros that are not zeros of f . \square

Lemma 4: Let $f \in \mathcal{M}^c(D)$. Suppose that there exists a sequence of intervals $[r'_n, r''_n]$ such that $\lim_{n \rightarrow +\infty} r'_n = +\infty$, $R < r'_1$ and such that $|(f' + f^m)|(r) = |f^m|(r) \forall r \in \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$ and that f admits

infinitely many zeros in the set $\Lambda = \bigcup_{n=1}^{\infty} \Delta(0, r'_n, r''_n)$. Let $m \in \mathbb{N}^*$ be $\neq 2$. Then $f' + f^m$ has infinitely many zeros in Λ that are not zeros of f .

Proof. Let $J = \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$. By Corollary B.13.6 in [16], we have

$$\nu^+(f' + f^m, \log r) = \nu^+(f^m, \log r), \quad \nu^-(f' + f^m, \log r) = \nu^-(f^m, \log r) \quad \forall r \in J.$$

Consequently, in each annulus $\Delta(0, r'_n, r)$, with $r'_n \leq r \leq r''_n$, f and $f' + f^m$ have the same difference between the number of zeros and poles (taking multiplicity into account).

Now, if $m \geq 3$ the poles of $f' + f^m$ and f^m in $\Delta(0, R, r''_n)$ are the same taking multiplicity into account. And when $m = 1$, each pole of f is a pole of $f' + f$ with a strictly greater order. Consequently, for each $r \in J$, the number of zeros of $f' + f^m$ in $\Delta(0, r'_n, r)$ is superior or equal to this of f^m .

Now, for each $n \in \mathbb{N}$, let s_n be the number of distinct zeros of f in the set $\Lambda_n = \bigcup_{j=1}^n \Delta(0, r'_j, r''_j)$.

Since f has infinitely many zeros in Λ , the sequence s_n is increasing and tends to $+\infty$. On the other hand, for each zero α of f of order u in $\bigcup_{j=1}^n \Delta(0, r'_j, r''_j)$, either α is not a zero of $f' + f^m$ (when $u = 1$), or it is a zero of order $u - 1$. Consequently, the difference between the sum of multiplicities of the number of zeros of $f' + f^m$ in Λ_n and the number of distinct zeros of f in Λ_n is at least s_n . Therefore, as in Lemma 2, the number of zeros of $f' + f^m$ in Λ_n which are not zeros of f is at least s_n . Thus, we have proved that $f' + f^m$ has infinitely many zeros that are not zeros of f in Λ . \square

Lemma 5: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ satisfy $\limsup_{r \rightarrow \infty} |f|(r) > 0$ and let $b \in \mathbb{K}^*$. Let $m \in \mathbb{N}^*$ be ≥ 3 . Then $f' + bf^m$ has infinitely many zeros that are not zeros of f .*

Lemma 6: *Let $f \in \mathcal{M}^c(D)$ satisfy $\limsup_{r \rightarrow \infty} |f|(r) > 0$ and let $b \in \mathbb{K}^*$. Let $m \in \mathbb{N}^*$ be ≥ 3 . Then $f' + bf^m$ has infinitely many zeros that are not zeros of f .*

Lemma 7: *Let $f \in \mathcal{M}_u(d(a, R^-))$ satisfy $\limsup_{r \rightarrow R^-} |f|(r) = +\infty$ and let $b \in \mathbb{K}^*$. Let $m \in \mathbb{N}^*$ be ≥ 3 . Then $f' + bf^m$ has infinitely many zeros that are not zeros of f .*

Proof. (Lemmas 5 and 7). Without loss of generality, we can assume $b = 1$ and when $f \in \mathcal{M}(d(a, R^-))$, we may assume $a = 0$. By hypotheses, there exists a sequence of intervals $[r'_n, r''_n]$ such that $\lim_{n \rightarrow +\infty} r'_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} r'_n = \lim_{n \rightarrow +\infty} r''_n = R$) and such that, putting $J = \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$, we have $\limsup_{\substack{r \rightarrow \infty, \\ r \in J}} |f|(r) > 0$ (resp. $\lim_{\substack{r \rightarrow R^-, \\ r \in J}} |f|(r) = +\infty$).

Suppose first we assume the hypothesis of Lemma 5. Let $M = \frac{\limsup_{r \rightarrow +\infty} |f|(r)}{2}$. We will prove that there exists $t > 0$ such that $|f' + f^m|(r) = |f^m|(r) \quad \forall r \in J \cap [t, +\infty[$. By Theorem C.2.10 in [16], we have $|f'|(r) \leq \frac{|f|(r)}{r}$. Consequently, when r lies in J , there exists $s > 0$ such that $|f|(r) \geq M \quad \forall r \in [s, +\infty[\cap J$, therefore

$$(|f|(r))^m \geq |f|(r)M^{m-1} \geq r|f'|(r)M^{m-1}.$$

Next, when r is big enough, rM^{m-1} is greater than 1, hence $(|f|(r))^m > |f'|(r)$. Thus there exists $t \geq s$ such that $(|f|(r))^m > |f'|(r) \forall r \in J \cap [t, +\infty[$. Let $J' = J \cap [t, +\infty[$. So we have $|f' + f^m|(r) = |f^m|(r) \forall r \in J'$.

Suppose now that we assume the hypothesis of Lemma 7. We have

$$|f'|(r) \leq \frac{|f|(r)}{r} \leq \frac{|f|(r)}{R}. \text{ Set } B = \frac{1}{R}. \text{ Then we have}$$

$$(|f|(r))^m \geq B|f'|(r)(|f|(r))^{m-1}.$$

Now, when r is close enough to R , $r \in J$, $B|f'(r)|^{m-1}$ is strictly greater than 1, hence $(|f|(r))^m > |f'|(r)$. Thus, there exists $t > 0$ such that $(|f|(r))^m > |f'|(r) \forall r \in [t, +\infty[\cap J$. We can set again $J' = J \cap [t, R[$ and then we have $|f' + f^m|(r) = |f^m|(r) \forall r \in J'$.

We can now conclude the proofs of Lemmas 5 and 7. For each $n \in \mathbb{N}$, let q_n be the number of zeros of f in $d(0, r''_n)$. Suppose the sequence $(q_n)_{n \in \mathbb{N}}$ is bounded. Then, f has finitely many zeros, hence it is of the form $\frac{P}{h}$ with $P \in \mathbb{K}[x]$ and $h \in \mathcal{A}(\mathbb{K})$ (resp. $h \in \mathcal{A}_u(d(0, R^-))$). Consequently, we have $\lim_{r \rightarrow +\infty} |f|(r) = 0$ (resp. $\lim_{r \rightarrow R^-} |f|(r) = 0$), a contradiction to the hypothesis in both theorems. Therefore, the sequence $(q_n)_{n \in \mathbb{N}}$ that is increasing by definition, tends to $+\infty$. Now, in each Lemmas 5 and 7, we may apply Lemma 2 showing that $f' + f^m$ has infinitely many zeros that are not zeros of f . \square

Proof. (Lemma 6). Without loss of generality, we can assume $b = 1$. By hypothesis, there exists a sequence of intervals $[r'_n, r''_n]$ such that $R < r_1$, $r''_n < r'_{n+1}$ and $\lim_{n \rightarrow +\infty} r'_n = +\infty$ and such that $\liminf_{n \rightarrow \infty} (\inf_{r'_n \leq r \leq r''_n} |f|(r)) > 0$.

Putting $J = \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$, then we have $\limsup_{\substack{r \rightarrow \infty \\ r \in J}} |f|(r) > 0$. Let $M = \frac{\limsup_{r \rightarrow +\infty} |f|(r)}{2}$. We will prove that there exists $t > 0$ such that $|f' + f^m|(r) = |f^m|(r) \forall r \in J \cap [t, +\infty[$. By Theorem C.2.10 in [16], we have $|f'|(r) \leq \frac{|f|(r)}{r}$. Consequently, when r lies in J , there exists $s > 0$ such that $|f|(r) \geq M \forall r \in [s, +\infty[\cap J$ therefore

$$(|f|(r))^m \geq |f|(r)M^{m-1} \geq r|f'|(r)M^{m-1}.$$

Next, when r is big enough, rM^{m-1} is greater than 1, hence $(|f|(r))^m > |f'|(r)$. Thus there exists $t \geq s$ such that $(|f|(r))^m > |f'|(r) \forall r \in J \cap [t, +\infty[$. Let $J' = J \cap [t, +\infty[$. So we have $|f' + f^m|(r) = |f^m|(r) \forall r \in J'$.

Consequently, we can apply Lemma 4 and conclude as in Lemmas 5 and 7. \square

3 Nevanlinna Theory

The Nevanlinna theory was made by Rolf Nevanlinna on complex functions [22]. It consists of defining counting functions of zeros and poles of a meromorphic function f and giving an upper bound for multiple zeros and poles of various functions $f - b$, $b \in \mathbb{C}$.

A similar theory for functions in a p -adic field was constructed and correctly proved by A. Boutabaa [8] in the field \mathbb{K} , after some previous work by Ha Huy Khoai [18]. The theory was also described in [25] with a nice description of the theorem on small functions. In [9] the theory was extended to functions in $\mathcal{M}(d(0, R^-))$ by taking into account Lazard's problem. A new extension to functions out of a hole was made in [15], see also [16].

Notations: Recall that given three functions ϕ , ψ , ζ defined in an interval $J =]a, +\infty[$ (resp. $J =]a, R[$), with values in $[0, +\infty[$, we shall write $\phi(r) \leq \psi(r) + O(\zeta(r))$ if there exists a constant $b \in \mathbb{R}$ such that $\phi(r) \leq \psi(r) + b\zeta(r)$. We shall write $\phi(r) = \psi(r) + O(\zeta(r))$ if $|\psi(r) - \phi(r)|$ is bounded by a function of the form $b\zeta(r)$.

