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# ALGEBRAIC DIFFERENTIAL OPERATORS ON ARITHMETIC AUTOMORPHIC FORMS, MODULAR DISTRIBUTIONS, P -ADIC INTERPOLATION OF THEIR CRITICAL L VALUES VIA BGG MODULES AND HECKE ALGEBRAS

Alain Escassut

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- 01 Algebraic differential operators on arithmetic automorphic forms, modular distributions, p-adic interpolation of their critical L values via BGG modules and Hecke algebras  
Alexei Pantchichkine
- 27 A survey on a few recent papers in p-adic value distribution  
Alain Escassut
- 57 Statistical Quantum Neural Networks  
Do Ngoc Diep
- 67 Optimality conditions for efficiency of constrained vector equilibrium problems  
Do Van Luu  
Tran Thi Mai
- 87 Information - leakage - free quantum dialogue via Greenberger - Horne - Zeilinger states  
Nguyen Ba An
- 103 A new class of unique range sets for meromorphic functions ignoring multiplicity with 15 elements  
Vu Hoai An
- 117 Nonautonomous attractors for Young differential equations driven by unbounded variation paths  
Phan Thanh Hong
- 135 On the equation  $F(n^2 + m^2 + k) = H(n) + H(m) + K$   
I. Katai  
B. M. M. Khanh  
B. M. Phong

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# ALGEBRAIC DIFFERENTIAL OPERATORS ON ARITHMETIC AUTOMORPHIC FORMS, MODULAR DISTRIBUTIONS, $P$ -ADIC INTERPOLATION OF THEIR CRITICAL $L$ VALUES VIA BGG MODULES AND HECKE ALGEBRAS

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(Received 19 July 2021; accepted 22 December 2021)

**Abstract.** The paper extends author’s method of modular distributions (2002, [75]) to arithmetic automorphic  $L$  functions on general classical groups. Main result gives a  $p$ -adic interpolation of their critical  $L$  values in the form of integrals of distributions constructed from a given eigenfunction of Hecke algebras by applying BGG modules, (see also preprints [78] and [79]).

In particular, algebraic differential operators are described acting on automorphic forms  $\varphi$  on unitary groups  $U(n, n)$  over an imaginary quadratic field  $\mathcal{K} = \mathbb{Q}(\sqrt{-D_{\mathcal{K}}}) \subset \mathbb{C}$ .

Applications are given to Shimura’s zeta functions  $L(s, \mathbf{f})$  [90] attached special  $L$ -values  $L(s, \varphi)$  attached to  $\varphi$ . and normalized in accordance with Deligne’s Gamma factors rule [21]. An explicit description of Shimura’s  $\Gamma$ -factors is used..

## Contents

1. The simplest case of modular forms for  $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ .
2. Algebraic differential operators on symplectic groups.
3. Unitary groups  $U(a, b)$  (of signature  $(a, b)$ ,  $a + b = n$ ), and the double group  $U(n, n)$ .
4. Analytic families of CM-abelian varieties and unitary groups.
5. Algebraic automorphic forms on unitary groups
6.  $C^\infty$ -differential operators via Shimura’s approach.
7. Algebraic differential operators for  $U(n, n)$ .
8. Applications to critical values of the standard zeta function  $\mathcal{L}(s, \varphi)$  in the unitary case.
9. Perspectives and examples for  $U(n, n)$ .

### 0.1. Algebraic differential operators in the simplest case of modular forms for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

Action of the derivative  $D = \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}$  (where  $q = e^{2\pi iz}$ ) on a mod-

ular form  $g = \sum_{n=0}^{\infty} b_n q^n$  is not a modular form, but it is quasi-modular ([96],

p.59, [66], p.67): the function  $f = D^r g = \sum_{n=0}^{\infty} n^r b_n q^n$  satisfies the following

transformation law:

*Key words and phrases:* Automorphic forms, classical groups,  $p$ -adic  $L$ -functions, differential operators, non-archimedean weight spaces, quasi-modular forms, Fourier coefficients.

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$$(cz + d)^{-\ell - 2r} D^r g(\gamma z) = \sum_{t=0}^r \binom{r}{t} \frac{\Gamma(r + \ell)}{\Gamma(t + \ell)} \left( \frac{1}{2\pi i} \frac{c}{cz + d} \right)^{r-t} D^t g(z)$$

for a modular form  $g \in \mathcal{M}_\ell(\Gamma)$  of weight  $\ell$ ,  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ .

In order to adjust it to the weight  $\ell + 2r$ , let us use  $S = \frac{1}{4\pi y}$ ,  $\text{Im } z = \frac{z - \bar{z}}{2i}$ , and  $\frac{1}{\text{Im } \gamma z} = \frac{|cz + d|^2}{\text{Im } z} = (cz + d) \left( -2ic + \frac{cz + d}{y} \right) :$

### 0.2. Maass-Shimura differential operator

If  $f = D^r g$  where  $g \in \mathcal{M}_\ell(\Gamma)$  is a modular form of weight  $\ell$ , then the transformation law produces also the Maass-Shimura differential operator  $\delta_\ell$  to the space of *nearly holomorphic forms* of weight  $\ell + 2r$ :

$$\delta_\ell^r g(z) = \sum_{t=0}^r \binom{r}{t} \frac{\Gamma(r + \ell)}{\Gamma(r - t + \ell)} (-S)^t D^{r-t} g(z), \text{ where } S = \frac{1}{4\pi y},$$

which preserves the rationality of the coefficients of  $S$  and  $g$ . It comes again from the above transformation law of  $D^r g$ . Notice:

$\delta_\ell(g) = \frac{1}{2\pi i} y^{-\ell} \frac{\partial}{\partial z} (y^\ell g) = \frac{1}{2\pi i} \left( \frac{\partial g(z)}{\partial z} + \frac{\ell}{2iy} g(z) \right) = (D - \ell S)(g)$ , which is of weight  $\ell + 2$  and its *degree of near holomorphy* (in the variable  $S$ ) is increased by one.

For an integer  $r \geq 0$ ,  $\delta_\ell^r := \delta_{\ell+2r-2} \circ \dots \circ \delta_\ell$  (see also [94]).

A conceptual explanation of the *algebraicity* comes from the Gauss-Manin connection (due to Grothendieck in higher dimensions see [34], [48]).

### 0.3. Algebraic differential operators on symplectic groups

On scalar-valued Siegel modular forms: Let  $Z = (z_{ij}) \in \text{GL}_n(\mathbb{C})$ ,  $Z = {}^t Z$ ,  $\partial_{ij} = \frac{1}{2\pi\sqrt{-1}} \begin{cases} \frac{\partial}{\partial z_{ij}} & i = j \\ \frac{1}{2} \frac{\partial}{\partial z_{ij}} & i \neq j \end{cases}$ , Maass operator  $\Delta = \det(\partial_{ij})$  acts by  $\Delta q^T = \det(T) q^T$  on  $q^T = \exp(2\pi i \text{tr}(TZ))$ .

The *Maass-Shimura operator*  $\delta_k f(Z) = (-4\pi)^{-n} \det(Z - \bar{Z})^{\frac{1+n}{2}-k} \Delta(\det(Z - \bar{Z})^{k - \frac{1+n}{2} + 1} f(Z))$

acts on  $q^T$  via the polynomial representations  $\rho_r : \mathrm{GL}_n(\mathbb{C}) \rightarrow \mathrm{GL}(\wedge^r \mathbb{C}^n)$  and its adjoint  $\rho_r^*$  (see [16])

$$\delta_k(q^T) = \sum_{\ell=0}^n (-1)^{n-\ell} c_{n-\ell}(k+1 - \frac{1+n}{2}) \mathrm{tr}({}^t \rho_{n-\ell}(S) \rho_\ell^*(T)) q^T,$$

where  $c_{n-\ell}(s) = s(s - \frac{1}{2}) \cdots (s - \frac{n-\ell-1}{2})$ ,  $S = (2\pi i(\bar{z} - z))^{-1}$ .

For a  $\mathbb{C}^d$ -valued Siegel modular form  $f$  this *algebraic operator* extends to a  $\mathbb{C}^d$ -valued smooth function of  $Z = (z_{ij}) = X + \sqrt{-1}Y$ .

Let  $S_e(\mathrm{Sym}^2(R^n), R^d)$  be the  $R$ -module of all polynomial maps of  $\mathrm{Sym}^2(R^n)$  into  $R^d$  homogeneous of degree  $e$ . Define inductively  $S_1(\mathrm{Sym}^2(\mathbb{C}^n), \mathbb{C}^d)$ -valued smooth functions:

$$(Df)(u) = \sum_{1 \leq i \leq j \leq n} u_{ij} \frac{\partial f}{\partial (2\pi \sqrt{-1} z_{ij})}, \quad (Cf)(u) = (Df)((Z - \bar{Z})u(Z - \bar{Z})),$$

$$(C^e(f))(u) = C(C^{e-1}(f))(u)$$

$$D_\rho^e(f) := (\rho \otimes \tau^e)(Z - \bar{Z})^{-1} C^e(\rho(Z - \bar{Z})f), \text{ where}$$

$$[(\rho \otimes \tau)(\alpha)(h)](u) := \rho(\alpha)h({}^t \alpha \cdot u \cdot \alpha).$$

Then  $D_\rho^e$  equals  $(2\sqrt{-1}\pi)^{-e}$  times the (vector-valued) *Maass-Shimura differential operator*.

#### 0.4. From symplectic case (Type C) to unitary case (Type A)

*Siegel modular forms of degree  $n$*  are holomorphic (vector-valued) functions on  $\mathbb{H}_n = \{Z = {}^t Z \in \mathbb{C}_n^n, \mathrm{Im}(Z) > 0\}$  (the Siegel space, (Type C) [90]).

*Automorphic forms on unitary groups (Type A) in [90]*

$U(a, b)$  (of degree  $n = a + b$ )  $\rightsquigarrow$  the double group  $U(n, n)$ ,

and the corresponding hermitian space of degree  $n$ :

$$\mathcal{H}_n = \{z \in \mathbb{C}_n^n \mid i(z^* - z) > 0\}$$

where  $z^* = {}^t \bar{z}$ ,  $x := (z + z^*)/2$  the hermitian part of  $z$ , and  $y := (z - z^*)/2$  the anti-hermitian part, such that  $i(z^* - z)/2 = iy$  is a positive hermitian matrix.

Note that  $z = x + iy$ , but  $x, y$  are not real: for a hermitian matrix  $h$ , the real matrices  $\dot{h} = \frac{\omega {}^t h - \bar{\omega} h}{\omega - \bar{\omega}}$ ,  $\ddot{h} = \frac{h - {}^t h}{\omega - \bar{\omega}}$  are used for  $\omega = \frac{1}{2}(\delta + \delta^{\frac{1}{2}})$ ,  $\delta$  the discriminant of  $\mathcal{K}$ , so that  $h = \dot{h} + \omega \ddot{h}$  (notation in [13]).

*Automorphic  $L$  functions on unitary groups and related geometric objects* where discussed by M. Harris (ICM 2014), *Automorphic Galois representations and the cohomology of Shimura varieties.*, [39], and by P.Scholze (ICM 2018),

Applications of  $p$ -adic geometry to automorphic Galois representations on unitary groups in [81].

## 1. Unitary groups and forms, [38],[24], [90]

### 1.1. Unitary groups $U(a, b)$ ( $a + b = n$ ) and $U(n, n)$ (the double group)

Let  $V$  be an  $n$ -dimensional space over an imaginary quadratic field  $\mathcal{K} = \mathbb{Q}(\sqrt{-D_{\mathcal{K}}})$ , and let  $\langle \cdot, \cdot \rangle$  be a *non degenerate hermitian pairing* of signature  $(a, b)$  on  $V$  relative to  $\mathcal{K} \subset \mathbb{C}$ .

Let us write  $-V$  for the vector space  $V$  with the pairing  $\langle \cdot, \cdot \rangle_{-V} = -\langle \cdot, \cdot \rangle_V$  (of signature  $(b, a)$ ).

Let  $2V$  denote the *double vector space*  $V \oplus V$  with the pairing  $\langle \cdot, \cdot \rangle_{2V}$  defined for all vectors  $v_1, v_2, w_1, w_2 \in V$  by  $\langle (v_1, v_2), (w_1, w_2) \rangle_{2V} := \langle (v_1, w_1) \rangle_V + \langle (v_2, w_2) \rangle_{-V}$  (of signature  $(b + a, a + b) = (n, n)$ ).

For a vector space  $W$  with hermitian pairing  $\langle \cdot, \cdot \rangle_W$ , and a  $\mathbb{Q}$ -algebra  $R$ , the *unitary groups are defined by*

$$U(W)(R) = \{g \in \mathrm{GL}(W \otimes R) \mid \forall v, v', \langle gv, gv' \rangle = \langle v, v' \rangle\}$$

$$GU(W)(R) = \{g \in \mathrm{GL}(W \otimes R) \mid \forall v, v', \exists \nu(g) \in R^*, \langle gv, gv' \rangle = \nu(g) \langle v, v' \rangle\}.$$

Then

$$U(2V)(R) \cong U(n, n)(R) = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{K} \otimes R) \mid M \eta_n M^* = \eta_n \right\},$$

where  $\eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix}$ . The group  $U(n, n)$  acts on the *hermitian space*

$$\mathcal{H}_n = \{z \in \mathbb{C}_n^n \mid i(z^* - z) > 0\}, \text{ with } z^* := {}^t \bar{z}.$$

## 1.2. Algebraic geometric approach: families of abelian varieties of CM-type and unitary groups

Main arithmetical applications of unitary groups  $U(n, n)$  use Shimura's analytic families of abelian varieties  $A$  of CM-type of  $\dim_{\mathbb{C}} A = 2n$ , that is, with fixed imbedding  $\iota : \mathcal{K} \hookrightarrow \text{End}(A) \otimes \mathbb{Q}$ , and other PEL-structures ("polarization, endomorphisms, level", following [24], §2).

Recall that elliptic curves  $E$  with complex multiplication by  $\mathcal{K}$  correspond to certain CM-points on the upper half plane  $\mathbb{H}$ , that is  $E \xrightarrow{\sim} \mathbb{C}/L$ , where  $L = \langle 1, \alpha \rangle \subset \mathcal{K} = \mathbb{Q}(\alpha)$  is a lattice in  $\mathbb{C}$  and  $\text{Im}(\alpha) > 0$  (only *special CM-points, not analytic families*).

*Families of 2n-dimensional CM-abelian varieties*  $A$  use the analytic parameter  $z \in \mathcal{H}_n$ . Any row vector  $x \in \mathcal{K}_{2n}^1$  defines a  $z$ -holomorphic  $\mathbb{C}^{2n}$ -valued function  $p_z(x)$  by

$$p_z(x) = ([z, 1_n] \cdot x^*, [{}^t z, 1_n] \cdot {}^t x)$$

For a fixed lattice  $L \subset \mathcal{K}^{2n} \subset \mathbb{C}^{2n}$ , denote by  $L_z = p_z(L)$  a  $4n$ -dimensional CM-lattice of analytic parameter  $z$ .

## 1.3. Explicit matrix description by the complex torus $\mathbb{C}^{2n}/L_z$

Any  $2n$ -dimensional abelian variety of CM-type is isomorphic to  $A_z$ , with the action of  $\mathcal{K}$  given by  $\iota_z(a) \cdot v = \text{diag}[\bar{a} \cdot 1_n, a \cdot 1_n] \cdot v$ .

*Universal analytic family*  $\mathcal{A}_{univ}$  over  $\mathcal{H}_n$  : taking  $L$  the lattice in  $\mathcal{K}_{2n}^1$  generated by the standard basis vectors  $e_1, \dots, e_{2n}$ , and the vectors  $\alpha \cdot e_1, \dots, \alpha \cdot e_{2n}$  with  $\alpha$  a generator of  $\mathcal{K}$  over  $\mathbb{Q}$ . Then the fiber  $A_z$  over each point  $z = (z_{ij}) \in \mathcal{H}_n$  is the abelian variety  $A_z \cong \mathbb{C}^{2n}/L$ , where  $L_z$  the  $\mathbb{Z}$ -lattice generated by  $4n$  rows:

$$\begin{aligned} z_j &= (z_{1j}, \dots, z_{nj}, z_{j1}, \dots, z_{jn}) \\ e_j &= \text{vector with } 1 \text{ in the } j\text{-th and } j+n\text{-th positions} \\ &\quad \text{and zeroes everywhere else,} \\ z'_j &= (\bar{\alpha}z_{1j}, \dots, \bar{\alpha}z_{nj}, \alpha z_{j1}, \dots, \alpha z_{jn}) \\ e'_j &= \text{vector with } \bar{\alpha} \text{ in the } j\text{-th, and } \alpha \text{ in the } j+n\text{-th positions} \\ &\quad \text{and zeroes everywhere else.} \end{aligned}$$



#### 1.4. Vector-valued automorphic forms on unitary groups, [24], p.18

Weight  $\rho$  of an automorphic form on  $G$  is a representation of the maximal compact subgroup  $K \subset G$ . Weights are constructed via the following polynomial representations  $\rho_{\kappa} : \mathrm{GL}_n \rightarrow \mathrm{GL}(V_{\kappa})$ .

For each set  $\kappa$  of ordered integers  $\kappa_1 \geq \dots \geq \kappa_n$  there is a representation  $(\rho_{\kappa}, \mathrm{GL}_n)$  of highest weight  $\kappa$ , constructed as  $V_{\kappa} = \mathrm{Sym}^{\kappa_1 - \kappa_2}(R^n) \otimes \mathrm{Sym}^{\kappa_2 - \kappa_3}(\wedge^2(R^n)) \otimes \dots \otimes \mathrm{Sym}^{\kappa_n}(\wedge^n(R^n))$  with the standard  $\mathrm{GL}_n$ -action, over any  $\mathbb{Q}$ -algebra  $R$ .

Vector valued modular forms  $\mathcal{M}_{\kappa}$  (symplectic case) and  $\mathcal{M}_{\kappa, \kappa'}$  (unitary case) can be attached to the representations with highest weight  $\rho = \rho_{\kappa}$  and  $\rho_{\kappa}^+ \otimes \rho_{\kappa'}^-$  of the maximal compact subgroups  $K \cong U(n) \subset \mathrm{Sp}_{2n}(\mathbb{R})$  and  $K \cong U(a) \times U(b) \subset U(a, b)$ .

These modular forms take values in  $V_{\kappa}$  and  $V_{\kappa, \kappa'}$ , and defined on the symmetric spaces  $G/K$ ,  $G = \mathrm{Sp}(\mathbb{R})$  or  $G = U(a, b)$ .

Some notation  $\alpha(z) = (az + b)(cz + d)^{-1}$ ,  $\lambda(z) = \bar{c} \cdot {}^t \bar{z} + \bar{d}$ ,  $\mu(z) = c \cdot z + d$  (used for the automorphy factors of weight  $\rho$ , and for the Eisenstein series).

#### 1.5. $C^\infty$ -differential operators via Shimura's approach

For each  $z \in \mathcal{H}_n$ , let  $\Xi(z) = (\xi(z), \eta(z)) = (i(\bar{z} - {}^t z), i(z^* - z))$ , so that  ${}^t \xi(z) = \eta(z) = i(z^* - z)$ . The tangent space  $T = \mathbb{C}^n$  over  $\mathbb{C}$  has a  $\mathbb{R}$ -rational basis  $\{e_\nu\}$ ,  $u := \sum_\nu u_\nu \varepsilon_\nu$ ,  $z := \sum_\nu z_\nu \varepsilon_\nu$ .

Let  $(\rho, V) = (\rho_- \otimes \rho_+, V_- \otimes V_+)$  be a finite dimensional representation of  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ , and  $e$  be a positive integer. For vector spaces  $X$  and  $Y$ , define  $S_e(Y, X)$  the vector space of degree  $e$  homogeneous polynomial maps of  $Y$  into  $X$ , i.e. the space of maps  $h$  from  $Y$  to  $X$  such that  $h(a \cdot y) = a^e h(y)$ ,  $S_e(Y) = S_e(Y, \mathbb{C})$ .

For  $f \in C^\infty(\mathcal{H}_n, V)$ , put  $\Xi = (\xi, \eta) \in S_1(T, \mathbb{C})$ , and define operators  $C, D : C^\infty(\mathcal{H}_n, V) \rightarrow C^\infty(\mathcal{H}_n, S_1(T, V))$  by

$$(Df)(u) = \sum_\nu u_\nu \frac{\partial f}{\partial z_\nu}$$

$$(Cf)(u) = (\tau^1(\Xi)Df)(u) := Df({}^t \xi u \eta).$$

For  $e > 1$  write  $D^e(f)$  and  $C^e(f)$  for  $D(D^{e-1}f)$  and  $C(C^{e-1}f)$ , viewed as  $C^\infty(\mathcal{H}_n, S_e(T, V))$ -valued.

### 1.5.1. Action on vector-values automorphic forms

Given  $g = (a, b) \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ ,  $(\rho, X)$  a polynomial representation, and  $h \in M\ell_e(T, X) = M\ell_e(T, \mathbb{C}) \otimes X$  (symmetric  $\mathbb{R}$ -multilinear map viewed also as element  $S_e(T, X)$ ), define  $[\tau^e(a, b)h](u_1, \dots, u_e) = ({}^t a u_1 b, \dots, {}^t a u_e b)$ , and a representation  $\rho \otimes \tau^e$  of  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$  on  $M\ell_e(T, \mathbb{C}) \otimes X$

$$[(\rho \otimes \tau^e)(g)](h(u) \otimes x) = \tau^e(g)h \otimes \rho(g)x$$

for each  $g \in \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ ,  $h \in M\ell_e(T, \mathbb{C})$ , and  $x \in X$ . For  $e > 1$  write  $D^e(f) = D(D^{e-1}(f))$  and  $C^e(f) = C(C^{e-1}(f))$ .

Such operators *take automorphic forms of weight  $\rho$  to automorphic forms of weight  $\rho \otimes \tau^e$*  as follows: define

$$(D_\rho f)(u) = \rho(\Xi)^{-1} D[\rho(\Xi)f](u) = (\rho \otimes \tau)(\Xi)^{-1} C[\rho(\Xi)f](u).$$

and  $(D_\rho^e f)(u) = (\rho \otimes \tau^e)(\Xi)^{-1} C^e[\rho(\Xi)f]$  for  $e > 1$ .

Then  $D_\rho^e$  maps automorphic forms of weight  $\rho$  to automorphic forms of weight  $\rho \otimes \tau^e$ .

