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A survey on a few recent papers in p -adic Value Distribution

by Alain Escassut

In memory of Professor Wolfgang Tutschke

In this article, we propose to present several recent results: a new proof of the p -adic Hermite-Lindemann Theorem, a new proof of the p -adic Gel'fond-Schneider Theorem, exceptional values of meromorphic functions and derivatives and the p -adic Nevanlinna theory applied to small functions. We first have to recall the definitions of the p -adic logarithm and exponential.

1 Logarithm and exponential in a p -adic field

Notations: We denote by \mathbb{Q}_p the completion of \mathbb{Q} with respect to the p -adic absolute value and by \mathbb{C}_p the completion of the algebraic closure of \mathbb{Q}_p , which is known to be algebraically closed [7]. In general, we denote by \mathbb{K} an algebraically closed field of characteristic 0 complete with respect to an ultrametric absolute value, such as \mathbb{C}_p . The ultrametric absolute value of \mathbb{K} is denoted $|\cdot|$ while the archimedean absolute value of \mathbb{C} is denoted $|\cdot|_\infty$.

Let $a \in \mathbb{K}$ and let $R \in \mathbb{R}_+$. We denote by $d(a, R)$ the "closed" disk $\{x \in \mathbb{K} \mid |x - a| \leq R\}$ and by $d(a, R^-)$ the "open" disk $\{x \in \mathbb{K} \mid |x - a| < R\}$.

We denote by $\mathcal{A}(\mathbb{K})$ the algebra of power series converging in all \mathbb{K} . Given $a \in \mathbb{K}$ and $R > 0$, we denote by $\mathcal{A}(d(a, R^-))$ the algebra of power series $\sum_{j=0}^{\infty} a_n(x - a)^n$ converging in $d(a, R^-)$ and by $\mathcal{A}_b(d(a, R^-))$ the subalgebra of functions $f(x) \in \mathcal{A}_b(d(a, R^-))$ that are bounded in $(d(a, R^-))$ and we put $\mathcal{A}_u(d(a, R^-)) = \mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$.

Moreover we denote by $H(d(a, R))$ the algebra of power series $\sum_{j=0}^{\infty} a_n(x - a)^n$ converging in $d(a, R)$ called *analytic elements in $d(a, R)$* . Given an element f of $H(d(0, R))$ we put $|f|(r) = \sup_{x \in d(0, R)} |f(x)|$.

We will define the p -adic logarithm and the p -adic exponential and will shortly study them, in connection with the study of the roots of 1. Here, as in [7], we compute the radius of convergence of the p -adic exponential by using results on injectivity.

The following lemma 1.a is easy:

Lemma 1.a: \mathbb{K} is supposed to have residue characteristic $p \neq 0$. Let $r \in]0, 1[$ and for each $n \in \mathbb{N}$, let $h_n(x) = (1 + x)^{p^n}$. The sequence h_n converges to 1 with respect to the uniform convergence on $d(0, r)$.

Notations: We denote by \log the real logarithm function of base e . Given a power series $\sum_{j=0}^{\infty} a_j x^j$ converging in $d(0, R^-)$ and given a number $\mu < \log(R)$ we denote by $\nu^+(f, \mu)$ the biggest integer q such that $\sup_{j \geq 0} \log(|a_j|) + j\mu = \log(|a_q|) + q\mu$.

For each $q \in \mathbb{N}^*$ we denote by R_q the positive number such that $\log_p(R_q) = -\frac{1}{p^q - 1(p-1)}$. We denote by $g(x)$ the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$.

The following lemma 1.b is well known (Theorem B.13.7 in [7]):

Lemma 1.b: *Let $f(x) = \sum_{j=0}^{\infty} a_j x^j$ be converging in $d(0, R^-)$ and let $r < R$.*

Then $\nu^+(f, \log(r))$ is the number of zeros of f in $d(0, r)$, taken multiplicity into account.

Theorem 1.1: *g has a radius of convergence equal to 1. If the residue characteristic of \mathbb{K} is $p \neq 0$, then g is unbounded in $d(0, 1^-)$. If the residue characteristic is zero, then $|g(x)|$ is bounded by 1 in $d(0, 1^-)$. The function defined in $d(1, 1^-)$ as $\text{Log}(x) = g(x-1)$ has a derivative equal to $\frac{1}{x}$ and satisfies $\text{Log}(ab) = \text{Log}(a) + \text{Log}(b)$ whenever $a, b \in d(1, 1^-)$.*

Proof. It is clearly seen that the radius of g is 1, because $|n| \geq \frac{1}{n}$ and $|n| \leq 1$ for all $n \in \mathbb{N}$. As in the Archimedean context, the property $\text{Log}(ab) = \text{Log}(a) + \text{Log}(b)$ comes from the fact that both Log and the function h_a defined as $h_a(x) = \text{Log}(ax)$ have the same derivative. The other statements are immediate. \square

Notation: When \mathbb{K} has residue characteristic $p \neq 0$, we introduce the group W of the p^s -th roots of 1, i.e., the set of the $u \in \mathbb{K}$ satisfying $u^{p^s} = 1$ for some $s \in \mathbb{N}$.

Recall that analytic elements were defined by M. Krasner and are defined in [7].

Theorem 1.2: *\mathbb{K} is supposed to have residue characteristic $p \neq 0$ (resp. 0). All zeros of Log are of order 1. The set of zeros of the function Log is equal to W , (resp. 1 is the only zero of Log). The restriction of Log to the disk $d(1, (R_1)^-)$ (resp. $d(1, 1^-)$) is injective and is a bijection from $d(1, (R_1)^-)$ onto $d(0, (R_1)^-)$ (resp. from $d(1, 1^-)$ onto $d(0, 1^-)$).*

Proof. It is obvious that the zeros of Log are of order 1 because the derivative of Log has no zero. First, we suppose \mathbb{K} to have residue characteristic $p \neq 0$. Each root of 1 in $d(1, 1^-)$ is a zero of Log . Moreover, by Theorem A.6.8 of [7], we know that the only roots of 1 in $d(1, 1^-)$ are the p^n -th roots. Now we can check that Log admits no zero other than the roots of 1. Indeed, suppose that a is a zero of Log but is not a root of 1, and for each $n \in \mathbb{N}$, let $b_n = a^{p^n}$.

Since b_n belongs to $d(1, 1^-)$, by Lemma B.16.1 of [7] we have $\lim_{n \rightarrow \infty} b_n = 1$. But obviously $\text{Log}(b_n) = 0$ for every $n \in \mathbb{N}$, hence this contradicts the fact that 1 is an isolated zero of Log .

Thus, Log has no zero in the disk $d(1, (R_1)^-)$, except 1 and therefore, by Lemma 1.b the series $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ satisfies $\nu^+(f, \log r) = 1$ for every $r \in]0, R_1[$, hence $r > \frac{r^n}{|n|}$ for all $r \in]0, R_1[$, for every $n \in \mathbb{N}^*$. Therefore, by Corollary B.14.10 of [7] it is injective in $d(0, R_1^-)$. Then, by Corollary B.13.10 of [7], we see that $\text{Log}(d(1, R_1^-)) = d(0, R_1^-)$.

Now we suppose that \mathbb{K} has residue characteristic zero. Then, the function $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ satisfies $\nu^+(f, \log r) = 1$ for every $r \in]0, 1[$, hence $r > \frac{r^n}{n}$ for all $r \in]0, 1[$, for every $n \in \mathbb{N}^*$. Therefore, f has no zero different from 1 in $d(0, 1^-)$ and, by Corollary B.14.10 of [7], is injective in $d(0, 1^-)$. Then by Corollary B.13.10 of [7] we see that $\text{Log}(d(1, 1^-)) = d(0, 1^-)$. This ends the proof. \square

Corollary 1.A: \mathbb{K} is supposed to have residue characteristic 0. There is no root of 1 in $d(1, 1^-)$, except 1.

Proof. Indeed any root of 1 should be a zero of Log in $d(1, 1^-)$. \square

Notations: If \mathbb{K} has residue characteristic $p \neq 0$, we first denote by \exp the inverse (or reciprocal) function of the restriction of Log to $d(1, R_1^-)$, which obviously is a function defined in $d(0, R_1^-)$, with values in $d(1, R_1^-)$. If \mathbb{K} has residue characteristic 0 we denote by \exp the inverse function of Log , which is obviously defined in $d(0, 1^-)$ and takes values in $d(1, 1^-)$.

Theorem 1.3: \mathbb{K} is supposed to have residue characteristic $p \neq 0$ (resp. $p = 0$). The function \exp belongs to $\mathcal{A}_b(d(0, R_1^-))$ (resp. $\mathcal{A}_b(d(0, 1^-))$), is a bijection from $d(0, R_1^-)$ onto $d(1, R_1^-)$ (resp. from $d(0, 1^-)$ onto $d(1, 1^-)$), and satisfies $\exp(x) = \exp'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ whenever $x \in d(0, R_1^-)$ (resp. $x \in d(0, 1^-)$). Moreover, the disk of convergence of its series is equal to $d(0, R_1^-)$ (resp. $d(0, 1^-)$). Further, if $p \neq 0$, then \exp is not an analytic element on $d(0, R_1^-)$.

Proof. By Corollary B.14.15 of [7] we know that the function \exp belongs to $\mathcal{A}_b(d(0, R_1^-))$ (resp. $\mathcal{A}_b(d(0, 1^-))$) and is obviously a bijection from $d(0, R_1^-)$ onto $d(1, R_1^-)$ (resp. from $d(0, 1^-)$ onto $d(1, 1^-)$). As it is the reciprocal of Log , it must satisfy $\exp(x) = \exp'(x)$ for all $x \in d(0, R_1^-)$ (resp. $x \in d(0, 1^-)$) and,

therefore, $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ whenever $x \in d(0, R_1^-)$ (resp. $x \in d(0, 1^-)$). Thus the radius of convergence r is at least R_1 (resp. 1). If the residue characteristic is 0, it is obviously seen that the series cannot converge for $|x| = 1$, hence the disk of convergence is $d(0, 1^-)$.