Similarly, we shall write $\phi(r) \leq \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]a, +\infty[$ (resp. from $J =]a, R[$) to \mathbb{R} such that $\lim_{r \rightarrow +\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) \leq \psi(r) + h(r)$. And we shall write $\phi(r) = \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]a, +\infty[$ (resp. from $J =]a, R[$) to \mathbb{R} such that $\lim_{r \rightarrow +\infty} \frac{h(r)}{\zeta(r)} = 0$, resp. $\lim_{r \rightarrow R} \frac{h(r)}{\zeta(r)} = 0$ and such that $\phi(r) = \psi(r) + h(r)$.

Throughout the next paragraphs, we will denote by I the interval $[t, +\infty[$ and by J an interval of the form $[t, R[$ with $t > 0$.

We have to introduce the counting function of zeros and poles of a meromorphic function f , counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Definitions: Let $f \in \mathcal{M}(d(0, R^-)$ and let $r \in]0, R[$. We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way.

Let (a_n) , $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n .

We set $Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|)$ and so, $Z(r, f)$ is called *the counting function of zeros of f in $d(0, r)$, counting multiplicity*.

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega}_0(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega}_0(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r, f)$ the counting function of zeros of f without multiplicity:

$\overline{Z}(r, f) = \overline{\omega}_0(f) \log r + \sum_{n=1}^{\sigma(r)} (\log r - \log |a_n|)$ and so, $\overline{Z}(r, f)$ is called *the counting function of zeros of f in $d(0, r)$ ignoring multiplicity*.

In the same way, considering the finite sequence (b_n) , $1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n , we put

$N(r, f) = \max(-\omega_0(f), 0) \log r + \sum_{n=1}^{\tau(r)} t_n (\log r - \log |b_n|)$ and then $N(r, f)$ is called *the counting function of the poles of f , counting multiplicity*.

Next, in order to define the counting function of poles of f without multiplicity, we put $\overline{\overline{\omega}}_0(f) = 0$ if $\omega_0(f) \geq 0$ and $\overline{\overline{\omega}}_0(f) = 1$ if $\omega_0(f) \leq -1$ and we set

$\overline{\overline{N}}(r, f) = \overline{\overline{\omega}}_0(f) \log r + \sum_{n=1}^{\tau(r)} (\log r - \log |b_n|)$ and then $\overline{\overline{N}}(r, f)$ is called *the counting function of the poles of f , ignoring multiplicity*.

Now we can define the Nevanlinna function $T(r, f)$ in I or J as

$$T(r, f) = \max(Z(r, f), N(r, f))$$

and the function $T(r, f)$ is called *characteristic function of f or Nevanlinna function of f* .

Finally, if S is a subset of \mathbb{K} we will denote by $Z_0^S(r, f')$ the counting function of zeros of f' , excluding those which are zeros of $f - a$ for any $a \in S$.

Remark: If we change the origin, the functions Z , N , T are not changed, up to an additive constant.

By Corollary B.13.2 in [16], Lemma 8 is easy.

Lemma 8: *Let $\widehat{\mathbb{K}}$ be a complete algebraically closed extension of \mathbb{K} whose absolute value extends that of \mathbb{K} and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$). Let $\widehat{d}(0, R) = \{x \in \widehat{\mathbb{K}} \mid |x| < R\}$. The meromorphic function \widehat{f} defined by f in $\widehat{d}(0, R^-)$ has the same Nevanlinna functions as f .*

In a p -adic field such as \mathbb{K} , the first Main theorem is almost immediate and is an immediate consequence of Corollary B.13.27 in [16].

Proposition 1 (C.4.2 in [16]): *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) have no zero and no pole at 0. Then $\log(|f|(r)) = \log(|f(0)|) + Z(r, f) - N(r, f)$.*

Lemma 9 (C.4.4 in [16]): *Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ be pairwise distinct, let $P(u) = \prod_{i=1}^n (u - \alpha_i)$ and*

let $f \in \mathcal{M}(d(0, R^-))$. Then $Z(r, P(f)) = \sum_{i=1}^n Z(r, f - \alpha_i)$ and $\overline{Z}(r, P(f)) = \sum_{i=1}^n \overline{Z}(r, f - \alpha_i)$.

We can now deduce Proposition 2 (C.4.5 in [16]):

Proposition 2: *Let $f \in \mathcal{M}(\mathbb{K})$. Then f belongs to $\mathbb{K}(x)$ if and only if $T(r, f) = O(\log r)$.*

Applying Lemma 8 and Theorem C.2.10 in [16] to $\frac{f'}{f}$, up to a change of origin, we can derive Corollary 1.

Corollary 1: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then*

$$Z(r, \frac{f'}{f}) - N(r, \frac{f'}{f}) \leq -\log r + O(1).$$

Proposition 3: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(0, R^-))$) and let $b \in \mathbb{K}$. Then $Z(r, f) = Z(r, f - b) + O(1)$ $r \in I$ (resp. $r \in J$).*

The following Proposition 4 gathers a lot of properties owned by ultrametric meromorphic functions, some of them corresponding to the First Main Theorem in complex analysis.

Proposition 4: *Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}(d(0, R^-))$). Then $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$, $T(r, f + b) = T(r, f) + O(1)$. Let h be a Moebius function. Then $T(r, f) = T(r, h \circ f) + O(1)$. Let $P(x) \in \mathbb{K}[x]$. Then $T(r, P(f)) = \deg(P)T(r, f) + O(1)$ and $T(r, f'P(f)) \geq T(r, P(f))$.*

Suppose now $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \mathcal{A}(d(0, R^-))$). Then $Z(r, fg) = Z(r, f) + Z(r, g)$, $T(r, f) = Z(r, f)$, $T(r, fg) = T(r, f) + T(r, g) + O(1)$, and $T(r, f + g) \leq \max(T(r, f), T(r, g))$. Moreover, if $\lim_{r \rightarrow +\infty} T(r, f) - T(r, g) = +\infty$ then $T(r, f + g) = T(r, f)$ when r is big enough.

Proposition 5: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). There exists $\phi, \psi \in \mathcal{A}(\mathbb{K})$ (resp. $\phi, \psi \in \mathcal{A}(d(0, R^-))$) such that $f = \frac{\phi}{\psi}$ and $\max(T(r, \phi), T(r, \psi)) \leq T(r, f) + O(1)$, $r \in I$ (resp. $r \in J$).

Proposition 6: Let $f \in \mathcal{M}(d(0, R^-))$. Then f belongs to $\mathcal{M}_b(d(0, R^-))$ if and only if $T(r, f)$ is bounded in $[0, R[$.

Corollary 2: Let $f \in \mathcal{M}_u(d(a, R^-))$ and let $h \in \mathcal{M}_b(d(a, R^-))$, $h \neq 0$. Then fh belongs to $\mathcal{M}_u(d(a, R^-))$.

By Propositions 4 and 6, we can also derive Corollary 3.

Corollary 3: Let $f \in \mathcal{M}(d(a, R^-))$ and let $P \in \mathbb{K}[x]$. Then $P(f)$ belongs to $\mathcal{M}_b(d(a, R^-))$ if and only if so does f .

Lemma 10 is classical and easily checked.

Lemma 10: Let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$ be pairwise distinct, let $S = \{\alpha_1, \dots, \alpha_q\}$ and let $P(x) = \prod_{j=1}^q (x - \alpha_j)$. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then

$$\sum_{j=1}^n Z(r, f - \alpha_j) = Z(r, P(f)), \quad \sum_{j=1}^n \bar{Z}(r, f - \alpha_j) = \bar{Z}(r, P(f)) \quad \forall r \in I$$

(resp. $\forall r \in J$). Moreover, we have

$$\sum_{j=1}^n \left(Z(r, f - \alpha_j) - \bar{Z}(r, f - \alpha_j) \right) = Z(r, f') - Z_0^S(r, f') \quad \forall r \in I \quad (\text{resp. } \forall r \in J).$$

Proposition 7: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then $Z(r, f') - N(r, f') \leq Z(r, f) - N(r, f) - \log r + O(1)$, $r \in I$ (resp. $r \in J$). Moreover, $N(r, f^{(k)}) = N(r, f) + k\bar{N}(r, f) + O(1)$, $r \in I$ and $Z(r, f^{(k)}) \leq Z(r, f) + k\bar{N}(r, f) - k \log r + O(1)$, $r \in I$ (resp. $r \in J$).

Corollary 4: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then $T(r, f^{(k)}) \leq (k+1)T(r, f) + O(1)$ ($r \in I$) (resp. $r \in J$).

Proposition 8: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then, $T(r, f) - Z(r, f) \leq T(r, f') - Z(r, f') + O(1)$. Further, given $\alpha \in \mathcal{M}(d(0, R^-))$, we have $T(r, \alpha f) - Z(r, \alpha f) \leq T(r, f) - Z(r, f) + T(r, \alpha)$.

Lemma 11 is an immediate consequence of Corollary B.13.27 and Theorem C.2.10 in [16].

Lemma 11: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) and let $G = \frac{f'}{f}$. Then, G satisfies $Z(r, G) \leq N(r, G) - \log r + O(1)$ $r \in I$ (resp. $r \in J$).

We can now prove the Second Main theorem under different forms. Lemma 12 is essential and directly leads to the theorems.

Lemma 12: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$). Suppose that there exists $\xi \in \mathbb{K}$ (resp. $\xi \in \mathcal{M}_b(d(0, R^-))$) and a sequence of intervals $I_n = [u_n, v_n]$ such that $u_n < v_n < u_{n+1}$, $\lim_{n \rightarrow +\infty} u_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} u_n = R$) and

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} T(r, f) - Z(r, f - \xi) \right) = +\infty$$

(resp. $\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} T(r, f) - Z(r, f - \xi) \right) = +\infty$).

Let $\tau \in \mathbb{K}$ (resp. let $\tau \in \mathcal{M}_b(d(0, R^-))$), $\tau \neq \xi$. Then $Z(r, f - \tau) = T(r, f) + O(1) \forall r \in I_n$ when n is big enough.

Now we can state a technical proposition that implies the famous 2nd Main Theorem.