### 1.5.2. General Shimura's differential operators $D_\rho^Z$ via $\varphi_Z$

The classification of the irreducible subspaces of polynomial representations of  $\mathrm{GL}_n(\mathbb{C})$  and of irreducible subspaces of  $\tau^e$  is studied in [90], Theorem 12.7, in terms of highest weights. Given a matrix  $a \in \mathbb{C}_n^n$ , let  $\det_j(a)$  denote the determinant of the upper left  $j \times j$  submatrix of  $a$ . If  $\rho$  and  $\sigma$  are irreducible representations of  $\mathrm{GL}_n(\mathbb{C})$ ,  $\rho \otimes \sigma$  occurs in  $\tau^e$  if and only if  $\rho$  and  $\sigma$  are representations of the same highest weights  $\kappa_1 \geq \dots \geq \kappa_n$  as each other  $\kappa_1 + \dots + \kappa_n = e$ , and the corresponding irreducible subspace of  $S_e(T)$  contains a polynomial  $p(x)$  defined by

$$\prod_{j=1}^n \det_j(x)^{e_j} \quad (x \in T = \mathbb{C}_n^n, e_j = \kappa_j - \kappa_{j+1}, 1 \leq j \leq n-1, e_n = \kappa_n)$$

If  $\rho$  is the representation of  $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ , there is a differential operator  $D^Z$  defined for a stable quotient of  $S_e(T)$  with the projection  $\varphi_Z$  of  $S_r(T) \otimes X$  onto  $Z \otimes X$ . Then the operator  $D_\rho^Z = \varphi_Z D_\rho^e$  is a map from the space of automorphic forms of weight  $\rho$  to automorphic forms of weight  $\rho \otimes \tau_Z$ , where  $\tau_Z$  denotes the restriction of  $\tau$  to  $Z$ . *There is a formula for the action of the algebraic differential operators  $\theta_\rho^Z$  on formal  $q$ -expansions on the double group  $G$  at a cusp (which is a certain formal object)  $f = \sum_{L_m \ni \beta > 0} a(\beta)q^\beta$ , where*

$L_m$  is the lattice in  $Herm_{\mathcal{K}}$  determined by  $m$ . If  $\zeta$  is a highest-weight vector in  $Z$ , then it follows from the formulas in [24], §9, that  $\theta(\zeta)(f) = \sum_{\beta} a(\beta)\zeta(\beta)q^{\beta}$ .

### 1.6. Holomorphic discrete series of $U(a, b)$

Following P.Garrett,[29] let us recall the structure of *holomorphic discrete series representations* of unitary groups  $U(a, b)$  for sufficiently high highest weight. For  $U(a, b)$ , the maximal compact is  $U(a)xU(b)$ , and for  $\rho$  with highest weight  $(\kappa_1, \dots, \kappa_a) \times (\kappa'_1, \dots, \kappa'_b)$  it is sufficient to assume that

$$\kappa_1 \geq \dots \geq \kappa_a \geq \frac{a+b-1}{2}, \quad \kappa'_1 \geq \dots \geq \kappa'_b \geq \frac{a+b-1}{2}$$

Let  $\mathfrak{g}$  be the Lie algebra of  $G = U(a, b)$  where the latter is the isometry group of the standard hermitian form given by  $(a+b) \times (a+b)$ -matrix  $H = \begin{pmatrix} 1_a & 0 \\ 0 & -1_b \end{pmatrix}$ . The copy  $K$  of  $U(a) \times U(b)$  in  $G$  is  $K = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in U(a), B \in U(b) \right\}$ , the center of  $K$  is  $Z = \left( \begin{smallmatrix} \lambda 1_a & 0 \\ 0 & \mu 1_b \end{smallmatrix} \right)$ ,  $\lambda, \mu \in U(1)$ ,  $\mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix} \right\}$ , with  $S$   $a$ -by- $b$ ,  $\mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ S & 0 \end{pmatrix} \right\}$ , with  $S$   $b$ -by- $a$ , and the Lie algebra of  $K$  denoted by  $\mathfrak{k}$ . The elements of  $\mathfrak{p}^+$  are the *raising* operators, the elements of  $\mathfrak{p}^-$  are the *lowering* operators, and  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$  is the Harish-Chandra decomposition.

## 2. Algebraic differential operators on automorphic forms on unitary groups.

Fix a  $\mathcal{O}_{\mathcal{K}}$ -algebra  $\mathcal{R}$  with inclusion  $\iota : \mathcal{R} \rightarrow \mathbb{C}$  and a weight representation  $\rho = (\rho^+, \rho^-)$  of the maximal compact subgroup  $K = U(n) \times U(n)$  of  $U = U(n, n)$ . Following §8 and 9 of [24], write an automorphic form in  $\mathcal{M}_{\rho}(\mathcal{R})$  with values in an  $\mathcal{R}$ -module  $V = V^{\rho}(\mathcal{R}^d)$  on the hermitian space  $\mathcal{H}_n = U/K$  as a formal  $q$ -expansion  $f(q) = \sum_{\beta \in H_{\geq 0}} c_{\beta}(\Xi)q^{\beta}$  with vector-valued polynomial coefficients  $c_{\beta}(\Xi) \in V^{\rho}$  of  $q^{\beta} = \exp(2\pi i \text{tr}(\beta z))$ ,  $z \in \mathcal{H}_n$ , where  $\Xi(z) = (i(\bar{z} - {}^t z), i(z^* - z)) = (\xi, \eta)$  (Shimura's notation),  $T = \mathbb{C}_n^n$ , and  $\{e_{\nu}\}$  a  $\mathbb{R}$ -rational basis of  $T$  over  $\mathbb{C}$ ,  $H_{\geq 0}$  is a lattice of hermitian semi-integral non-negative matrices.

Then a general algebraic operator  $\theta(f)$  is defined as above via  $\theta(\zeta)(f)$ , using  $\beta$  and  $\Xi$  as formal variables over a cusp:  $\theta(\zeta)(f)(q) = \sum_{\beta \in H_{\geq 0}} \zeta(\beta) c_{\beta}(\Xi) q^{\beta}$

more general formal  $q$ -expansions:  $f(q) = \sum_{\beta \in H_{\geq 0}} c_{\beta}(\Xi; T_1, \dots, T_n) q^{\beta}$  with additional polynomial variables  $T_1, \dots, T_n$ , and define

$$\theta(f) = \sum_{\beta \in H_{\geq 0}} d_{\beta}(\Xi; T_1, \dots, T_n) q^{\beta}$$

, where  $T_1, \dots, T_n \in T(\mathcal{R}^n)$  in the tensor algebra of  $n$  letters,

$$d_{\beta} = \sum_{\beta_{i,j} \in H_{\geq 0}} \beta_{i,j} c(\beta) \cdot (T_i \otimes T_j).$$

This construction allows to treat *vector-valued modular forms as polynomial-valued*, and to prove congruences between them monomial-by-monomial.

## 2.1. Classical setting: arithmetic differential operators

In the Unitary case such operators were studied in [24]; we may write  $\beta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}$  in the  $q$  expansion on the double group, with hermitian matrices  $\beta_1, \beta_4$ , and  $\beta_2^* = \beta_3$ . In the Sp-case such operator studied in [8] and [23] are compositions Shimura-type operators, described then via its action on the  $q$ -expansions.

For  $\nu \in \mathbb{N}$ , we put

$$\begin{aligned} \mathfrak{D}_{n,\alpha}^{\nu} &= \mathfrak{D}_{n,\alpha+\nu-1} \circ \dots \circ \mathfrak{D}_{n,\alpha} \\ \overset{\circ}{\mathfrak{D}}_{n,\alpha}^{\nu} &= (\mathfrak{D}_{n,\alpha}^{\nu})|_{z_2=0}. \end{aligned}$$

The arithmetic applications of this differential operator is due to its explicit action on the exponentials in the Fourier expansion as follows: for  $\mathcal{T} \in \mathbb{C}_{\text{sym}}^{2n,2n}$ , we recall a polynomial  $\mathfrak{P}_{n,\alpha}^{\nu}(\mathcal{T})$  defined by S. Böcherer in the entries  $t_{ij} (1 \leq i \leq j \leq 2n)$  of  $\mathcal{T}$  by

$$\overset{\circ}{\mathfrak{D}}_{n,\alpha}^{\nu}(e^{\text{tr}(\mathcal{T}Z)}) = \mathfrak{P}_{n,\alpha}^{\nu}(\mathcal{T}) e^{\text{tr}(\mathcal{T}_1 z_1 + \mathcal{T}_4 z_4)}, \mathcal{T} = \begin{pmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ t\mathcal{T}_2 & \mathcal{T}_4 \end{pmatrix}, \mathfrak{Z} = \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}$$

that is, it represents "action of differential operator on exponential function". The  $\mathfrak{P}_{n,\alpha}^{\nu}$  are homogenous polynomials of degree  $n\nu$ .

## 2.2. Applications to critical values

of the *standard zeta function*  $L(\varphi, \chi, s)$  of vector-valued automorphic forms  $\varphi$  on unitary groups, see [38], [25].

More generally, take a unitary group  $U$  of a  $n$ -dimensional  $\mathcal{K}$ -vector space with a non-degenerate hermitian form  $\langle \cdot, \cdot \rangle_V : V \times V \rightarrow \mathcal{K}$  of signature  $(a, b)$ ,  $a + b = n$ . Then a vector-valued automorphic (Hecke eigenform)  $\varphi$  on  $U$  generates a cuspidal automorphic representation  $\pi = \pi_\varphi$  of the adelic group  $U(\mathbb{A})$ .

The *standard zeta function*  $L(\varphi, \chi, s) = L(\pi_\varphi, \chi, s)$  with a Hecke character  $\chi : \mathbb{A}_{\mathcal{K}}^\times \rightarrow \mathbb{C}^\times$  of allowed type  $\chi_\infty$  is a certain Euler product  $L(\varphi, \chi, s) = \prod_{\mathfrak{q}} L_{\mathfrak{q}}(\varphi, \chi, s)$ , where  $L_{\mathfrak{q}}(\varphi, \chi, s)^{-1} = L_{\mathfrak{q}}(\varphi, X)$  is a polynomial of  $\deg = 2n$  of  $X = N(\mathfrak{q})^{-s} \chi(\mathfrak{q})$  given by the Satake parameters  $t_{\mathfrak{q}, i}$  ( $i = 1, \dots, n$ ) of  $\pi_{\mathfrak{q}, \varphi}$  (for  $\mathfrak{q}$  outside a finite set  $S$ ). The signature  $(a, b)$  is such that  $n = a + b$  and  $s = \frac{n-1}{2}$  is critical for the  $L$ -function  $L(\pi, \chi, s) = L(\pi_\varphi, \chi, s)$ .

## 3. The integral representation for the $L$ -function $L(\varphi, \chi, s)$

is on the double group  $G = U(a + b, a + b) \supset U \times U$  of type

$$\int_{U \times U} E((g_1, g_2), f) \chi^{-1}(\det g_2) \varphi_1(g_1) \varphi_2(g_2) dg_1 dg_2$$

$$= Z_S(s) L^S(\pi_\varphi, \chi, s + \frac{n-1}{2}) \langle \varphi_1, \varphi_2 \rangle$$

where  $E((g_1, g_2), f_{s, \chi})$  denotes the restriction to  $(g_1, g_2)$  of an *Eisenstein series on the double adelic group*  $G = U(a + b, a + b)$ , the series defined from a suitably chosen section  $f = f_{s, \chi} \in \text{Ind}_{P_{\text{Siegel}}}^G$ ,  $\varphi_1 \in \pi$ ,  $\varphi_2 \in \tilde{\pi}$ , with  $P_{\text{Siegel}} = \begin{pmatrix} * & * \\ 0_{a+b} & * \end{pmatrix}$  is the Siegel parabolic in  $G$ ,  $E(g, f) = \sum_{\gamma \in P(\mathcal{K}) \backslash G(\mathcal{K})} f(\gamma g)$ ,  $f_{k, \chi} =$

$$\chi(\det(c)) \det(cz + d)^{-k}, \langle \varphi_1, \varphi_2 \rangle = \int_{U(a, b)} \varphi_1(g) \varphi_2(g) dg.$$

The section  $f$  is an automorphic form on  $U(n, n)$  has a weight, which is a representation  $\rho$  of  $\text{GL}_n \times \text{GL}_n$ . In the special case where this representation is of the form  $\rho(a, b) = \det(a)^{k+\nu} \det(b)^{-\nu}$   $f$  is said an automorphic form of weight  $k, \nu$ . For the critical values  $s = s_*, \dots, s^*$  we use certain algebraic

operators  $\theta_{s^*-s}$  to move the Eisenstein series from  $s^*$  to  $s$  by acting on the section  $f_{s^*,\chi}$  to get  $f_{s,\chi}$ . This allows to compare their  $q$ -expansions and get congruences for the critical values.

### 3.1. Classical setting: pull-back identity

This integral representation takes the form of a double Petersson product.

In the Sp case (see [8]) it becomes a *double integral representation* (pull-back identity) for the normalized  $L$ -function  $\mathcal{D}(\mathbf{f}, s, \chi)$  and its critical values at  $t$  with  $k + t = \ell$ ,

$$\mathcal{F}(g) = \frac{\left\langle \left\langle \mathbf{f}_1^0(w), g(*, *) \right\rangle^w, \mathbf{f}_2^0(z) \right\rangle^z}{\langle \mathbf{f}_1^0, \mathbf{f}_2^0 \rangle}$$

From test functions  $g = g_{\chi_i, s_i}(*, *)$  to normalized critical  $L$ -values  $\mathcal{D}(\mathbf{f}, t_i, \chi_i) = \mathcal{F}(g_{\chi_i, s_i}) = L_{geom}^*(\boldsymbol{\pi}, s_i, \chi_i)$  at  $t_i$  with  $k_i + t_i = \ell$

Here  $g(z, w) = \mathcal{H}_{t, \chi}(-\bar{z}, w)$  is a function in the tensor product of certain spaces of automorphic forms

$$\mathcal{H}_{t, \chi} \in C^\infty M_n^\ell(\Gamma_0(M), \varphi)|_z \otimes_{\mathbb{C}} C^\infty M_n^\ell(\Gamma_0(M), \varphi)|_w,$$

obtained from a double Eisenstein series  $E_{k_i, \chi_i}$  on  $U(n, n)$  of the above type, with  $\mathbf{f}_1^0, \mathbf{f}_2^0$  suitably chosen eigenfunctions of Atkin's type operator

$$U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$$

(the Hermitian Fourier expansion): .

This analytic properties of the  $L$ -function indicate that the representation  $\pi_\infty$  eventually produces a geometric object of a certain Hodge type, described in [25], (4.4.19) at p.66 in terms of its Hodge polygon. The existence of such objects was proved by P.Scholze via geometric  $p$ -adic Galois representations of Fontaine-Mazur type ([80]).

### 3.2. Eisenstein series and congruences (Unitary case)

The (Siegel-Hermite) Eisenstein series  $E_{2\ell, n, K}(Z)$  of weight  $2\ell$ , character  $\det^{-\ell}$ , is defined in [27] by  $E_{2\ell, n, K}(Z) = \sum_{g \in \Gamma_{n, K, \infty} \backslash \Gamma_{n, K}} (\det g)^\ell j(g, Z)^{-2\ell}$  (con-

verges for  $\ell > n$ ). The normalized Eisenstein series is given by  $\mathcal{E}_{2\ell, n, K}(Z) = 2^{-n} \prod_{i=1}^n L(i - 2\ell, \theta^{i-1}) \cdot E_{2\ell, n, K}(Z)$ .

If  $H \in \Lambda_n(\mathcal{O})^+$ , then the  $H$ -th Fourier coefficient of  $\mathcal{E}_{2\ell}^{(n)}(Z)$  is *polynomial* over  $\mathbb{Z}$  in variables  $\{p^{\ell-(n/2)}\}_p$ , and equals

$$|\gamma(H)|^{\ell-(n/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, p^{-\ell+(n/2)}), \gamma(H) = (-D_K)^{[n/2]} \det H.$$

Here,  $\tilde{F}_p(H, X)$  is a certain Laurent polynomial in the variables  $\{X_p = p^{-s}, X_p^{-1}\}_p$  over  $\mathbb{Z}$ . This polynomial is a *key point in proving congruences* for the modular forms in both the *pull-back double integral representation* and *Rankin-Selberg integral*.

### 3.3. Strategy of the construction of $p$ -adic $L$ -functions

It slightly differs from that on [25] and uses our *method of automorphic distributions* on the  $p$ -adic weight space  $X_\pi$  in [75], [76]. This method allows to treat a general non-ordinary case.

- The integral representation for the normalized critical values  $L^*(\boldsymbol{\pi}, \chi_i, s_i)$  via the doubling method:  $Z_S(s_i) L^S(\boldsymbol{\pi}, \chi_i, s_i + \frac{n-1}{2}) \times \langle \boldsymbol{\varphi}_{i,1}, \boldsymbol{\varphi}_{i,2} \rangle$   

$$= \int_{U \times U_-} E((g_1, g_2), f_{s_i, \chi_i}) \chi_i^{-1}(\det g_2) \boldsymbol{\varphi}_{i,1}(g_1) \boldsymbol{\varphi}_{i,2}(g_2) dg_2$$
where  $\boldsymbol{\varphi}_{i,1} \in \boldsymbol{\pi}$ ,  $\boldsymbol{\varphi}_{i,2} \in \tilde{\boldsymbol{\pi}}$  are chosen functions in dual spaces (factorizable adelic Schwartz functions on the group  $U(n)(\mathbb{A})$ ),  $E((g_1, g_2), f_{s_i, \chi_i})$  the pull-back of the Eisenstein series on  $U(n, n)$ ,  $f = f_{s_i, \chi_i}$  its Siegel section  $f \in I_P^U = \text{Ind}_{P_{\text{Siegel}}}^{U(n, n)}$ ,  $E(g, f) = \sum_{\gamma \in P(\mathcal{K}) \backslash G(\mathcal{K})} f(\gamma g)$ .

- From Siegel sections  $f_{\chi_i, s_i}$  to critical values  $L_{\text{geom}}^*(\boldsymbol{\pi}, s_i, \chi_i)$ .  
Families of automorphic distributions  $\{\boldsymbol{\mu}_r\}$ ,  $0 \leq r \leq s^* - s_*$  on the weight space  $X$  attached to  $U(a, b)$ . They produce  $\bar{\mathbb{Q}}$ -valued distributions  $\boldsymbol{\mu}_i$  on  $X$  such that  $\int_X \chi_i(x_p) d\boldsymbol{\mu}_{s^* - s_i} = L_{\text{geom}}^*(\boldsymbol{\pi}, s_i, \chi_i)$ , where  $X_\pi \rightarrow \mathbb{Z}_p^*$  is a  $p$ -part projection. Fixing embeddings  $\bar{\mathbb{Q}} \xrightarrow{i_\infty} \mathbb{C}$ ,  $\bar{\mathbb{Q}} \xrightarrow{i_p} \mathbb{C}_p = \widehat{\mathbb{Q}}_p$  produces  $p$ -adic-valued distributions.

### 3.4. Constructing $p$ -adic measures via congruences

- Proving Kummer type congruences in the form

Definition. Let  $M$  be a  $\mathcal{O}$ -module of finite rank where  $\mathcal{O} \subset \mathbb{C}_p$ . For  $h \geq 1$ , consider the following  $\mathbb{C}_p$ -vector spaces of functions on  $\mathbb{Z}_p^*$ :  $\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$ . Then a *continuous homomorphism*  $\mu : \mathcal{C} \rightarrow M$  is called a *(bounded)  $M$ -valued measure* on  $\mathbb{Z}_p^*$ . Let us define a measure with given integrals.

Take a dense family of continuous functions  $\{\varphi_i = \varphi_{s_i, \chi_i}\}$  in  $\mathcal{C}(X_\pi, \mathbb{C}_p)$  on the  $p$ -adic space  $X_\pi$ . Then Kummer says:

$$\sum_i \beta_i \varphi_i \equiv 0 \pmod{p^N} \implies \sum_i \beta_i L_{geom}^*(\pi, s_i, \chi_i) \equiv 0 \pmod{p^N}.$$

Each  $\varphi \in \mathcal{C}(X_\pi, \mathbb{C}_p)$  can be approximated by  $\{\varphi_i\}_i$ , and a measure  $\mu_\pi(\varphi)$  with given  $\mu_\pi(\varphi_i) = L_{geom}^*(\pi, s_i, \chi_i)$  is a well-defined limit over all approximations of  $\varphi$ .

- From bounded measures on  $X$  to admissible measures using  $h_{\pi, p} = P_{Newton, p}(d/2) - P_{Hodge}(d/2) \geq 0$ .

Computing critical values at  $s = s_*, \dots, s^*$  and prove admissibility congruences for them as follows

A  $\mathbb{C}_p$ -linear mapping  $\mu : \mathcal{C}^h \rightarrow M$  is called an  *$h$  admissible  $M$ -valued measure* on  $\mathbb{Z}_p^*$  if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

for  $j = 0, 1, \dots, h-1$ . Such  $\mu$  extends to  $\mathcal{C}^{loc-an}$  (and to  $\mathcal{Y}_p = Hom_{cont}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$ , the space of definition of  *$p$ -adic Mellin transform*)

### 3.5. Perspectives and applications

1. The case  $U(n, n)$ : a striking analogue of Manin-Mazur's result on  $p$ -adic analytic interpolation of critical values, [62], [68], to any imaginary quadraic  $\mathcal{K}$ , a hermitian Hecke-eigenform of weight  $\ell > 2n$ ,  $s_* = n$ ,  $s^* = \ell - n$ .
2. Using the Hodge and Newton polygons of an Euler product with a functional equation, for its geometric recognition



3. Link to a new revolutionary tool – Prisms and Prismatic cohomology (by P.Scholze-B.Bhatt [4], via Kisin-Fargue-Wach-modules and Iwasawa cohomology, using the obtained Iwasawa series,.

Given a formally smooth  $\mathbb{Z}_p$ -scheme  $X$ , this cohomology yields a universal  $q$ -deformation of the de Rham cohomology of  $X/\mathbb{Z}_p$  across the map  $\mathbb{Z}_p[[q-1]] \xrightarrow{q \rightarrow 1} \mathbb{Z}_p$ , and the Iwasawa algebra  $\mathbb{Z}_p[[q-1]]$  provides a description.

4. Special hypergeometric motives and their L-functions: Asai recognition, see [22] The generalized hypergeometric functions are often used in arithmetic and algebraic geometry. They come as periods of certain algebraic varieties, and consequently they encode important information about the invariants of these varieties. Euler factors, Newton and Hodge polygons attached to them, provide a tool for their geometric recognition.

**4. The case  $U(n, n)$ . Hermitian modular group  $\Gamma_{n,K}$  and the standard zeta function  $\mathcal{Z}(s, \mathbf{f})$  (definitions)**

The following function  $\mathcal{Z}(s, \mathbf{f})$  is a special case of Euler products constructed by G. Shimura. Let  $\theta = \theta_K$  be the quadratic character attached to  $K = \mathbb{Q}(\sqrt{-D_K})$ ,  $n' = \lfloor \frac{n}{2} \rfloor$ .

$$\Gamma_{n,K} = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_K) \mid M\eta_n M^* = \eta_n \right\}, \quad \eta_n = \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix},$$

$$\mathcal{Z}(s, \mathbf{f}) = \left( \prod_{i=1}^{2n} L(2s - i + 1, \theta^{i-1}) \right) \sum_{\mathfrak{a}} \lambda(\mathfrak{a}) N(\mathfrak{a})^{-s},$$

(defined via Hecke's eigenvalues:  $\mathbf{f}|T(\mathfrak{a}) = \lambda(\mathfrak{a})\mathbf{f}$ ,  $\mathfrak{a} \subset \mathcal{O}_K$ )

$$= \prod_{\mathfrak{q}} \mathcal{Z}_{\mathfrak{q}}(N(\mathfrak{q})^{-s})^{-1} \text{ (an Euler product over primes } \mathfrak{q} \subset \mathcal{O}_K \text{)},$$

with  $\deg \mathcal{Z}_{\mathfrak{q}}(X) = 2n$ , the Satake parameters  $t_{i,\mathfrak{q}}, i = 1, \dots, n$ ,

$$\boxed{\mathcal{D}(s, \mathbf{f}) = \mathcal{Z}\left(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f}\right)} \quad \text{(Geometrically normalized standard zeta function}$$

with a functional equation  $s \mapsto \ell - s$ ;  $\mathrm{rk} = 4n$ , and geometric weight  $\ell - 1$ ),

$$\Gamma_{\mathcal{D}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s - i)^2.$$

Main result in the lifted case: Assuming  $\ell > 2n$ , a  $p$ -adic interpolation is constructed of all critical values  $\mathcal{D}(s, \mathbf{f}, \chi)$  normalized by  $\times \Gamma_{\mathcal{D}}(s)/\Omega_{\mathbf{f}}$ , in the critical strip  $n \leq s \leq \ell - n$  for all  $\chi \bmod p^r$  in both *bounded or unbounded case*, i.e. when the product  $\alpha_{\mathbf{f}} = \left( \prod_{q|p} \prod_{i=1}^n t_{q,i} \right) p^{-n(n+1)}$  is *not a  $p$ -adic unit*.