Now we suppose that the residue characteristic is $p \neq 0$. Suppose that the power series of \exp converges in $d(0, R_1)$. Then \exp has continuation to an analytic element on $d(0, R_1)$. On the other hand, since $\nu(f, \log r) = 1$ for all $r \in]0, R_1[$, we have $\nu^-(f, \log R_1) = 1$ and then by Theorem B.13.9 of [7] $\text{Log}(d(1, R_1))$ is equal to $d(0, R_1)$. Hence, we can consider $\exp(\text{Log}(x))$ in all the disk $d(0, R_1)$. By Corollary B.3.3 of [7] this is an analytic element on $d(1, R_1)$. But this element is equal to the identity in all of $d(1, R_1^-)$ and, therefore, in all of $d(1, R_1)$. Of course this contradicts the fact that Log is not injective in the circle $C(1, R_1)$. This finishes proving that the disk of convergence of \exp is just $d(0, R_1^-)$. \square

Notations: Henceforth, we put $e^x = \exp(x)$.

Theorem 1.4: \mathbb{K} is supposed to have residue characteristic $p \neq 0$. Let $x \in d(0, R_1^-)$. Then e^x is algebraic over \mathbb{Q}_p if and only if so is x . Let $u \in d(0, 1^-)$. Then $\log(1+u)$ is algebraic over \mathbb{Q}_p if and only if so is u .

Proof. By Theorem B.5.24 of [7], if x is algebraic over \mathbb{Q}_p , so is e^x . Similarly, if u is algebraic over \mathbb{Q}_p , so is $\log(1+u)$. Consequently, suppose that e^x is algebraic over \mathbb{Q}_p . Then e^x is of the form $1+t$ with $|t| < 1$, hence $\log(1+t)$ is algebraic over \mathbb{Q}_p . But then, $\log(1+t) = \log(e^x) = x$, hence x is algebraic over \mathbb{Q}_p . Now, more generally, suppose $\log(1+u)$ is algebraic over \mathbb{Q}_p , with $|u| < 1$. Take $q \in \mathbb{N}$ such that $|p^q \log(1+u)| < R_1$. We have $p^q \log(1+u) = \log((1+u)^{p^q})$. Since $|p^q \log(1+u)| < R_1$, we have $|\log((1+u)^{p^q})| < R_1$, hence $\exp(\log((1+u)^{p^q})) = (1+u)^{p^q}$. Consequently, $(1+u)^{p^q}$ is algebraic over \mathbb{Q}_p and hence so is u . \square

We can show a similar result when $p = 0$.

Theorem 1.5: \mathbb{K} is supposed to have residue characteristic 0. Let $x \in d(0, 1^-)$. Then e^x is algebraic over \mathbb{Q}_p if and only if so is x . Let $u \in d(0, 1^-)$. Then $\log(1+u)$ is algebraic over \mathbb{Q}_p if and only if so is u .

The following proposition 1.6 will be used in the poof of Theorem 2.3 and is proven by induction, similarly as (1.4.2) in [16].

Proposition 1.6: Let $P_1, \dots, P_q \in \mathbb{K}[X]$ different from 0 and let $w_1, \dots, w_q \in \mathbb{K}$ be pairwise distinct. Let $F(x) = \sum_{j=1}^q P_j(x)e^{w_j x}$. Then F is not identically zero.

2 Hermite-Lindemann's and Gel'fond-Schneider's Theorems in ultrametric fields

We will use the following classical notation:

Notation: We will denote by \mathcal{K} an algebraically closed complete ultrametric extension of \mathbb{Q} of residue characteristic 0.

We will denote by U the disk $d(0, 1)$ and by D_0 the disk $d(0, 1^-)$ in the field \mathbb{K} no matter what the residue characteristic.

If the residue characteristic of \mathbb{K} is $p > 0$ we put $R_1 = p^{\frac{-1}{p-1}}$ and denote by D_1 the disk $d(0, R_1^-)$.

Given an algebraic number $a \in \mathbb{C}_p$ (resp. $a \in \mathcal{K}$) and a_1, a_2, \dots, a_q its conjugates over \mathbb{Q} (with $a_1 = a$), we put $\overline{|a|} = \max_{1 \leq j \leq q} |a_j|$ and we denote by $den(a)$ its *smallest denominator*, i.e. the smallest positive integer q such that qa is an algebraic integer. Then we put $s(a) = \max(\log \overline{|a|}, \log(den(a)))$ and $s(a)$ is called *the size* of a . More generally we call denominator of a number a all positive integer multiple of its smallest denominator.

Given a polynomial $P(X_1, \dots, X_q) \in \mathbb{Z}[X_1, \dots, X_q]$, we denote by $H(P)$ the supremum of the archimedean absolute values of its coefficients.

Given a positive real number a , we denote by $[a]$ the largest integer n such that $n \leq a$.

Hermite-Lindemann's theorem is well known in complex analysis. The same holds in p-adic analysis. The first proof was presented in 1930 by K. Malher [13]. This proof given in [13] is written in German and uses symbols which are not currently known. Here we present a new proof using classical methods in transcendental processes that are maybe easier to understand.

We will need Siegel's Lemma in all the following theorems of this chapter. We will choose a particular form of this famous lemma [16] whose formulation is due to M. Mignotte:

Lemma 2.a (Siegel): *Let E be a finite extension of \mathbb{Q} of degree q and let $\lambda_{i,j}$ $1 \leq i \leq m$, $1 \leq j \leq n$ be elements of E integral over \mathbb{Z} . Let $M = \max(\overline{|\lambda_{i,j}|} \ 1 \leq i \leq m, \ 1 \leq j \leq n)$ and let (\mathcal{S}) be the linear system $\{\sum_{j=1}^n \lambda_{i,j} x_j = 0, \ 1 \leq i \leq m\}$. There exists solutions (x_1, \dots, x_n) of (\mathcal{S}) such that $x_j \in \mathbb{Z} \ \forall j = 1, \dots, n$ and*

$$\log(|x_j|_\infty) \leq \log(M) \frac{qm}{n - qm} + \frac{\log(2)}{2} \ \forall j = 1, \dots, n.$$

Lemma 2.b will be necessary in the proof of Theorem 2.4 and is easily proven in [16] since its proof implies no change in the field \mathbb{K} since it only concerns algebraic numbers

Lemma 2.b: *Let $a_1, \dots, a_q \in \mathbb{K}$ be algebraic over \mathbb{Q} , let $P(X_1, \dots, X_q) \in \mathbb{Z}[X_1, \dots, X_q]$ be such that $\deg_{X_j}(P) \leq r_j \ 1 \leq j \leq q$ and let $\beta = P(a_1 \dots a_q)$.*

Then β is algebraic over \mathbb{Q} , $d(a_1)^{r_1} \dots d(a_q)^{r_q}$ is a multiple of $\text{den}(\beta)$ and we have

$$s(\beta) \leq \log H(P) + \sum_{j=1}^q (r_j s(a_j) + \log(r_j) + 1)$$

Theorem 2.1 (Hermite-Lindemann): *Suppose that \mathbb{K} has residue characteristic $p > 0$. Let $\alpha \in D_1$ be algebraic. Then e^α is transcendental.*

Proof. We suppose that α and e^α are algebraic. Let $h = |\alpha|$. Let E be the field $\mathbb{Q}[\alpha, e^\alpha]$, let $q = [E : \mathbb{Q}]$ and let w be a common denominator of α and e^α . We will construct a sequence of polynomials $(P_N(X, Y))_{N \in \mathbb{N}}$ in two variables such that $\deg_X(P_N) = \lfloor \frac{N}{\log(N)} \rfloor$, $\deg_Y(P_N) = \lfloor (\log N)^3 \rfloor$ and such that the function $F_N(x) = P_N(x, e^x)$ satisfy further, for every $s = 0, \dots, N-1$ and for every $j = 0, \dots, \lfloor \log(N) \rfloor$

$$\frac{d^s}{dx^s} F_N(j\alpha) = 0.$$

According to formal computations in the proof of Hermite Lindemann's Theorem in the complex context, (Theorem 3.1.1 in [16]) we have

$$(1) \quad \frac{d^M F_N(\gamma_N)}{dx^M} = \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{u_1(N)} \left(\frac{u_1(N)!}{\sigma!(u_1(N) - \sigma)!} \right) \left(\frac{l!}{(u_1(N) - \sigma)!} \right) m^{u_1(N) - \sigma} j^{u_1(N) - \sigma} (\alpha)^{u_1(N) - \sigma} (e^\alpha)^{ju_2(N)}.$$

We put $u_1(N) = \deg_X(P_N)$, $u_2(N) = \deg_Y(P_N)$. We will solve the system

$$w^{u_1(N) + u_2(N)} \frac{d^s}{dx^s} F_N(j\alpha) = 0, \quad 0 \leq s \leq N-1, \quad j = 0, \dots, \lfloor \log(N) \rfloor$$

where the undeterminates are the coefficients $b_{l,m,N}$ of P_N . We then write the system under the form

$$(2) \quad \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{\min(s,l)} \left(\frac{s!}{\sigma!(s - \sigma)!} \right) \left(\frac{l!}{(l - \sigma)!} \right) m^{s - \sigma} j^{l - \sigma} (w\alpha)^{l - \sigma} (we^\alpha)^{jm} w^{u_1(N) - (l - \sigma) + u_2(N) - jm} = 0.$$

That represents a system of $N \lfloor \log(N) \rfloor$ equations of at least $N(\lfloor \log(N) \rfloor)^2$ undeterminates, with coefficients in E , integral over \mathbb{Z} .