Proposition 9: Let $f \in \mathcal{M}(\mathbb{K})$ and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct. Then

$$(q-1)T(r, f) \leq \max_{1 \leq k \leq q} \left(\sum_{j=1, j \neq k}^q Z(r, f - a_j) \right) + O(1).$$

Corollary 5: Let $f \in \mathcal{M}(\mathbb{K})$ and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct. Then

$$(q-1)T(r, f) \leq \sum_{j=1}^q Z(r, f - a_j) + O(1).$$

Proposition 10: Let $f \in \mathcal{M}(d(0, R^-))$ and let $\tau_1, \dots, \tau_q \in \mathcal{M}_b(d(0, R^-))$ be distinct. Then

$$(q-1)T(r, f) \leq \max_{1 \leq k \leq q} \left(\sum_{j=1, j \neq k}^q Z(r, f - \tau_j) \right) + O(1).$$

Corollary 6: Let $f \in \mathcal{M}(d(0, R^-))$ and let $\tau_1, \dots, \tau_q \in \mathcal{M}_b(d(0, R^-))$ be distinct. Then

$$(q-1)T(r, f) \leq \sum_{j=1}^q Z(r, f - \tau_j) + O(1).$$

Remark: Proposition 9 does not hold in complex analysis. Indeed, let f be a meromorphic function in \mathbb{C} omitting two values a and b , such as $f(x) = \frac{e^x}{e^x - 1}$. Then $Z(r, f - a) + Z(r, f - b) = 0$.

Proposition 11: Let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$, with $q \geq 2$, let $S = \{\alpha_1, \dots, \alpha_q\}$, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then

$$(q-1)T(r, f) \leq \sum_{j=1}^q \bar{Z}(r, f - \alpha_j) + Z(r, f') - Z_0^S(r, f') + O(1) \quad \forall r \in I \quad (\text{resp. } \forall r \in J).$$

Moreover, if f belongs to $f \in \mathcal{A}(\mathbb{K})$ (resp. $\mathcal{A}(d(0, R^-))$), then

$$qT(r, f) \leq \sum_{j=1}^q \bar{Z}(r, f - \alpha_j) + Z(r, f') - Z_0^S(r, f') + O(1) \quad \forall r \in I \quad (\text{resp. } \forall r \in J).$$

Proposition 12 (Second Main Theorem): Let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$, with $q \geq 2$, let $S = \{\alpha_1, \dots, \alpha_q\}$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$). Then

$$(q-1)T(r, f) \leq \sum_{j=1}^q \bar{Z}(r, f - \alpha_j) + \bar{N}(r, f) - Z_0^S(r, f') - \log r + O(1) \quad \forall r \in I \quad (\text{resp. } \forall r \in J).$$

Remark: In Proposition 10, in the hypothesis $f \in \mathcal{M}(d(0, R^-))$, the term $-\log r$ has no veritable meaning since r is bounded.

Corollary 7: Let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$, with $q \geq 2$, let $S = \{\alpha_1, \dots, \alpha_q\}$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then

$$\sum_{j=1}^q (Z(r, f - \alpha_j) - \bar{Z}(r, f - \alpha_j)) \leq T(r, f) + \bar{N}(r, f) - Z_0^S(r, f') - \log r + O(1) \quad \forall r \in I \quad (\text{resp. } \forall r \in J).$$

4 Meromorphic Functions out of a Hole

We will now describe the behaviour of meromorphic functions out of a hole. We denote by $\| \cdot \|$ the norm of uniform convergence on bounded functions $f \in \mathcal{M}^c(D)$.

Proposition 13 (E. Motzkin's Theorem): [4], [20] *Let $f \in \mathcal{M}(D)$. There exist $h \in H(D)$ such that $\|h - 1\| < 1$, $\lim_{|x| \rightarrow +\infty} h(x) = 1$, $\lambda \in \mathbb{K}$ and $q \in \mathbb{Z}$ such $f(x) = \lambda x^q h(x)$.*

Definitions and notations : According to the factorization due to f Motzkin's Theorem, h is called *the Motzkin factor of f with respect to the hole T* and q is called *the Motzkin index with respect to the hole T* and is denoted by $m(f, T)$.

We denote by S the disk $d(0, R^-)$. Let $f \in \mathcal{M}(D)$. Then f admits a unique factorization in the form $f^0 f$ [4], [20].

We denote by $H_0(D)$ the \mathbb{K} -vector space of analytic elements f in D such that $\lim_{|x| \rightarrow \infty} f(x) = 0$.

Given $f \in \mathcal{M}(D)$, for $r > R$, here we will denote by $Z_R(r, f)$ the counting function of zeros of f between R and r , i.e., if $\alpha_1, \dots, \alpha_m$ are the distinct zeros of f in $\Delta(0, R, r)$, with respective multiplicity u_j , $1 \leq j \leq m$, then $Z_R(r, f) = \sum_{j=1}^m u_j (\log(r) - \log(|\alpha_j|))$. Similarly, we denote by $N_R(r, f)$ the counting function of poles of f between R and r , i.e., if β_1, \dots, β_n are the distinct poles of f in $\Delta(0, R, r)$, with respective multiplicity v_j , $1 \leq j \leq n$, then $N_R(r, f) = \sum_{j=1}^n v_j (\log(r) - \log(|\beta_j|))$.

Finally, we put $T_R(r, f) = \max(Z_R(r, f), N_R(r, f))$ [15].

Next, we denote by $\bar{Z}_R(r, f)$ the counting function of zeros without counting multiplicity: if $\alpha_1, \dots, \alpha_m$ are the distinct zeros of f in $\Delta(0, R, r)$, then we put

$$\bar{Z}_R(r, f) = \sum_{j=1}^m \log(r) - \log(|\alpha_j|).$$

Similarly, we denote by $\bar{N}_R(r, f)$ the counting function of poles without counting multiplicity: if β_1, \dots, β_n are the distinct poles of f in $\Delta(0, R, r)$, then we put

$$\bar{N}_R(r, f) = \sum_{j=1}^n \log(r) - \log(|\beta_j|).$$

Finally, taking $W = \{a_1, \dots, a_q \in \mathbb{K}\}$, we denote by $Z_R^W(r, f')$ the counting function of zeros of f' on points x where $f(x) \notin W$.

Given two functions defined in an interval $I = [b, +\infty[$, we will write $\phi(r) = \psi(r) + O(\log(r))$ (resp. $\phi(r) \leq \psi(r) + O(\log(r))$) if there exists a constant $B > 0$ such that $|\phi(r) - \psi(r)|_\infty \leq B \log(r)$, $r \in I$ (resp. $\phi(r) - \psi(r) \leq B \log(r)$, $r \in I$).

We will write $\phi(r) = o(\psi(r))$, $r \in I$ if $\lim_{r \rightarrow +\infty} \frac{\phi(r)}{\psi(r)} = 0$.

Lemma 13: *Let $f \in \mathcal{M}(D)$. Then $\log(|f|(r)) - \log(|f|(R)) = Z_R(r, f) - N_R(r, f) + m(f, S)(\log r - \log R)$ ($r \in I$).*

Proof. By Theorem C.3.6 in [16], we have $f = f^S f^0$. Since f^S has no zero and no pole in D , by Theorem C.3.3 in [16] it satisfies $|f^S|(r) = r^{m(f, S)} \forall r \in I$, hence $\log(|f^S|(r)) - \log(|f^S|(R)) = m(f, S)(\log r - \log R)$ ($r \in I$). Next, since f^0 has no zero and no pole in S , we have $\log(|f^0|(r)) - \log(|f^0|(R)) = Z_R(r, f^0) - N_R(r, f^0)$ ($r \in I$), therefore the statement is clear. \square

Corollary 8: *Let $f \in \mathcal{M}(D)$. Then $T_R(r, f)$ is identically zero if and only if f is a Motzkin factor.*

Corollary 9: *Let $f \in \mathcal{M}(D)$. Then $T_R(r, f^0) = O(\log(r))$.*

Corollary 10: *Let $f \in \mathcal{A}(D)$ and let $\phi \in H_0(D)$. Then $Z_R(r, f + \phi) = Z_R(r, f) + O(\log(r))$ ($r \in I$).*

Proof. Indeed, since ϕ is bounded and tends to zero at infinite, we have $\log |f|(r) = \log |f + \phi|(r)$ when r is big enough. \square

Corollary 11: *Let $f, g \in \mathcal{A}(D)$ satisfy $\log(|f|(r)) \leq \log(|g|(r)) \forall r \geq R$ ($r \in I$). Then $Z_R(r, f) \leq Z_R(r, g) + (m(g, S) - m(f, S))(\log(r) - \log(R))$, ($r \in I$).*

Lemma 14: *Let $f \in \mathcal{A}(D)$. Then $Z_R(r, f') \leq Z_R(r, f) + O(\log(r))$ ($r \in I$).*

Proof. Indeed, by Theorem B.9.2 in [16] we have $|f'|(r) \leq \frac{|f|(r)}{r}$. Therefore, the conclusion comes from Lemma 13. \square

We can now characterize the set $\mathcal{M}^c(D)$:

Proposition 14: *Let $f \in \mathcal{M}(D)$. The following three statements are equivalent:*

- (i) $\lim_{r \rightarrow +\infty} \frac{T_R(r, f)}{\log(r)} = +\infty$ ($r \in I$),
- (ii) $\frac{T_R(r, f)}{\log(r)}$ is unbounded, and
- (iii) f belongs to $\mathcal{M}^c(D)$.

Proof. Consider an increasing sequence $(u_n)_{n \in \mathbb{N}}$ in \mathbb{R}_+ such that $\lim_{n \rightarrow +\infty} u_n = +\infty$ and let $(k_n)_{n \in \mathbb{N}}$ be a sequence of \mathbb{N}^* . Clearly, we have

$$\lim_{r \rightarrow +\infty} \frac{\sum_{u_n \leq r} k_n (\log(r) - \log(u_n))}{\log(r)} = +\infty.$$

Consequently, if a function $f \in \mathcal{M}^c(D)$ has infinitely many zeros (resp. infinitely many poles in D),

then $\lim_{n \rightarrow +\infty} \frac{Z_R(r, f)}{\log(r)} = +\infty$ (resp. $\lim_{n \rightarrow +\infty} \frac{N_R(r, f)}{\log(r)} = +\infty$), hence in both cases, $\lim_{n \rightarrow +\infty} \frac{T_R(r, f)}{\log(r)} = +\infty$.