#### 4.1. The Hodge and Newton polygons of $\mathcal{D}(s)$

are used in order to state our Main result. The Hodge polygon  $P_H(t) : [0, d] \rightarrow \mathbb{R}$  of the function  $\mathcal{D}(s)$  and the Newton polygon  $P_{N,p}(t) : [0, d] \rightarrow \mathbb{R}$  at  $p$  are piecewise linear:

The *Hodge polygon* of (weak) pure weight  $w$  has the slopes  $j$  of *length* $_j = h^{j,w-j}$  given by Serre's Gamma factors of the functional equation of the form  $s \mapsto w + 1 - s$ , relating  $\Lambda_{\mathcal{D}}(s, \chi) = \Gamma_{\mathcal{D}}(s)\mathcal{D}(s, \chi)$  and  $\Lambda_{\mathcal{D}^\rho}(w + 1 - s, \bar{\chi})$ , where  $\rho$  is the complex conjugation of  $a_n$ , and  $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathcal{D}^\rho}(s)$  equals to the product  $\Gamma_{\mathcal{D}}(s) = \prod_{j \leq \frac{w}{2}} \Gamma_{j,w-j}(s)$ , where

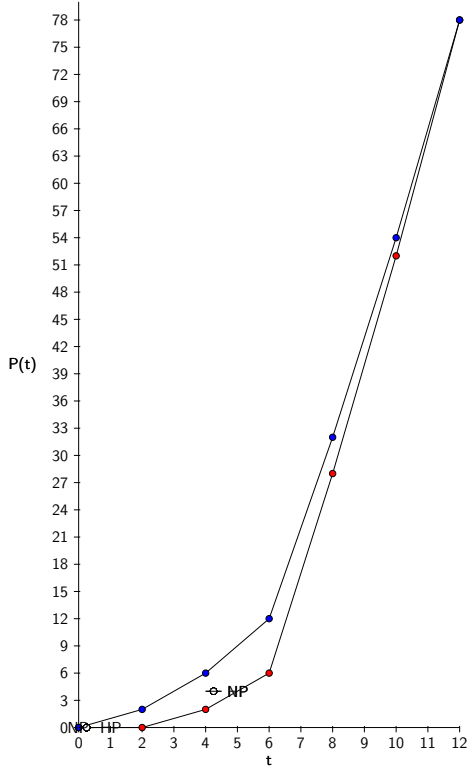
$$\Gamma_{j,w-j}(s) = \begin{cases} \Gamma_{\mathbb{C}}(s-j)^{h^{j,w-j}}, & \text{if } j < w, \\ \Gamma_{\mathbb{R}}(s-j)^{h_+^{j,j}} \Gamma_{\mathbb{R}}(s-j+1)^{h_-^{j,j}}, & \text{if } 2j = w, \text{ where} \end{cases}$$

$\Gamma_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$ ,  $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$ ,  $h^{j,j} = h_+^{j,j} + h_-^{j,j}$ ,  $\sum_j h^{j,w-j} = d$ , see [20] for the various examples with Gamma factors.

The *Newton polygon at  $p$*  is the convex hull of points  $(i, \text{ord}_p(a_i))$  ( $i = 0, \dots, d$ ); its slopes  $\lambda$  are the  $p$ -adic valuations  $\text{ord}_p(\alpha_i)$  of the inverse roots  $\alpha_i$  of  $\mathcal{D}_p(X) \in \mathbb{Q}[X] \subset \mathbb{C}_p[X]$ : *length* $_\lambda = \#\{i \mid \text{ord}_p(\alpha_i) = \lambda\}$ . According to [9], Th8.36,  $P_{Newton,p}(t) \geq P_{Hodge}(t)$  on  $[0, d]$ , see also [12].

#### 4.2. Hodge/Newton polygons for $\mathbf{f} = \text{Lift}(\Delta), n = 3, U(3, 3)$

Let us draw  $P_{Hodge}(t)$  (slopes 0, 1, 2, 1, 12,13), and  $P_{Newton,p}(t)$  (slopes 1,2,3,10,11,12), symmetry for slopes:  $j \mapsto 13 - j$ , for  $p = 7$ ,  $\mathbf{f} = \text{Lift}(\Delta)$ ,  $k = 12$ ,  $n' = 1$ ,  $\ell = 14 = k + 2n'$ ,  $d = 4n = 12$ ,  $\Gamma_{\mathcal{D}}(s) = \Gamma_{\mathbb{C}}(s)^2 \Gamma_{\mathbb{C}}(s-1)^2 \Gamma_{\mathbb{C}}(s-2)^2$ , symmetry  $s \mapsto 14 - s$ .  $P_{Newton,p}(6) = 12$ ,  $P_{Hodge}(6) = 6$ ,  $h = 6$  ("the Hasse invariant")



### 4.3. Description of the Main theorem

Let  $\Omega_{\mathbf{f}}$  be a period attached to an Hermitian cusp eigenform  $\mathbf{f}$ ,  $\mathcal{D}(s, \mathbf{f}) = \zeta(s - \frac{\ell}{2} + \frac{1}{2}, \mathbf{f})$  the standard zeta function, and

$$\alpha_{\mathbf{f}} = \alpha_{\mathbf{f},p} = \left( \prod_{q|p} \prod_{i=1}^n t_{q,i} \right) p^{-n(n+1)}, \quad h = \text{ord}_p(\alpha_{\mathbf{f},p}),$$

The number  $\alpha_{\mathbf{f}}$  turns out to be an eigenvalue of Atkin's type operator  $U_p : \sum_H A_H q^H \mapsto \sum_H A_{pH} q^H$  (the Hermitian Fourier expansion) on some  $\mathbf{f}_0$ , and  $h = P_N(\frac{d}{2}) - P_H(\frac{d}{2})$ ,  $d = 4n$ ,  $\frac{d}{2} = 2n$ .

**Definition.** Let  $M$  be a  $\mathcal{O}$ -module of finite rank where  $\mathcal{O} \subset \mathbb{C}_p$ . For  $h \geq 1$ , consider the following  $\mathbb{C}_p$ -vector spaces of functions on  $\mathbb{Z}_p^*$ :  $\mathcal{C}^h \subset \mathcal{C}^{loc-an} \subset \mathcal{C}$ . Then

- a continuous homomorphism  $\mu : \mathcal{C} \rightarrow M$  is called a (*bounded*) *measure*  $M$ -valued measure on  $\mathbb{Z}_p^*$ .

-  $\mu : \mathcal{C}^h \rightarrow M$  is called an  $h$  admissible measure  $M$ -valued measure on  $\mathbb{Z}_p^*$  measure if the following growth condition is satisfied

$$\left| \int_{a+(p^v)} (x-a)^j d\mu \right|_p \leq p^{-v(h-j)}$$

for  $j = 0, 1, \dots, h-1$ , and let  $\mathcal{Y}_p = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^*, \mathbb{C}_p^*)$  be the space of definition of  $p$ -adic Mellin transform

**Theorem** ([2], [68]) For an  $h$ -admissible measure  $\mu$ , the Mellin transform  $\mathcal{L}_\mu : \mathcal{Y}_p \rightarrow \mathbb{C}_p$  exists and has growth  $o(\log^h)$  (with infinitely many zeros).

#### 4.4. Main Theorem.

Let  $\mathbf{f}$  be a Hermitian cusp eigenform of degree  $n \geq 2$  and of weight  $\ell > 2n$ . There exist distributions  $\mu_{\mathcal{D},s}$  for  $s = n, \dots, \ell - n$  with the properties:

i) for all pairs  $(s, \chi)$  such that  $s \in \mathbb{Z}$  with  $n \leq s \leq \ell - n$ ,

$$\int_{\mathbb{Z}_p^*} \chi d\mu_{\mathcal{D},s} = A_p(s, \chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \bar{\chi})}{\Omega_{\mathbf{f}}}$$

(under the inclusion  $i_p$ ), with elementary factors  $A_p(s, \chi) = \prod_{q|p} A_q(s, \chi)$  including a finite Euler product, Satake parameters  $t_{q,i}$ , gaussian sums, the conductor of  $\chi$ ; the integral is a finite sum.

(ii) if  $\text{ord}_p \left( \left( \prod_{q|p} \prod_{i=1}^n t_{q,i} \right) p^{-n(n+1)} \right) = 0$  then the above distributions  $\mu_{\mathcal{D},s}$  are bounded measures, we set  $\mu_{\mathcal{D}} = \mu_{\mathcal{D},s^*}$  and the integral is defined for all continuous characters  $y \in \text{Hom}(\mathbb{Z}_p^*, \mathbb{C}_p^*) =: \mathcal{Y}_p$ .

Their Mellin transforms  $\mathcal{L}_{\mu_{\mathcal{D},s}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D},s}$ ,  $\mathcal{L}_{\mu_{\mathcal{D}}} : \mathcal{Y}_p \rightarrow \mathbb{C}_p$ , give bounded  $p$ -adic analytic interpolation of the above  $L$ -values to on the  $\mathbb{C}_p$ -analytic group  $\mathcal{Y}_p$ ; and these distributions are related by:  $\int_X \chi d\mu_{\mathcal{D},s} = \int_X \chi x^{s^*-s} \mu_{\mathcal{D},s^*}$ ,  $X = \mathbb{Z}_p^*$ , where  $s^* = \ell - n$ ,  $s_* = n$ .

Main theorem (continued)

(iii) in the admissible case assume that  $0 < h \leq s^* - s_* + 1 = \ell + 1 - 2n$ , where  $h = \text{ord}_p \left( \left( \prod_{q|p} \prod_{i=1}^n t_{q,i} \right) p^{-n(n+1)} \right) > 0$ , Then there exists an  $h$ -admissible

measure  $\mu_{\mathcal{D}}$  whose integrals  $\int_{\mathbb{Z}_p^*} \chi x_p^s d\mu_{\mathcal{D}}$  are given by  $i_p \left( A_p(s, \chi) \frac{\mathcal{D}^*(s, \mathbf{f}, \bar{\chi})}{\Omega_{\mathbf{f}}} \right) \in \mathbb{C}_p$  with  $A_p(s, \chi)$  as in (i); their Mellin transforms  $\mathcal{L}_{\mathcal{D}}(y) = \int_{\mathbb{Z}_p^*} y d\mu_{\mathcal{D}}$ , belong to the type  $o(\log x_p^h)$ . (iv) the functions  $\mathcal{L}_{\mathcal{D}}$  are determined by (i)-(iii).

**Remarks.** (a) Interpretation of  $s^*$ : the smallest of the "big slopes" of  $P_H$   
 (b) Interpretation of  $s_* - 1$ : the biggest of the "small slopes" of  $P_H$ .

## A. Appendix . Recovering geometric objects from automorphic forms and special functions

For an irreducible automorphic representation  $\pi = \pi_{\varphi}$  of a  $\mathbb{Q}$ -algebraic group  $G(\mathbb{A})$ , the eventual geometric type of  $\pi$  is determined by the component  $\pi_{\infty}$ , where  $\pi = \otimes_v \pi_v$ ,  $v$  the set of valuations.

- (Wiles) Elliptic curves  $E/\mathbb{Q} \leftrightarrow$  Hecke cusp eigenforms  $f = \sum_{n=1}^{\infty} a_n q^n$  of weight  $w = 2$  and  $a_n \in \mathbb{Q}$  (where  $q = e^{2\pi iz}$ ).
- (Deligne, Serre, Scholl, Carayol) Holomorphic modular forms of higher weight  $w \geq 2 \rightsquigarrow X_f$ , certain  $(w - 1)$ -dimensional parts  $X_f$  (called "motives") of a Kuga-Sato variety  $E_{univ}^{w-2}$ , such that  $L_f(s) = \sum_{n=1}^{\infty} a_n n^{-s} = L(H^{w-1}(X_f), s)$
- (Manin-Shimura-Mazur) Periods and modular symbols  $\int_x^{i\infty} f(z) z^r dz \rightsquigarrow$  Normalized special values  $L_f^*(r + 1, \chi)$ , where  $L_f^*(s, \chi) := \Gamma(s) L_f(s, \chi)$ , for any Dirichlet character  $\chi$ ,  $0 \leq r \leq w - 2$ ,  $x \in \mathbb{Q}$ . That is, the integrals on the left give linear forms on homology classes of geodesics  $\{x, i\infty\}$ , i.e. elements of certain cohomology groups  $H^{w-1}(X_f)$ , producing  $X_f$  and  $L(X_f, s)$ .
- The use of the Iwasawa algebra  $\Lambda = \mathbb{Z}_p[[T]] = \text{Dist}(\mathbb{Z}_p, \mathbb{Z}_p)$ ,  $\Lambda \ni \mu \longleftrightarrow A_{\mu}(T) = \sum_{k \geq 0} A_k T^k$ , where  $A_k = \int_{\mathbb{Z}_p} \binom{x}{k} d\mu$ .

The integral  $I = \int_{\mathbb{Z}_p} \varphi(x) d\mu(x)$  of any continuous function  $\varphi = \sum_{k \geq 0} a_k \binom{x}{k} \in \mathcal{C}(\mathbb{Z}_p, \mathbb{Z}_p)$  becomes  $I = \sum_{k \geq 0} a_k A_k$ .

## B. Appendix. Prisms and Prismatic cohomology [4]

This new tool in the theory of geometric  $p$ -adic Galois representations appeared since [80], [81] and can be used for the study of  $q$ -universal deformation the De Rham cohomology of locally-symmetric hermitian spaces (or Shimura varieties of PEL-type). The above example of unitary groups  $U_{\mathcal{K}}(n, n)$  describes analytic families of abelian varieties  $A$  with imbedding  $\iota : \mathcal{K} \hookrightarrow \text{End}_{\mathcal{K}}(A)$ . Thus obtained  $p$ -adic schemes  $X_{\pi, p}$  produce de Rham cohomology groups as above, and their universal deformations can be described using prisms [4] as certain Iwasawa-type modules, notably,  $\mathbb{Z}_p[[q-1]]$ -modules, where  $T = q-1$  is the Iwasawa variable attached to the quantum variable  $q$ .

According to [4], the notion of a prism substitutes in applications the notion of a perfectoid ring. Using prisms, one may attach a ringed site - the prismatic site - to a formal  $\mathbb{Z}_p$ -scheme. The resulting cohomology theory specializes to most known integral  $p$ -adic cohomology theories (étale, crystalline, de Rham). As application, a co-ordinate free description of  $q$ -de Rham cohomology is given.

Given a formally smooth  $\mathbb{Z}_p$ -scheme  $X$ , this cohomology yields a deformation of the de Rham cohomology of  $X/\mathbb{Z}_p$  across the map  $\mathbb{Z}_p[[q-1]] \xrightarrow{q \rightarrow 1} \mathbb{Z}_p$ .

## C. Appendix . Ikeda's lifting $f \rightsquigarrow \mathbf{f} = \text{Lift}(f)$

Its  $L$ -function gives a crucial motivation for both complex and  $p$ -adic theory of  $L$ -functions on unitary groups, and extends to a general (not necessarily

lifted) case. Recall that in [27]

$$\begin{aligned}
S_{2k+1}(\Gamma_0(D), \theta) \ni f \rightsquigarrow \mathbf{f} = \text{Lift}(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}), \text{ if } n = 2n' \text{ is even (E)} \\
S_{2k}(\text{SL}(\mathbb{Z})) \ni f \rightsquigarrow \mathbf{f} = \text{Lift}(f) \in \mathcal{S}_{2k+2n'}(\Gamma_{K,n}), \text{ if } n = 2n' + 1 \text{ is odd (O)} \\
\text{the standard } L\text{-function of } \mathbf{f} = \text{Lift}^{(n)}(f) \text{ is a nice product: } \mathcal{Z}(s, \mathbf{f}) = \\
\prod_{i=1}^n L(s+k+n'-i+(1/2), f) L(s+k+n'-i+(1/2), f, \theta) \quad [27] \\
= \prod_{i=0}^{n-1} L(s+\ell/2-i-(1/2), f) L(s+\ell/2-i-(1/2), f, \theta).
\end{aligned}$$

Notice  $k+n' = \ell/2$ , then the Gamma factor of the standard zeta function with the symmetry  $s \mapsto 1-s$  becomes  $\Gamma_{\mathcal{Z}}(s) = \prod_{i=0}^{n-1} \Gamma_{\mathbb{C}}(s+\ell/2-i-(1/2))^2$ .

#### D. Appendix . Special hypergeometric motives and their L-functions: Asai recognition, [22]

The generalized hypergeometric functions are a familiar player in arithmetic and algebraic geometry. They come quite naturally as periods of certain algebraic varieties, and consequently they encode important information about the invariants of these varieties.

Euler factors, Newton and Hodge polygons attached to them, provide a tool for their geometric recognition.

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## A SURVEY ON A FEW RECENT PAPERS IN P-ADIC VALUE DISTRIBUTION

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**Abstract.** In this article, we propose to present several recent results: a new proof of the p-adic Hermite-Lindemann Theorem, a new proof of the p-adic Gel'fond-Schneider Theorem, exceptional values of meromorphic functions and derivatives and the p-adic Nevanlinna theory applied to small functions. We first have to recall the definitions of the p-adic logarithm and exponential.

### 1 Logarithm and exponential in a p-adic field

**Notations:** We denote by  $\mathbb{Q}_p$  the completion of  $\mathbb{Q}$  with respect to the p-adic absolute value and by  $\mathbb{C}_p$  the completion of the algebraic closure of  $\mathbb{Q}_p$ , which is known to be algebraically closed [7]. In general, we denote by  $\mathbb{K}$  an algebraically closed field of characteristic 0 complete with respect to an ultrametric absolute value, such as  $\mathbb{C}_p$ . The ultrametric absolute value of  $\mathbb{K}$  is denoted  $|\cdot|$  while the archimedean absolute value of  $\mathbb{C}$  is denoted  $|\cdot|_\infty$ .

Let  $a \in \mathbb{K}$  and let  $R \in \mathbb{R}_+$ . We denote by  $d(a, R)$  the "closed" disk  $\{x \in \mathbb{K} \mid |x - a| \leq R\}$  and by  $d(a, R^-)$  the "open" disk  $\{x \in \mathbb{K} \mid |x - a| < R\}$ .

We denote by  $\mathcal{A}(\mathbb{K})$  the algebra of power series converging in all  $\mathbb{K}$ . Given  $a \in \mathbb{K}$  and  $R > 0$ , we denote by  $\mathcal{A}(d(a, R^-))$  the algebra of power series  $\sum_{j=0}^{\infty} a_n(x - a)^n$  converging in  $d(a, R^-)$  and by  $\mathcal{A}_b(d(a, R^-))$  the subalgebra of

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functions  $f(x) \in \mathcal{A}_b(d(a, R^-))$  that are bounded in  $(d(a, R^-))$  and we put  $\mathcal{A}_u(d(a, R^-)) = \mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$ .

Moreover we denote by  $H(d(a, r))$  the algebra of power series  $\sum_{j=0}^{\infty} a_j(x-a)^j$  converging in  $d(a, R)$  called *analytic elements in  $d(a, R)$* . Given an element  $f$  of  $H(d(0, R))$  we put  $|f|(r) = \sup_{x \in d(0, R)} |f(x)|$ .

We will define the  $p$ -adic logarithm and the  $p$ -adic exponential and will shortly study them, in connection with the study of the roots of 1. Here, as in [7], we compute the radius of convergence of the  $p$ -adic exponential by using results on injectivity.

The following lemma 1.a is easy:

**Lemma 1.a:**  $\mathbb{K}$  is supposed to have residue characteristic  $p \neq 0$ . Let  $r \in ]0, 1[$  and for each  $n \in \mathbb{N}$ , let  $h_n(x) = (1+x)^{p^n}$ . The sequence  $h_n$  converges to 1 with respect to the uniform convergence on  $d(0, r)$ .

**Notations:** We denote by  $\log$  the real logarithm function of base  $e$ . Given a power series  $\sum_{j=0}^{\infty} a_j x^j$  converging in  $d(0, R^-)$  and given a number  $\mu < \log(R)$  we denote by  $\nu^+(f, \mu)$  the biggest integer  $q$  such that  $\sup_{j \geq 0} \log(|a_j|) + j\mu = \log(|a_q|) + q\mu$ .

For each  $q \in \mathbb{N}^*$  we denote by  $R_q$  the positive number such that  $\log_p(R_q) = -\frac{1}{p^{q-1}(p-1)}$ . We denote by  $g(x)$  the series  $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$ .

The following lemma 1.b is well known (Theorem B.13.7 in [7]):

**Lemma 1.b:** Let  $f(x) = \sum_{j=0}^{\infty} a_j x^j$  be converging in  $d(0, R^-)$  and let  $r < R$ . Then  $\nu^+(f, \log(r))$  is the number of zeros of  $f$  in  $d(0, r)$ , taken multiplicity into account.

**Theorem 1.1:**  $g$  has a radius of convergence equal to 1. If the residue characteristic of  $\mathbb{K}$  is  $p \neq 0$ , then  $g$  is unbounded in  $d(0, 1^-)$ . If the residue characteristic is zero, then  $|g(x)|$  is bounded by 1 in  $d(0, 1^-)$ . The function defined in  $d(1, 1^-)$  as  $\text{Log}(x) = g(x-1)$  has a derivative equal to  $\frac{1}{x}$  and satisfies  $\text{Log}(ab) = \text{Log}(a) + \text{Log}(b)$  whenever  $a, b \in d(1, 1^-)$ .

**Proof.** It is clearly seen that the radius of  $g$  is 1, because  $|n| \geq \frac{1}{n}$  and  $|n| \leq 1$  for all  $n \in \mathbb{N}$ . As in the Archimedean context, the property  $\text{Log}(ab) =$

$\text{Log}(a) + \text{Log}(b)$  comes from the fact that both  $\text{Log}$  and the function  $h_a$  defined as  $h_a(x) = \text{Log}(ax)$  have the same derivative. The other statements are immediate.

**Notation:** When  $\mathbb{K}$  has residue characteristic  $p \neq 0$ , we introduce the group  $W$  of the  $p^s$ -th roots of 1, i.e., the set of the  $u \in \mathbb{K}$  satisfying  $u^{p^s} = 1$  for some  $s \in \mathbb{N}$ .

Recall that analytic elements were defined by M. Krasner and are defined in [7].