According to formal computations of Hermite-Lindemann's Theorem in the complex context (Theorem 3.1.1 in [16]), it appears that in the system (2), each factor $\left(\frac{s!}{\sigma!(s - \sigma)!} \right)$, $\left(\frac{l!}{(l - \sigma)!} \right)$, $m^{s - \sigma}$, $j^{l - \sigma}$, $(w\alpha)^{l - \sigma}$, $(we^\alpha)^{jm}$, $w^{u_1(N) - (l - \sigma) + u_2(N) - jm}$

admits a bounding of the form $SN(\log(\log(N)))$ when N goes to $+\infty$. On one hand $w^{u_1(N)+u_2(N)}$ is a common denominator and we have

$$\log(w^{u_1(N)+u_2(N)}) \leq \log(\omega) \left(\frac{N}{\log(N)} + (\log(N))^3 \right)$$

and hence we have a constant $T > 0$ such that

$$(3) \quad \log(w^{u_1(N)+u_2(N)}) \leq \frac{TM}{\log M}.$$

Next we notice that

$$(4) \quad \log \left(\frac{u_1(N)!}{\sigma!(u_1(N) - \sigma)!} \right) \leq u_1(N) \log(u_1(N)) \leq \frac{N}{\log(N)} \log \left(\frac{N}{\log(N)} \right) \leq N$$

and similarly,

$$(5) \quad \log \left(\frac{l!}{(u_1(N) - \sigma)!} \right) \leq u_1(N) \log(u_1(N)) \leq N.$$

and

$$(6) \quad \log(m^{u_1(N)-\sigma}) \leq \frac{3N}{\log(N)} \log(\log(N)).$$

Now, we check that

$$\log \left(j^{u_1(N)-\sigma} \cdot (|\bar{\alpha}|)^{u_1(N)-\sigma} \cdot (|e^{\bar{\alpha}}|)^{ju_2(N)} \right) \leq N + \frac{N}{\log(N)} \log(|\bar{\alpha}|) + \log(N) (\log(N))^3 \log(|e^{\bar{\alpha}}|)$$

and hence there exists a constant $L > 0$ such that

$$(7) \quad \log \left(j^{u_1(N)-\sigma} \cdot (|\bar{\alpha}|)^{u_1(N)-\sigma} \cdot (|e^{\bar{\alpha}}|)^{ju_2(N)} \right) \leq LN.$$

Therefore by (2), (3), (4), (5), (6) and (7) we have a constant $C > 0$ such that each coefficient a of the system satisfies

$$(8) \quad s(a) \leq CN(\log(\log(N))).$$

By Siegel's Lemma 2.a and by (8) there exist integers $b_{l,m,N}$, $0 \leq l \leq u_1(N)$, $0 \leq m \leq u_2(N)$ in \mathbb{Z} such that

$$(9) \quad 0 < \max_{l \leq u_1(N), m \leq u_2(N)} \log(|b_{l,m,N}|_\infty) \leq \frac{qN \log(N)}{N(\log(N))^2 - qN \log(N)} (CN \log(\log(N)))$$

and such that the function

$$(10) \quad F_N(x) = \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} x^l e^{mx}$$

satisfies

$$\frac{d^s}{dx^s} F_N(j\alpha) = 0, \quad 0 \leq s \leq N-1, \quad j = 0, 1, \dots, [\log(N)].$$

Now, by (9), we can check that there exists a constant $G > 0$ such that

$$(11) \quad \max_{l \leq u_1(N), m \leq u_2(N)} (\log(|b_{l,m,N}|_\infty)) \leq \frac{GN \log(\log(N))}{\log(N)}.$$

The function F_N defined in (10) belongs to $\mathcal{A}(D_1)$ and is not identically zero, hence at least one of the numbers $\frac{d^s}{dx^s} F_N(0)$ is not null. Let M be the biggest of the integers such that $\frac{d^s}{dx^s} F_N(j\alpha) = 0 \quad \forall s = 0, \dots, M-1, \quad j = 0, 1, 2, \dots, [\log(N)]$. Thus we have $M \geq N$ and there exists $j_0 \in \{0, 1, \dots, [\log(N)]\}$ such that $\frac{d^M}{dx^M} F_N(j_0\alpha) \neq 0$. We put $\gamma_N = \frac{d^M}{dx^M} F_N(j_0\alpha)$.

Let us now give an upper bound of $s(\gamma_N)$. On one hand $w^{u_1(N)+u_2(N)}$ is a common denominator and by (2) we have a constant $T > 0$ such that

$$\log(w^{u_1(N)+u_2(N)}) \leq \frac{TM}{\log M}.$$

On the other hand, by (1) we have

$$\begin{aligned} \frac{d^M F_N(\gamma_N)}{dx^M} &= \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{u_1(N)} \left(\frac{u_1(N)!}{\sigma!(u_1(N)-\sigma)!} \right) \left(\frac{l!}{(u_1(N)-\sigma)!} \right) m^{u_1(N)-\sigma} \\ &\quad j^{u_1(N)-\sigma} (\alpha)^{u_1(N)-\sigma} (e^\alpha)^{ju_2(N)}. \end{aligned}$$

Now, by (2), (3), (6), (7), (8), (10) and taking into account that the number of terms is bounded by $N(\log N)^2$, we can check that there exists a constant B such that

$$(12) \quad s(\gamma_N) \leq BN.$$

Let us now give an upper bound of $|\gamma_N|$. For convenience, we first suppose that $j_0 = 0$, hence $\frac{d^M}{dx^M} F_N(0) \neq 0$. Set $h = |\alpha|$. Then by Theorem B.9.1 of [7] we have $|\gamma_N| \leq \frac{|F_N|(h)}{h^M}$. Moreover, we notice that F_N admits at least $M[\log(M)]$ zeros in $d(0, h)$ and therefore by Corollary B.13.30 of [7] we have $|F_N|(h) \leq \left(\frac{h}{R_1} \right)^{M[\log(M)]}$ because $|F_N|(r) \leq 1 \quad \forall r < R_1$. Consequently, $|\gamma_N| \leq \frac{h^{M(\log(M)-1)}}{(R_1)^{M \log M}}$ and hence

$$\log(|\gamma_N|) \leq M(\log(M)-1)(\log(h)) - M \log(M)(\log(R_1)).$$

Let $\lambda = \log(h) - \log(R_1)$. Then $\lambda < 0$. And we have $\log(|\gamma_N|) \leq \lambda M \log(M) - M \log(h)$, therefore there exists a constant $A > 0$ such that

$$(13) \quad \log(|\gamma_N|) \leq -AM \log(M).$$

Let us now stop assuming that $j_0 = 0$. Putting $z = x - j\alpha$ and $g(z) = f(x)$, since all points $j\alpha$ belong to $d(0, h)$, it is immediate to go back to the case $j_0 = 0$, which confirms (13) in the general case. But now, by Lemma A.8.10 in [7], relations (12) and (13) make a contradiction to the relation $-2qs(\gamma_N) \leq \log(|\gamma_N|)$ satisfied by algebraic numbers and show that γ_N is transcendental. But then, so is e^α . □

Example: Let $Q(x) \in \mathbb{Z}[x]$. Then $e^{pQ(p)}$ is transcendental. Moreover, if Q is monic, and if α is a zero of Q , then $|p\alpha| \leq \frac{1}{p}$ because Q is monic and obviously $p\alpha$ is algebraic, hence $e^{p\alpha}$ is transcendental.

In the field of characteristic 0, \mathcal{K} such as Levi-Civita's field [15], we have a similar version:

Theorem 2.2: *Let $\alpha \in \mathcal{K}$ be algebraic, such that $|\alpha| < 1$. Then e^α is transcendental over \mathbb{Q} .*

Proof. Everything works in \mathcal{K} as in a field of residue characteristic $p \neq 0$ up to Relation (8) in the proof of Theorem 2.1. Here we can replace R_1 by 1 and therefore the conclusion is the same as in Theorem 2.1. □

Similarly as Hermite-Lindemann's Theorem, Gelfond-Schneider's Theorem is well known in the field \mathbb{C} and has an analogue in an ultrametric field.

In the proof of Theorem 2.4 we will need the following theorem:

Theorem 2.3: *Let $b_1, \dots, b_n \in D_1$ (resp. in D_0). the functions $x, e^{b_1x}, \dots, e^{b_nx}$ are algebraically independant over \mathbb{K} (resp. over \mathcal{K}) if and only if b_1, \dots, b_n are \mathbb{Q} -linearly independant.*

Theorem 2.4 (Gel'fond-Schneider): *\mathbb{K} is supposed to have residue characteristic $p \neq 0$. Let $\ell \in D_1$, $\ell \neq 0$, and let $b \notin \mathbb{Q}$ belong to \mathbb{K} be such that $b\ell \in D_1$. Then at least one of the three numbers $a = e^\ell$, b , $e^{b\ell}$ is transcendental.*

Proof. A large part of the proof does not involve the topology of the field \mathbb{K} and hence is similar to the proof in the field \mathbb{C} [16] where we can copy many technical relations. We suppose that $a = e^\ell$, b and $e^{b\ell}$ are algebraic over \mathbb{Q} . Let $L = \mathbb{Q}[e^\ell, b, e^{b\ell}]$ and let $\delta = [L : \mathbb{Q}]$ and let d be a common denominator of $b, e^\ell, e^{b\ell}$.

Put $S = \max(1, |b|)$, $T \in]S, \frac{R_1}{|\ell}|[$, $\sigma = \log(\frac{T}{S})$, $\tau = \log T$, $\Lambda = d(0, S)$ and $\Delta = d(0, T)$. We will consider integers N of the form q^2 , with $q \in \mathbb{N}$ and we will first show that there exists a non-identically zero polynomial $P_N(X, Y) \in \mathbb{Z}[X, Y]$ such that $\deg_X(P_N) \leq N^{\frac{3}{2}}$, and $\deg_Y(P_N) \leq 2\delta N^{\frac{1}{2}}$ such that the function $F_N(x)$ defined in Δ by $F_N(x) = P_N(x, e^{\ell x})$ satisfy

$$F_N(i + jb) = 0 \quad \forall i = 1, \dots, N, \quad \forall j = 1, \dots, N.$$

In order to find P_N , let us write it

$$\sum_{h=0}^{N^{\frac{3}{2}}-1} \sum_{k=0}^{2\delta N^{\frac{1}{2}}-1} C_{h,k}(N) X^h Y^k$$

with $C_{h,k}(N) \in \mathbb{Z}$ and consider the system of equations where the $C_{h,k}(N)$ are the undeterminates:

$$d^{(4\delta+1)N^{\frac{3}{2}}} \cdot F_N(i + jb) = 0 \quad (1 \leq i \leq N; 1 \leq j \leq N).$$

Thus, we obtain a system of N^2 equations of $2\delta N^2$ undeterminates in \mathbb{Z} , with coefficients in L . By Lemma 2.b, these coefficients have size bounded by

$$N^{\frac{3}{2}} \log(N) + N^{\frac{3}{2}} (8\delta + 2) \log(d) + \log(1 + |b|) + 2\delta \log(|e^{\ell+b\ell}|) \leq \frac{3}{2} N^{\frac{3}{2}} \log(N).$$

By Lemma 2.a we can find in \mathbb{Z} a family of integers not all equal to zero, $(C_{h,k}(N), 0 \leq h \leq N^{\frac{3}{2}} - 1, 0 \leq k \leq 2\delta N^{\frac{1}{2}} - 1)$ satisfying

$$\log \left(\max_{h,k} |C_{h,k}(N)|_{\infty} \right) \leq 2N^{\frac{3}{2}} \log N \left(\frac{\delta N^2}{2\delta N^2 - \delta N^2} \right) = 2N^{\frac{3}{2}} \log N$$

such that the function F_N defined by $F_N(x) = P_N(x, e^{\ell x})$ satisfies $F_N(i + jb) = 0 \quad \forall i = 1, \dots, N, \quad j = 1, \dots, N$.