Conversely, if f has finitely many zeros and finitely many poles in D , then we check that

$\lim_{n \rightarrow +\infty} \frac{T_R(r, f)}{\log(r)} < +\infty$. Thus, the equivalence of the three statements is clear. \square

Operations on $\mathcal{M}(D)$ work almost like for meromorphic functions in the whole field, thanks to the use of Motzkin factors.

Proposition 15 (First Main Theorem out of a hole) [15] : *Let $f, g \in \mathcal{M}(D)$. Then for every $b \in \mathbb{K}$, we have $T_R(r, f + b) = T_R(r, f) + O(\log(r))$, ($r \in I$) $T_R(r, f.g) \leq T_R(r, f) + T_R(r, g) +$*

$O(\log(r))$ ($r \in I$), $T_R(r, \frac{1}{f}) = T_R(r, f)$, $T_R(r, f+g) \leq T_R(r, f) + T_R(r, g) + O(\log(r))$ ($r \in I$) and $T_R(r, f^n) = nT_R(r, f)$. Let h be a Moebius function. Then $T_R(r, h \circ f) = T_R(r, f) + O(\log(r))$ ($r \in I$).

Moreover, if both f and g belong to $\mathcal{A}(D)$, then

$$T_R(r, f+g) \leq \max(T_R(r, f), T_R(r, g)) + O(\log(r)) \quad (r \in I)$$

and $T_R(r, fg) = T_R(r, f) + T_R(r, g)$, ($r \in I$). Particularly, if $f \in \mathcal{A}^c(D)$, then $T_R(r, f+b) = T_R(r, f) + O(1)$ ($r \in I$). Given a polynomial $P(X) \in \mathbb{K}[X]$, then $T_R(r, P \circ f) = qT_R(r, f) + O(\log(r))$.

Corollary 12 Let $h_1, h_2 \in \mathcal{A}(D)$, let g be a Motzkin factor. Then

$$T_R(r, gh_1 + gh_2) \leq \max(T_R(r, h_1), T_R(r, h_2)) + O(\log(r)).$$

Similarly to Proposition 7 here we have the following Proposition

Proposition 16: Let $f \in \mathcal{M}(D)$. Then $N_R(r, f^{(k)}) = N_R(r, f) + k\bar{N}_R(r, f)$, ($r \in I$) and $Z_R(r, f^{(k)}) \leq Z_R(r, f) + k\bar{N}_R(r, f) + O(\log(r))$, ($r \in I$).

The following Lemma 15 will be necessary in the proof of Proposition 17.

Lemma 15 [16]: Let $f \in \mathcal{M}(D)$. Suppose that there exists $\xi \in \mathbb{K}$ and a sequence of intervals $J_n = [u_n, v_n]$ such that $u_n < v_n < u_{n+1}$, $\lim_{n \rightarrow +\infty} u_n = +\infty$, and

$$\lim_{n \rightarrow +\infty} \left[\inf_{r \in J_n} \frac{T_R(r, f) - Z_R(r, f - \xi)}{\log(r)} \right] = +\infty.$$

Let $\zeta \in \mathbb{K}$ $\zeta \neq \xi$. Then $Z_R(r, f - \zeta) = T_R(r, f) + O(\log(r)) \quad \forall r \in J_n$ when n is big enough.

Proof. Without loss of generality, we can obviously suppose that $\xi = 0$. By Lemma 13, f is of the form $f^S f^0$ and f^0 is of the form $\frac{g}{h}$ with $g, h \in \mathcal{A}(D)$, having no zero in S . Set $w = f^S$. Thus we have

$$\lim_{n \rightarrow +\infty} \left[\inf_{r \in J_n} \frac{Z_R(r, h) - Z_R(r, g)}{\log(r)} \right] = +\infty.$$

Consequently, by Lemma 13,

$$(7) \quad \lim_{n \rightarrow +\infty} \left[\inf_{r \in J_n} \frac{\log(|h|(r)) - \log(|g|(r))}{\log(r)} \right] = +\infty.$$

Consider now $f - \zeta$. We have $f - \zeta = \frac{wg - \zeta h}{h}$, hence

$$\log(|f|(r)) = \log(|wg - \zeta h|(r) - \log(|h|(r))).$$

But by (7), we have $\log(|\zeta h|(r)) > \log(|wg|(r))$ because $\log(|w|(r)) = O(\log(r))$, therefore $\log(|wg - \zeta h|(r)) = \log(|\zeta h|(r)) \quad \forall r \in J_n$ when n is big enough and hence

$$(8) \quad \lim_{n \rightarrow +\infty} \left[\sup_{r \in J_n} \frac{\log(|\zeta h - wg|(r) - \log(|h|(r)))}{\log(r)} \right] = 0.$$

Consequently, by (8) and by Lemma 13,

$$\lim_{n \rightarrow +\infty} \left[\sup_{r \in J_n} \frac{Z_R(r, \zeta h - wg) - Z_R(r, h)}{\log(r)} \right] = 0,$$

i.e.,

$$\lim_{n \rightarrow +\infty} \left[\sup_{r \in J_n} \frac{Z_R(r, f - \zeta) - T_R(r, f)}{\log(r)} \right] = 0,$$

which proves the claim. \square

Proposition 17: *Let $f \in \mathcal{M}(D)$ and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct. Then*

$$(q-1)T_R(r, f) \leq \max_{1 \leq k \leq q} \left(\sum_{j=1, j \neq k}^q Z_R(r, f - a_j) \right) + O(\log(r)) \quad (r \in I).$$

Proof. Suppose Proposition 17 is wrong. Thus, there exists $f \in \mathcal{M}(D)$ and $a_1, \dots, a_q \in \mathbb{K}$ such that $(q-1)T_R(r, f) - \max_{1 \leq k \leq q} \left(\sum_{j=1, j \neq k}^q Z_R(r, f - a_j) \right)$ admits no superior bound in $]0, +\infty[$.

So, there exists a sequence of intervals $J_s = [w_s, y_s]$ such that $w_s < y_s < w_{s+1}$, $\lim_{s \rightarrow +\infty} w_s = +\infty$ and two distinct indices $m \leq q$ and $t \leq q$ such that

$$\lim_{s \rightarrow +\infty} \left[\inf_{r \in J_s} \frac{(T_R(r, f) - Z_R(r, f - a_m))}{\log(r)} \right] = +\infty$$

and

$$\lim_{s \rightarrow +\infty} \left[\inf_{r \in J_s} \frac{(T_R(r, f) - Z_R(r, f - a_t))}{\log(r)} \right] = +\infty.$$

But by Lemma 15, that is impossible. \square

We can now state and prove the Second Main Theorem for $\mathcal{M}(D)$.

Proposition 18 (Second Main Theorem out of a hole): *Let $f \in \mathcal{M}(D)$, let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$, with $q \geq 2$ and let $W = \{\alpha_1, \dots, \alpha_q\}$. Then*

$$(q-1)T_R(r, f) \leq \sum_{j=1}^q \bar{Z}_R(r, f - \alpha_j) + \bar{N}_R(r, f) - Z_R^W(r, f') + O(\log(r)) \quad (r \in I).$$

Proof. By Proposition 17 there exists a constant $B > 0$ and for each $r > R$ there exists $k(r) \in \mathbb{N}$, $k(r) \leq q$, such that

$$(q-1)T_R(r, f) \leq \sum_{j=1, j \neq k(r)}^q Z_R(r, f - a_j) + B \log(r),$$

i.e., $(q-1)T_R(r, f) \leq \sum_{j=1}^q Z_R(r, f - a_j) - Z_R(r, a_{k(r)}) + O(\log(r))$. Now,

$$\sum_{j=1}^q Z_R(r, f - a_j) = \sum_{j=1}^q \bar{Z}_R(r, f - a_j) + Z_R(r, f') - Z_R^W(r, f') + B \log(r).$$

Consequently,

$$(9) \quad (q-1)T_R(r, f) \leq \sum_{j=1}^q \bar{Z}_R(r, f - a_j, \dots) + Z_R(r, f') - Z_R^W(r, f') - Z_R(r, f - a_{k(r)}) + O(\log(r))$$

Particularly, if $f \in \mathcal{A}(D)$ then we have $Z_R(r, f - a_j) = T_R(r, f - a_j) = T_R(r, f) + O(\log(r)) \forall j = 1, \dots, q$, hence $Z_R(r, f - a_{k(r)}) = T_R(r, f) + O(\log(r))$ and therefore

$$qT_R(r, f) \leq \sum_{j=1}^q \bar{Z}_R(r, f - a_j) + Z_R(r, f') - Z_R^W(r, f') + O(\log(r)),$$

By Proposition 7, for each $j = 1, \dots, q$, there exists a constant $B_j > 0$ such that $Z_R(r, f') \leq Z_R(r, f - a_j) + \bar{N}_R(r, f - a_j) + B_j \log(r)$. Consequently, there exists a constant $C > 0$ such that $Z_R(r, f') \leq Z_R(r, f - a_{k(r)}) + \bar{N}_R(r, f - a_{k(r)}) + C \log(r) \forall r > R$.

Therefore, by Relation (9) that remains true in Proposition 18, we can derive

$$(q-1)T_R(r, f) \leq \sum_{j=1}^q \bar{Z}_R(r, f - \alpha_j) + \bar{N}_R(r, f) - Z_R^W(r, f') + O(\log(r)) \quad \forall r \in I.$$

□

5 Small Functions

Small functions with respect to a meromorphic functions are well known in the general theory of complex functions. Particularly, one knows the Nevanlinna theorem on 3 small functions. Here we will construct a similar theory.