**Theorem 1.2:**  $\mathbb{K}$  is supposed to have residue characteristic  $p \neq 0$  (resp. 0). All zeros of  $\text{Log}$  are of order 1. The set of zeros of the function  $\text{Log}$  is equal to  $W$ , (resp. 1 is the only zero of  $\text{Log}$ ). The restriction of  $\text{Log}$  to the disk  $d(1, (R_1)^-)$  (resp.  $d(1, 1^-)$ ) is injective and is a bijection from  $d(1, (R_1)^-)$  onto  $d(0, (R_1)^-)$  ( resp. from  $d(1, 1^-)$  onto  $d(0, 1^-)$ ).

**Proof.** It is obvious that the zeros of  $\text{Log}$  are of order 1 because the derivative of  $\text{Log}$  has no zero. First, we suppose  $\mathbb{K}$  to have residue characteristic  $p \neq 0$ . Each root of 1 in  $d(1, 1^-)$  is a zero of  $\text{Log}$ . Moreover, by Theorem A.6.8 of [7], we know that the only roots of 1 in  $d(1, 1^-)$  are the  $p^n$ -th roots. Now we can check that  $\text{Log}$  admits no zero other than the roots of 1. Indeed, suppose that  $a$  is a zero of  $\text{Log}$  but is not a root of 1, and for each  $n \in \mathbb{N}$ , let  $b_n = a^{p^n}$ . Since  $b_n$  belongs to  $d(1, 1^-)$ , by Lemma B.16.1 of [7] we have  $\lim_{n \rightarrow \infty} b_n = 1$ . But obviously  $\text{Log}(b_n) = 0$  for every  $n \in \mathbb{N}$ , hence this contradicts the fact that 1 is an isolated zero of  $\text{Log}$ .

Thus,  $\text{Log}$  has no zero in the disk  $d(1, (R_1)^-)$ , except 1 and therefore, by Lemma 1.b the series  $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$  satisfies  $\nu^+(f, \log r) = 1$  for every  $r \in ]0, R_1[$ , hence  $r > \frac{r^n}{|n|}$  for all  $r \in ]0, R_1[$ , for every  $n \in \mathbb{N}^*$ . Therefore, by Corollary B.14.10 of [7] it is injective in  $d(0, R_1^-)$ . Then, by Corollary B.13.10 of [7], we see that  $\text{Log}(d(1, R_1^-)) = d(0, R_1^-)$ .

Now we suppose that  $\mathbb{K}$  has residue characteristic zero. Then, the function  $f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$  satisfies  $\nu^+(f, \log r) = 1$  for every  $r \in ]0, 1[$ , hence  $r > \frac{r^n}{n}$  for all  $r \in ]0, 1[$ , for every  $n \in \mathbb{N}^*$ . Therefore,  $f$  has no zero different from 1 in  $d(0, 1^-)$  and, by Corollary B.14.10 of [7], is injective in  $d(0, 1^-)$ . Then by Corollary B.13.10 of [7] we see that  $\text{Log}(d(1, 1^-)) = d(0, 1^-)$ . This ends the proof.



**Corollary 1.A:**  $\mathbb{K}$  is supposed to have residue characteristic 0. There is no root of 1 in  $d(1, 1^-)$ , except 1. **Proof.** Indeed any root of 1 should be a zero of  $Log$  in  $d(1, 1^-)$ .

**Notations:** If  $\mathbb{K}$  has residue characteristic  $p \neq 0$ , we first denote by  $exp$  the inverse (or reciprocal) function of the restriction of  $Log$  to  $d(1, R_1^-)$ , which obviously is a function defined in  $d(0, R_1^-)$ , with values in  $d(1, R_1^-)$ . If  $\mathbb{K}$  has residue characteristic 0 we denote by  $exp$  the inverse function of  $Log$ , which is obviously defined in  $d(0, 1^-)$  and takes values in  $d(1, 1^-)$ .

**Theorem 1.3:**  $\mathbb{K}$  is supposed to have residue characteristic  $p \neq 0$  (resp.  $p = 0$ ). The function  $exp$  belongs to  $\mathcal{A}_b(d(0, R_1^-))$  (resp.  $\mathcal{A}_b(d(0, 1^-))$ ), is a bijection from  $d(0, R_1^-)$  onto  $d(1, R_1^-)$  (resp. from  $d(0, 1^-)$  onto  $d(1, 1^-)$ ), and satisfies  $exp(x) = exp'(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  whenever  $x \in d(0, R_1^-)$  (resp.  $x \in d(0, 1^-)$ ). Moreover, the disk of convergence of its series is equal to  $d(0, R_1^-)$  (resp.  $d(0, 1^-)$ ). Further, if  $p \neq 0$ , then  $exp$  is not an analytic element on  $d(0, R_1^-)$ .

**Proof.** By Corollary B.14.15 of [7] we know that the function  $exp$  belongs to  $\mathcal{A}_b(d(0, R_1^-))$  (resp.  $\mathcal{A}_b(d(0, 1^-))$ ) and is obviously a bijection from  $d(0, R_1^-)$  onto  $d(1, R_1^-)$  (resp. from  $d(0, 1^-)$  onto  $d(1, 1^-)$ ). As it is the reciprocal of  $Log$ , it must satisfy  $exp(x) = exp'(x)$  for all  $x \in d(0, R_1^-)$  (resp.  $x \in d(0, 1^-)$ ) and, therefore,  $exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$  whenever  $x \in d(0, R_1^-)$  (resp.  $x \in d(0, 1^-)$ ). Thus the radius of convergence  $r$  is at least  $R_1$  (resp. 1). If the residue characteristic is 0, it is obviously seen that the series cannot converge for  $|x| = 1$ , hence the disk of convergence is  $d(0, 1^-)$ .

Now we suppose that the residue characteristic is  $p \neq 0$ . Suppose that the power series of  $exp$  converges in  $d(0, R_1)$ . Then  $exp$  has continuation to an analytic element on  $d(0, R_1)$ . On the other hand, since  $\nu(f, \log r) = 1$  for all  $r \in ]0, R_1[$ , we have  $\nu^-(f, \log R_1) = 1$  and then by Theorem B.13.9 of [7]  $Log(d(1, R_1))$  is equal to  $d(0, R_1)$ . Hence, we can consider  $exp(Log(x))$  in all the disk  $d(0, R_1)$ . By Corollary B.3.3 of [7] this is an analytic element on  $d(1, R_1)$ . But this element is equal to the identity in all of  $d(1, R_1^-)$  and, therefore, in all of  $d(1, R_1)$ . Of course this contradicts the fact that  $Log$  is not injective in the circle  $C(1, R_1)$ . This finishes proving that the disk of convergence of  $exp$  is just  $d(0, R_1^-)$ .

**Notations:** Henceforth, we put  $e^x = exp(x)$ .

**Theorem 1.4:**  $\mathbb{K}$  is supposed to have residue characteristic  $p \neq 0$ . Let  $x \in d(0, R_1^-)$ . Then  $e^x$  is algebraic over  $\mathbb{Q}_p$  if and only if so is  $x$ . Let  $u \in d(0, 1^-)$ . Then  $\log(1+u)$  is algebraic over  $\mathbb{Q}_p$  if and only if so is  $u$ .

**Proof.** By Theorem B.5.24 of [7], if  $x$  is algebraic over  $\mathbb{Q}_p$ , so is  $e^x$ . Similarly, if  $u$  is algebraic over  $\mathbb{Q}_p$ , so is  $\log(1+u)$ . Consequently, suppose that  $e^x$  is algebraic over  $\mathbb{Q}_p$ . Then  $e^x$  is of the form  $1+t$  with  $|t| < 1$ , hence  $\log(1+t)$  is algebraic over  $\mathbb{Q}_p$ . But then,  $\log(1+t) = \log(e^x) = x$ , hence  $x$  is algebraic over  $\mathbb{Q}_p$ . Now, more generally, suppose  $\log(1+u)$  is algebraic over  $\mathbb{Q}_p$ , with  $|u| < 1$ . Take  $q \in \mathbb{N}$  such that  $|p^q \log(1+u)| < R_1$ . We have  $p^q \log(1+u) = \log((1+u)^{p^q})$ . Since  $|p^q \log(1+u)| < R_1$ , we have  $|\log((1+u)^{p^q})| < R_1$ , hence  $\exp(\log((1+u)^{p^q})) = (1+u)^{p^q}$ . Consequently,  $(1+u)^{p^q}$  is algebraic over  $\mathbb{Q}_p$  and hence so is  $u$ .

We can show a similar result when  $p = 0$ .

**Theorem 1.5:**  $\mathbb{K}$  is supposed to have residue characteristic 0. Let  $x \in d(0, 1^-)$ . Then  $e^x$  is algebraic over  $\mathbb{Q}_p$  if and only if so is  $x$ . Let  $u \in d(0, 1^-)$ . Then  $\log(1+u)$  is algebraic over  $\mathbb{Q}_p$  if and only if so is  $u$ .

The following proposition 1.6 will be used in the poof of Theorem 2.3 and is proven by induction, similarly as (1.4.2) in [16].

**Proposition 1.6:** Let  $P_1, \dots, P_q \in \mathbb{K}[X]$  different from 0 and let  $w_1, \dots, w_q \in \mathbb{K}$  be pairwise distinct. Let  $F(x) = \sum_{j=1}^q P_j(x)e^{w_j x}$ . Then  $F$  is not identically zero.

## 2 Hermite-Lindemann's and Gel'fond-Schneider's Theorems in ultrametric fields

We will use the following classical notation:

**Notation:** We will denote by  $\mathcal{K}$  an algebraically closed complete ultrametric extension of  $\mathbb{Q}$  of residue characteristic 0.

We will denote by  $U$  the disk  $d(0, 1)$  and by  $D_0$  the disk  $d(0, 1^-)$  in the field  $\mathbb{K}$  no matter what the residue characteristic.

If the residue characteristic of  $\mathbb{K}$  is  $p > 0$  we put  $R_1 = p^{\frac{-1}{p-1}}$  and denote by  $D_1$  the disk  $d(0, R_1^-)$ .

Given an algebraic number  $a \in \mathbb{C}_p$  (resp.  $a \in \mathcal{K}$ ) and  $a_1, a_2, \dots, a_q$  its conjugates over  $\mathbb{Q}$  (with  $a_1 = a$ ), we put  $\overline{|a|} = \max_{1 \leq j \leq q} |a_j|$  and we denote by

$\text{den}(a)$  its *smallest denominator*, i.e. the smallest positive integer  $q$  such that  $qa$  is an algebraic integer. Then we put  $s(a) = \max(\log [a], \log(\text{den}(a)))$  and  $s(a)$  is called *the size* of  $a$ . More generally we call denominator of a number  $a$  all positive integer multiple of its smallest denominator.

Given a polynomial  $P(X_1, \dots, X_q) \in \mathbb{Z}[X_1, \dots, X_q]$ , we denote by  $H(P)$  the supremum of the archimedean absolute values of its coefficients.

Given a positive real number  $a$ , we denote by  $[a]$  the largest integer  $n$  such that  $n \leq a$ .

Hermite-Lindemann's theorem is well known in complex analysis. The same holds in  $p$ -adic analysis. The first proof was presented in 1930 by K. Malher [13]. This proof given in [13] is written in German and uses symbols which are not currently known. Here we present a new proof using classical methods in transcendental processes that are maybe easier to understand.

We will need Siegel's Lemma in all the following theorems of this chapter. We will choose a particular form of this famous lemma [16] whose formulation is due to M. Mignotte:

**Lemma 2.a (Siegel):** *Let  $E$  be a finite extension of  $\mathbb{Q}$  of degree  $q$  and let  $\lambda_{i,j}$   $1 \leq i \leq m$ ,  $1 \leq j \leq n$  be elements of  $E$  integral over  $\mathbb{Z}$ . Let  $M = \max(|\lambda_{i,j}| \mid 1 \leq i \leq m, \mid 1 \leq j \leq n)$  and let  $(\mathcal{S})$  be the linear system  $\{\sum_{j=1}^n \lambda_{i,j} x_j = 0, 1 \leq i \leq m\}$ . There exists solutions  $(x_1, \dots, x_n)$  of  $(\mathcal{S})$  such that  $x_j \in \mathbb{Z} \forall j = 1, \dots, n$  and*

$$\log(|x_j|_\infty) \leq \log(M) \frac{qm}{n - qm} + \frac{\log(2)}{2} \quad \forall j = 1, \dots, n.$$

Lemma 2.b will be necessary in the proof of Theorem 2.4 and is easily proven in [16] since its proof implies no change in the field  $\mathbb{K}$  since it only concerns algebraic numbers

**Lemma 2.b:** *Let  $a_1, \dots, a_q \in \mathbb{K}$  be algebraic over  $\mathbb{Q}$ , let  $P(X_1, \dots, X_q) \in \mathbb{Z}[X_1, \dots, X_q]$  be such that  $\deg_{X_j}(P) \leq r_j$   $1 \leq j \leq q$  and let  $\beta = P(a_1 \dots a_q)$ . Then  $\beta$  is algebraic over  $\mathbb{Q}$ ,  $d(a_1)^{r_1} \dots d(a_q)^{r_q}$  is a multiple of  $\text{den}(\beta)$  and we have*

$$s(\beta) \leq \log H(P) + \sum_{j=1}^q (r_j s(a_j) + \log(r_j) + 1)$$

**Theorem 2.1 (Hermite-Lindemann):** *Suppose that  $\mathbb{K}$  has residue characteristic  $p > 0$ . Let  $\alpha \in D_1$  be algebraic. Then  $e^\alpha$  is transcendental.*

**Proof.** We suppose that  $\alpha$  and  $e^\alpha$  are algebraic. Let  $h = |\alpha|$ . Let  $E$  be the field  $\mathbb{Q}[\alpha, e^\alpha]$ , let  $q = [E : \mathbb{Q}]$  and let  $w$  be a common denominator of  $\alpha$  and  $e^\alpha$ . We will construct a sequence of polynomials  $(P_N(X, Y))_{N \in \mathbb{N}}$  in two variables such that  $\deg_X(P_N) = \lfloor \frac{N}{\log(N)} \rfloor$ ,  $\deg_Y(P_N) = \lfloor (\log N)^3 \rfloor$  and such that the function  $F_N(x) = P_N(x, e^x)$  satisfy further, for every  $s = 0, \dots, N-1$  and for every  $j = 0, \dots, \lfloor \log(N) \rfloor$

$$\frac{d^s}{dx^s} F_N(j\alpha) = 0.$$

According to formal computations in the proof of Hermite Lindemann's Theorem in the complex context, (Theorem 3.1.1 in [16]) we have

$$\frac{d^M F_N(\gamma_N)}{dx^M} = \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{u_1(N)} \left( \frac{u_1(N)!}{\sigma!(u_1(N)-\sigma)!} \right) \left( \frac{l!}{(u_1(N)-\sigma)!} \right) \cdot m^{u_1(N)-\sigma} (1)^j j^{u_1(N)-\sigma} (\alpha)^{u_1(N)-\sigma} (e^\alpha)^{ju_2(N)}.$$

We put  $u_1(N) = \deg_X(P_N)$ ,  $u_2(N) = \deg_Y(P_N)$ . We will solve the system

$$w^{u_1(N)+u_2(N)} \frac{d^s}{dx^s} F_N(j\alpha) = 0, \quad 0 \leq s \leq N-1, \quad j = 0, \dots, \lfloor \log(N) \rfloor$$

where the undeterminates are the coefficients  $b_{l,m,N}$  of  $P_N$ . We then write the system under the form

$$\sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{\min(s,l)} \left( \frac{s!}{\sigma!(s-\sigma)!} \right) \left( \frac{l!}{(l-\sigma)!} \right) m^{s-\sigma} j^{l-\sigma}.$$

$$(2) \quad (w\alpha)^{l-\sigma} (we^\alpha)^{jm} w^{u_1(N)-(l-\sigma)+u_2(N)-jm} = 0.$$

That represents a system of  $N \lfloor \log(N) \rfloor$  equations of at least  $N(\lfloor \log(N) \rfloor)^2$  undeterminates, with coefficients in  $E$ , integral over  $\mathbb{Z}$ .

According to formal computations of Hermite-Lindemann's Theorem in the complex context (Theorem 3.1.1 in [16]), it appears that in the system (2), each factor  $\left( \frac{s!}{\sigma!(s-\sigma)!} \right)$ ,  $\left( \frac{l!}{(l-\sigma)!} \right)$ ,  $m^{s-\sigma}$ ,  $j^{l-\sigma}$ ,  $(w\alpha)^{l-\sigma}$ ,  $(we^\alpha)^{jm}$ ,  $w^{u_1(N)-(l-\sigma)+u_2(N)-jm}$  admits a bounding of the form  $SN(\log(\log(N)))$  when  $N$  goes to  $+\infty$ . On one hand  $w^{u_1(N)+u_2(N)}$  is a common denominator and we have

$$\log(w^{u_1(N)+u_2(N)}) \leq \log(\omega) \left( \frac{N}{\log(N)} + (\log(N))^3 \right)$$

and hence we have a constant  $T > 0$  such that

$$(3) \quad \log(w^{u_1(N)+u_2(N)}) \leq \frac{TM}{\log M}.$$

Next we notice that

$$(4) \quad \log\left(\frac{u_1(N)!}{\sigma!(u_1(N)-\sigma)!}\right) \leq u_1(N) \log(u_1(N)) \leq \frac{N}{\log(N)} \log\left(\frac{N}{\log(N)}\right) \leq N$$

and similarly,

$$(5) \quad \log\left(\frac{l!}{(u_1(N)-\sigma)!}\right) \leq u_1(N) \log(u_1(N)) \leq N.$$

and

$$(6) \quad \log(m^{u_1(N)-\sigma}) \leq \frac{3N}{\log(N)} \log(\log(N)).$$

Now, we check that

$$\log\left(j^{u_1(N)-\sigma} \cdot (|\bar{\alpha}|)^{u_1(N)-\sigma} \cdot (|e^{\bar{\alpha}}|)^{ju_2(N)}\right) \leq N + \frac{N}{\log(N)} \log(|\bar{\alpha}|) + \log(N)(\log(N))^3 \log(|e^{\bar{\alpha}}|)$$

and hence there exists a constant  $L > 0$  such that

$$(7) \quad \log\left(j^{u_1(N)-\sigma} \cdot (|\bar{\alpha}|)^{u_1(N)-\sigma} \cdot (|e^{\bar{\alpha}}|)^{ju_2(N)}\right) \leq LN.$$

Therefore by (2), (3), (4), (5), (6) and (7) we have a constant  $C > 0$  such that each coefficient  $a$  of the system satisfies

$$(8) \quad s(a) \leq CN(\log(\log(N))).$$

By Siegel's Lemma 2.a and by (8) there exist integers  $b_{l,m,N}$ ,  $0 \leq l \leq u_1(N)$ ,  $0 \leq m \leq u_2(N)$  in  $\mathbb{Z}$  such that

$$(9) \quad 0 < \max_{l \leq u_1(N), m \leq u_2(N)} \log(|b_{l,m,N}|_\infty) \leq \frac{qN \log(N)}{N(\log(N))^2 - qN \log(N)} (CN \log(\log(N)))$$

and such that the function

$$(10) \quad F_N(x) = \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m;N} x^l e^{mx}$$

satisfies

$$\frac{d^s}{dx^s} F_N(j\alpha) = 0, \quad 0 \leq s \leq N-1, \quad j = 0, 1, \dots, [\log(N)].$$

Now, by (9), we can check that there exists a constant  $G > 0$  such that

$$(11) \quad \max_{l \leq u_1(N), m \leq u_2(N)} (\log(|b_{l,m,N}|_\infty)) \leq \frac{GN \log(\log(N))}{\log(N)}.$$

The function  $F_N$  defined in (10) belongs to  $\mathcal{A}(D_1)$  and is not identically zero, hence at least one of the numbers  $\frac{d^s}{dx^s} F_N(0)$  is not null. Let  $M$  be the biggest of the integers such that  $\frac{d^s}{dx^s} F_N(j\alpha) = 0 \forall s = 0, \dots, M-1, j = 0, 1, 2, \dots, [\log(N)]$ . Thus we have  $M \geq N$  and there exists  $j_0 \in \{0, 1, \dots, [\log(N)]\}$  such that  $\frac{d^M}{dx^M} F_N(j_0\alpha) \neq 0$ . We put  $\gamma_N = \frac{d^M}{dx^M} F_N(j_0\alpha)$ .

Let us now give an upper bound of  $s(\gamma_N)$ . On one hand  $w^{u_1(N)+u_2(N)}$  is a common denominator and by (2) we have a constant  $T > 0$  such that

$$\log(w^{u_1(N)+u_2(N)}) \leq \frac{TM}{\log M}.$$

On the other hand, by (1) we have

$$\frac{d^M F_N(\gamma_N)}{dx^M} = \sum_{l=0}^{u_1(N)} \sum_{m=0}^{u_2(N)} b_{l,m,N} \sum_{\sigma=0}^{u_1(N)} \left( \frac{u_1(N)!}{\sigma!(u_1(N)-\sigma)!} \right) \left( \frac{l!}{(u_1(N)-\sigma)!} \right) \cdot m^{u_1(N)-\sigma} \cdot j^{u_1(N)-\sigma} \cdot (\alpha)^{u_1(N)-\sigma} \cdot (e^\alpha)^{ju_2(N)}.$$

Now, by (2), (3), (6), (7), (8), (10) and taking into account that the number of terms is bounded by  $N(\log N)^2$ , we can check that there exists a constant  $B$  such that

$$(12) \quad s(\gamma_N) \leq BN.$$

Let us now give an upper bound of  $|\gamma_N|$ . For convenience, we first suppose that  $j_0 = 0$ , hence  $\frac{d^M}{dx^M} F_N(0) \neq 0$ . Set  $h = |\alpha|$ . Then by Theorem B.9.1 of [7] we have  $|\gamma_N| \leq \frac{|F_N|(h)}{h^M}$ . Moreover, we notice that  $F_N$  admits at least  $M[\log(M)]$  zeros in  $d(0, h)$  and therefore by Corollary B.13.30 of [7] we have  $|F_N|(h) \leq \left(\frac{h}{R_1}\right)^{M[\log(M)]}$  because  $|F_N|(r) \leq 1 \forall r < R_1$ . Consequently,  $|\gamma_N| \leq \frac{h^{M(\log(M)-1)}}{(R_1)^{M \log M}}$  and hence

$$\log(|\gamma_N|) \leq M(\log(M) - 1)(\log(h)) - M \log(M)(\log(R_1)).$$

Let  $\lambda = \log(h) - \log(R_1)$ . Then  $\lambda < 0$ . And we have  $\log(|\gamma_N|) \leq \lambda M \log(M) - M \log(h)$ , therefore there exists a constant  $A > 0$  such that

$$(13) \quad \log(|\gamma_N|) \leq -AM \log(M).$$

Let us now stop assuming that  $j_0 = 0$ . Putting  $z = x - j\alpha$  and  $g(z) = f(x)$ , since all points  $j\alpha$  belong to  $d(0, h)$ , it is immediate to go back to the case  $j_0 = 0$ , which confirms (13) in the general case. But now, by Lemma A.8.10 in [7], relations (12) and (13) make a contradiction to the relation  $-2qs(\gamma_N) \leq \log(|\gamma_N|)$  satisfied by algebraic numbers and show that  $\gamma_N$  is transcendental. But then, so is  $e^\alpha$ .

**Example:** Let  $Q(x) \in \mathbb{Z}[x]$ . Then  $e^{pQ(p)}$  is transcendental. Moreover, if  $Q$  is monic, and if  $\alpha$  is a zero of  $Q$ , then  $|p\alpha| \leq \frac{1}{p}$  because  $Q$  is monic and obviously  $p\alpha$  is algebraic, hence  $e^{p\alpha}$  is transcendental.