Now we can check the function F_N is an analytic element in every disk of the form $d(0, r)$ such that $r|\ell| < R_1$ and hence in $\Delta = d(0, T)$ [7]. Since the power of x in the various terms is at most $N^{\frac{3}{2}}$ and since all coefficients are integers, we can check that $\log(|F_N|(T)) \leq \tau N^{\frac{3}{2}}$. On the other hand, since the polynomial P_N is not identically zero, by Proposition 1.6 F_N is not identically zero and then, by classical results [7], the function F_N has finitely many zeros in Λ . Particularly, there exists a point of the form $i + jb$ such that $F_N(i + jb) \neq 0$. Consequently there exists $M \geq N$ such that $F_N(i + jb) = 0 \quad \forall i \leq M, \quad \forall j \leq M$ and there exists a point γ_N of the form $i_0 + j_0 b$ such that $F_N(\gamma_N) \neq 0$ with $M < i_0 \leq M + 1, \quad M < j_0 \leq M + 1$. Consequently the number of zeros of F_N in Λ is at least M^2 . Then by Corollary B.13.30 in [7] we have $\log(|F_N(\gamma_N)|) \leq \tau N^{\frac{3}{2}} - \sigma M^2$, hence there exists $\lambda > 0$ such that

$$(1) \quad \log(|F_N(\gamma_N)|) \leq -\lambda M^2 \quad \forall N \in \mathbb{N}.$$

By definition neither σ nor τ depend on N , hence neither does λ .

On the other hand, by Lemma 2.b we can check that $s(F_N(\gamma_N))$ satisfies an inequality of the form $s(F_N(\gamma_N)) \leq AM^{\frac{3}{2}} \log(M)$ which by (1) contradicts the inequality $-2\delta s(F_N(\gamma_N)) \leq \log(|F_N(\gamma_N)|)$ and this ends the proof. \square

Example: Let $u = pe^p$ and let $\ell = pe^u$. Let $b \notin \mathbb{Q}$ be such that $|b| \leq 1$. Then at least one of the 3 numbers e^ℓ , b , $e^{b\ell}$ is transcendental. In particular, if b is algebraic, then one of the numbers e^ℓ and $e^{b\ell}$ is transcendental.

Theorem 2.5 (Gel'fond-Schneider in zero residue characteristic): *Let \mathcal{K} be an algebraically closed complete ultrametric field whose residue characteristic is 0. Let $\ell \in D_0$, $\ell \neq 0$, and let $b \notin \mathbb{Q}$ belong to \mathcal{K} and be such that $b\ell \in D_0$. Then at least one of the three numbers $a = e^\ell$, b , $e^{b\ell}$ is transcendental.*

Proof. The proof is identical to the proof of Theorem 2.4 except that T now belongs to $]S, \frac{1}{|\ell}[$. \square

3 Nevanlinna Theory in \mathbb{K} and in an open disk

Notations: We denote by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$. Let $d(a, R^-)$ be a disk in \mathbb{K} . We denote by $\mathcal{M}(d(a, R^-))$ the field of fractions $\mathcal{A}(d(a, R^-))$ and by $\mathcal{M}_b(d(a, R^-))$ the field of fractions $\mathcal{A}_b(d(a, R^-))$. Finally we put $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

Given two meromorphic functions $f, g \in \mathcal{M}(\mathbb{K})$ or $f, g \in \mathcal{M}(d(a, R^-))$ ($a \in \mathbb{K}$, $R > 0$), we will denote by $W(f, g)$ the Wronskian of f and g : $f'g - fg'$.

Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. Let $f \in \mathcal{M}_u(d(\alpha, R^-))$). A value $b \in \mathbb{K}$ will be called a *quasi-exceptional value* for f if $f - b$ has finitely many zeros in \mathbb{K} (resp. in (α, R^-)) and it will be called an *exceptional value* for f if $f - b$ has no zero in \mathbb{K} (resp. in $d(\alpha, R^-)$).

We have the following result:

Theorem 3.1: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(a, R^-))$). Then f admits at most one quasi-exceptional value. Moreover, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(a, R^-))$) then f admits no quasi-exceptional value*

The Nevanlinna Theory was made by Rolf Nevanlinna on complex functions [14], and widely used by many specialists of complex functions, particularly Walter Hayman [10]. It consists of defining counting functions of zeros and poles of a meromorphic function f and giving an upper bound for multiple zeros and poles of various functions $f - b$, $b \in \mathbb{C}$.

A similar theory for functions in a p-adic field was constructed and correctly proved by A. Boutabaa [5] in the field \mathbb{K} , after some previous work by Ha Huy Khoai [9]. See also [11]. In [6] the theory was extended to functions in $\mathcal{M}(d(0, R^-))$ by taking into account Lazard's problem [12]. A new extension to functions out of a hole was made in [7] but we won't describe it because we would miss place. Here we will only give an abstract of the ultrametric Nevanlinna Theory in order to give the new theorems on q small functions.

Notations: Recall that given three functions ϕ, ψ, ζ defined in an interval $J =]a, +\infty[$ (resp. $J =]a, R[$), with values in $[0, +\infty[$, we shall write $\phi(r) \leq \psi(r) + O(\zeta(r))$ if there exists a constant $b \in \mathbb{R}$ such that $\phi(r) \leq \psi(r) + b\zeta(r)$. We shall write $\phi(r) = \psi(r) + O(\zeta(r))$ if $|\psi(r) - \phi(r)|$ is bounded by a function of the form $b\zeta(r)$.

Similarly, we shall write $\phi(r) \leq \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]a, +\infty[$ (resp. from $J =]a, R[$) to \mathbb{R} such that $\lim_{r \rightarrow +\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) \leq \psi(r) + h(r)$. And we shall write $\phi(r) = \psi(r) + o(\zeta(r))$ if there exists a function h from $J =]a, +\infty[$ (resp. from $J =]a, R[$) to \mathbb{R} such that $\lim_{r \rightarrow +\infty} \frac{h(r)}{\zeta(r)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{h(r)}{\zeta(r)} = 0$) and such that $\phi(r) = \psi(r) + h(r)$.

Throughout the next paragraphs, we will denote by I the interval $[t, +\infty[$ and by J an interval of the form $[t, R[$ with $t > 0$.

We have to introduce the counting function of zeros and poles of f , counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Definitions: Next, let $f = \frac{h}{l} \in \mathcal{M}(\mathbb{K})$ (resp. $f = \frac{h}{l} \in \mathcal{M}(d(a, R^-))$). The order of a zero α of f will be denoted by $\omega_\alpha(f)$. Next, given any point $\alpha \in \mathbb{K}$ (resp. $\alpha \in d(a, R^-)$), the number $\omega_\alpha(h) - \omega_\alpha(l)$ does not depend on the functions h, l choosed to make $f = \frac{h}{l}$. Thus, we can generalize the notation by setting $\omega_\alpha(f) = \omega_\alpha(h) - \omega_\alpha(l)$. We then denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way.

Let (a_n) , $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n .

We set $Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|)$ and so, $Z(r, f)$

is called *the counting function of zeros of f in $d(0, r)$, counting multiplicity*.

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega}_0(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega}_0(f) = 1$ if $\omega_0(f) \geq 1$.

Now, we denote by $\overline{Z}(r, f)$ the counting function of zeros of f without multiplicity:

$\overline{Z}(r, f) = \overline{\omega}_0(f) \log r + \sum_{n=1}^{\sigma(r)} (\log r - \log |a_n|)$ and so, $\overline{Z}(r, f)$ is called *the counting function of zeros of f in $d(0, r)$ ignoring multiplicity*.

In the same way, considering the finite sequence (b_n) , $1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n , we put

$N(r, f) = \max(-\omega_0(f), 0) \log r + \sum_{n=1}^{\tau(r)} t_n (\log r - \log |b_n|)$ and then $N(r, f)$ is called *the counting function of the poles of f , counting multiplicity*

Next, in order to define the counting function of poles of f without multiplicity, we put $\overline{\omega}_0(f) = 0$ if $\omega_0(f) \geq 0$ and $\overline{\omega}_0(f) = 1$ if $\omega_0(f) \leq -1$ and we set

$\overline{N}(r, f) = \overline{\omega}_0(f) \log r + \sum_{n=1}^{\tau(r)} (\log r - \log |b_n|)$ and then $\overline{N}(r, f)$ is called *the counting function of the poles of f , ignoring multiplicity*

Now we can define the the Nevanlinna function $T(r, f)$ in I or J as $T(r, f) = \max(Z(r, f), N(r, f))$ and the function $T(r, f)$ is called *characteristic function of f or Nevanlinna function of f* .

Finally, if S is a subset of \mathbb{K} we will denote by $Z_0^S(r, f')$ the counting function of zeros of f' , excluding those which are zeros of $f - a$ for any $a \in S$.

Remark: If we change the origin, the functions Z , N , T are not changed, up to an additive constant.

In a p -adic field such as \mathbb{K} , the first Main Theorem is almost immediate.