Definitions and notation: Throughout the chapter we set $a \in K$ and $R \in]0, +\infty[$ and we still denote by D the set $\mathbb{K} \setminus d(0, R^-)$. For each $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a, R^-))$, resp. $f \in \mathcal{M}(D)$) we denote by $\mathcal{M}_f(\mathbb{K})$, (resp. $\mathcal{M}_f(d(a, R^-))$, resp. $\mathcal{M}_f(D)$) the set of functions $h \in \mathcal{M}(\mathbb{K})$, (resp. $h \in \mathcal{M}(d(a, R^-))$, resp. $\mathcal{M}(D)$) such that $T(r, h) = o(T(r, f))$ when r tends to $+\infty$ (resp. when r tends to R , resp. when r tends to $+\infty$). Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(a, R^-))$, $f \in \mathcal{A}(D)$) we shall denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(a, R^-))$, resp. $\mathcal{A}_f(D)$) the set $\mathcal{M}_f(\mathbb{K}) \cap \mathcal{A}(\mathbb{K})$, (resp. $\mathcal{M}_f(d(a, R^-)) \cap \mathcal{A}(d(a, R^-))$, resp. $\mathcal{M}_f(D) \cap \mathcal{A}(D)$).

The elements of $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(a, R^-))$, resp. $\mathcal{M}_f(D)$) are called *small meromorphic functions with respect to f* , *small functions* in brief. Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(a, R^-))$, resp. $f \in \mathcal{A}(D)$) the elements of $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(a, R^-))$, resp. $\mathcal{A}_f(D)$) are called *small analytic functions with respect to f* *small functions* in brief.

Propositions 19 and 20 are immediate consequences of Proposition 8:

Proposition 19: *Let $a \in \mathbb{K}$ and $r > 0$. $\mathcal{A}_f(\mathbb{K})$ is a \mathbb{K} -subalgebra of $\mathcal{A}(\mathbb{K})$, $\mathcal{A}_f(d(a, R^-))$ is a \mathbb{K} -subalgebra of $\mathcal{A}(d(a, R^-))$, $\mathcal{A}_f(D)$ is a \mathbb{K} -subalgebra of $\mathcal{A}(D)$, $\mathcal{M}_f(\mathbb{K})$ is a subfield field of $\mathcal{M}(\mathbb{K})$, $\mathcal{M}_f(d(a, R^-))$ is a subfield of field of $\mathcal{M}(a, R^-)$ and $\mathcal{M}_f(D)$ is a subfield field of $\mathcal{M}(D)$. Moreover, $\mathcal{A}_b(d(a, R^-))$ is a sub-algebra of $\mathcal{A}_f(d(a, R^-))$ and $\mathcal{M}_b(d(a, R^-))$ is a subfield of $\mathcal{M}_f(d(a, R^-))$.*

Proposition 20: Let $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(d(0, R^-))$), (resp. $f \in \mathcal{M}(D)$) and let $g \in \mathcal{M}_f(\mathbb{K})$, (resp. $g \in \mathcal{M}_f(d(0, R^-))$), (resp. $g \in \mathcal{M}_f(D)$). Then $T(r, fg) = T(r, f) + o(T(r, f))$ and $T(r, \frac{f}{g}) = T(r, f) + o(T(r, f))$, (resp. $T(r, fg) = T(r, f) + o(T(r, f))$ and $T(r, \frac{f}{g}) = T(r, f) + o(T(r, f))$), (resp. $T_R(r, fg) = T_R(r, f) + o(T_R(r, f))$ and $T_R(r, \frac{f}{g}) = T_R(r, f) + o(T_R(r, f))$).

Proposition 21 is known as Second Main Theorem on Three Small Functions. It holds as well as in complex analysis, where it was showed first [18]. Notice that this theorem was generalized to any finite set of small functions by K. Yamanoi in complex analysis [24], through methods that have no equivalent on a p-adic field. The Second Main Theorem on Three Small Functions holds in p-adic analysis as well as in Complex analysis and is proven particularly in [18] (see also [16]). The most precise form is given in [18].

Proposition 21: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$), (resp. $f \in \mathcal{M}^c(D)$) and let $w_1, w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$ (resp. $w_1, w_2, w_3 \in \mathcal{M}_f(d(0, R^-))$), (resp. $w_1, w_2, w_3 \in \mathcal{M}_f(D)$) be pairwise distinct and let $S(r) = \max(T(r, w_1), T(r, w_2), T(r, w_3))$, $r > 0$, (resp. $S(r) = \max(T(r, w_1), T(r, w_2), T(r, w_3))$, $r < R$), (resp. $S(r) = \max(T_R(r, w_1), T_R(r, w_2), T_R(r, w_3))$, $r \geq R$).

Then $T(r, f) \leq \sum_{j=1}^3 \bar{Z}(r, f - w_j) + O(S(r))$, (resp. $T(r, f) \leq \sum_{j=1}^3 \bar{Z}(r, f - w_j) + 13S(r)$), (resp. $T_R(r, f) \leq \sum_{j=1}^3 \bar{Z}_R(r, f - w_j) + 13S(r)$).

Proposition 22: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$), (resp. $f \in \mathcal{M}^c(D)$) and let $w_1, w_2 \in \mathcal{M}_f(\mathbb{K})$ (resp. $w_1, w_2 \in \mathcal{M}_f(d(0, R^-))$), (resp. $w_1, w_2 \in \mathcal{M}_f(D)$) be distinct and let $S(r) = \max(T(r, w_1), T(r, w_2))$, $r > 0$, (resp. $S(r) = \max(T(r, w_1), T(r, w_2))$, $r < R$), (resp. $S(r) = \max(T_R(r, w_1), T_R(r, w_2))$, $r \geq R$). Then $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{N}(r, f) + 13S(r) + O(1)$, (resp. $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{N}(r, f) + 13S(r) + O(1)$), (resp. $T_R(r, f) \leq \bar{Z}_R(r, f - w_1) + \bar{Z}_R(r, f - w_2) + \bar{N}_R(r, f) + 13S(r) + O(1)$).

Next, by setting $g = f - w_1$ and $w = w_1 + w_2$, we can write Corollary 13:

Corollary 13: Let $g \in \mathcal{M}(\mathbb{K})$ (resp. $g \in \mathcal{M}_u(d(0, R^-))$), (resp. $g \in \mathcal{M}^c(D)$) and let $w \in \mathcal{M}_g(\mathbb{K})$ (resp. $w \in \mathcal{M}_g(d(0, R^-))$), (resp. $w \in \mathcal{M}_g(D)$). and let $S(r) = T(r, w)$ $r > 0$, (resp. $S(r) = T(r, w)$ $r < R$), (resp. $S(r) = T_R(r, w)$ $r \geq R$). Then $T(r, g) \leq \bar{Z}(r, g) + \bar{Z}(r, g - w) + \bar{N}(r, g) + 13S(r) + O(1)$, (resp. $T(r, g) \leq \bar{Z}(r, g) + \bar{Z}(r, g - w) + \bar{N}(r, g) + 13S(r) + O(1)$), (resp. $T_R(r, g) \leq \bar{Z}_R(r, g) + \bar{Z}_R(r, g - w) + \bar{N}_R(r, g) + 13S(r) + O(1)$).

Corollary 14: Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$), (resp. $f \in \mathcal{A}^c(D)$) and let $w_1, w_2 \in \mathcal{A}_f(\mathbb{K})$ (resp. $w_1, w_2 \in \mathcal{A}_f(d(0, R^-))$), (resp. $w_1, w_2 \in \mathcal{A}_f(D)$) be distinct and let $S(r) = \max(T(r, w_1), T(r, w_2))$ $r > 0$, (resp. $S(r) = \max(T(r, w_1), T(r, w_2))$ $r < R$), (resp. $S(r) = \max(T_R(r, w_1), T_R(r, w_2))$ $r \geq R$). Then $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + 13S(r) + O(1)$, (resp. $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + 13S(r) + O(1)$), (resp. $T_R(r, f) \leq \bar{Z}_R(r, f - w_1) + \bar{Z}_R(r, f - w_2) + 13S(r) + O(1)$).

And similarly to Corollary 13, we get Corollary 15:

Corollary 15: Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$), (resp. $f \in \mathcal{A}^c(D)$) and let $w \in \mathcal{A}_f(\mathbb{K})$ (resp. $w \in \mathcal{A}_f(d(0, R^-))$), (resp. $w \in \mathcal{A}_f(D)$). Let $S(r) = T(r, w)$, $r \in]0, +\infty[$, (resp. $S(r) = T(r, w)$, $r < R$), (resp. $S(r) = T_R(r, w)$, $r \geq R$).

Then $T(r, f) \leq \overline{Z}(r, f) + \overline{Z}(r, f - w) + 13S(r) + O(1)$, (resp. $T(r, f) \leq \overline{Z}(r, f) + \overline{Z}(r, f - w) + 13S(r) + O(1)$), resp. $T_R(r, f) \leq \overline{Z}_R(r, f) + \overline{Z}_R(r, f - w) + 13S(r) + O(1)$.

Lemma 16: Let $f \in \mathcal{M}(\mathbb{K})$, (resp. let $f \in \mathcal{M}(d(0, R^-))$, $R > 0$, (resp. let $f \in \mathcal{M}^c(D)$)), suppose that f admits infinitely many zeros and suppose that there exists a sequence of intervals $[r'_n, r''_n]$ such that $\lim_{n \rightarrow +\infty} r'_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} r'_n = \lim_{n \rightarrow +\infty} r''_n = R$, resp. $\lim_{n \rightarrow +\infty} r'_n = +\infty$, $R \leq r_1$) and such that $|(f' + f^m)|(r) = |f^m|(r) \forall r \in \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$. Let $m \in \mathbb{N}^*$ be $\neq 2$. Then $f' + f^m$ has infinitely many zeros that are not zeros of f .

Proof. Let $J = \bigcup_{n \in \mathbb{N}} [r'_n, r''_n]$. By Corollary B.13.6 we have

$$\nu^+(f' + f^m, \log r) = \nu^+(f^m, \log r), \quad \nu^-(f' + f^m, \log r) = \nu^-(f^m, \log r) \quad \forall r \in J.$$

Consequently, in each circle $C(0, r)$ with $r \in J$, f and $f' + f^m$ have the same difference between the number of zeros and poles. Now, if $m \geq 3$ the poles of $f' + f^m$ and f^m are the same taking multiplicity into account. And when $m = 1$, each pole of f is a pole of $f' + f$ with a strictly greater order. Consequently, for each $r \in J$, the number of zeros of $f' + f^m$ in $C(0, r)$ is superior or equal to this of f^m .