In the field of characteristic 0,  $\mathcal{K}$  such as Levi-Civita's field [15], we have a similar version:

**Theorem 2.2:** *Let  $\alpha \in \mathcal{K}$  be algebraic, such that  $|\alpha| < 1$ . Then  $e^\alpha$  is transcendental over  $\mathbb{Q}$ .*

**Proof.** Everything works in  $\mathcal{K}$  as in a field of residue characteristic  $p \neq 0$  up to Relation (8) in the proof of Theorem 2.1. Here we can replace  $R_1$  by 1 and therefore the conclusion is the same as in Theorem 2.1.

Similarly as Hermite-Lindemann's Theorem, Gelfond-Schneider's Theorem is well known in the field  $\mathbb{C}$  and has an analogue in an ultrametric field.

In the proof of Theorem 2.4 we will need the following theorem:

**Theorem 2.3:** *Let  $b_1, \dots, b_n \in D_1$  (resp. in  $D_0$ ). the functions  $x, e^{b_1 x}, \dots, e^{b_n x}$  are algebraically independent over  $\mathbb{K}$  (resp. over  $\mathcal{K}$ ) if and only if  $b_1, \dots, b_n$  are  $\mathbb{Q}$ -linearly independent.*

**Theorem 2.4 (Gel'fond-Schneider):**  *$\mathbb{K}$  is supposed to have residue characteristic  $p \neq 0$ . Let  $\ell \in D_1$ ,  $\ell \neq 0$ , and let  $b \notin \mathbb{Q}$  belong to  $\mathbb{K}$  be such that  $b\ell \in D_1$ . Then at least one of the three numbers  $a = e^\ell$ ,  $b$ ,  $e^{b\ell}$  is transcendental.*

**Proof.** A large part of the proof does not involve the topology of the field  $\mathbb{K}$  and hence is similar to the proof in the field  $\mathbb{C}$  [16] where we can copy many technical relations. We suppose that  $a = e^\ell$ ,  $b$  and  $e^{b\ell}$  are algebraic over  $\mathbb{Q}$ . Let  $L = \mathbb{Q}[e^\ell, b, e^{b\ell}]$  and let  $\delta = [L : \mathbb{Q}]$  and let  $d$  be a common denominator of  $b, e^\ell, e^{b\ell}$ .

Put  $S = \max(1, |b|)$ ,  $T \in ]S, \frac{R_1}{|\ell}[$ ,  $\sigma = \log(\frac{T}{S})$ ,  $\tau = \log T$ ,  $\Lambda = d(0, S)$  and  $\Delta = d(0, T)$ . We will consider integers  $N$  of the form  $q^2$ , with  $q \in \mathbb{N}$  and we will first show that there exists a non-identically zero polynomial  $P_N(X, Y) \in \mathbb{Z}[X, Y]$  such that  $\deg_X(P_N) \leq N^{\frac{3}{2}}$ , and  $\deg_Y(P_N) \leq 2\delta N^{\frac{1}{2}}$  such that the function  $F_N(x)$  defined in  $\Delta$  by  $F_N(x) = P_N(x, e^{\ell x})$  satisfy

$$F_N(i + jb) = 0 \quad \forall i = 1, \dots, N, \quad \forall j = 1, \dots, N.$$

In order to find  $P_N$ , let us write it

$$\sum_{h=0}^{N^{\frac{3}{2}}-1} \sum_{k=0}^{2\delta N^{\frac{1}{2}}-1} C_{h,k}(N) X^h Y^k$$

with  $C_{h,k}(N) \in \mathbb{Z}$  and consider the system of equations where the  $C_{h,k}(N)$  are the undeterminates:

$$d^{(4\delta+1)N^{\frac{3}{2}}} \cdot F_N(i + jb) = 0 \quad (1 \leq i \leq N; 1 \leq j \leq N).$$

Thus, we obtain a system of  $N^2$  equations of  $2\delta N^2$  undeterminates in  $\mathbb{Z}$ , with coefficients in  $L$ . By Lemma 2.b, these coefficients have size bounded by

$$N^{\frac{3}{2}} \log(N) + N^{\frac{3}{2}} (8\delta + 2) \log(d) + \log(1 + |\bar{b}|) + 2\delta \log(|e^{\ell} + b\ell|) \leq \frac{3}{2} N^{\frac{3}{2}} \log(N).$$

By Lemma 2.a we can find in  $\mathbb{Z}$  a family of integers not all equal to zero,  $(C_{h,k}(N), 0 \leq N^{\frac{3}{2}} - 1, 0 \leq k \leq 2\delta N^{\frac{1}{2}} - 1)$  satisfying

$$\log \left( \max_{h,k} |C_{h,k}(N)|_{\infty} \right) \leq 2N^{\frac{3}{2}} \log N \left( \frac{\delta N^2}{2\delta N^2 - \delta N^2} \right) = 2N^{\frac{3}{2}} \log N$$

such that the function  $F_N$  defined by  $F_N(x) = P_N(x, e^{\ell x})$  satisfies  $F_N(i + jb) = 0 \quad \forall i = 1, \dots, N, \quad j = 1, \dots, N$ .

Now we can check the function  $F_N$  is an analytic element in every disk of the form  $d(0, r)$  such that  $r|\ell| < R_1$  and hence in  $\Delta = d(0, T)$  [7]. Since the power of  $x$  in the various terms is at most  $N^{\frac{3}{2}}$  and since all coefficients are integers, we can check that  $\log(|F_N|(T)) \leq \tau N^{\frac{3}{2}}$ . On the other hand, since the polynomial  $P_N$  is not identically zero, by Proposition 1.6  $F_N$  is not identically zero and then, by classical results [7], the function  $F_N$  has finitely many zeros in  $\Lambda$ . Particularly, there exists a point of the form  $i + jb$  such that  $F_N(i + jb) \neq 0$ . Consequently there exists  $M \geq N$  such that  $F_N(i + jb) = 0 \quad \forall i \leq M, \quad \forall j \leq M$  and there exists a point  $\gamma_N$  of the form  $i_0 + j_0 b$  such that  $F_N(\gamma_N) \neq 0$  with  $M < i_0 \leq M + 1, \quad M < j_0 \leq M + 1$ . Consequently the number of zeros of  $F_N$



in  $\Lambda$  is at least  $M^2$ . Then by Corollary B.13.30 in [7] we have  $\log(|F_N(\gamma_N)|) \leq \tau N^{\frac{3}{2}} - \sigma M^2$ , hence there exists  $\lambda > 0$  such that

$$(1) \quad \log(|F_N(\gamma_N)|) \leq -\lambda M^2 \quad \forall N \in \mathbb{N}.$$

By definition neither  $\sigma$  nor  $\tau$  depend on  $N$ , hence neither does  $\lambda$ .

On the other hand, by Lemma 2.b we can check that  $s(F_N(\gamma_N))$  satisfies an inequality of the form  $s(F_N(\gamma_N)) \leq AM^{\frac{3}{2}} \log(M)$  which by (1) contradicts the inequality  $-2\delta s(F_N(\gamma_N)) \leq \log(|F_N(\gamma_N)|)$  and this ends the proof.

**Example:** Let  $\ell = pe^p$  and let  $b \notin \mathbb{Q}$  be such that  $|b| \leq 1$ . Then at least one of the 3 numbers  $\ell$ ,  $b$ ,  $e^{b\ell}$  is transcendental.

**Theorem 2.5 (Gel'fond-Schneider in zero residue characteristic):** *Let  $\mathcal{K}$  be an algebraically closed complete ultrametric field whose residue characteristic is 0. Let  $\ell \in D_0$ ,  $\ell \neq 0$ , and let  $b \notin \mathbb{Q}$  belong to  $\mathcal{K}$  and be such that  $b\ell \in D_0$ . Then at least one of the three numbers  $a = e^\ell$ ,  $b$ ,  $e^{b\ell}$  is transcendental.*

**Proof.** The proof is identical to the proof of Theorem 2.4 except that  $T$  now belongs to  $]S, \frac{1}{|\ell}[$ .

### 3 Nevanlinna Theory in $\mathbb{K}$ and in an open disk

**Notations:** We denote by  $\mathcal{M}(\mathbb{K})$  the field of meromorphic functions in  $\mathbb{K}$  i.e. the field of fractions of  $\mathcal{A}(\mathbb{K})$ . Let  $d(a, R^-)$  be a disk in  $\mathbb{K}$ . We denote by  $\mathcal{M}(d(a, R^-))$  the field of fractions  $\mathcal{A}(d(a, R^-))$  and by  $\mathcal{M}_b(d(a, R^-))$  the field of fractions  $\mathcal{A}_b(d(a, R^-))$ . Finally we put  $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$ .

Given two meromorphic functions  $f, g \in \mathcal{M}(\mathbb{K})$  or  $f, g \in \mathcal{M}(d(a, R^-))$  ( $a \in \mathbb{K}$ ,  $R > 0$ ), we will denote by  $W(f, g)$  the Wronskian of  $f$  and  $g$ :  $f'g - fg'$ .

Let  $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$  (resp.  $f \in \mathcal{M}_u(d(\alpha, R^-))$ ). A value  $b \in \mathbb{K}$  will be called a *quasi-exceptional value for  $f$*  if  $f - b$  has finitely many zeros in  $\mathbb{K}$  (resp. in  $(\alpha, R^-)$ ) and it will be called an *exceptional value for  $f$*  if  $f - b$  has no zero in  $\mathbb{K}$  (resp. in  $d(\alpha, R^-)$ ).

We have the following result:

**Theorem 3.1:** *Let  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}_u(d(a, R^-))$ ). Then  $f$  admits at most one quasi-exceptional value. Moreover, if  $f \in \mathcal{A}(\mathbb{K})$  (resp.  $f \in \mathcal{A}_u(d(a, R^-))$ ) then  $f$  admits no quasi-exceptional value*

The Nevanlinna Theory was made by Rolf Nevanlinna on complex functions [14], and widely used by many specialists of complex functions, particularly Walter Hayman [10]. It consists of defining counting functions of zeros and poles of a meromorphic function  $f$  and giving an upper bound for multiple zeros and poles of various functions  $f - b$ ,  $b \in \mathbb{C}$ .

A similar theory for functions in a p-adic field was constructed and correctly proved by A. Boutabaa [5] in the field  $\mathbb{K}$ , after some previous work by Ha Huy Khoai [9]. See also [11]. In [6] the theory was extended to functions in  $\mathcal{M}(d(0, R^-))$  by taking into account Lazard's problem [12]. A new extension to functions out of a hole was made in [7] but we won't describe it because we would miss place. Here we will only give an abstract of the ultrametric Nevanlinna Theory in order to give the new theorems on  $q$  small functions.

**Notations:** Recall that given three functions  $\phi$ ,  $\psi$ ,  $\zeta$  defined in an interval  $J = ]a, +\infty[$  (resp.  $J = ]a, R[$ ), with values in  $[0, +\infty[$ , we shall write  $\phi(r) \leq \psi(r) + O(\zeta(r))$  if there exists a constant  $b \in \mathbb{R}$  such that  $\phi(r) \leq \psi(r) + b\zeta(r)$ . We shall write  $\phi(r) = \psi(r) + O(\zeta(r))$  if  $|\psi(r) - \phi(r)|$  is bounded by a function of the form  $b\zeta(r)$ .

Similarly, we shall write  $\phi(r) \leq \psi(r) + o(\zeta(r))$  if there exists a function  $h$  from  $J = ]a, +\infty[$  (resp. from  $J = ]a, R[$ ) to  $\mathbb{R}$  such that  $\lim_{r \rightarrow +\infty} \frac{h(r)}{\zeta(r)} = 0$  (resp.  $\lim_{r \rightarrow R} \frac{h(r)}{\zeta(r)} = 0$ ) and such that  $\phi(r) \leq \psi(r) + h(r)$ . And we shall write  $\phi(r) = \psi(r) + o(\zeta(r))$  if there exists a function  $h$  from  $J = ]a, +\infty[$  (resp. from  $J = ]a, R[$ ) to  $\mathbb{R}$  such that  $\lim_{r \rightarrow +\infty} \frac{h(r)}{\zeta(r)} = 0$  (resp.  $\lim_{r \rightarrow R} \frac{h(r)}{\zeta(r)} = 0$ ) and such that  $\phi(r) = \psi(r) + h(r)$ .

Throughout the next paragraphs, we will denote by  $I$  the interval  $[t, +\infty[$  and by  $J$  an interval of the form  $[t, R[$  with  $t > 0$ .

We have to introduce the counting function of zeros and poles of  $f$ , counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

**Definitions:** Next, let  $f = \frac{h}{l} \in \mathcal{M}(\mathbb{K})$  (resp.  $f = \frac{h}{l} \in \mathcal{M}(d(a, R^-))$ ). The order of a zero  $\alpha$  of  $f$  will be denoted by  $\omega_\alpha(f)$ . Next, given any point  $\alpha \in \mathbb{K}$  (resp.  $\alpha \in d(a, R^-)$ ), the number  $\omega_\alpha(h) - \omega_\alpha(l)$  does not depend on the functions  $h$ ,  $l$  chosen to make  $f = \frac{h}{l}$ . Thus, we can generalize the notation by setting  $\omega_\alpha(f) = \omega_\alpha(h) - \omega_\alpha(l)$ . We then denote by  $Z(r, f)$  the counting function of zeros of  $f$  in  $d(0, r)$  in the following way.

Let  $(a_n)$ ,  $1 \leq n \leq \sigma(r)$  be the finite sequence of zeros of  $f$  such that  $0 < |a_n| \leq r$ , of respective order  $s_n$ .

We set  $Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|)$  and so,  $Z(r, f)$  is called *the counting function of zeros of  $f$  in  $d(0, r)$ , counting multiplicity*.

In order to define the counting function of zeros of  $f$  without multiplicity, we put  $\overline{\omega}_0(f) = 0$  if  $\omega_0(f) \leq 0$  and  $\overline{\omega}_0(f) = 1$  if  $\omega_0(f) \geq 1$ .

Now, we denote by  $\overline{Z}(r, f)$  the counting function of zeros of  $f$  without multiplicity:

$\overline{Z}(r, f) = \overline{\omega}_0(f) \log r + \sum_{n=1}^{\sigma(r)} (\log r - \log |a_n|)$  and so,  $\overline{Z}(r, f)$  is called *the counting function of zeros of  $f$  in  $d(0, r)$  ignoring multiplicity*.

In the same way, considering the finite sequence  $(b_n)$ ,  $1 \leq n \leq \tau(r)$  of poles of  $f$  such that  $0 < |b_n| \leq r$ , with respective multiplicity order  $t_n$ , we put

$N(r, f) = \max(-\omega_0(f), 0) \log r + \sum_{n=1}^{\tau(r)} t_n (\log r - \log |b_n|)$  and then  $N(r, f)$  is called *the counting function of the poles of  $f$ , counting multiplicity*

Next, in order to define the counting function of poles of  $f$  without multiplicity, we put  $\overline{\omega}_0(f) = 0$  if  $\omega_0(f) \geq 0$  and  $\overline{\omega}_0(f) = 1$  if  $\omega_0(f) \leq -1$  and we set

$\overline{N}(r, f) = \overline{\omega}_0(f) \log r + \sum_{n=1}^{\tau(r)} (\log r - \log |b_n|)$  and then  $\overline{N}(r, f)$  is called *the counting function of the poles of  $f$ , ignoring multiplicity*

Now we can define the the Nevanlinna function  $T(r, f)$  in  $I$  or  $J$  as

$T(r, f) = \max(Z(r, f), N(r, f))$  and the function  $T(r, f)$  is called *characteristic function of  $f$  or Nevanlinna function of  $f$* .

Finally, if  $S$  is a subset of  $\mathbb{K}$  we will denote by  $Z_0^S(r, f')$  the counting function of zeros of  $f'$ , excluding those which are zeros of  $f - a$  for any  $a \in S$ .

**Remark:** If we change the origin, the functions  $Z$ ,  $N$ ,  $T$  are not changed, up to an additive constant.

In a  $p$ -adic field such as  $\mathbb{K}$ , the first Main Theorem is almost immediate.

**Theorem 3.2:** *Let  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}(d(0, R^-))$ ) have no zero and no pole at 0. Then  $\log(|f|(r)) = \log(|f(0)|) + Z(r, f) - N(r, f)$ .*

Then we can derive Theorem 3.3 (Theorem C.4.3 in [7])

**Theorem 3.3:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  (resp.  $f, g \in \mathcal{M}(d(0, R^-))$ ). Then  $Z(r, fg) \leq Z(r, f) + Z(r, g)$ ,  $N(r, fg) \leq N(r, f) + N(r, g)$ ,  $T(r, fg) \leq T(r, f) + T(r, g)$ ,  $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$ ,  $T(r, cf) = T(r, f) \forall c \in \mathbb{K}^*$ ,  $T(r, \frac{1}{f}) = T(r, f)$ ,  $T(r, \frac{f}{g}) \leq T(r, f) + T(r, g)$ .*

*Suppose now  $f, g \in \mathcal{A}(\mathbb{K})$  (resp.  $f, g \in \mathcal{A}(d(0, R^-))$ ). Then  $Z(r, fg) = Z(r, f) + Z(r, g)$ ,  $T(r, f) = Z(r, f)$ ,  $T(r, fg) = T(r, f) + T(r, g) + O(1)$  and  $T(r, f + g) \leq \max(T(r, f), T(r, g))$ . Moreover, if  $\lim_{r \rightarrow +\infty} T(r, f) - T(r, g) = +\infty$  then  $T(r, f + g) = T(r, f)$  when  $r$  is big enough.*

**Corollary 3.A:** *Let  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}(d(0, R^-))$ ). Then*

$$Z(r, \frac{f'}{f}) - N(r, \frac{f'}{f}) \leq -\log r + O(1).$$

Thus we have Theorem 3.4 (Theorem C.4.8 in [7])

**Theorem 3.4 (First Main Fundamental Theorem):** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  (resp. let  $f, g \in \mathcal{M}(d(0, R^-))$ ). Then  $T(r, f + b) = T(r, f) + O(1)$ . Let  $h$  be a Moebius function. Then  $T(r, f) = T(r, h \circ f) + O(1)$ . Let  $P(X) \in \mathbb{K}[X]$ . Then  $T(r, P(f)) = \deg(P)T(r, f) + O(1)$  and  $T(r, f'P(f)) \geq T(r, P(f))$ .*

*Suppose now  $f, g \in \mathcal{A}(\mathbb{K})$  (resp.  $f, g \in \mathcal{A}(d(0, R^-))$ ). Then  $Z(r, fg) = Z(r, f) + Z(r, g)$ ,  $T(r, f) = Z(r, f)$ ,  $T(r, fg) = T(r, f) + T(r, g) + O(1)$  and  $T(r, f + g) \leq \max(T(r, f), T(r, g))$ . Moreover, if  $\lim_{r \rightarrow +\infty} T(r, f) - T(r, g) = +\infty$  then  $T(r, f + g) = T(r, f)$  when  $r$  is big enough.*

The following Theorem 3.5 is a good way to obtain the famous Second Main Theorem (Theorem C.4.24 in [7]).

**Theorem 3.5:** *Let  $f \in \mathcal{M}(\mathbb{K})$  and let  $a_1, \dots, a_q \in \mathbb{K}$  be distinct. Then*

$$(q-1)T(r, f) \leq \max_{1 \leq k \leq q} \left( \sum_{j=1, j \neq k}^q Z(r, f - a_j) \right) + O(1).$$

**Theorem 3.6 (Second Main Theorem, Theorem C.4.24 in [7]):** *Let  $\alpha_1, \dots, \alpha_q \in \mathbb{K}$ , with  $q \geq 2$ , let  $S = \{\alpha_1, \dots, \alpha_q\}$  and let  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}_u(d(0, R^-))$ ). Then*

$$(q-1)T(r, f) \leq \sum_{j=1}^q \overline{Z}(r, f - \alpha_j) + \overline{N}(r, f) - Z_0^S(r, f') - \log r + O(1) \quad \forall r \in I$$

(resp.  $\forall r \in J$ ).

Now we can easily deduce the following corollaries:

**Corollary 3.B:** *Let  $a_1, a_2 \in \mathbb{K}$  ( $a_1 \neq a_2$ ) and let  $f, g \in \mathcal{A}(\mathbb{K})$  satisfy  $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$  ( $i = 1, 2$ ). Then  $f = g$ .*

**Remark:** Corollary 3.B does not hold in complex analysis. Indeed, let  $f(z) = e^z$ ,  $g(z) = e^{-z}$ , let  $a_1 = 1$ ,  $a_2 = -1$ . Then  $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$  ( $i = 1, 2$ ), though  $f \neq g$ .

**Corollary 3.C:** *Let  $a_1, a_2, a_3 \in \mathbb{K}$  ( $a_i \neq a_j \forall i \neq j$ ) and let  $f, g \in \mathcal{A}_u(d(a, R^-))$  (resp.  $f, g \in \mathcal{A}_u(D)$ ) satisfy  $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$  ( $i = 1, 2, 3$ ). Then  $f = g$ .*

**Corollary 3.D:** *Let  $a_1, a_2, a_3, a_4 \in \mathbb{K}$  ( $a_i \neq a_j \forall i \neq j$ ) and let  $f, g \in \mathcal{M}(\mathbb{K})$  satisfy  $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$  ( $i = 1, 2, 3, 4$ ). Then  $f = g$ .*

**Corollary 3.E:** *Let  $a_1, a_2, a_3, a_4, a_5 \in \mathbb{K}$  ( $a_i \neq a_j \forall i \neq j$ ) and let  $f, g \in \mathcal{M}_u(d(a, R^-))$  (resp.  $f, g \in \mathcal{M}_u(D)$ ) satisfy  $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$  ( $i = 1, 2, 3, 4, 5$ ). Then  $f = g$ .*

**Remark:** Let  $f(x) = \frac{x}{3x-1}$ ,  $g(x) = \frac{x^2}{x^2+2x-1}$ . Let  $a_0 = 0$ ,  $a_1 = 1$ ,  $a_2 = \frac{1}{2}$ . Then we can check that  $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ,  $i = 1, 2, 3$ . So, Corollary 3.D is sharp.

#### 4 Exceptional values of meromorphic functions and derivatives

The paragraph is aimed at studying various properties of derivatives of meromorphic functions, particularly their sets of zeros [2], [3], [4]. Many important results are due to Jean-Paul Bézivin [1], [2].

We will first notice a general property concerning quasi-exceptional values of meromorphic functions and derivatives.

**Theorem 4.1:** *Let  $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$  (resp. Let  $f \in \mathcal{M}_u(d(\alpha, R^-))$ ). If  $f$  admits a quasi-exceptional value, then  $f'$  has no quasi-exceptional value different from 0. **Proof.** Without loss of generality, we may assume  $\alpha = 0$  and that  $f$  has no zero and no pole at 0. Let  $b \in \mathbb{K}$  and suppose that  $b$  is a quasi-exceptional value of  $f$ . There exist  $P \in \mathbb{K}[x]$  and  $l \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$  (resp. and  $l \in \mathcal{A}_u(d(0, R^-))$ ) without common zeros, such that  $f = b + \frac{P}{l}$ .*

Let  $c \in \mathbb{K}^*$ . Remark that  $f' - c = \frac{P'l - Pl' - cl^2}{l^2}$ . Let  $a \in \mathbb{K}$  (resp. let  $a \in d(0, R^-)$ ). If  $a$  is a pole of  $f$ , it is a pole of  $f' - c$  and we can check that

$$(1) \quad \omega_a(P'l - Pl' - cl^2) = \omega_a(l') = \omega_a(l) - 1$$

because  $a$  is not a zero of  $P$ .