Theorem 3.2: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) have no zero and no pole at 0. Then $\log(|f|(r)) = \log(|f(0)|) + Z(r, f) - N(r, f)$.*

Then we can derive Theorem 3.3 (Theorem C.4.3 in [7])

Theorem 3.3: *Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(0, R^-))$). Then $Z(r, fg) \leq Z(r, f) + Z(r, g)$, $N(r, fg) \leq N(r, f) + N(r, g)$, $T(r, fg) \leq T(r, f) + T(r, g)$, $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$, $T(r, cf) = T(r, f) \forall c \in \mathbb{K}^*$, $T(r, \frac{1}{f}) = T(r, f)$, $T(r, \frac{f}{g}) \leq T(r, f) + T(r, g)$.*

Suppose now $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \mathcal{A}(d(0, R^-))$). Then $Z(r, fg) = Z(r, f) + Z(r, g)$, $T(r, f) = Z(r, f)$, $T(r, fg) = T(r, f) + T(r, g) + O(1)$ and $T(r, f + g) \leq \max(T(r, f), T(r, g))$. Moreover, if $\lim_{r \rightarrow +\infty} T(r, f) - T(r, g) = +\infty$ then $T(r, f + g) = T(r, f)$ when r is big enough.

Corollary 3.A: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Then*

$$Z(r, \frac{f'}{f}) - N(r, \frac{f'}{f}) \leq -\log r + O(1).$$

Thus we have Theorem 3.4 (Theorem C.4.8 in [7])

Theorem 3.4 (First Main Fundamental Theorem): *Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}(d(0, R^-))$). Then $T(r, f + b) = T(r, f) + O(1)$. Let h be a Moebius function. Then $T(r, f) = T(r, h \circ f) + O(1)$. Let $P(X) \in \mathbb{K}[X]$. Then $T(r, P(f)) = \deg(P)T(r, f) + O(1)$ and $T(r, f'P(f)) \geq T(r, P(f))$.*

Suppose now $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \mathcal{A}(d(0, R^-))$). Then $Z(r, fg) = Z(r, f) + Z(r, g)$, $T(r, f) = Z(r, f)$, $T(r, fg) = T(r, f) + T(r, g) + O(1)$ and $T(r, f + g) \leq \max(T(r, f), T(r, g))$. Moreover, if $\lim_{r \rightarrow +\infty} T(r, f) - T(r, g) = +\infty$ then $T(r, f + g) = T(r, f)$ when r is big enough.

The following Theorem 3.5 is a good way to obtain the famous Second Main Theorem (Theorem C.4.24 in [7]).

Theorem 3.5: *Let $f \in \mathcal{M}(\mathbb{K})$ and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct. Then*

$$(q-1)T(r, f) \leq \max_{1 \leq k \leq q} \left(\sum_{j=1, j \neq k}^q Z(r, f - a_j) \right) + O(1).$$

Theorem 3.6 (Second Main Theorem, Theorem C.4.24 in [7]): *Let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$, with $q \geq 2$, let $S = \{\alpha_1, \dots, \alpha_q\}$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$). Then*

$$(q-1)T(r, f) \leq \sum_{j=1}^q \bar{Z}(r, f - \alpha_j) + \bar{N}(r, f) - Z_0^S(r, f') - \log r + O(1) \quad \forall r \in I$$

(resp. $\forall r \in J$).

Now we can easily deduce the following corollaries:

Corollary 3.B: *Let $a_1, a_2 \in \mathbb{K}$ ($a_1 \neq a_2$) and let $f, g \in \mathcal{A}(\mathbb{K})$ satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2$). Then $f = g$.*

Remark: Corollary 3.B does not hold in complex analysis. Indeed, let $f(z) = e^z$, $g(z) = e^{-z}$, let $a_1 = 1$, $a_2 = -1$. Then $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2$), though $f \neq g$.

Corollary 3.C: *Let $a_1, a_2, a_3 \in \mathbb{K}$ ($a_i \neq a_j \forall i \neq j$) and let $f, g \in \mathcal{A}_u(d(a, R^-))$ (resp. $f, g \in \mathcal{A}_u(D)$) satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3$). Then $f = g$.*

Corollary 3.D: *Let $a_1, a_2, a_3, a_4 \in \mathbb{K}$ ($a_i \neq a_j \forall i \neq j$) and let $f, g \in \mathcal{M}(\mathbb{K})$ satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3, 4$). Then $f = g$.*

Corollary 3.E: *Let $a_1, a_2, a_3, a_4, a_5 \in \mathbb{K}$ ($a_i \neq a_j \forall i \neq j$) and let $f, g \in \mathcal{M}_u(d(a, R^-))$ (resp. $f, g \in \mathcal{M}_u(D)$) satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3, 4, 5$). Then $f = g$.*

Remark: Let $f(x) = \frac{x}{3x-1}$, $g(x) = \frac{x^2}{x^2+2x-1}$. Let $a_0 = 0$, $a_1 = 1$, $a_2 = \frac{1}{2}$. Then we can check that $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$, $i = 1, 2, 3$. So, Corollary 3.D is sharp.

4 Exceptional values of meromorphic functions and derivatives

The paragraph is aimed at studying various properties of derivatives of meromorphic functions, particularly their sets of zeros [2], [3], [4]. Many important results are due to Jean-Paul Bézivin [1], [2].

We will first notice a general property concerning quasi-exceptional values of meromorphic functions and derivatives.

Theorem 4.1: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ (resp. $f \in \mathcal{M}_u(d(\alpha, R^-))$). If f admits a quasi-exceptional value, then f' has no quasi-exceptional value different from 0.*

Proof. Without loss of generality, we may assume $\alpha = 0$ and that f has no zero and no pole at 0. Let $b \in \mathbb{K}$ and suppose that b is a quasi-exceptional value of f . There exist $P \in \mathbb{K}[x]$ and $l \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ (resp. and $l \in \mathcal{A}_u(d(0, R^-))$) without common zeros, such that $f = b + \frac{P}{l}$.

Let $c \in \mathbb{K}^*$. Remark that $f' - c = \frac{P'l - Pl' - cl^2}{l^2}$. Let $a \in \mathbb{K}$ (resp. let $a \in d(0, R^-)$). If a is a pole of f , it is a pole of $f' - c$ and we can check that

$$(1) \quad \omega_a(P'l - Pl' - cl^2) = \omega_a(l') = \omega_a(l) - 1$$

because a is not a zero of P .

Now suppose that a is not a pole of f . Then

$$(2) \quad \omega_a(f' - c) = \omega_a(P'l - Pl' - cl^2)$$

Consequently, $Z(r, f' - c) = Z(r, (P'l - Pl' - cl^2) \mid l(x) \neq 0)$. But, by (1) we have

$$(3) \quad Z(r, (P'l - Pl' - cl^2) \mid l(x) = 0) < Z(r, l).$$

and therefore by (2) and (3) we obtain

$$(4) \quad Z(r, f' - c) = Z(r, (P'l - Pl' - cl^2) \mid l(x) \neq 0) > Z(r, P'l - Pl' - cl^2) - Z(r, l)$$

Now, let us examine $Z(r, P'l - Pl' - cl^2)$. Let $r \in]0, +\infty[$ (resp. let $r \in]0, R[$). Since $l \in \mathcal{A}(\mathbb{K})$ is transcendental (resp. since $l \in \mathcal{A}_u(d(0, R^-))$), we can check that when r is big enough, we have $|P'l|(r) < |c|(|l|(r))^2$ and $|Pl|(r) < |c|(|l|(r))^2$, hence clearly $|P'l - Pl'|(r) < |c|(|l|(r))^2$ and hence $|P'l - Pl' - cl^2|(r) = |c|(|l|(r))^2$. Consequently, when r is big enough, by Theorem C.4.2 in [7] we have $Z(r, P'l - Pl' - cl^2) = Z(r, l^2) + O(1)$. But $Z(r, l^2) = 2Z(r, l)$, hence $Z(r, P'l - Pl' - cl^2) = 2Z(r, l) + O(1)$ and therefore by (4) we check that when r is big enough, we obtain

$$(5) \quad Z(r, f' - c) > Z(r, l).$$

Now, if $l \in \mathcal{A}(\mathbb{K})$, since l is transcendental, by (5), for every $q \in \mathbb{N}$, we have $Z(r, f' - c) > Z(r, l) > q \log r$, when r is big enough, hence $f' - c$ has infinitely many zeros in \mathbb{K} . And similarly if $l \in \mathcal{A}_u(d(0, R^-))$, then by (5), $Z(r, f' - c)$ is unbounded when r tends to R , hence $f' - c$ has infinitely many zeros in $d(0, R^-)$. \square

We will now notice a property of differential equations of the form $y^{(n)} - \psi y = 0$ that is almost classical.

The problem of a constant Wronskian is involved in several questions.

Theorem 4.2: *Let $h, l \in \mathcal{A}(\mathbb{K})$ (resp. $h, l \in \mathcal{A}(d(\alpha, R^-))$) and satisfy $h'l - hl' = c \in \mathbb{K}$, with h non-affine. If h, l belong to $\mathcal{A}(\mathbb{K})$, then $c = 0$ and $\frac{h}{l}$ is a constant. If $c \neq 0$ and if $h, l \in \mathcal{A}(d(\alpha, R^-))$, there exists $\phi \in \mathcal{A}(d(\alpha, R^-))$ such that $h'' = \phi h$, $l'' = \phi l$.*

Proof. Suppose $c \neq 0$. If $h(a) = 0$, then $l(a) \neq 0$. Next, h and l satisfy

$$(1) \quad \frac{h''}{h} = \frac{l''}{l}.$$

Remark first that since h is not affine, h'' is not identically zero. Next, every zero of h or l of order ≥ 2 is a trivial zero of $h'l - hl'$, which contradicts $c \neq 0$. So we can assume that all zeros of h and l are of order 1.

Now suppose that a zero a of h is not a zero of h'' . Since a is a zero of h of order 1, $\frac{h''}{h}$ has a pole of order 1 at a and so does $\frac{l''}{l}$, hence $l(a) = 0$, a contradiction. Consequently, each zero of h is a zero of order 1 of h and is a zero of h'' and hence, $\frac{h''}{h}$ is an element ϕ of $\mathcal{M}(\mathbb{K})$ (resp. of $\mathcal{M}(d(\alpha, R^-))$) that has no pole in \mathbb{K} (resp. in $d(\alpha, R^-)$). Therefore ϕ lies in $\mathcal{A}(\mathbb{K})$ (resp. in $\mathcal{A}(d(\alpha, R^-))$).

The same holds for l and so, l'' is of the form ψl with $\psi \in \mathcal{A}(\mathbb{K})$ (resp. in $\mathcal{A}(d(\alpha, R^-))$). But since $\frac{h''}{h} = \frac{l''}{l}$, we have $\phi = \psi$.