Now, for each $n \in \mathbb{N}$, let s_n be the number of distinct zeros of f in $\bigcup_{j=1}^n \Delta(r'_1, r''_n)$. Since f has infinitely many zeros, the sequence s_n is increasing and tends to $+\infty$. On the other hand, for each zero α of order u of f , either α is not a zero of $f' + f^m$ (when $u = 1$), or it is a zero of order $u - 1$. Consequently, the number of zeros of $f' + f^m$ in $\bigcup_{j=1}^n \Delta(r'_1, r''_n)$ which are not zeros of f is at least s_n . Thus we have proved that $f' + f^m$ has infinitely many zeros that are not zeros of f . \square

6 Proofs of the Theorems

Proposition 23: Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ satisfy $\limsup_{r \rightarrow \infty} |f|(r) > 0$ and let $b \in \mathbb{K}^*$. Let $m \in \mathbb{N}^*$ be ≥ 3 . Then $f' + bf^m$ has infinitely many zeros that are not zeros of f .

Proposition 24: Let $f \in \mathcal{M}_u(d(a, R^-))$ satisfy $\limsup_{r \rightarrow R} |f|(r) = +\infty$ and let $b \in \mathbb{K}^*$. Let $m \in \mathbb{N}^*$ be ≥ 3 . Then $f' + bf^m$ has infinitely many zeros that are not zeros of f .

Proof. (Propositions 23 and 24). Without loss of generality, we can assume $b = 1$ and when $f \in \mathcal{M}(d(a, R^-))$, we may assume $a = 0$. By hypotheses, there exists a sequence of intervals $[r'_n, r''_n]$ such that $\lim_{n \rightarrow +\infty} r'_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} r'_n = \lim_{n \rightarrow +\infty} r''_n = R$) and such that, putting $J =$

$$\bigcup_{n \in \mathbb{N}} [r'_n, r''_n], \text{ we have } \limsup_{\substack{r \rightarrow \infty, \\ r \in J}} |f|(r) > 0 \text{ (resp. } \lim_{\substack{r \rightarrow R^-, \\ r \in J}} |f|(r) = +\infty).$$

Suppose first we assume the hypothesis of Proposition 23. Let $M = \frac{\limsup_{r \rightarrow +\infty} |f|(r)}{2}$. We will prove that there exists $t > 0$ such that $|f' + f^m|(r) = |f^m|(r) \forall r \in J \cap [t, +\infty[$. By Theorem C.2.10 in [16], we have $|f'|(r) \leq \frac{|f|(r)}{r}$. Consequently, when r lies in J , there exists $s > 0$ such that $|f|(r) \geq M \forall r \in [s, +\infty[\cap J$.

$$(|f|(r))^m \geq |f|(r)M^{m-1} \geq r|f'(r)M^{m-1}.$$

Next, when r is big enough, rM^{m-1} is greater than 1, hence $(|f|(r))^m > |f'(r)$. Thus there exists $t \geq s$ such that $(|f|(r))^m > |f'(r) \forall r \in J \cap [t, +\infty[$. Let $J' = J \cap [t, +\infty[$. So we have $|f' + f^m|(r) = |f^m|(r) \forall r \in J'$.

Suppose now that we assume the hypothesis of Proposition 24. We have

$$|f'(r) \leq \frac{|f|(r)}{r} \leq \frac{|f|(r)}{R}. \text{ Set } B = \frac{1}{R}. \text{ Then we have}$$

$$(|f|(r))^m \geq B|f'(r)(|f|(r))^{m-1}.$$

Now, when r is close enough to R , $r \in J$, $B|f'(r)|^{m-1}$ is strictly greater than 1, hence $(|f|(r))^m > |f'(r)$. Thus, there exists $t > 0$ such that $(|f|(r))^m > |f'(r) \forall r \in [t, +\infty[\cap J$. We can set again $J' = J \cap [t, R[$ and then we have $|f' + f^m|(r) = |f^m|(r) \forall r \in J'$.

We can now conclude Propositions 23 and 24. For each $n \in \mathbb{N}$, let q_n be the number of zeros of f in $d(0, r_n'')$. Suppose the sequence $(q_n)_{n \in \mathbb{N}}$ is bounded. Then, f has finitely many zeros, hence it is of the form $\frac{P}{h}$ with $P \in \mathbb{K}[x]$ and $h \in \mathcal{A}(\mathbb{K})$ (resp. $h \in \mathcal{A}_u(d(0, R^-))$). Consequently, we have $\lim_{r \rightarrow +\infty} |f|(r) = 0$ (resp. $\lim_{r \rightarrow R^-} |f|(r) = 0$), a contradiction to the hypothesis in both theorems. Therefore, the sequence $(q_n)_{n \in \mathbb{N}}$ that is increasing by definition, tends to $+\infty$. Now, in each Propositions 23 and 24, we may apply Lemma 16 showing that $f' + f^m$ has infinitely many zeros that are not zeros of f . \square

Theorem D : *let $a \in \mathbb{K}$ and $r > 0$. Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$, (resp. $f \in \mathcal{M}_u(d(a, R^-))$, resp. $f \in \mathcal{M}^c(D)$). There exists at most one function $w \in \mathcal{M}_f(\mathbb{K})$, (resp. $w \in \mathcal{M}_f(d(a, R^-))$, resp. $w \in \mathcal{M}_f(D)$) such that $f - w$ have finitely many zeros. Moreover, if f belongs to $\mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. to $\mathcal{A}_u(d(a, R^-))$, resp. to $\mathcal{A}^c(\mathbb{K})$) then there exists no function $w \in \mathcal{M}_f(\mathbb{K}) \setminus \mathbb{K}(x)$, (resp. $w \in \mathcal{M}_f(d(a, R^-))$, resp. $w \in \mathcal{M}_f(D)$) such that $f - w$ have finitely many zeros.*

Proof. Concerning claims on $\mathcal{M}_u(d(a, R^-))$ we can obviously assume $a = 0$. Suppose that there exist two distinct functions $w_1, w_2 \in \mathcal{M}_f(\mathbb{K})$, (resp. $w_1, w_2 \in \mathcal{M}_f(d(0, R^-))$) such that $f - w_k$ has finitely many zeros. So, there exist $P_1, P_2 \in \mathbb{K}[x]$ and $h_1, h_2 \in \mathcal{A}(\mathbb{K})$ (resp. $h_1, h_2 \in \mathcal{A}(d(0, R^-))$) such that $f - w_k = \frac{P_k}{h_k}$, $k = 1, 2$ and hence we notice that

$$(1) \quad T(r, f) = T(r, \frac{P_k}{h_k}) + o(T(r, f)) = T(r, h_k) + o(T(r, f)) \quad k = 1, 2.$$

Consequently, putting $w = q_2 - w_1$, we have

$$\frac{P_1}{h_1} = \frac{P_2}{h_2} + w$$

and by Proposition 19, w belongs to $\mathcal{M}_f(\mathbb{K})$ (resp. to $\mathcal{M}_f(d(0, R^-))$). Therefore $P_1 h_2 - P_2 h_1 = g h_1 h_2$ and hence

$$(2) \quad T(r, P_1 h_2 - P_2 h_1) = T(r, g h_1 h_2).$$

Now, by Proposition 8 we have

$$T(r, P_1 h_2 - P_2 h_1) \leq \max(T(r, P_1 h_2), T(r, P_2 h_1)) \leq \max(T(r, h_1), T(r, h_2)) + o(T(r, f))$$

and hence by (1), we obtain

$$(3) \quad T(r, P_1 h_2 - P_2 h_1) \leq T(r, f) + o(T(r, f)).$$

On the other hand, by Proposition 20, we have

$$T(r, w h_1 h_2) = T(r, h_1 h_2) + o(T(r, h_1 h_2)) = 2T(r, f) + o(T(r, f)),$$

which by (2), contradicts (3).

Suppose now that f belongs to $\mathcal{M}^c(D)$. Suppose that there exist two distinct functions $w_1, w_2 \in \mathcal{M}_f^c(D)$, such that $f - w_k$ has finitely many zeros. So, there exist $P_1, P_2 \in \mathbb{K}[x]$ and $h_1, h_2 \in \mathcal{A}^c(D)$ such that $f - w_k = w_k^0 \frac{P_k}{h_k}$, $k = 1, 2$ and hence we notice that

$$(1bis) \quad T_R(r, f) \leq T_R(w_k^0) + T_R(r, \frac{P_k}{h_k}) + o(T(r, f)) = T_R(w_k^0) + T(r, h_k) + o(T_R(r, f)) \quad k = 1, 2.$$

Consequently, putting $w = w_2 - w_1$, we have

$$\frac{w_1^0 P_1}{h_1} = \frac{w_2^0 P_2}{h_2} + w$$

and by Proposition 19, w belongs to $\mathcal{M}_f(D)$. Therefore $w_1^0 P_1 h_2 - w_2^0 P_2 h_1 = w h_1 h_2$ and hence

$$T_R(r, w_1^0 P_1 h_2 - w_2^0 P_2 h_1) = T_R(r, g h_1 h_2)$$

and hence by Corollary 12

$$(2bis) \quad T_R(r, g h_1 h_2) \leq \max(T_R(r, P_1 h_2), T_R(r, P_2 h_1)) + O(\log(r)) \leq T_R(r, f) + o(T_R(r, f)).$$

On the other hand, by Proposition 20, we have

$$T_R(r, g h_1 h_2) = T_R(r, h_1 h_2) + o(T_R(r, h_1 h_2)) = 2T_R(r, f) + o(T_R(r, f)),$$

which contradicts (2bis).

Suppose now that f belongs to $\mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ and that there exists a function $w \in \mathcal{M}_f(\mathbb{K})$ such that $f - w$ has finitely many zeros.