Now suppose that  $a$  is not a pole of  $f$ . Then

$$(2) \quad \omega_a(f' - c) = \omega_a(P'l - Pl' - cl^2)$$

Consequently,  $Z(r, f' - c) = Z(r, (P'l - Pl' - cl^2) \mid l(x) \neq 0)$ . But, by (1) we have

$$(3) \quad Z(r, (P'l - Pl' - cl^2) \mid l(x) = 0) < Z(r, l).$$

and therefore by (2) and (3) we obtain

$$(4) \quad Z(r, f' - c) = Z(r, (P'l - Pl' - cl^2) \mid l(x) \neq 0) > Z(r, P'l - Pl' - cl^2) - Z(r, l)$$

Now, let us examine  $Z(r, P'l - Pl' - cl^2)$ . Let  $r \in ]0, +\infty[$  (resp. let  $r \in ]0, R[$ ). Since  $l \in \mathcal{A}(\mathbb{K})$  is transcendental (resp. since  $l \in \mathcal{A}_u(d(0, R^-))$ ), we can check that when  $r$  is big enough, we have  $|P'l|(r) < |c|(|l|(r))^2$  and  $|Pl|(r) < |c|(|l|(r))^2$ , hence clearly  $|P'l - Pl'|(r) < |c|(|l|(r))^2$  and hence  $|P'l - Pl' - cl^2|(r) = |c|(|l|(r))^2$ . Consequently, when  $r$  is big enough, by Theorem C.4.2 in [7] we have  $Z(r, P'l - Pl' - cl^2) = Z(r, l^2) + O(1)$ . But  $Z(r, l^2) = 2Z(r, l)$ , hence  $Z(r, P'l - Pl' - cl^2) = 2Z(r, l) + O(1)$  and therefore by (4) we check that when  $r$  is big enough, we obtain

$$(5) \quad Z(r, f' - c) > Z(r, l).$$

Now, if  $l \in \mathcal{A}(\mathbb{K})$ , since  $l$  is transcendental, by (5), for every  $q \in \mathbb{N}$ , we have  $Z(r, f' - c) > Z(r, l) > q \log r$ , when  $r$  is big enough, hence  $f' - c$  has infinitely many zeros in  $\mathbb{K}$ . And similarly if  $l \in \mathcal{A}_u(d(0, R^-))$ , then by (5),  $Z(r, f' - c)$  is unbounded when  $r$  tends to  $R$ , hence  $f' - c$  has infinitely many zeros in  $d(0, R^-)$ .

We will now notice a property of differential equations of the form  $y^{(n)} - \psi y = 0$  that is almost classical.

The problem of a constant Wronskian is involved in several questions.

**Theorem 4.2:** *Let  $h, l \in \mathcal{A}(\mathbb{K})$  (resp.  $h, l \in \mathcal{A}(d(\alpha, R^-))$ ) and satisfy  $h'l - hl' = c \in \mathbb{K}$ , with  $h$  non-affine. If  $h, l$  belong to  $\mathcal{A}(\mathbb{K})$ , then  $c = 0$  and  $\frac{h}{l}$  is a constant. If  $c \neq 0$  and if  $h, l \in \mathcal{A}(d(\alpha, R^-))$ , there exists  $\phi \in \mathcal{A}(d(\alpha, R^-))$  such that  $h'' = \phi h$ ,  $l'' = \phi l$ . **Proof.** Suppose  $c \neq 0$ . If  $h(a) = 0$ , then  $l(a) \neq 0$ . Next,  $h$  and  $l$  satisfy*

$$(1) \quad \frac{h''}{h} = \frac{l''}{l}.$$

Remark first that since  $h$  is not affine,  $h''$  is not identically zero. Next, every zero of  $h$  or  $l$  of order  $\geq 2$  is a trivial zero of  $h'l - hl'$ , which contradicts  $c \neq 0$ . So we can assume that all zeros of  $h$  and  $l$  are of order 1.

Now suppose that a zero  $a$  of  $h$  is not a zero of  $h''$ . Since  $a$  is a zero of  $h$  of order 1,  $\frac{h''}{h}$  has a pole of order 1 at  $a$  and so does  $\frac{l''}{l}$ , hence  $l(a) = 0$ , a contradiction. Consequently, each zero of  $h$  is a zero of order 1 of  $h$  and is a zero of  $h''$  and hence,  $\frac{h''}{h}$  is an element  $\phi$  of  $\mathcal{M}(\mathbb{K})$  (resp. of  $\mathcal{M}(d(\alpha, R^-))$ ) that has no pole in  $\mathbb{K}$  (resp. in  $d(\alpha, R^-)$ ). Therefore  $\phi$  lies in  $\mathcal{A}(\mathbb{K})$  (resp. in  $\mathcal{A}(d(\alpha, R^-))$ ).

The same holds for  $l$  and so,  $l''$  is of the form  $\psi l$  with  $\psi \in \mathcal{A}(\mathbb{K})$  (resp. in  $\mathcal{A}(d(\alpha, R^-))$ ). But since  $\frac{h''}{h} = \frac{l''}{l}$ , we have  $\phi = \psi$ .

Now, suppose  $h, l$  belong to  $\mathcal{A}(\mathbb{K})$ . Since  $h''$  is of the form  $\phi h$  with  $\phi \in \mathcal{A}(\mathbb{K})$ , we have  $|h''|(r) = |\phi|(r)|h|(r)$ . But by Theorem C.2.10 in [7], we know that  $|h''|(r) \leq \frac{1}{r^2}|h|(r)$ , a contradiction when  $r$  tends to  $+\infty$ . Consequently,  $c = 0$ . But then  $h'l - hl' = 0$  implies that the derivative of  $\frac{h}{l}$  is identically zero, hence  $\frac{h}{l}$  is constant.

**Corollary 4.A :** *Let  $h, l \in \mathcal{A}(\mathbb{K})$  with coefficients in  $\mathbb{Q}$ , also be entire functions in  $\mathbb{C}$ , with  $h$  non-affine. If  $h'l - hl'$  is a constant  $c$ , then  $c = 0$ .*

**Theorem 4.3:** *Let  $\psi \in \mathcal{M}(\mathbb{K})$  (resp. let  $\psi \in \mathcal{M}_u(d(\alpha, R^-))$ ) and let  $(\mathcal{E})$  be the differential equations  $y'' - \psi y = 0$ . Let  $E$  be the sub-vector space of  $\mathcal{A}(\mathbb{K})$  (resp. of  $\mathcal{A}(d(\alpha, R^-))$ ) of the solutions of  $(\mathcal{E})$ . Then, the dimension of  $E$  is 0 or 1. **Proof.** Suppose  $E$  is not  $\{0\}$ . Let  $h, l \in E$  be non-identically zero. Then  $h''l - hl'' = 0$  and therefore  $h'l - hl'$  is a constant  $c$ . On the other hand, since  $h, l$  are not identically zero, neither are  $h'', l''$ . Therefore,  $h, l$  are not affine functions.*

Suppose  $\psi$  belongs to  $\mathcal{M}(\mathbb{K})$  and that  $h, l$  belong to  $\mathcal{A}(\mathbb{K})$ . By Theorem 4.2, we have  $c = 0$  and hence  $\frac{h}{l}$  is a constant, which proves that  $E$  is of

dimension 1.

Suppose now that  $\psi$  lies in  $\mathcal{M}_u(d(\alpha, R^-))$  and that  $h, l$  belong to  $\mathcal{A}(d(\alpha, R^-))$ . If  $\psi$  lies in  $\mathcal{A}(d(\alpha, R^-))$ , then by Theorem 4.1,  $E = \{0\}$ . Finally, suppose that  $\psi$  lies in  $\mathcal{M}_u(d(\alpha, R^-)) \setminus \mathcal{A}(d(\alpha, R^-))$ . If  $c \neq 0$ , by Theorem 4.2, there exists  $\phi \in \mathcal{A}(d(\alpha, R^-))$  such that  $h'' = \phi h$ ,  $l'' = \phi l$ . Consequently,  $\phi = \psi$ , hence  $\psi \in \mathcal{A}(\mathbb{K})$  and therefore  $c = 0$ . Hence  $h'l - hl' = 0$  again and hence  $\frac{h}{l}$  is a constant. Thus, we see that  $E$  is at most of dimension 1.

**Remark:** The hypothesis  $\psi$  unbounded in  $d(\alpha, R^-)$  is indispensable to show that the space  $E$  is of dimension 0 or 1, as shows the example given again by the p-adic hyperbolic functions  $h(x) = \cosh(x)$  and  $l(x) = \sinh(x)$ . The radius of convergence of both  $h, l$  is  $p^{\frac{-1}{p-1}}$  when  $\mathbb{K}$  has residue characteristic  $p$  and is 1 when  $\mathbb{K}$  has residue characteristic 0. Of course, both functions are solutions of  $y'' - y = 0$  but they are bounded.

The following Theorem 4.4 is an improvement of Theorem 4.2. It follows previous results [1].

**Theorem 4.4 [2]:** *Let  $f, g \in \mathcal{A}(\mathbb{K})$  be such that  $W(f, g)$  is a non-identically zero polynomial. Then both  $f, g$  are polynomials. **Proof.** First, by Theorem 4.2 we check that the claim is satisfied when  $W(f, g)$  is a polynomial of degree 0. Now, suppose the claim holds when  $W(f, g)$  is a polynomial of certain degree  $n$ . We will show it for  $n + 1$ . Let  $f, g \in \mathcal{A}(\mathbb{K})$  be such that  $W(f, g)$  is a non-identically zero polynomial  $P$  of degree  $n + 1$*

Thus, by hypothesis, we have  $f'g - fg' = P$ , hence  $f''g - fg'' = P'$ . We can extract  $g'$  and get  $g' = \frac{(f'g - P)}{f}$ . Now consider the function  $Q = f''g' - f'g''$  and replace  $g'$  by what we just found: we can get  $Q = f'(\frac{(f''g - fg'')}{f}) - \frac{Pf''}{f}$ .

Now, we can replace  $f''g - fg''$  by  $P'$  and obtain  $Q = \frac{(f'P' - Pf'')}{f}$ . Thus, in that expression of  $Q$ , we can write  $|Q|(R) \leq \frac{|f|(R)|P'(R)|}{R^2|f|(R)}$ , hence  $|Q|(R) \leq \frac{|P'(R)|}{R^2} \forall R > 0$ . But by definition,  $Q$  belongs to  $\mathcal{A}(\mathbb{K})$ . Consequently,  $Q$  is a polynomial of degree  $t \leq n - 1$ .

Now, suppose  $Q$  is not identically zero. Since  $Q = W(f', g')$  and since  $\deg(Q) < n$ , by the induction hypothesis  $f'$  and  $g'$  are polynomials and so are  $f, g$ . Finally, suppose  $Q = 0$ . Then  $P'f' - Pf'' = 0$  and therefore  $f', P$  are two solutions of the differential equation of order 1 for meromorphic functions in  $\mathbb{K} : (\mathcal{E}) y' = \psi y$  with  $\psi = \frac{P'}{P}$ , whereas  $y$  belongs to  $\mathcal{A}(\mathbb{K})$ . By Theorem 4.3, the space of solutions of  $(\mathcal{E})$  is of dimension 0 or 1. Consequently, there exists  $\lambda \in \mathbb{K}$  such that  $f' = \lambda P$ , hence  $f$  is a polynomial. The same holds for  $g$ .



Here we can find again the following result that is known and may be proved without ultrametric properties:

*Let  $F$  be an algebraically closed field and let  $P, Q \in F[x]$  be such that  $PQ' - P'Q$  is a constant  $c$ , with  $\deg(P) \geq 2$ . Then  $c = 0$ .*

**Notation:** Let  $f \in \mathcal{A}(\mathbb{K})$ . We can factorize  $f$  in the form  $\bar{f}\tilde{f}$  where the zeros of  $\bar{f}$  are the distinct zeros of  $f$  each with order 1. Moreover, if  $f(0) \neq 0$  we will take  $\bar{f}(0) = 1$ .

**Lemma 4.a:** *Let  $U, V \in \mathcal{A}(\mathbb{K})$  have no common zero and let  $f = \frac{U}{V}$ . If  $f'$  has finitely many zeros, there exists a polynomial  $P \in \mathbb{K}[x]$  such that  $U'V - UV' = P\tilde{V}$ .* **Proof.** If  $V$  is a constant, the statement is obvious. So, we assume that  $V$  is not a constant. Now  $\tilde{V}$  divides  $V'$  and hence  $V'$  factorizes in the way  $V' = \tilde{V}Y$  with  $Y \in \mathcal{A}(\mathbb{K})$ . Then no zero of  $Y$  can be a zero of  $V$ . Consequently, we have

$$f'(x) = \frac{U'V - UV'}{V^2} = \frac{U'\bar{V} - UY}{\bar{V}^2\tilde{V}}.$$

The two functions  $U'\bar{V} - UY$  and  $\bar{V}^2\tilde{V}$  have no common zero since neither have  $U$  and  $V$ . So, the zeros of  $f'$  are those of  $U'\bar{V} - UY$  which therefore has finitely many zeros and consequently is a polynomial.

**Theorem 4.5:** *Let  $f \in \mathcal{M}(\mathbb{K})$  have finitely many multiple poles, such that for certain  $b \in \mathbb{K}$ ,  $f' - b$  has finitely many zeros. Then  $f$  belongs to  $\mathbb{K}(x)$ .*

**Proof.** Suppose first  $b = 0$ . Let us write  $f = \frac{U}{V}$  with  $U, V \in \mathcal{A}(\mathbb{K})$ , having no common zeros. By Lemma 4.a, there exists a polynomial  $P \in \mathbb{K}[x]$  such that  $U'V - UV' = P\tilde{V}$ . Since  $f$  has finitely many multiple poles,  $\tilde{V}$  is a polynomial, hence so is  $U'V - UV'$ . But then by Theorem 4.4, both  $U, V$  are polynomials, which ends the proof when  $b = 0$ . Consider now the general case.  $f' - b$  is the derivative of  $f - bx$  that satisfies the same hypothesis, so the conclusion is immediate.

**Notation:** For each  $n \in \mathbb{N}^*$ , we set  $\lambda_n = \max\{\frac{1}{|k|}, 1 \leq k \leq n\}$ . Given positive integers  $n, q$ , we denote by  $C_n^q$  the combination  $\frac{n!}{q!(n-q)!}$ . Let us recall that  $\log$  is the Neperian logarithm, we denote by  $e$  the number such that  $\log(e) = 1$  and  $\text{Exp}$  is the real exponential function.

**Remark:** For every  $n \in \mathbb{N}^*$ , we have  $\lambda_n \leq n$  because  $k|k| \geq 1 \forall k \in \mathbb{N}$ . The equality holds for all  $n$  of the form  $p^h$ .

Lemmas 4.b and 4.c are due to Jean-Paul Bézivin [1]:

**Lemma 4.b:** *Let  $U, V \in \mathcal{A}(d(0, R^-))$ . Then for all  $r \in ]0, R[$  and  $n \geq 1$  we have*

$$|U^{(n)}V - UV^{(n)}|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{r^{n-1}}.$$

*More generally, given  $j, l \in \mathbb{N}$ , we have*

$$|U^{(j)}V^{(l)} - U^{(l)}V^{(j)}|(r) \leq |(j!)(l!)|\lambda_{j+l} \frac{|U'V - UV'|(r)}{r^{j+l-1}}$$

**Lemma 4.c:** *Let  $U, V \in \mathcal{A}(\mathbb{K})$  and let  $r, R \in ]0, +\infty[$  satisfy  $r < R$ . For all  $x, y \in \mathbb{K}$  with  $|x| \leq R$  and  $|y| \leq r$ , we have the inequality:*

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|(R)}{e(\log R - \log r)}$$

**Notation:** Let  $f \in \mathcal{M}(d(0, R^-))$ . For each  $r \in ]0, R[$ , we denote by  $\zeta(r, f)$  the number of zeros of  $f$  in  $d(0, r)$ , taking multiplicity into account and set  $\xi(r, f) = \zeta(r, \frac{1}{f})$ . Similarly, we denote by  $\beta(r, f)$  the number of multiple zeros of  $f$  in  $d(0, r)$ , each counted with its multiplicity and we set  $\gamma(r, f) = \beta(r, \frac{1}{f})$ .

**Theorem 4.6** [2] *Let  $f \in \mathcal{M}(\mathbb{K})$  be such that for some  $c, q \in ]0, +\infty[$ ,  $\gamma(r, f)$  satisfies  $\gamma(r, f) \leq cr^q$  in  $[1, +\infty[$ . If  $f'$  has finitely many zeros, then  $f \in \mathbb{K}(x)$ .*

**Proof.** Suppose  $f'$  has finitely many zeros and set  $f = \frac{U}{V}$ . If  $V$  is a constant, the statement is immediate. So, we suppose  $V$  is not a constant and hence it admits at least one zero  $a$ . By Lemma 4.a, there exists a polynomial  $P \in \mathbb{K}[x]$  such that  $U'V - UV' = P\tilde{V}$ . Next, we take  $r, R \in [1, +\infty[$  such that  $|a| < r < R$  and  $x \in d(0, R)$ ,  $y \in d(0, r)$ . By Lemma 4.c we have

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|(R)}{e(\log R - \log r)}.$$

Notice that  $U(a) \neq 0$  because  $U$  and  $V$  have no common zero. Now set  $l = \max(1, |a|)$  and take  $r \geq l$ . Setting  $c_1 = \frac{1}{e|U(a)|}$ , we have

$$|V(a+y)| \leq c_1 \frac{R|P|(R)|\tilde{V}|(R)}{\log R - \log r}.$$

Then taking the supremum of  $|V(a+y)|$  inside the disk  $d(0, r)$ , we can derive

$$(1) \quad |V|(r) \leq c_1 \frac{R|P|(R)|\tilde{V}|(R)}{\log R - \log r}.$$

Let us apply Corollary B.13.30 in [7], by taking  $R = r + \frac{1}{r^q}$ , after noticing that the number of zeros of  $\tilde{V}(R)$  is bounded by  $\beta(R, V)$ . So, we have

$$(2) \quad |\tilde{V}|(R) \leq \left(1 + \frac{1}{r^{q+1}}\right)^{\beta((r+\frac{1}{r^q}), V)} |\tilde{V}|(r).$$

Now, due to the hypothesis:  $\beta(r, V) = \gamma(r, f) \leq cr^q$  in  $[1, +\infty[$ , we have

$$(3) \quad \left(1 + \frac{1}{r^{q+1}}\right)^{\beta((r+\frac{1}{r^q}), V)} \leq \left(1 + \frac{1}{r^{q+1}}\right)^{[c(r+\frac{1}{r^q})^m]} = \\ \text{Exp}\left[c\left(r + \frac{1}{r^q}\right)^q \log\left(1 + \frac{1}{r^{q+1}}\right)\right].$$

The function  $h(r) = c\left(r + \frac{1}{r^m}\right)^m \log\left(1 + \frac{1}{r^{m+1}}\right)$  is continuous on  $]0, +\infty[$  and equivalent to  $\frac{c}{r}$  when  $r$  tends to  $+\infty$ . Consequently, it is bounded on  $[l, +\infty[$ . Therefore, by (2) and (3) there exists a constant  $M > 0$  such that, for all  $r \in [l, +\infty[$  by (3) we obtain

$$(4) \quad |\tilde{V}|(r + \frac{1}{r^q}) \leq M|\tilde{V}|(r).$$

On the other hand,  $\log\left(r + \frac{1}{r^q}\right) - \log r = \log\left(1 + \frac{1}{r^{q+1}}\right)$  clearly satisfies an inequality of the form  $\log\left(1 + \frac{1}{r^{q+1}}\right) \geq \frac{c_2}{r^{q+1}}$  in  $[l, +\infty[$  with  $c_2 > 0$ . Moreover,

we can find positive constants  $c_3, c_4$  such that  $\left(r + \frac{1}{r^q}\right)|P|\left(r + \frac{1}{r^q}\right) \leq c_3 r^{c_4}$ .

Consequently, by (1) and (4) we can find positive constants  $c_5, c_6$  such that  $|V|(r) \leq c_5 r^{c_6} |\tilde{V}|(r) \forall r \in [l, +\infty[$ . Thus, writing again  $V = \bar{V}\tilde{V}$ , we have  $|\bar{V}|(r)|\tilde{V}|(r) \leq c_5 r^{c_6} |\tilde{V}|(r)$  and hence  $|\bar{V}|(r) \leq c_5 r^{c_6} \forall r \in [l, +\infty[$ . Consequently, by Corollary B.13.31 in [7],  $\bar{V}$  is a polynomial of degree  $\leq c_6$  and hence it has finitely many zeros and so does  $V$ . But then, by Theorem 4.5,  $f$  must be a rational function.

**Corollary 4.B:** *Let  $f$  be a meromorphic function on  $\mathbb{K}$  such that, for some  $c, q \in ]0, +\infty[$ ,  $\gamma(r, f)$  satisfies  $\gamma(r, f) \leq cr^q$  in  $[1, +\infty[$ . If for some  $b \in \mathbb{K}$   $f' - b$  has finitely many zeros, then  $f$  is a rational function. **Proof.** Suppose  $f' - b$  has finitely many zeros. Then  $f - bx$  satisfies the same hypothesis as  $f$ , hence it is a rational function and so is  $f$ .*

**Corollary 4.C:** *Let  $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$  be such that  $\xi(r, f) \leq cr^q$  in  $[1, +\infty[$  for some  $c, q \in ]0, +\infty[$ . Then for each  $k \in \mathbb{N}^*$ ,  $f^{(k)}$  has no quasi-exceptional value.*

**Proof.** Indeed, if  $k = 1$ , the statement just comes from Corollary 4.B Now suppose  $k \geq 2$ . Each pole  $a$  of order  $n$  of  $f$  is a pole of order  $n + k$  of  $f^{(k)}$  and  $f^{(k)}$  has no other pole. Consequently, we have  $\gamma(r, f^{(k-1)}) = \xi(r, f^{(k-1)}) \leq kcr^q$ . So, we can apply Corollary 4.B to  $f^{(k-1)}$  to show the claim.

Theorem 4.6 suggests us the following conjecture:

**Conjecture:** *Let  $f \in \mathcal{M}(\mathbb{K})$  be such that  $f'$  admits finitely many zeros. Then  $f \in \mathbb{K}(x)$ .*

In other words, the conjecture suggests that the derivative of a meromorphic function in  $\mathbb{K}$  has no quasi-exceptional value, except if it is a rational function.

**Remark:** Of course, there exist meromorphic functions in  $\mathbb{K}$  having no zero but not satisfying the hypotheses of Theorem 4.6, hence such a function cannot have primitives. For example, consider an entire function  $f$  having an infinity of zeros  $(a_n)_{n \in \mathbb{N}}$  of order 2 such that  $|a_n| < |a_{n+1}|$ ,  $\lim_{n \rightarrow +\infty} |a_n| = +\infty$  and  $2n \leq |a_n|$ . Then the meromorphic function  $g = \frac{1}{f}$  has no zeros but does not satisfy the hypotheses of Theorem 4.6 hence it has no primitives.