Now, suppose h, l belong to $\mathcal{A}(\mathbb{K})$. Since h'' is of the form ϕh with $\phi \in \mathcal{A}(\mathbb{K})$, we have $|h''|(r) = |\phi|(r)|h|(r)$. But by Theorem C.2.10 in [7], we know that $|h''|(r) \leq \frac{1}{r^2}|h|(r)$, a contradiction when r tends to $+\infty$. Consequently, $c = 0$.

But then $h'l - hl' = 0$ implies that the derivative of $\frac{h}{l}$ is identically zero, hence $\frac{h}{l}$ is constant. \square

Corollary 4.A : *Let $h, l \in \mathcal{A}(\mathbb{K})$ with coefficients in \mathbb{Q} , also be entire functions in \mathbb{C} , with h non-affine. If $h'l - hl'$ is a constant c , then $c = 0$.*

Theorem 4.3: *Let $\psi \in \mathcal{M}(\mathbb{K})$ (resp. let $\psi \in \mathcal{M}_u(d(\alpha, R^-))$) and let (\mathcal{E}) be the differential equations $y'' - \psi y = 0$. Let E be the sub-vector space of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{A}(d(\alpha, R^-))$) of the solutions of (\mathcal{E}) . Then, the dimension of E is 0 or 1.*

Proof. Suppose E is not $\{0\}$. Let $h, l \in E$ be non-identically zero. Then $h''l - hl'' = 0$ and therefore $h'l - hl'$ is a constant c . On the other hand, since h, l are not identically zero, neither are h'', l'' . Therefore, h, l are not affine functions.

Suppose ψ belongs to $\mathcal{M}(\mathbb{K})$ and that h, l belong to $\mathcal{A}(\mathbb{K})$. By Theorem 4.2, we have $c = 0$ and hence $\frac{h}{l}$ is a constant, which proves that E is of dimension 1.

Suppose now that ψ lies in $\mathcal{M}_u(d(\alpha, R^-))$ and that h, l belong to $\mathcal{A}(d(\alpha, R^-))$. If ψ lies in $\mathcal{A}(d(\alpha, R^-))$, then by Theorem 4.1, $E = \{0\}$. Finally, suppose that ψ lies in $\mathcal{M}_u(d(\alpha, R^-)) \setminus \mathcal{A}(d(\alpha, R^-))$. If $c \neq 0$, by Theorem 4.2, there exists $\phi \in \mathcal{A}(d(\alpha, R^-))$ such that $h'' = \phi h, l'' = \phi l$. Consequently, $\phi = \psi$, hence $\psi \in \mathcal{A}(\mathbb{K})$ and therefore $c = 0$. Hence $h'l - hl' = 0$ again and hence $\frac{h}{l}$ is a constant. Thus, we see that E is at most of dimension 1. \square

Remark: The hypothesis ψ unbounded in $d(\alpha, R^-)$ is indispensable to show that the space E is of dimension 0 or 1, as shows the example given again by the p-adic hyperbolic functions $h(x) = \cosh(x)$ and $l(x) = \sinh(x)$. The radius of convergence of both h, l is $p^{\frac{-1}{p-1}}$ when \mathbb{K} has residue characteristic p and is 1 when \mathbb{K} has residue characteristic 0. Of course, both functions are solutions of $y'' - y = 0$ but they are bounded.

The following Theorem 4.4 is an improvement of Theorem 4.2. It follows previous results [1].

Theorem 4.4 [2]: *Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $W(f, g)$ is a non-identically zero polynomial. Then both f, g are polynomials.*

Proof. First, by Theorem 4.2 we check that the claim is satisfied when $W(f, g)$ is a polynomial of degree 0. Now, suppose the claim holds when $W(f, g)$ is a polynomial of certain degree n . We will show it for $n + 1$. Let $f, g \in \mathcal{A}(\mathbb{K})$ be such that $W(f, g)$ is a non-identically zero polynomial P of degree $n + 1$

Thus, by hypothesis, we have $f'g - fg' = P$, hence $f''g - fg'' = P'$. We can extract g' and get $g' = \frac{(f'g - P)}{f}$. Now consider the function $Q = f''g' - f'g''$ and replace g' by what we just found: we can get $Q = f' \left(\frac{(f''g - fg'')}{f} \right) - \frac{P'f''}{f}$.

Now, we can replace $f''g - fg''$ by P' and obtain $Q = \frac{(f'P' - Pf'')}{f}$. Thus, in that expression of Q , we can write $|Q|(R) \leq \frac{|f|(R)|P|(R)}{R^2|f|(R)}$, hence $|Q|(R) \leq$

$\frac{|P|(R)}{R^2} \forall R > 0$. But by definition, Q belongs to $\mathcal{A}(\mathbb{K})$. Consequently, Q is a polynomial of degree $t \leq n - 1$.

Now, suppose Q is not identically zero. Since $Q = W(f', g')$ and since $\deg(Q) < n$, by the induction hypothesis f' and g' are polynomials and so are f, g . Finally, suppose $Q = 0$. Then $P'f' - Pf'' = 0$ and therefore f', P are two solutions of the differential equation of order 1 for meromorphic functions in \mathbb{K} : $(\mathcal{E}) y' = \psi y$ with $\psi = \frac{P'}{P}$, whereas y belongs to $\mathcal{A}(\mathbb{K})$. By Theorem 4.3, the space of solutions of (\mathcal{E}) is of dimension 0 or 1. Consequently, there exists $\lambda \in \mathbb{K}$ such that $f' = \lambda P$, hence f is a polynomial. The same holds for g . \square

Here we can find again the following result that is known and may be proved without ultrametric properties:

Let F be an algebraically closed field and let $P, Q \in F[x]$ be such that $PQ' - P'Q$ is a constant c , with $\deg(P) \geq 2$. Then $c = 0$.

Notation: Let $f \in \tilde{\mathcal{A}}(\mathbb{K})$. We can factorize f in the form $\bar{f}\tilde{f}$ where the zeros of \bar{f} are the distinct zeros of f each with order 1. Moreover, if $f(0) \neq 0$ we will take $\bar{f}(0) = 1$.

Lemma 4.a: *Let $U, V \in \mathcal{A}(\mathbb{K})$ have no common zero and let $f = \frac{U}{V}$. If f has finitely many zeros, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\tilde{V}$*

Proof. If V is a constant, the statement is obvious. So, we assume that V is not a constant. Now \tilde{V} divides V and hence V' factorizes in the way $V' = \tilde{V}Y$ with $Y \in \mathcal{A}(\mathbb{K})$. Then no zero of Y can be a zero of V . Consequently, we have

$$f'(x) = \frac{U'V - UV'}{V^2} = \frac{U'\bar{V} - UY}{\bar{V}^2\tilde{V}}.$$

The two functions $U'\bar{V} - UY$ and $\bar{V}^2\tilde{V}$ have no common zero since neither have U and V . So, the zeros of f' are those of $U'\bar{V} - UY$ which therefore has finitely many zeros and consequently is a polynomial. \square

Theorem 4.5: *Let $f \in \mathcal{M}(\mathbb{K})$ have finitely many multiple poles, such that for certain $b \in \mathbb{K}$, $f' - b$ has finitely many zeros. Then f belongs to $\mathbb{K}(x)$.*

Proof. Suppose first $b = 0$. Let us write $f = \frac{U}{V}$ with $U, V \in \mathcal{A}(\mathbb{K})$, having no common zeros. By Lemma 4.a, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\tilde{V}$. Since f has finitely many multiple poles, \tilde{V} is a polynomial, hence so is $U'V - UV'$. But then by Theorem 4.4, both U, V are polynomials, which ends the proof when $b = 0$. Consider now the general case. $f' - b$ is the derivative of $f - bx$ that satisfies the same hypothesis, so the conclusion is immediate. \square

Notation: For each $n \in \mathbb{N}^*$, we set $\lambda_n = \max\{\frac{1}{|k|}, 1 \leq k \leq n\}$. Given positive integers n, q , we denote by C_n^q the combination $\frac{n!}{q!(n-q)!}$. Let us recall that \log is the Neperian logarithm, we denote by e the number such that $\log(e) = 1$ and Exp is the real exponential function.

Remark: For every $n \in \mathbb{N}^*$, we have $\lambda_n \leq n$ because $k|k| \geq 1 \forall k \in \mathbb{N}$. The equality holds for all n of the form p^h .

Lemmas 4.b and 4.c are due to Jean-Paul Bézivin [1]:

Lemma 4.b: *Let $U, V \in \mathcal{A}(d(0, R^-))$. Then for all $r \in]0, R[$ and $n \geq 1$ we have*

$$|U^{(n)}V - UV^{(n)}|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{r^{n-1}}.$$

More generally, given $j, l \in \mathbb{N}$, we have

$$|U^{(j)}V^{(l)} - U^{(l)}V^{(j)}|(r) \leq |(j!)(l!)|\lambda_{j+l} \frac{|U'V - UV'|(r)}{r^{j+l-1}}$$

Lemma 4.c: *Let $U, V \in \mathcal{A}(\mathbb{K})$ and let $r, R \in]0, +\infty[$ satisfy $r < R$. For all $x, y \in \mathbb{K}$ with $|x| \leq R$ and $|y| \leq r$, we have the inequality:*

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|(R)}{e(\log R - \log r)}$$

Notation: Let $f \in \mathcal{M}(d(0, R^-))$. For each $r \in]0, R[$, we denote by $\zeta(r, f)$ the number of zeros of f in $d(0, r)$, taking multiplicity into account and set $\xi(r, f) = \zeta(r, \frac{1}{f})$. Similarly, we denote by $\beta(r, f)$ the number of multiple zeros of f in $d(0, r)$, each counted with its multiplicity and we set $\gamma(r, f) = \beta(r, \frac{1}{f})$.