Set $w = \frac{l}{t}$ where l and t belong to $\mathcal{A}_f(\mathbb{K})$ and have no common zeros. Thus, $f - w = \frac{tf - l}{t}$ and each zero of $tf - l$ cannot be a zero of t hence is zero of $f - w$. Consequently, since $f - w$ has finitely many zeros, $tf - l$ has finitely many zeros and hence is a polynomial. But since l belongs to $\mathcal{A}_f(\mathbb{K})$, when r is big enough we have $|f|(r) > |l|(r)$ and hence $|tf|(r) > |l|(r)$, therefore $|tf - l|(r) = |tf|(r)$. And since f is transcendental, by Corollary B.13.23 in [16] for every fixed $q \in \mathbb{N}$, $|f|(r) > r^q$ when r is big enough. Similarly, $|tf - l|(r) > r^q$ when r is big enough. Consequently, by Corollary B.13.23 in [16], $tf - l$ is not a polynomial, which proves that w does not exist.

Suppose now that f belongs to $\mathcal{A}_u(d(0, R^-))$ and that there exists a function $w \in \mathcal{M}_f(d(0, R^-))$ such that $f - w$ has finitely many zeros. Without loss of generality, we can assume that the field \mathbb{K} is spherically complete because both f and w have continuation to an algebraically closed spherically complete extension of \mathbb{K} where their zeros are the same as in \mathbb{K} . Consequently,

we can write $w = \frac{l}{t}$ where l and t have no common zeros. Now, the zeros of $f - w$ are those of $tf - l$, hence $tf - l$ has finitely many zeros and hence, is bounded in $d(0, R^-)$. But since w belongs to $\mathcal{M}_f(d(0, R^-))$, so does l and hence $|tf|(r) > |l|(r)$ when r tends to R . Consequently, $|tf - l|(r) = |tf|(r)$ is not bounded in $d(0, R^-)$, a contradiction proving again that w does not exist.

Suppose finally that f belongs to $\mathcal{A}^c(\mathbb{K})$ and that there exists a function $w \in \mathcal{M}_f(D)$ such that $f - w$ has finitely many zeros. Thus, $f - w = (f - w)^0 \left(\frac{tf - l}{t} \right)$ where both $tf - l$ and l belong to $\mathcal{A}(\mathbb{K})$ and have no zero in S .

Then each zero of $tf - l$ cannot be a zero of t , hence is zero of $f - w$. Consequently, since $f - w$ has finitely many zeros, $tf - l$ has finitely many zeros and hence is a polynomial. But since l belongs to $\mathcal{A}_f(\mathbb{K})$, when r is big enough we have $|f|(r) > |l|(r)$ and hence $|tf|(r) > |l|(r)$, therefore $|tf - l|(r) = |tf|(r)$. And since f is transcendental, by Corollary B.13.23 in [16] for every fixed $q \in \mathbb{N}$, $|f|(r) > r^q$ when r is big enough. Similarly, $|tf - l|(r) > r^q$ when r is big enough. Consequently, by Corollary B.13.23 in [16], $tf - l$ is not a polynomial, which proves that w does not exist. \square

Theorem B: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. For each $b \in \mathbb{K}^*$, $f' + bf$ has infinitely many zeros that are not zeros of f .*

Proof. Without loss of generality, we can assume again $b = 1$. By Theorem C.2.10 in [16], we have $|f'|(r) < |f|(r)$ when r is big enough and hence $|f' + f|(r) = |f|(r)$ in an interval $I = [s, +\infty[$. Suppose first that f has infinitely many zeros. We can then apply Lemma 4 and get the conclusion.

Suppose now that f has finitely many zeros. Then f has infinitely many poles c_n of respective order t_n . Since \mathbb{K} has characteristic zero, f' admits each c_n as a pole of order $t_n + 1$ and similarly, $f' + f$ also admits each c_n as a pole of order $t_n + 1$. Thus, we have $N(r, f' + f) = N(r, f) + \bar{N}(r, f)$. But since $|f' + f|(r) = |f|(r)$ holds in I , we have $\nu(f' + f, \log r) = \nu(f, \log r) \forall r \in I$ and hence $Z(r, f' + f) - N(r, f' + f) = Z(r, f) - N(r, f)$, therefore $Z(r, f' + f) - (N(r, f) + \bar{N}(r, f)) = Z(r, f) - N(r, f)$ and hence $Z(r, f' + f) = Z(r, f) + \bar{N}(r, f)$. Since we have supposed that f has finitely many zeros and since f has infinitely many poles, $f' + f$ has infinitely many zeros and all but finitely many are not zeros of f . \square

Concerning functions $f' + bf^2$, we can obtain a first conclusion when f is analytic.

Theorem E: *Let $a \in \mathbb{K}$, let $R \in]0, +\infty[$ and let $f \in \mathcal{A}_u(d(a, R^-))$, (resp. $f \in \mathcal{A}^c(D)$). For each $b \in \mathbb{K}^*$, $f' + bf^2$ has infinitely many zeros that are not zeros of f .*

Proof. Without loss of generality, we can assume $b = 1$ and $a = 0$. Clearly, when r is big enough, in $]0, +\infty[$ (resp. in $]0, R[$), we have $|f' + f^2|(r) = |f^2|(r)$ therefore, by Corollary B.13.6 in [16], f^2 and $f' + f^2$ have the same number of zeros in $C(0, r)$ (taking multiplicity into account). Let $\alpha \in C(0, r)$ be a zero of f of order q . When r is big enough, it is a zero of order $2q$ for f^2 and it is a zero of order $q - 1$ for $f' + f^2$. Consequently, by Corollary B.13.6 in [16], $f' + f^2$ has at least $q + 1$ zero in $C(0, r)$ that are not zeros of f (taking multiplicity into account). This is true for every such zeros of f and hence $f' + f^2$ has infinitely many zeros that are not zeros of f . \square

Corollary 16: *Let $m \in \mathbb{N}$ $m \geq 2$, let $a \in \mathbb{K}$, let $R \in]0, +\infty[$, and let $f \in \mathcal{A}_u(d(a, R^-))$, (resp. $f \in \mathcal{A}^c(D)$). For each $b \in \mathbb{K}^*$, $f' + bf^m$ has infinitely many zeros that are not zeros of f .*

Proposition 25 was published in [7] and partially in [22]. In its proof, we will get the proof of Theorem C.

Proposition 25: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. let $a \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$ and let $f \in \mathcal{M}_u(d(a, R^-)$, resp. let $f \in \mathcal{M}^c(R)$) and let $m \in \mathbb{N}$. If $m \geq 5$, then for each $b \in \mathbb{K}^*$, $f' + bf^m$ has infinitely many zeros that are not zeros of f . If $m = 4$, if $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$, and if f admits at least s multiple zeros and at least t multiple poles, then $f' + bf^4$ admits a number of zeros that are not zeros of f (taken account of multiplicity), which is superior or equal to $\frac{s+t}{26}$.*

Proof. By Corollary B.13.2 in [16], the zeros of $f' + bf^m$ in \mathbb{K} are the same as in a spherically complete algebraically closed extension $\widehat{\mathbb{K}}$ of \mathbb{K} . So, for simplicity, we can suppose that the field \mathbb{K} is spherically complete without loss of generality. We can also suppose that $b = 1$ and $a = 0$. Then if $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$, (resp. if $f \in \mathcal{M}^c(R)$) we can obviously write $f = \frac{h}{l}$ with $h, l \in \mathcal{A}(\mathbb{K})$ (resp. $h, l \in \mathcal{A}(R)$), having no common zeros and if $f \in \mathcal{M}(d(0, R^-)$, since \mathbb{K} is spherically complete, we can write $f = \frac{h}{l}$ with $h, l \in \mathcal{A}(d(0, R^-))$, having no common zeros again.

Let $g = \frac{1}{f}$ and let $n = m - 2$. So, by Lemma 1, the problem is reduced to show that $g'g^n - 1$ has infinitely many zeros. So we suppose that the lemma is wrong and hence, $g'g^n - 1$ has q zeros (counting multiplicity).

Then, $g'g^n - 1 = \frac{(l'h - h'l)l^n - h^{n+2}}{h^{n+2}}$ and since h, l have no common zeros, this is of the form $\frac{P}{h^{n+2}}$ where P is a polynomial of degree q . Now, set $F = (l'h - h'l)l^n$.

Then, assuming that $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}(d(0, R^-)$, by Corollary 15,

$$T(r, F) = Z(r, F) + O(1) \leq \overline{Z}(r, F) + \overline{Z}(r, F - P) + 13(T(r, P)) \text{ hence}$$

$$(1) \quad Z(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, F - P) + 13(T(r, P)),$$

hence

$$Z(r, l'h - h'l) + nZ(r, l) \leq \overline{Z}(r, l'h - h'l) + \overline{Z}(r, l) + \overline{Z}(F - P) + 13T(r, P) + O(1). \text{ Actually, } \overline{Z}(r, F - P) = \overline{Z}(r, h), \text{ hence } nZ(r, l) \leq \overline{Z}(r, l) + \overline{Z}(r, h) + 13T(r, P) + O(1) \text{ and hence } (n-1)Z(r, l) \leq Z(r, h) + 13(T(r, P)) + O(1) \text{ and then}$$

$$(2) \quad (n-1)Z(r, l) \leq Z(r, h) + 13T(r, P) + O(1).$$

Similarly, when $f \in \mathcal{M}^c(D)$, we have

$$(3) \quad (n-1)Z_R(r, l) \leq Z_R(r, h) + 13T(r, P) + O(1).$$

Now, suppose that $f \in \mathcal{M}_u(d(0, R^-)$. By Proposition 24, if $\lim_{r \rightarrow R^-} |f|(r) = +\infty$, i.e., if $\liminf_{r \rightarrow R^-} Z(r, f) - N(r, f) = +\infty$, the claim is proved. Consequently, since the claim is not true, we can assume $\liminf_{r \rightarrow R^-} Z(r, f) - N(r, f) < +\infty$, hence we have

$$(4) \quad \liminf_{r \rightarrow R^-} Z(r, h) - Z(r, l) < +\infty. \text{ But by (2), we see that (4) is impossible whenever } n \geq 3, \text{ i.e., } m \geq 5.$$

And now, suppose $f \in \mathcal{M}^c(R)$. By Proposition 24, if $\lim_{r \rightarrow +\infty} |f|(r) = +\infty$, i.e., if $\liminf_{r \rightarrow +\infty} Z_R(r, f) - N_R(r, f) = +\infty$, then the claim is proved again. Consequently, since the claim is not true, we can assume $\liminf_{r \rightarrow +\infty} Z_R(r, f) - N_R(r, f) < +\infty$ and get

$$\liminf_{r \rightarrow +\infty} Z_R(r, h) - Z_R(r, l) < +\infty$$

and then by (3), the conclusion is the same.