## 5 Small functions

Small functions with respect to a meromorphic function are well known in the general theory of complex functions. Particularly, one knows the Nevanlinna theorem on 3 small functions. Here we will recall the construction of a similar theory.

**Definitions and notation:** Throughout the chapter we set  $a \in \mathbb{K}$  and  $R \in ]0, +\infty[$ . For each  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}(d(a, R^-))$ ) we denote by  $\mathcal{M}_f(\mathbb{K})$ , (resp.  $\mathcal{M}_f(d(a, R^-))$ ) the set of functions  $h \in \mathcal{M}(\mathbb{K})$ , (resp.  $h \in \mathcal{M}(d(a, R^-))$ ) such that  $T(r, h) = o(T(r, f))$  when  $r$  tends to  $+\infty$  (resp. when  $r$  tends to  $R$ ). Similarly, if  $f \in \mathcal{A}(\mathbb{K})$  (resp.  $f \in \mathcal{A}(d(a, R^-))$ ) we shall denote by  $\mathcal{A}_f(\mathbb{K})$  (resp.  $\mathcal{A}_f(d(a, R^-))$ ) the set  $\mathcal{M}_f(\mathbb{K}) \cap \mathcal{A}(\mathbb{K})$ , (resp.  $\mathcal{M}_f(d(a, R^-)) \cap \mathcal{A}(d(a, R^-))$ ).

The elements of  $\mathcal{M}_f(\mathbb{K})$  (resp.  $\mathcal{M}_f(d(a, R^-))$ ) are called *small meromorphic functions with respect to  $f$* , (*small functions* in brief). Similarly, if  $f \in \mathcal{A}(\mathbb{K})$  (resp.  $f \in \mathcal{A}(d(a, R^-))$ ) the elements of  $\mathcal{A}_f(\mathbb{K})$  (resp.  $\mathcal{A}_f(d(a, R^-))$ ) are called *small analytic functions with respect to  $f$* , (*small functions* in brief).

Theorems 5.1 and Theorem 5.2 are immediate consequences of Theorems C.9.1 and C.9.2 in [7]:

**Theorem 5.1:** *Let  $a \in \mathbb{K}$  and  $r > 0$ . Then  $\mathcal{A}_f(\mathbb{K})$  is a  $\mathbb{K}$ -subalgebra of  $\mathcal{A}(\mathbb{K})$ ,  $\mathcal{A}_f(d(a, R^-))$  is a  $\mathbb{K}$ -subalgebra of  $\mathcal{A}(d(a, R^-))$   $\mathcal{M}_f(\mathbb{K})$  is a subfield of  $\mathcal{M}(\mathbb{K})$ ,  $\mathcal{M}_f(d(a, R^-))$  is a subfield of field of  $\mathcal{M}(a, R^-)$ . Moreover,  $\mathcal{A}_b(d(a, R^-))$  is a sub-algebra of  $\mathcal{A}_f(d(a, R^-))$  and  $\mathcal{M}_b(d(a, R^-))$  is a subfield of  $\mathcal{M}_f(d(a, R^-))$ .*

**Theorem 5.2 :** *Let  $f \in \mathcal{M}(\mathbb{K})$ , (resp.  $f \in \mathcal{M}(d(0, R^-))$ ) and let  $g \in \mathcal{M}_f(\mathbb{K})$ , (resp.  $g \in \mathcal{M}_f(d(0, R^-))$ ). Then  $T(r, fg) = T(r, f) + o(T(r, f))$  and  $T(r, \frac{f}{g}) = T(r, f) + o(T(r, f))$ , (resp.  $T(r, fg) = T(r, f) + o(T(r, f))$  and  $T(r, \frac{f}{g}) = T(r, f) + o(T(r, f))$ ).*

Theorem 5.3 is known as Second Main Theorem on Three Small Functions in  $p$ -adic analysis [7] and [10]. It holds as well as in complex analysis, where it was showed first and it is proven in the same way.

**Theorem 5.3:** *Let  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}_u(d(0, R^-))$ ) and let  $w_1, w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$  (resp.  $w_1, w_2, w_3 \in \mathcal{M}_f(d(0, R^-))$ ) be pairwise distinct. Then  $T(r, f) \leq \sum_{j=1}^3 \bar{Z}(r, f - w_j) + o(T(r, f))$ , resp  $T(r, f) \leq \sum_{j=1}^3 \bar{Z}(r, f - w_j) + o(T(r, f))$ , resp.  $T_R(r, f) \leq \sum_{j=1}^3 \bar{Z}_R(r, f - w_j) + o(T(r, f))$ .*

**Theorem 5.4:** *Let  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}_u(d(0, R^-))$ ) and let  $w_1, w_2 \in \mathcal{M}_f(\mathbb{K})$  (resp.  $w_1, w_2 \in \mathcal{M}_f(d(0, R^-))$ ) be distinct. Then  $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{N}(r, f) + o(T(r, f))$ , (resp.  $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{N}(r, f) + o(T(r, f))$ ).*

**Proof.** Suppose first  $f \in \mathcal{M}(\mathbb{K})$  or  $f \in \mathcal{M}_u(d(0, R^-))$ . Let  $g = \frac{1}{f}$ ,  $h_j = \frac{1}{w_j}$ ,  $j = 1, 2$ ,  $h_3 = 0$ . Clearly,

$$T(r, g) = T(r, f) + O(1), \quad T(r, h) = T(r, w_j), \quad j = 1, 2,$$

so we can apply Theorem 5.3 to  $g, h_1, h_2, h_3$ . Thus we have:  $T(r, g) \leq \bar{Z}(r, g - h_1) + \bar{Z}(r, g - h_2) + \bar{Z}(r, g) + o(T(r, g))$ .

But we notice that  $\bar{Z}(r, g - h_j) = \bar{Z}(r, f - w_j)$  for  $j = 1, 2$  and  $\bar{Z}(r, g) = \bar{N}(r, f)$ . Moreover, we know that  $o(T(r, g)) = o(T(r, f))$ . Consequently, the claim is proved when  $w_1 w_2$  is not identically zero.

Now, suppose that  $w_1 = 0$ . Let  $\lambda \in \mathbb{K}^*$ , let  $l = f + \lambda$  and  $\tau_j = u_j + \lambda$ , ( $j = 1, 2, 3$ ). Thus, we have  $T(r, l) = T(r, f) + O(1)$ ,  $T(r, \tau_j) = T(r, w_j) +$

$O(1)$ , ( $j = 1, 2$ ),  $\bar{N}(r, l) = \bar{N}(r, f)$ . By the claim already proven whenever  $w_1 w_2 \neq 0$  we may write  $T(r, l) \leq \bar{Z}(r, l - \tau_1) + \bar{Z}(r, l - \tau_2) + \bar{N}(r, l) + o(T(r, l))$  hence

$$T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{N}(r, l) + o(T(r, f)).$$

Next, by setting  $g = f - w_1$  and  $w = w_1 + w_2$ , we can write Corollary 5.A:

**Corollary 5.A:** *Let  $g \in \mathcal{M}(\mathbb{K})$  (resp.  $g \in \mathcal{M}_u(d(0, R^-))$ ) and let  $w \in \mathcal{M}_g(\mathbb{K})$ . Then  $T(r, g) \leq \bar{Z}(r, g) + \bar{Z}(r, g - w) + \bar{N}(r, g) + o(T(r, g))$ , (resp.  $T(r, g) \leq \bar{Z}(r, g) + \bar{Z}(r, g - w) + \bar{N}(r, g) + o(T(r, g))$ ).*

**Corollary 5.B:** *Let  $f \in \mathcal{A}(\mathbb{K})$  (resp.  $f \in \mathcal{A}_u(d(0, R^-))$ ) and let  $w_1, w_2 \in \mathcal{A}_f(\mathbb{K})$  (resp.  $w_1, w_2 \in \mathcal{A}_f(d(0, R^-))$ ) be distinct. Then  $T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + o(T(r, f))$  ( $r \rightarrow +\infty$ ), resp. ( $r \rightarrow R^-$ ).*

And similarly to Corollary 5.A, we can get Corollary 5.C:

**Corollary 5.C:** *Let  $f \in \mathcal{A}(\mathbb{K})$  (resp.  $f \in \mathcal{A}_u(d(0, R^-))$ ), resp.  $f \in \mathcal{A}^c(D)$  ) and let  $w \in \mathcal{A}_f(\mathbb{K})$ . Then  $T(r, f) \leq \bar{Z}(r, f) + \bar{Z}(r, f - w) + o(T(r, f))$ , (resp.  $T(r, f) \leq \bar{Z}(r, f) + \bar{Z}(r, f - w) + o(T(r, f))$ ).*

We are now able to state a theorem on  $q$  small functions that is not as good as Yamanoi's Theorem [17] in complex analysis, but seems the best possible in ultrametric analysis;

**Theorem 5.5 [8] (A. Escassut, C.C. Yang):** *Let  $f \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f \in \mathcal{M}_u(d(0, R^-))$ ) and let  $w_j \in \mathcal{M}_f(\mathbb{K})$  ( $j = 1, \dots, q$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-))$ ) be  $q$  distinct small functions other than the constant  $\infty$ . Then*

$$qT(r, f) \leq 3 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)),$$

(resp.

$$qT(r, f) \leq 3 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f))),$$

Moreover, if  $f$  has finitely many poles in  $\mathbb{K}$  (resp. in  $d(0, R^-)$ ), then

$$qT(r, f) \leq 2 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)),$$

(resp.

$$qT(r, f) \leq 2 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)),$$

**Proof.** By Theorem 5.3, for every triplet  $(i, j, k)$  such that  $1 \leq i \leq j \leq k \leq q$ , we can write

$$T(r, f) \leq \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + \bar{Z}(r, f - w_k) + o(T(r, f)).$$

The number of such inequalities is  $C_q^3$ . Summing up, we obtain

$$(1) \quad C_q^3 T(r, f) \leq \sum_{(i,j,k), 1 \leq i \leq j \leq k \leq q} \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + \bar{Z}(r, f - w_k) + o(T(r, f)).$$

In this sum, for each index  $i$ , the number of terms  $\bar{Z}(r, f - w_i)$  is clearly  $C_{q-1}^2$ . Consequently, by (1) we obtain

$$C_q^3 T(r, f) \leq C_{q-1}^2 \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(T(r, f))$$

and hence

$$\frac{q}{3} T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(T(r, f)).$$

Suppose now that  $f$  has finitely many poles. By Theorem 5.4, for every pair  $(i, j)$  such that  $1 \leq i \leq j \leq q$ , we have

$$T(r, f) \leq \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + o(T(r, f)).$$

The number of such inequalities is then  $C_q^2$ . Summing up we now obtain

$$(2) \quad C_q^2 T(r, f) \leq \sum_{(i,j), 1 \leq i \leq j \leq q} \bar{Z}(r, f - w_i) + \bar{Z}(r, f - w_j) + o(T(r, f)).$$

In this sum, for each index  $i$ , the number of terms  $\bar{Z}(r, f - w_i)$  is clearly  $C_{q-1}^1 = q - 1$ . Consequently, by (2) we obtain

$$C_q^2 T(r, f) \leq (q - 1) \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(T(r, f))$$

and hence

$$\frac{q}{2} T(r, f) \leq \sum_{i=1}^q \bar{Z}(r, f - w_i) + o(T(r, f)).$$

**Definition:** Let  $f, g \in \mathcal{M}(\mathbb{K})$  (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ). Then  $f$  and  $g$  will be to share a small function, I.M.  $w \in \mathcal{M}(\mathbb{K})$  (resp.  $w \in \mathcal{M}(d(a, R^-))$ ) if  $f(x) = w(x)$  implies  $g(x) = w(x)$  and if  $g(x) = w(x)$  implies  $f(x) = w(x)$ .

**Theorem 5.6:** Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ) be distinct and share  $q$  distinct small functions I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, q$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, \dots, q$ )) other than the constant  $\infty$ . Then

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

**Proof.** Suppose that  $f$  and  $g$  belong to  $\mathcal{M}(\mathbb{K})$ , are distinct and share  $q$  distinct small functions I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, q$ ).

Let  $b$  be a zero of  $f - w_i$  for a certain index  $i$ . Then it is also a zero of  $g - w_i$ . Suppose that  $b$  is counted several times in the sum  $\sum_{j=1}^q \bar{Z}(r, f - w_j)$ , which means that it is a zero of another function  $f - w_h$  for a certain index  $h \neq i$ . Then we have  $w_i(b) = w_h(b)$  and hence  $b$  is a zero of the function  $w_i - w_h$  which belongs to  $\mathcal{M}_f(\mathbb{K})$ . Now, put  $\tilde{Z}(r, f - w_1) = \bar{Z}(r, f - w_1)$  and for each  $j > 1$ , let  $\tilde{Z}(r, f - w_j)$  be the counting function of zeros of  $f - w_j$  in the disk  $d(0, r^-)$  ignoring multiplicity and avoiding the zeros already counted as zeros of  $f - w_h$  for some  $h < j$ . Consider now the sum  $\sum_{j=1}^q \tilde{Z}(r, f - w_j)$ . Since the functions  $w_i - w_j$  belong to  $\mathcal{M}_f(\mathbb{K})$ , clearly, we have

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) = \sum_{j=1}^q \tilde{Z}(r, f - w_j) = o(T(r, f)).$$

It is clear, from the assumption, that  $f(x) - w_j(x) = 0$  implies  $g(x) - w_j(x) = 0$  and hence  $f(x) - g(x) = 0$ . Since  $f - g$  is not the identically zero function, it follows that

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g).$$

Consequently,

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) \leq \bar{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

Now, if  $f$  and  $g$  belong to  $\mathcal{M}(d(0, R^-))$  the proof is exactly the same.



**Theorem 5.7 [8] (A. Escassut, C.C. Yang):** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ) be distinct and share 7 distinct small functions (other than the constant  $\infty$ ) I.M.  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, 7$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ ), resp.  $w_j \in \mathcal{M}_f(D) \cap \mathcal{M}_g(D)$  ( $j = 1, \dots, 7$ ), ). Then  $f = g$ .*

*Moreover, if  $f$  and  $g$  have finitely many poles and share 3 distinct small functions (other than the constant  $\infty$ ) I.M. then  $f = g$ .*

**Proof.** We put  $M(r) = \max(T(r, f), T(r, g))$ . Suppose that  $f$  and  $g$  are distinct and share  $q$  small function I.M.  $w_j$ , ( $1 \leq j \leq q$ ). By Theorem 5.5, we have

$$qT(r, f) \leq 3 \sum_{j=1}^q \bar{Z}(r, f - w_j) + o(T(r, f)).$$

But thanks to Theorem 5.6, we can derive

$$qT(r, f) \leq 3T(r, f - g) + o(T(r, f))$$

and similarly

$$qT(r, g) \leq 3T(r, f - g) + o(T(r, g))$$

hence

$$(1) \quad qM(r) \leq 3T(r, f - g) + o(M(r)).$$

By Theorem C.4.8 in [7], we can derive that

$$qM(r) \leq 3(T(r, f) + T(r, g)) + o(M(r))$$

and hence  $qM(r) \leq 6M(r) + o(M(r))$ . That applies to the situation when  $f$  and  $g$  belong to  $\mathcal{M}(\mathbb{K})$  as well as when  $f$  and  $g$  belong to  $\mathcal{M}_u(d(0, R^-))$ . Consequently, it is impossible if  $q \geq 7$  and hence the first statement of Theorem 5.7 is proved.

Suppose now that  $f$  and  $g$  have finitely many poles. By Theorems C.4.8 in [7], Relation (1) gives us

$$qM(r) \leq 2M(r) + o(M(r))$$

which is obviously absurd whenever  $q \geq 3$  and proves that  $f = g$  when  $f$  and  $g$  belong to  $\mathcal{M}(\mathbb{K})$  as well as when  $f$  and  $g$  belong to  $\mathcal{M}_u(d(0, R^-))$ .

**Corollary 5.D:** *Let  $f, g \in \mathcal{A}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{A}_u(d(a, R^-))$ ) be distinct and share 3 distinct small functions (other than the constant  $\infty$ ) I.M.  $w_j \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$  ( $j = 1, 2, 3$ ) (resp.  $w_j \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$ ), ( $j = 1, 2, 3$ ). Then  $f = g$ .*

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## STATISTICAL QUANTUM NEURAL NETWORKS

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**Abstract.** We explain a new idea of how to use the high probability interval thresholds for neurons in quantum neural networks. Some basic quantum neural networks were analyzed and constructed in a recent work of the author. In particular the Least Square Error Problem (LSEP) and the Linear Regression Problem (LRP) was discussed. In this paper we analyze a new look on the threshold rules for neurons, taking the intervals of high probability in place of classical sigmoid half-line threshold and then we construct the least-square quantum neural network (LS-QNN), the polynomial interpolation quantum neural network (PI-QNN), the polynomial regression quantum neural network (PR-QNN) and chi-squared quantum neural network ( $\chi^2$ -QNN). We use the corresponding solutions or statistical tests as the threshold for the corresponding training rules.

### 1. Introduction

The classical machine learning (ML) [9],[5] theory was created in 1950, but only 9 years later in 1959 Arthur Samuel gave a definition of ML being “... computers learning without being explicitly programmed”. It should understand that the unknown functions (inputs-outputs) are deduced from a set of training data. The classical ML is characterized by the types: 1) *supervised learning*, i.e. classes of inputs corresponds to different classes, (2) *unsupervised learning*, i.e. the large data are summarized into a few stereotypes, and (3) *reinforcement learning*, i.e. one rewards, reinforces the current strategy. Normally the classical MLs are working with *big data*, see [7],[3],[8].

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The quantum Machine Learning (QML) are characterized by using quantum computing into the ML theory. One uses the ordinary interpretation of qubits, 1-qubit quantum gates, such as the Pauli matrices,

$$\mathbf{1} = -\boxed{\text{Id}}- \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X = -\boxed{\text{X}}- \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$Y = -\boxed{\text{Y}}- \sim \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$Z = -\boxed{\text{Z}}- \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then the 2-qubit gates like

$$\text{XOR} = -\boxed{\text{XOR}}- \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\text{SWAP} = -\boxed{\text{SWAP}}- \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

and finally, Measurements

$$M = -\boxed{\text{M}}-,$$

etc.

One uses the quantum algorithms to solve the ML problems with use of quantum computing. The most important ingredients in QML are: - choices of *training sets*, i.e. finite sets of given vectors in order to then find some value corresponding to another input, - *pattern completion*, i.e. adding missing informations to incomplete inputs, and - *associative memory*, i.e. retrieving stored memory vectors upon an input.

This paper is the second part of the paper [2], in which we continue to treat the cases of polynomial regression with the high probability region bounds used as the corresponding thresholds. In Section 2, we analyze the conceptions of classical artificial neural networks (ANN) and quantum neural networks (QNN). We explain also how to introduce some approximation of sigmoid functions by normal probability distributions. This let us to use a lot of test and confidential intervals and criteria from statistics. The next Section 3 is devoted

to the problem of training the least square quantum neural networks (LS-QNN), like the least square interpolation, the general polynomial regression quantum neural network (PR-QNN) and the chi-squared test training ( $\chi^2$ -QNN) in the next Section 4. We look at the problem of least square problem (LSP) solution of the general polynomial regression and propose to use the quantum Gauss-Jordan Elimination (GJE) Code to solve the LSP equation. This let us to make the network works outperform the classical approaches. The paper is finished with a conclusion in Section 5.

## 2. Quantum Neural Networks

### 2.1. Thresholds

Following the model of Deutsch, a quantum neural network  $QNN(s, d)$  is a set of all quantum circuits of size  $s$  and depth  $d$  with thresholds bounded by  $w$ . Quantum gates are interconnected by wires, preserve the sources and sink gates (measured the qubits and removed the entanglements with the maining qubits. Examples of QNNs are the implementation of NAND gate, dissipative  $D(m, \delta)$  and sink gates [4].

A threshold circuit is a boolean function  $Th^{n, \Delta} : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$  of  $n$  integral variables  $x_1, \dots, x_n$  such that  $Th^{n, \Delta}(x_1, \dots, x_n) = 1$  if and only if  $\sum x_i \geq \Delta$ . The class  $TC(s(n), d(n))$  of threshold circuits of size  $s(n)$  and depth  $d(n)$ , weighted by weight bound  $w$  can be approximated by elementary functions.

An equality threshold circuit is a boolean function  $Et_{w_1, \dots, w_n}^n : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$  of  $n$  integral variables  $x_1, \dots, x_n$  such that  $Et_{w_1, \dots, w_n}^n(x_1, \dots, x_n) = 0$  if and only if  $\sum x_i = 0$ . The class  $EC(s(n), d(n))$  of equality threshold circuits of size  $s(n)$  and depth  $d(n)$ , weighted by weight bound  $w$  can be approximated by elementary functions.

It was proven that  $TC(s(n), d(n)) \subseteq EC(O(s^2(n)), 2d(n))$  of weight bound  $O(s(n))$  and  $TC(s(n), d(n)) \subseteq EC(O(s^2(n), d(n)+1)$  of weight bound  $O(s^2(n))$ . And finally,  $EC(s(n), d(n)) \subseteq QNN(O)d(n). \log s(n), 2d(n))$  of precision  $O(\log w + d(n) \log s(n))$ . (Theorem 4.6 from [4]).

The question is whether a QNN can be implemented on Quantum Turing Machine (QTM) (Church-Turing Thesis) is difficult to answer: Quantum computing showed that the answer is No, but physicists speculate that it is Yes.

## 2.2. High Probability Thresholds for Neurons

Let us remind that the standard neuron thresholds are defined by some sigmoid function

$$f(x) = \frac{1}{1 + e^{-cx}},$$

for some constant  $c > 0$ . The sigmoid function has the value  $1/2$  at  $x = 0$ , monotonically increasing behaviour, the horizontal asymptotes  $y = 0$  for  $x \rightarrow -\infty$  and  $y = +1$  for  $x \rightarrow +\infty$ . Often using this function to define a threshold of type

$$\text{Th}_f^\delta(x) : \mathbb{Z}_2 \rightarrow \mathbb{Z}_2, \quad \text{Th}_f^\delta(x) = \begin{cases} 1 & \text{if } f(x) \geq \delta, \\ 0 & \text{if } f(x) < \delta \end{cases}$$

for some  $0 \leq \delta \leq 1$ .

Similarly, for functions of several variables, one uses some multivariable  $\mathbf{x} = (x_1, \dots, x_n)$  sigmoid functions, namely

$$f(\mathbf{x}) = \prod_{i=1}^n \frac{1}{1 + e^{-c_i x_i}},$$

for some constants  $c_i > 0$  which has the value  $1/2$  at  $x = 0$ , monotonically increasing behaviour, with horizontal asymptotes  $y = 0$  for  $x_i \rightarrow -\infty$  and  $y = +1$  for  $x_i \rightarrow +\infty$ . Often, one uses this function to define a threshold of type

$$\text{Th}_f^{n,\delta}(\mathbf{x}) : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2, \quad \text{Th}_f^{n,\delta}(\mathbf{x}) = \begin{cases} 1 & \text{if } f(\mathbf{x}) \geq \delta, \\ 0 & \text{if } f(\mathbf{x}) < \delta \end{cases}$$

for some threshold  $0 \leq \delta \leq 1$ .

We therefore have the following observation.