Theorem 4.6 [2] *Let $f \in \mathcal{M}(\mathbb{K})$ be such that for some $c, q \in]0, +\infty[$, $\gamma(r, f)$ satisfies $\gamma(r, f) \leq cr^q$ in $[1, +\infty[$. If f' has finitely many zeros, then $f \in \mathbb{K}(x)$.*

Proof. Suppose f' has finitely many zeros and set $f = \frac{U}{V}$. If V is a constant, the statement is immediate. So, we suppose V is not a constant and hence it admits at least one zero a . By Lemma 4.a, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\tilde{V}$. Next, we take $r, R \in [1, +\infty[$ such that $|a| < r < R$ and $x \in d(0, R)$, $y \in d(0, r)$. By Lemma 4.c we have

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|(R)}{e(\log R - \log r)}.$$

Notice that $U(a) \neq 0$ because U and V have no common zero. Now set $l = \max(1, |a|)$ and take $r \geq l$. Setting $c_1 = \frac{1}{e|U(a)|}$, we have

$$|V(a+y)| \leq c_1 \frac{R|P|(R)|\tilde{V}|(R)}{\log R - \log r}.$$

Then taking the supremum of $|V(a+y)|$ inside the disk $d(0, r)$, we can derive

$$(1) \quad |V|(r) \leq c_1 \frac{R|P|(R)|\tilde{V}|(R)}{\log R - \log r}.$$

Let us apply Corollary B.13.30 in [7], by taking $R = r + \frac{1}{r^q}$, after noticing that the number of zeros of $\tilde{V}(R)$ is bounded by $\beta(R, V)$. So, we have

$$(2) \quad |\tilde{V}|(R) \leq \left(1 + \frac{1}{r^{q+1}}\right)^{\beta((r+\frac{1}{r^q}), V)} |\tilde{V}|(r).$$

Now, due to the hypothesis: $\beta(r, V) = \gamma(r, f) \leq cr^q$ in $[1, +\infty[$, we have

$$(3) \quad \left(1 + \frac{1}{r^{q+1}}\right)^{\beta((r+\frac{1}{r^q}), V)} \leq \left(1 + \frac{1}{r^{q+1}}\right)^{[c(r+\frac{1}{r^q})^m]} = \text{Exp} \left[c \left(r + \frac{1}{r^q}\right)^q \log \left(1 + \frac{1}{r^{q+1}}\right) \right].$$

The function $h(r) = c \left(r + \frac{1}{r^m}\right)^m \log \left(1 + \frac{1}{r^{m+1}}\right)$ is continuous on $]0, +\infty[$ and equivalent to $\frac{c}{r}$ when r tends to $+\infty$. Consequently, it is bounded on $[l, +\infty[$. Therefore, by (2) and (3) there exists a constant $M > 0$ such that, for all $r \in [l, +\infty[$ by (3) we obtain

$$(4) \quad |\tilde{V}| \left(r + \frac{1}{r^q}\right) \leq M |\tilde{V}|(r).$$

On the other hand, $\log \left(r + \frac{1}{r^q}\right) - \log r = \log \left(1 + \frac{1}{r^{q+1}}\right)$ clearly satisfies an inequality of the form $\log \left(1 + \frac{1}{r^{q+1}}\right) \geq \frac{c_2}{r^{q+1}}$ in $[l, +\infty[$ with $c_2 > 0$. Moreover, we can find positive constants c_3, c_4 such that $\left(r + \frac{1}{r^q}\right) |P| \left(r + \frac{1}{r^q}\right) \leq c_3 r^{c_4}$. Consequently, by (1) and (4) we can find positive constants c_5, c_6 such that $|V|(r) \leq c_5 r^{c_6} |\tilde{V}|(r) \forall r \in [l, +\infty[$. Thus, writing again $V = \bar{V} \tilde{V}$, we have $|\bar{V}|(r) |\tilde{V}|(r) \leq c_5 r^{c_6} |\tilde{V}|(r)$ and hence $|\bar{V}|(r) \leq c_5 r^{c_6} \forall r \in [l, +\infty[$. Consequently, by Corollary B.13.31 in [7], \bar{V} is a polynomial of degree $\leq c_6$ and hence it has finitely many zeros and so does V . But then, by Theorem 4.5, f must be a rational function. \square

Corollary 4.B: *Let f be a meromorphic function on \mathbb{K} such that, for some $c, q \in]0, +\infty[$, $\gamma(r, f)$ satisfies $\gamma(r, f) \leq cr^q$ in $[1, +\infty[$. If for some $b \in \mathbb{K}$ $f' - b$ has finitely many zeros, then f is a rational function.*

Proof. Suppose $f' - b$ has finitely many zeros. Then $f - bx$ satisfies the same hypothesis as f , hence it is a rational function and so is f . \square

Corollary 4.C: *Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$ be such that $\xi(r, f) \leq cr^q$ in $[1, +\infty[$ for some $c, q \in]0, +\infty[$. Then for each $k \in \mathbb{N}^*$, $f^{(k)}$ has no quasi-exceptional value.*

Proof. Indeed, if $k = 1$, the statement just comes from Corollary 4.B Now suppose $k \geq 2$. Each pole a of order n of f is a pole of order $n + k$ of $f^{(k)}$ and $f^{(k)}$ has no other pole. Consequently, we have $\gamma(r, f^{(k-1)}) = \xi(r, f^{(k-1)}) \leq kcr^q$. So, we can apply Corollary 4.B to $f^{(k-1)}$ to show the claim. \square

Theorem 4.6 suggests us the following conjecture:

Conjecture: *Let $f \in \mathcal{M}(\mathbb{K})$ be such that f' admits finitely many zeros. Then $f \in \mathbb{K}(x)$.*

In other words, the conjecture suggests that the derivative of a meromorphic function in \mathbb{K} has no quasi-exceptional value, except if it is a rational function.

Remark: Of course, there exist meromorphic functions in \mathbb{K} having no zero but not satisfying the hypotheses of Theorem 4.6, hence such a function cannot have primitives. For example, consider an entire function f having an infinity of zeros $(a_n)_{n \in \mathbb{N}}$ of order 2 such that $|a_n| < |a_{n+1}|$, $\lim_{n \rightarrow +\infty} |a_n| = +\infty$ and

$2n \leq |a_n|$. Then the meromorphic function $g = \frac{1}{f}$ has no zeros but does not satisfy the hypotheses of Theorem 4.6 hence it has no primitives.

5 Small functions

Small functions with respect to a meromorphic function are well known in the general theory of complex functions. Particularly, one knows the Nevanlinna theorem on 3 small functions. Here we will recall the construction of a similar theory.

Definitions and notation: Throughout the chapter we set $a \in \mathbb{K}$ and $R \in]0, +\infty[$. For each $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a, R^-))$) we denote by $\mathcal{M}_f(\mathbb{K})$, (resp. $\mathcal{M}_f(d(a, R^-))$) the set of functions $h \in \mathcal{M}(\mathbb{K})$, (resp. $h \in \mathcal{M}(d(a, R^-))$) such that $T(r, h) = o(T(r, f))$ when r tends to $+\infty$ (resp. when r tends to R). Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(a, R^-))$) we shall denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(a, R^-))$) the set $\mathcal{M}_f(\mathbb{K}) \cap \mathcal{A}(\mathbb{K})$, (resp. $\mathcal{M}_f(d(a, R^-)) \cap \mathcal{A}(d(a, R^-))$).

The elements of $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(a, R^-))$) are called *small meromorphic functions with respect to f* , (*small functions* in brief). Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(a, R^-))$) the elements of $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(a, R^-))$) are called *small analytic functions with respect to f* , (*small functions* in brief).

Theorems 5.1 and Theorem 5.2 are immediate consequences of Theorems C.9.1 and C.9.2 in [7]:

Theorem 5.1: Let $a \in \mathbb{K}$ and $r > 0$. Then $\mathcal{A}_f(\mathbb{K})$ is a \mathbb{K} -subalgebra of $\mathcal{A}(\mathbb{K})$, $\mathcal{A}_f(d(a, R^-))$ is a \mathbb{K} -subalgebra of $\mathcal{A}(d(a, R^-))$, $\mathcal{M}_f(\mathbb{K})$ is a subfield of $\mathcal{M}(\mathbb{K})$, $\mathcal{M}_f(d(a, R^-))$ is a subfield of $\mathcal{M}(d(a, R^-))$. Moreover, $\mathcal{A}_b(d(a, R^-))$ is a sub-algebra of $\mathcal{A}_f(d(a, R^-))$ and $\mathcal{M}_b(d(a, R^-))$ is a subfield of $\mathcal{M}_f(d(a, R^-))$.

Theorem 5.2 : Let $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(d(0, R^-))$) and let $g \in \mathcal{M}_f(\mathbb{K})$, (resp. $g \in \mathcal{M}_f(d(0, R^-))$). Then $T(r, fg) = T(r, f) + o(T(r, f))$ and $T(r, \frac{f}{g}) = T(r, f) + o(T(r, f))$, (resp. $T(r, fg) = T(r, f) + o(T(r, f))$ and $T(r, \frac{f}{g}) = T(r, f) + o(T(r, f))$).

Theorem 5.3 is known as Second Main Theorem on Three Small Functions in p -adic analysis [7] and [10]. It holds as well as in complex analysis, where it was showed first and it is proven in the same way.

Theorem 5.3: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$) and let $w_1, w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$ (resp. $w_1, w_2, w_3 \in \mathcal{M}_f(d(0, R^-))$) be pairwise distinct. Then $T(r, f) \leq \sum_{j=1}^3 \bar{Z}(r, f - w_j) + o(T(r, f))$, resp. $T(r, f) \leq \sum_{j=1}^3 \bar{Z}(r, f - w_j) + o(T(r, f))$, resp. $T_R(r, f) \leq \sum_{j=1}^3 \bar{Z}_R(r, f - w_j) + o(T(r, f))$.

Theorem 5.4: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$) and let $w_1, w_2 \in \mathcal{M}_f(\mathbb{K})$ (resp. $w_1, w_2 \in \mathcal{M}_f(d(0, R^-))$) be distinct. Then $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{N}(r, f) + o(T(r, f))$, (resp. $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{N}(r, f) + o(T(r, f))$).

Proof. Suppose first $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}_u(d(0, R^-))$. Let $g = \frac{1}{f}$, $h_j = \frac{1}{w_j}$, $j = 1, 2$, $h_3 = 0$. Clearly,

$$T(r, g) = T(r, f) + O(1), \quad T(r, h) = T(r, w_j), \quad j = 1, 2,$$

so we can apply Theorem 5.3 to g, h_1, h_2, h_3 . Thus we have: $T(r, g) \leq \bar{Z}(r, g - h_1) + \bar{Z}(r, g - h_2) + \bar{Z}(r, g) + o(T(r, g))$.