This finishes the proof of Theorem C.

Now, consider the hypothesis $f \in \mathcal{M}(\mathbb{K})$. By Proposition 23, if $\liminf_{r \rightarrow +\infty} |f|(r) > 0$, i.e., if $\liminf_{r \rightarrow +\infty} Z(r, f) - N(r, f) > -\infty$ the claim is proved. Consequently, since we suppose that the claim is not true, we can assume $\liminf_{r \rightarrow +\infty} Z(r, f) - N(r, f) = -\infty$, i.e.,

$$(5) \quad \limsup_{r \rightarrow +\infty} Z(r, l) - Z(r, h) = +\infty.$$

We notice that Relation (4) holds again. Since f is transcendental, by (4) we notice that l is transcendental. Consequently, (2) is impossible whenever $n \geq 3$, i.e., $m \geq 5$.

Now, suppose $m = 4$, i.e., $n = 2$. More precisely, $\bar{Z}(r, l) \leq Z(r, l) - \frac{s \log r}{2}$ and $\bar{Z}(r, h) \leq Z(r, h) - \frac{t \log r}{2}$. Then, by Corollary 15 and by Relation (1) we have

$$\begin{aligned} Z(r, F) &= Z(r, h'l - hl') + 2Z(l) \leq \bar{Z}(r, h'l - l'h) + \bar{Z}(r, l) + \bar{Z}(r, F - P) + 13T(r, P) + O(1) \\ &= \bar{Z}(r, h'l - l'h) + \bar{Z}(r, l) + \bar{Z}(r, h) + 13T(r, P) + O(1). \end{aligned}$$

Therefore

$$\begin{aligned} 2Z(r, l) &\leq \bar{Z}(r, l) + \bar{Z}(r, h) + 13T(r, P) + O(1) \\ &\leq Z(r, l) + Z(r, h) - \frac{s+t}{2} \log(r) + 13T(r, P) + O(1). \end{aligned}$$

Now, $T(r, P) \leq q \log(r)$, therefore

$$\begin{aligned} Z(r, l) &\leq Z(r, h) + (13 - \frac{s+t}{2}) \log(r) + O(1). \\ (6) \quad Z(r, l) &\leq Z(r, h) + (13 - \frac{s+t}{2}) \log(r) + O(1). \end{aligned}$$

Now, Relation (4) implies $13q - \frac{s+t}{2} > 0$ and hence $f'f^n$ admits a number of zeros strictly superior to $\frac{s+t}{26}$. □

Corollary 17: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ have finitely multiple zeros and finitely multiple poles. Let $b \in \mathbb{K}^*$. Then if f has infinitely many multiple zeros or poles, then $f' + bf^4$ has infinitely many zeros that are not zeros of f .*

We will now thoroughly examine the situation when $m = 4$, i.e., $n = 2$, as made in [14]. This requires several basic lemmas.

Lemma 13: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and such that f' has finitely many multiple zeros. Then $\frac{f''f}{(f')^2}$ has no quasi-exceptional value.*

Proof. Let $g = \frac{f}{f'}$. A pole of g is a zero of f' , hence by hypothesis, g has finitely many multiple poles. Consequently, by Theorem C.8.7 in [16], g' has no quasi-exceptional value. And hence

neither has $1 - g'$. But $g' = \frac{(f')^2 - f''f}{(f')^2} = 1 - \frac{f''f}{(f')^2}$. Therefore, $\frac{f''f}{(f')^2}$ has no quasi-exceptional value. \square

Lemma 14: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and have finitely many multiple zeros. Then $f''f + 2(f')^2$ has infinitely many zeros that are not zeros of f .*

Proof. Suppose first that f' has infinitely many multiple zeros. Since f has finitely many multiple zeros, the zeros of f' are not zeros of f except at most finitely many. Hence, f' has infinitely many multiple zeros that are not zeros of f . And then, they are zeros of f'' , hence of $f''f + 2(f')^2$, which proves the statement.

So we are now led to assume that f' has finitely many multiple zeros. By Lemma 13 $\frac{f''f + 2(f')^2}{(f')^2}$ has infinitely many zeros. Let $c \in \mathbb{K}$ be a pole of order q of f . Without loss of generality, we can suppose $c = 0$. The beginning of the Laurent development of f at 0 is of the form $\frac{a_{-q}}{x^q} + \frac{\varphi(x)}{x^{q-1}}$, whereas $\varphi \in \mathcal{M}(\mathbb{K})$ has no pole at 0. Consequently, $\frac{f''f + 2(f')^2}{(f')^2}$ is of the form

$$\frac{(a_{-q})^2(3q^2 + q) + x\phi(x)}{(a_{-q})^2(q^2) + x\psi(x)}$$

whereas $\phi, \psi \in \mathcal{M}(\mathbb{K})$ have no pole at 0. So, the function $\frac{f''f + 2(f')^2}{(f')^2}$ has no zero at 0.

Therefore, each zero of $\frac{f''f + 2(f')^2}{(f')^2}$ is a zero of $f''f + 2(f')^2$ and hence $f''f + 2(f')^2$ has infinitely many zeros.

Now, let us show that the zeros of $f''f + 2(f')^2$ are not zeros of f , except maybe finitely many. Let c be a zero of $f''f + 2(f')^2$ and suppose that c is a zero of f . Then, it is a zero of f' and hence it is a multiple zero of f . But by hypotheses, f has finitely many multiple zeros, hence the zeros of $f''f + 2(f')^2$ are not zeros of f , except at most finitely many. That finishes proving the claim. \square

Lemma 15: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and let $b \in \mathbb{K}^*$ be such that $f^2f' - b$ has finitely many zeros. Then, $N(r, f) \leq Z(r, f) + O(1)$.*

Proof. Let $F = f^2f'$. Since $F - b$ is transcendental and has finitely many zeros, it is of the form $\frac{P(x)}{h(x)}$ with $h \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$. Consequently, $|F|(r)$ is a constant when r is big enough and therefore, by Proposition 1 we have $Z(r, F) = N(r, F) + O(1)$ when r is big enough. Now, $Z(r, F) = 2Z(r, f) + Z(r, f')$ and, by Proposition 7 $Z(r, f') \leq Z(r, f) + \overline{N}(r, f) - \log r + O(1)$. On the other hand, by Proposition 7 again, we have $N(r, F) = 3N(r, f) + \overline{N}(r, f)$. Consequently, $3N(r, f) + \overline{N}(r, f) \leq 3Z(r, f) + \overline{N}(r, f) - \log r + O(1)$, which proves the claim. \square

Theorem A was published in [14]. Here we will add some precisions in the proof.

Theorem A: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. Then for each $b \in \mathbb{K}^*$, $f'f^2 - b$ has infinitely many zeros.*

Proof. Let $b \in \mathbb{K}^*$ and suppose that the claim is wrong, i.e., $f^2 f' - b$ has q zeros, taking multiplicity into account. By Proposition 25, we may assume that f has finitely many multiple zeros and finitely multiple poles. Set $F = f^2 f'$. Then $F' = f(f'' f + 2(f')^2)$. By Lemma 14, $f'' f + 2(f')^2$ has infinitely many zeros that are not zeros of f . Consequently, F' admits for zeros: the zeros of f and the zeros of $f'' f + 2(f')^2$. And by Lemma 14, there exists a sequence of zeros of $f'' f + 2(f')^2$ that are not zeros of f .

Let $S = \{0, b\}$ and let $Z_0^S(r, F')$ be the counting function of zeros of F' when $F(x)$ is different from 0 and b . Since $F - b$ has finitely many zeros, the zeros c of F' , which are not zeros of f , cannot satisfy $F(c) = b$ except at most finitely many. Consequently, there are infinitely many zeros of F' counted by the counting function $Z_0^S(r, F')$ and hence for every fixed integer $M \in \mathbb{N}$, we have

$$(1) \quad Z_0^S(r, F') \geq M \log r + O(1).$$

Let us apply Theorem C.4.24 in [16] to F . We have

$$(2) \quad T(r, F) \leq \bar{Z}(r, F) + \bar{Z}(r, F - b) + \bar{N}(r, F) - Z_0^S(r, F') - \log(r) + O(1).$$

Now, we have

$$(3) \quad \bar{Z}(r, F) \leq Z(r, f) + Z(r, f')$$

$$(4) \quad \bar{N}(r, F) = \bar{N}(r, f)$$

and since the number of zeros of $F - b$ is q , taking multiplicity into account, then:

$$(5) \quad \bar{Z}(r, F - b) \leq s \log r + O(1).$$

Consequently, by (2), (3), (4), and (5) we obtain

$$(6) \quad T(r, F) \leq Z(r, f) + Z(r, f') + \bar{N}(r, f) - Z_0^S(r, F') + (q - 1) \log r + O(1).$$

On the other hand, by construction, $T(r, F) \geq Z(r, F) = 2Z(r, f) + Z(r, f')$, hence by (6) we obtain (7)

$$(7) \quad Z(r, f) \leq \bar{N}(r, f) - Z_0^S(r, F') + (q - 1) \log r + O(1).$$

Now, by Lemma 15, we have $N(r, f) \leq Z(r, f) + O(1)$, hence by (7) we obtain $0 \leq (s - 1) \log r - Z_0^S(r, F') + O(1)$ and hence by (1), fixing $M > q - 1$ we can derive $0 \leq (q - 1) \log r - M \log r + O(1)$, a contradiction. This finishes the proof of Theorem A. \square

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