But we notice that $\bar{Z}(r, g - h_j) = \bar{Z}(r, f - w_j)$ for $j = 1, 2$ and $\bar{Z}(r, g) = \bar{N}(r, f)$. Moreover, we know that $o(T(r, g)) = o(T(r, f))$. Consequently, the claim is proved when $w_1 w_2$ is not identically zero.

Now, suppose that $w_1 = 0$. Let $\lambda \in \mathbb{K}^*$, let $l = f + \lambda$ and $\tau_j = w_j + \lambda$, ($j = 1, 2, 3$). Thus, we have $T(r, l) = T(r, f) + O(1)$, $T(r, \tau_j) = T(r, w_j) + O(1)$, ($j = 1, 2$), $\bar{N}(r, l) = \bar{N}(r, f)$. By the claim already proven whenever $w_1 w_2 \neq 0$ we may write $T(r, l) \leq \bar{Z}(r, l - \tau_1) + \bar{Z}(r, l - \tau_2) + \bar{N}(r, l) + o(T(r, l))$ hence $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{N}(r, l) + o(T(r, f))$. □

Next, by setting $g = f - w_1$ and $w = w_1 + w_2$, we can write Corollary 5.A:

Corollary 5.A: *Let $g \in \mathcal{M}(\mathbb{K})$ (resp. $g \in \mathcal{M}_u(d(0, R^-))$) and let $w \in \mathcal{M}_g(\mathbb{K})$. Then $T(r, g) \leq \bar{Z}(r, g) + \bar{Z}(r, g - w) + \bar{N}(r, g) + o(T(r, g))$, (resp. $T(r, g) \leq \bar{Z}(r, g) + \bar{Z}(r, g - w) + \bar{N}(r, g) + o(T(r, g))$).*

Corollary 5.B: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$) and let $w_1, w_2 \in \mathcal{A}_f(\mathbb{K})$ (resp. $w_1, w_2 \in \mathcal{A}_f(d(0, R^-))$) be distinct. Then $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + o(T(r, f))$ ($r \rightarrow +\infty$), resp. ($r \rightarrow R^-$).*

And similarly to Corollary 5.A, we can get Corollary 5.C:

Corollary 5.C: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$, resp. $f \in \mathcal{A}^e(D)$) and let $w \in \mathcal{A}_f(\mathbb{K})$. Then $T(r, f) \leq \bar{Z}(r, f) + \bar{Z}(r, f - w) + o(T(r, f))$, (resp. $T(r, f) \leq \bar{Z}(r, f) + \bar{Z}(r, f - w) + o(T(r, f))$).*

We are now able to state a theorem on q small functions that is not as good as Yamanoi's Theorem [17] in complex analysis, but seems the best possible in ultrametric analysis;

Theorem 5.5 [8] (A. Escassut, C.C. Yang): *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f \in \mathcal{M}_u(d(0, R^-))$) and let $w_j \in \mathcal{M}_f(\mathbb{K})$ ($j = 1, \dots, q$) (resp. $w_j \in \mathcal{M}_f(d(a, R^-))$) be q distinct small functions other than the constant ∞ . Then*

$$qT(r, f) \leq 3 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)),$$

(resp.

$$qT(r, f) \leq 3 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f))),$$

Moreover, if f has finitely many poles in \mathbb{K} (resp. in $d(0, R^-)$), then

$$qT(r, f) \leq 2 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)),$$

(resp.

$$qT(r, f) \leq 2 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)).),$$

Proof. By Theorem 5.3, for every triplet (i, j, k) such that $1 \leq i \leq j \leq k \leq q$, we can write

$$T(r, f) \leq \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + \bar{Z}(r, f - w_k) + o(T(r, f)).$$

The number of such inequalities is C_q^3 . Summing up, we obtain

$$(1) \quad C_q^3 T(r, f) \leq \sum_{(i,j,k), 1 \leq i \leq j \leq k \leq q} \bar{Z}(r, f-w_i) + \bar{Z}(r, f-w_j) + \bar{Z}(r, f-w_k) + o(T(r, f)).$$

In this sum, for each index i , the number of terms $\bar{Z}(r, f-w_i)$ is clearly C_{q-1}^2 . Consequently, by (1) we obtain

$$C_q^3 T(r, f) \leq C_{q-1}^2 \sum_{i=1}^q \bar{Z}(r, f-w_i) + o(T(r, f))$$

and hence

$$\frac{q}{3} T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f-w_i) + o(T(r, f)).$$

Suppose now that f has finitely many poles. By Theorem 5.4, for every pair (i, j) such that $1 \leq i < j \leq q$, we have

$$T(r, f) \leq \bar{Z}(r, f-w_i) + \bar{Z}(r, f-w_j) + o(T(r, f)).$$

The number of such inequalities is then C_q^2 . Summing up we now obtain

$$(2) \quad C_q^2 T(r, f) \leq \sum_{(i,j), 1 \leq i < j \leq q} \bar{Z}(r, f-w_i) + \bar{Z}(r, f-w_j) + o(T(r, f)).$$

In this sum, for each index i , the number of terms $\bar{Z}(r, f-w_i)$ is clearly $C_{q-1}^1 = q-1$. Consequently, by (1) we obtain

$$C_q^2 T(r, f) \leq (q-1) \sum_{i=1}^q \bar{Z}(r, f-w_i) + o(T(r, f))$$

and hence

$$\frac{q}{2} T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f-w_i) + o(T(r, f)).$$

□

Definition: Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}_u(d(a, R^-))$). Then f and g will be to share a small function, I.M. $w \in \mathcal{M}(\mathbb{K})$ (resp. $w \in \mathcal{M}(d(a, R^-))$) if $f(x) = w(x)$ implies $g(x) = w(x)$ and if $g(x) = w(x)$ implies $f(x) = w(x)$.

Theorem 5.6: Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(d(a, R^-))$) be distinct and share q distinct small functions I.M. $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ ($j = 1, \dots, q$) (resp. $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) ($j = 1, \dots, q$) other than the constant ∞ . Then

$$\sum_{j=1}^q \bar{Z}(r, f-w_j) \leq \bar{Z}(r, f-g) + o(T(r, f)) + o(T(r, g)).$$

Proof. Suppose that f and g belong to $\mathcal{M}(\mathbb{K})$, are distinct and share q distinct small functions I.M. $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ ($j = 1, \dots, q$).

Let b be a zero of $f - w_i$ for a certain index i . Then it is also a zero of $g - w_i$. Suppose that b is counted several times in the sum $\sum_{j=1}^q \bar{Z}(r, f - w_j)$,

which means that it is a zero of another function $f - w_h$ for a certain index $h \neq i$. Then we have $w_i(b) = w_h(b)$ and hence b is a zero of the function $w_i - w_h$ which belongs to $\mathcal{M}_f(\mathbb{K})$. Now, put $\tilde{Z}(r, f - w_1) = \bar{Z}(r, f - w_1)$ and for each $j > 1$, let $\tilde{Z}(r, f - w_j)$ be the counting function of zeros of $f - w_j$ in the disk $d(0, r^-)$ ignoring multiplicity and avoiding the zeros already counted as zeros of $f - w_h$ for some $h < j$. Consider now the sum $\sum_{j=1}^q \tilde{Z}(r, f - w_j)$. Since the functions $w_i - w_j$ belong to $\mathcal{M}_f(\mathbb{K})$, clearly, we have

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) = \sum_{j=1}^q \tilde{Z}(r, f - w_j) = o(T(r, f))$$

It is clear, from the assumption, that $f(x) - w_j(x) = 0$ implies $g(x) - w_j(x) = 0$ and hence $f(x) - g(x) = 0$. Since $f - g$ is not the identically zero function, it follows that

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g).$$

Consequently,

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

Now, if f and g belong to $\mathcal{M}(d(0, R^-))$ the proof is exactly the same. \square

Theorem 5.7 [8] (A. Escassut, C.C. Yang): *Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(d(a, R^-))$) be distinct and share 7 distinct small functions (other than the constant ∞) I.M. $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ ($j = 1, \dots, 7$) (resp. $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$), resp. $w_j \in \mathcal{M}_f(D) \cap \mathcal{M}_g(D)$ ($j = 1, \dots, 7$),). Then $f = g$.*

Moreover, if f and g have finitely many poles and share 3 distinct small functions (other than the constant ∞) I.M. then $f = g$.

Proof. We put $M(r) = \max(T(r, f), T(r, g))$. Suppose that f and g are distinct and share q small function I.M. w_j , ($1 \leq j \leq q$). By Theorem 5.5, we have

$$qT(r, f) \leq 3 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)).$$

But thanks to Theorem 5.6, we can derive

$$qT(r, f) \leq 3T(r, f - g) + o(T(r, f))$$

and similarly

$$qT(r, g) \leq 3T(r, f - g) + o(T(r, g))$$

hence

$$(1) \quad qM(r) \leq 3T(r, f - g) + o(M(r)).$$

By Theorem C.4.8 in [7], we can derive that

$$qM(r) \leq 3(T(r, f) + T(r, g)) + o(M(r))$$

and hence $qM(r) \leq 6M(r) + o(M(r))$. That applies to the situation when f and g belong to $\mathcal{M}(\mathbb{K})$ as well as when f and g belong to $\mathcal{M}_u(d(0, R^-))$. Consequently, it is impossible if $q \geq 7$ and hence the first statement of Theorem 5.7 is proved.

Suppose now that f and g have finitely many poles. By Theorems C.4.8 in [7], Relation (1) gives us

$$qM(r) \leq 2M(r) + o(M(r))$$

which is obviously absurd whenever $q \geq 3$ and proves that $f = g$ when f and g belong to $\mathcal{M}(\mathbb{K})$ as well as when f and g belong to $\mathcal{M}_u(d(0, R^-))$. \square

Corollary 5.D: *Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{A}_u(d(a, R^-))$) be distinct and share 3 distinct small functions (other than the constant ∞) I.M. $w_j \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ ($j = 1, 2, 3$) (resp. $w_j \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$, ($j = 1, 2, 3$)). Then $f = g$.*

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