**Proposition 2.1.** *The one variable sigmoid function  $f(x)$  has the normal curve  $(x, \mathcal{N}(0, 1)(x))$  of the standard normal distribution*

$$\mathcal{N}(0, 1)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{1}{2}t^2} dt$$

as some curved asymptote and  $f(0) = \mathcal{N}(0, 1)(0) = \frac{1}{2}$ .

Remark that following these sigmoid functions the thresholds are defined by the accessible intervals  $[\delta, +\infty)$ . We can then approximate the sigmoid functions by normal distribution functions. For normal distributions we have many tests, namely  $z$ -test,  $t$ -tests,  $F$ -tests, and corresponding confidential intervals

and criteria to define the high probability region. We use the tests and confidential intervals to make thresholds in form of high probability regions in place of the half-line intervals  $[\delta, +\infty)$  for sigmoid functions. The difference error between two monotonically increasing functions: the sigmoid functions and the normal distribution functions are rather small. We have therefore the following new idea, that didn't have been introduced before.

**Corollary 2.1.** *We can use the normal distributions in place of sigmoid functions to define the thresholds of neurons by using the tests and confidential intervals to find the high probability region.*

### 3. Least Square QNN and Polynomial Regression QNN

First we remind that many problem, including the least squared problem and polynomial interpolation problems are reduced to solving systems of linear equations. In the previous work [1] we had showed that the Gauss-Jordan elimination procedure is consisting of an application of searching the pivot columns, which is reduced to use the Grover's Search Algorithm and by the way necessary arithmetic operations over rows. The following lemma [2] is fundamental in many problems of namely the least square or the polynomial interpolation quantum neural networks.

**Lemma 3.1.** *The quantum Gauss-Jordan Elimination Code can be implemented in QNN.*

Let us consider the polynomial  $f(\mathbf{x}) = \sum_{|\alpha|=0}^N a_\alpha \mathbf{x}^\alpha$  of degree  $N$  on  $n$  variables, with unknown coefficients  $a_\alpha$ , those we want to interpolate, and let  $(\{\mathbf{x}_{(j)}^\alpha\}_{|\alpha|=0}^N, y_j)$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n \leq N$  be the  $N + 1$  interpolating points of the polynomial,  $x = (x_1, \dots, x_n)$  be the unknown variables,  $\mathbf{x}_{(j)}^\alpha = \prod_{i=1}^n x_{i,(j)}^{\alpha_i}$ ,  $j = 0, \dots, N$ . The system of interpolating equations is a system of  $N + 1$  equation on  $N + 1$  unknown variables  $a_\alpha$ ,  $|\alpha| = 0, \dots, N$ :

$$f(\mathbf{x}_{(j)}) = \sum_{|\alpha|=0}^N a_\alpha \mathbf{x}_{(j)}^\alpha = y_j; j = 0, \dots, N.$$

The determinant of the system is of the Vandermonde type and of size  $(N +$



$1) \times (N + 1)$

$$|A| = \begin{vmatrix} 1 & x_{(0)}^{(1,\dots,0)} & \dots & x_{(0)}^{(0,\dots,N)} \\ 1 & x_{(1)}^{(1,\dots,0)} & \dots & x_{(1)}^{(0,\dots,N)} \\ \cdot & \dots & \dots & \dots \\ 1 & x_N^{(1,\dots,0)} & \dots & x_N^{(0,\dots,N)} \end{vmatrix},$$

then the system can be written as

$$A^\dagger A[a_\alpha] = A^\dagger[\mathbf{y}_j]. \quad (3.1)$$

The matrix of the system is nondegenerate if the interpolating points are in a generic position. In that case the solution of the system is  $[a_\alpha]_{\alpha=0}^N = (A^\dagger A)^{-1} A^\dagger \mathbf{b}$ , where  $\mathbf{b} = [\mathbf{y}_j]_{j=0}^N$ .

In general case the matrix can not be invertible, but the system is consistent. Based on Lemma 3.1, we can use the Gauss-Jordan elimination procedure on quantum neural networks to find out a basis of the null-space of the augmented matrix of the system (3.1). If the system satisfies the consistency conditions, then there exists at least one solution. Let  $(A^\dagger A)_{psi}^{-1}$  be the Moore-Penrose pseudoinverse of  $A^\dagger A$ , then the solution to the interpolation problem is  $[a_\alpha] = (A^\dagger A)_{psi}^{-1} A^\dagger \mathbf{b}$ , where  $\mathbf{b} = \text{proj}_{\text{col}(A^\dagger A)} A^\dagger[\mathbf{y}_j]$  is the projection on the column space of the matrix  $A^\dagger A$ .

The general interpolated solution is

$$\hat{f}(\mathbf{x}_{(j)}) = \sum_{|\alpha|=0}^N a_\alpha \mathbf{x}_{(j)}^\alpha = \hat{y}_j; j = 0, \dots, N. \quad (3.2)$$

Let us now discuss about the implementation method on quantum neural networks. The XOR gate is implemented in neural network (see [7]), and then the Fourier transform is implemented on neural networks ([2], Figs 2.3). Recently we used those to implement the quantum Gauss-Jordan elimination on neural networks. The algorithms are applied to our situation and we have an implementation of our least square quantum neural networks.

We have therefore the following result

**Theorem 3.1.** *The Least Square Quantum Neural Network (LS-QNN) and Polynomial Interpolation Quantum Neural Networks (PI-QNN) are implementable on QNN, with complexity  $O(\sqrt{N})$ .*

We now apply the Least Square Method to the problem of (general) regression (GRP). Let us remind that the Grover's Search Code can be implemented in QNN because the basic step is to repeatedly use the XOR quantum network

gate [2]. The method of QGJE [1], [6] is based on use of the Quantum Grover's Search to find the pivot columns in the matrix  $A^\dagger A$ .

We have therefore the following result

**Theorem 3.2.** *The Polynomial Regression Quantum Neural Network (PR-QNN) is implementable, i.e. the general regression problem GRP can be solved by a QNN, with complexity  $O(\sqrt{N})$ .*

Let us analyze how to train the GRP code in QNN. With the above interpolating quantum code, we can divide the data  $y_j$  into two treatments: regression treatment  $Y_{regr} = [\hat{y}_j]$  and residual treatment  $Y_{resid} = [y_j - \hat{y}_j]$ , where

$$\hat{y}_j = f_{regr}(\mathbf{x}_{(j)}) = \hat{f}(\mathbf{x}_{(j)}). \quad (3.3)$$

Let us denote by

$$F = \frac{MS_{regr}}{MS_{resid}} = \frac{\frac{(r^2 SS_Y)}{1}}{\frac{(1-r^2)SS_{resid}}{N-2}} = \frac{(N-2)r^2}{1-r^2}, \quad (3.4)$$

where  $r$  is the Pearson correlation,  $r = \text{Cor}(X, Y)$ . We may fix a level  $\alpha$  of explained proportion of variance and define the  $F$ -ratio  $F_{(1, N-2), \alpha}$ . Therefore we define the *training threshold* as if the  $F$ -ratio is in the high probability  $1 - \alpha$  region

$$F < F_{(1, N-2), \alpha}. \quad (3.5)$$

□

#### 4. Chi-Squared QNN

In the nonparametric statistics, the  $\chi^2$ -test plays important roles in many problems like contingency tables, homogeneity, ..... Let us consider the corresponding quantum code in QNN. Denote by  $\mathbf{e} = [e_{ij}]_{n \times r}$  be a contingency matrix of expected values  $e_{ij}$ . The random distribution  $X = [x_{ij}]$  is a matrix of size  $n \times r$ . The degree of freedom is

$$df_X = \begin{cases} (n-1) \times (r-1), & \text{if } r > 1 \\ (n-1), & \text{if } r = 1 \end{cases}$$

. The chi-squared statistic is of form

$$\chi_X^2 = \sum_{i=1}^n \sum_{j=1}^r \frac{(x_{ij} - e_{ij})^2}{e_{ij}}. \quad (4.1)$$

Our aim is to implement the  $\chi^2$ -test in a QNN and use the chi-squared test as the rule of training.

**Theorem 4.1.** *The Chi-Squared Quantum Neural Network ( $\chi^2$ -QNN) is implementable, i.e. the  $\chi^2$ -tests can be solved by a QNN, with complexity  $O(\sqrt{(n-1) \times (r-1)})$ .*

Indeed, the high probability  $1 - \alpha$  region is

$$\chi_X^2 < \chi_{df,\alpha} \quad (4.2)$$

for a fixed  $\alpha$ -level of confidence and

the *training rule* is to sink the network if the constraint is failed to be satisfied. If the constraint holds, it passes to the next layer of QNN.  $\square$

## 5. Conclusion

We implemented the quantum neural networks: the least square quantum neural network (LS-QNN) and the polynomial interpolation quantum neural networks (PI-QNN), the Polynomial Regression Quantum Network (PR-QNN) and the Chi-Squared Quantum Neural Network ( $\chi^2$ -QNN). The training rules are provided with the corresponding tests from Statistics.

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## OPTIMALITY CONDITIONS FOR EFFICIENCY OF CONSTRAINED VECTOR EQUILIBRIUM PROBLEMS

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**Abstract.** Fritz John necessary conditions for local Henig and global efficient solutions of vector equilibrium problems involving equality, inequality and set constraints with nonsmooth functions are established via convexifiers. Under suitable constraint qualifications, Kuhn–Tucker necessary conditions for local Henig and globally efficient solutions are derived. Note that Henig and global efficient solutions of (VEP) are studied with respect to a closed convex cone. Sufficient condition for Henig and globally efficient solutions are derived under some assumptions on asymptotic semiinvexity-infinite of the problem. Some illustrative examples are also given.

### 1. Introduction

In recent years, vector equilibrium problems have been extensively studied with many applications. Vector equilibrium problems include a lot of other problems as special cases such as vector variational inequalities, vector optimization problems, vector saddle point problems, vector complementarity problems, vector Nash equilibrium problems. Optimality conditions for weakly efficient solutions, efficient solutions, Henig efficient solutions, globally efficient solutions and superefficient solutions of vector equilibrium problems have been

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studied by many authors (see, e.g., [2–5], [8–14], and references therein). There are lot of works to dealt with optimality conditions for Henig and global efficient solutions of vector equilibrium problems. Gong [5] derived optimality conditions for Henig and global efficient solutions of vector equilibrium problems with a set constraint. Long et al. [8] established optimality conditions for Henig efficient solutions of vector equilibrium problems involving a cone-constraint and a set constraint with subconvexlike functions. Recently, Luu–Hang [11] derived optimality conditions for efficient solutions of vector equilibrium problems involving equality and inequality constraints with locally Lipschitz functions in terms of the Clarke subdifferentials on using the notion of quasirelative interior of a convex set in infinite dimensional spaces, but not for Henig and global efficient solutions. Necessary and sufficient conditions for efficiency of nonsmooth constrained vector optimization problems via convexificators are established by Luu [12–14].

Motivated by the works [8, 14], in this paper we establish Fritz John and Kuhn–Tucker necessary conditions for local Henig and global efficient solutions of vector equilibrium problems involving equality, inequality and set constraints with nonsmooth functions via convexificators. Under suitable constraint qualifications, Kuhn–Tucker necessary conditions for local Henig efficient solutions are derived. Sufficient conditions for Henig and global efficient solutions are derived under some assumptions on asymptotic semiinvexity-infine of the considering problem.

The remainder of the paper is organized as follows. After some preliminaries, in Section 3, based on a Fritz John necessary condition by Luu [14], we derive Fritz John necessary conditions for local Henig and global efficient solutions with respect to a closed convex cone of vector equilibrium problems involving equality, inequality and set constraints with nonsmooth functions. In Section 4, Kuhn–Tucker necessary conditions for local Henig and global efficient solutions are derived under some suitable constraint qualifications. Note that Henig and global efficient solutions of (VEP) are studied with respect to a closed convex cone. Kuhn–Tucker necessary conditions via convexificators can be sharper than those expressed in terms of the Clarke subdifferentials and the Michel–Penot subdifferentials. Observe that the results obtained in this paper are more general than those obtained by Gong [5] for vector equilibrium problems with only a set constraint, and those obtained by Long et al. [8] for vector equilibrium problems with subconvexlike functions. Section 5 presents sufficient conditions for Henig and global efficient solutions under some assumptions on asymptotic semiinvexity-infine of the problem.

## 2. Preliminaries

Let  $X$  be a Banach space,  $X^*$  topological dual of  $X$ ,  $\bar{x} \in X$ . We recall some notions on convexificators in [7]. The lower (upper) Dini directional derivatives of  $f : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$  at  $\bar{x} \in X$  in a direction  $v \in X$  is defined as

$$f^-(\bar{x}; v) := \liminf_{t \downarrow 0} \frac{f(\bar{x} + tv) - f(\bar{x})}{t}$$

$$\left( \text{resp. } f^+(\bar{x}; v) := \limsup_{t \downarrow 0} \frac{f(\bar{x} + tv) - f(\bar{x})}{t} \right).$$

In case  $f^+(\bar{x}; v) = f^-(\bar{x}; v)$ , their common value is denoted by  $f'(\bar{x}; v)$ , which is called Dini derivative of  $f$  at  $\bar{x}$  in the direction  $v$ . The function  $f$  is called Dini differentiable at  $\bar{x}$  iff its Dini derivative at  $\bar{x}$  exists in all directions.

Recall [7] that the function  $f$  is said to have an upper (lower) convexificator  $\partial^* f(\bar{x})$  (resp.  $\partial_* f(\bar{x})$ ) at  $\bar{x}$  iff  $\partial^* f(\bar{x}) \subseteq X^*$  (resp.  $\partial_* f(\bar{x}) \subseteq X^*$ ) is weakly\* closed, and for all  $v \in X$ ,

$$f^-(\bar{x}; v) \leq \sup_{\xi \in \partial^* f(\bar{x})} \langle \xi, v \rangle$$

$$\left( \text{resp. } f^+(\bar{x}; v) \geq \inf_{\xi \in \partial_* f(\bar{x})} \langle \xi, v \rangle \right).$$

A weakly\* closed set  $\partial^* f(\bar{x}) \subseteq X^*$  is said to be a convexificator of  $f$  at  $\bar{x}$  iff it is both upper and lower convexificators of  $f$  at  $\bar{x}$ .

The function  $f$  is said to have an upper (lower) semi-regular convexificator  $\partial^* f(\bar{x})$  (resp.  $\partial_* f(\bar{x})$ ) at  $\bar{x}$  iff  $\partial^* f(\bar{x})$  (resp.  $\partial_* f(\bar{x})$ ) is weakly\* closed and for all  $v \in X$ ,

$$f^+(\bar{x}; v) \leq \sup_{\xi \in \partial^* f(\bar{x})} \langle \xi, v \rangle$$

$$\left( \text{resp. } f^-(\bar{x}; v) \geq \inf_{\xi \in \partial_* f(\bar{x})} \langle \xi, v \rangle \right).$$

If equality holds in these inequalities, then  $\partial^* f(\bar{x})$  (resp.  $\partial_* f(\bar{x})$ ) is called an upper (resp. lower) regular convexificator of  $f$  at  $\bar{x}$ .

Following [1], the Clarke generalized directional derivative of  $f$  at  $\bar{x}$ , with respect to a direction  $v$ , is defined as

$$f^0(\bar{x}; v) = \limsup_{x \rightarrow \bar{x}, t \downarrow 0} \frac{f(x + tv) - f(x)}{t}.$$

The Clarke subdifferential of  $f$  at  $\bar{x}$  is

$$\partial f(\bar{x}) = \left\{ \xi \in X^* : \langle \xi, v \rangle \leq f^0(\bar{x}; v), \forall v \in X \right\}.$$

For a locally Lipschitz function  $f$  at  $\bar{x}$ ,  $\partial f(\bar{x})$  is a convexificator of  $f$  at  $\bar{x}$  (see [7]).

The Michel–Penot directional derivative of  $f$  at  $\bar{x}$  in a direction  $v \in X$  is defined as follows

$$f^\diamond(\bar{x}; v) := \sup_{w \in X} \limsup_{t \downarrow 0} \frac{f(\bar{x} + t(v + w)) - f(\bar{x} + tw)}{t}.$$

The Michel–Penot subdifferential of  $f$  at  $\bar{x}$  is

$$\partial^\diamond f(\bar{x}) := \left\{ \xi \in X^* : \langle \xi, v \rangle \leq f^\diamond(\bar{x}; v), \forall v \in X \right\}.$$

If  $f$  is locally Lipschitz at  $\bar{x}$ , then  $\partial^\diamond f(\bar{x})$  is also a convexificator of  $f$  at  $\bar{x}$ . The convex hull of a convexificator of a locally Lipschitz function may be strictly contained in both the Clarke and Michel–Penot subdifferentials (see [7], Example 2.1). It is obvious that for a function  $f$  which is locally Lipschitz at  $\bar{x}$ ,

$$\begin{aligned} f^\diamond(\bar{x}; v) &\leq f^0(\bar{x}; v) \quad (\forall v \in X), \\ \partial^\diamond f(\bar{x}) &\subseteq \partial f(\bar{x}). \end{aligned}$$

The following example shows that the subdifferentials  $\partial^\diamond f(\bar{x})$  and  $\partial f(\bar{x})$  may be greatly different, but they are convexificators of  $f$  at  $\bar{x}$ .

**Example 2.1.** The function  $f$  be defined on  $\mathbb{R}$  as

$$f(x) = \begin{cases} x^2 |\cos \frac{\pi}{x}|, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then,  $\partial^\diamond f(0) = \{0\}$ ,  $f^\diamond(0; v) = 0$  ( $\forall v \in \mathbb{R}$ ),  $f^0(0; v) = \pi|v|$  ( $\forall v \in \mathbb{R}$ ),  $\partial f(0) = [-\pi, \pi]$ . Thus,  $\partial^\diamond f(0) \subsetneq \partial f(0)$ , but  $\{0\}$  and  $[-\pi, \pi]$  are convexificators of  $f$  at  $\bar{x}$ .

Recall [1] that the Clarke tangent cone to a set  $C \subseteq X$  at a point  $\bar{x} \in C$  is defined as

$$\begin{aligned} T(C; \bar{x}) := \left\{ v \in X : \forall x_n \in C, x_n \rightarrow \bar{x}, \forall t_n \downarrow 0, \exists v_n \rightarrow v \right. \\ \left. \text{such that } x_n + t_n v_n \in C, \forall n \right\}. \end{aligned}$$



The Clarke normal cone to  $C$  at  $\bar{x}$  is

$$N(C; \bar{x}) := \left\{ \xi \in X^* : \langle \xi, v \rangle \leq 0, \forall v \in T(C; \bar{x}) \right\}.$$

Thus  $N(C; \bar{x}) = T(C; \bar{x})^\circ = -T(C; \bar{x})^*$ , where  $T(C; \bar{x})^\circ$  is the polar of  $T(C; \bar{x})$ , and  $T(C; \bar{x})^*$  is the dual cone of  $T(C; \bar{x})$ . Note that the cones  $T(C; \bar{x})$  and  $N(C; \bar{x})$  are nonempty convex,  $T(C; \bar{x})$  is closed and  $N(C; \bar{x})$  is weakly\* closed.

### 3. Fritz John necessary conditions for efficiency

This section deals with Fritz John necessary conditions for local Henig and globally efficient solutions of vector equilibrium problems via convexificators. The Kuhn–Tucker necessary conditions obtained here via convexificators can be sharper than those expressed in terms of the Clarke and Michel–Penot sub-differentials.

Let  $K$  a nonempty closed subset of a Banach space  $X$ . Let  $F$  be a mapping from  $K \times K$  to  $\mathbb{R}^r$  and  $Q$  a pointed closed convex cone in  $\mathbb{R}^r$ . Let us consider the following vector equilibrium problem (VEP): Finding a point  $x \in K$  such that

$$F(x, y) \notin -Q \setminus \{0\} \quad (\forall y \in K). \quad (1)$$

A vector  $\bar{x}$  solved (1) will be called efficient solution of (VEP).

A vector  $x \in K$  is called a globally efficient solution to Problem (VEP) iff there exists a pointed convex cone  $M \subset \mathbb{R}^r$  with  $Q \setminus \{0\} \subset \text{int}M$  such that

$$F(x, y) \cap ((-M) \setminus \{0\}) = \emptyset \quad (\forall y \in K). \quad (2)$$

For  $\bar{x} \in K$ , denoting  $F_{\bar{x}}(y) := F(\bar{x}, y)$ , we have that

$$F_{\bar{x}}(y) = (F_{1, \bar{x}}(y), \dots, F_{r, \bar{x}}(y)).$$

Denote the dual cone of  $Q$  by  $Q^* := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \forall y \in Q\}$ . A nonempty convex subset  $B$  of  $Q$  is called a base of  $Q$ , if  $Q = \text{cone}B$  and  $0 \notin \text{cl}B$ , where  $\text{cl}$  stands for the closure,  $\text{cone}B$  denotes the cone hull of  $B$ :  $\text{cone}B = \{tb : t \geq 0, b \in B\}$ . Denote the quasi-interior of  $Q^*$  by  $Q^\# := \{y^* \in \mathbb{R}^r : \langle y^*, y \rangle > 0, \forall y \in Q \setminus \{0\}\}$ . We set  $Q^\Delta(B) := \{y^* \in Q^\# : \exists t > 0 \text{ such that } \langle y^*, b \rangle \geq t, \forall b \in B\}$ . Then  $Q^\Delta(B)$  is a cone in  $\mathbb{R}^r$  and  $Q^\Delta(B) \subseteq Q^\#$ . Moreover, if  $y^* \in Q^\Delta(B)$ , then  $y^* \neq 0$ . If  $B$  is a base of the cone  $Q$ , by a separation theorem see, e.g., Theorem 3.6 [6]), there exists  $y^* \in \mathbb{R}^r \setminus \{0\}$  such that

$$\alpha := \inf\{\langle y^*, b \rangle : b \in B\} > y^*(0) = 0.$$

Then the set  $V_B := \{y \in \mathbb{R} : |\langle y^*, y \rangle| < \frac{\alpha}{2}\}$  is an absolutely convex open neighborhood of  $0 \in Y$  (see [5]), and  $\inf\{\langle y^*, y \rangle : y \in B + V_B\} \geq \frac{\alpha}{2}$ . For each convex neighborhood  $U$  of  $0$ ,  $U \subseteq V_B$ , one has  $0 \notin \text{cl}(B + U)$ . Hence, the set  $Q_U(B) := \text{cone}(U + B)$  is a pointed convex cone, and

$$Q \setminus \{0\} \subseteq \text{int}Q_U B. \quad (3)$$

A vector  $\bar{x} \in K$  is called Henig efficient solution of (VEP) if there is an absolutely convex neighborhood  $U$  of  $0$ ,  $U \subseteq V_B$  such that

$$\text{cone}F_{\bar{x}}(K) \cap (-\text{int}Q_U(B)) = \emptyset,$$

where  $F_{\bar{x}}(K) = \bigcup_{y \in K} F_{\bar{x}}(y)$ . Since  $Q_U(B)$  is a pointed convex cone,  $\bar{x}$  is a Henig solution if and only if

$$F_{\bar{x}}(K) \cap (-\text{int}Q_U(B)) = \emptyset. \quad (4)$$

Note that  $\bar{x} \in K$  is a Henig efficient solution of (VEP) if and only if there is an absolutely convex neighborhood  $U$  of  $0$ ,  $U \subseteq V_B$  such that (see [5])

$$\text{cone}F_{\bar{x}}(K) \cap (U - B) = \emptyset. \quad (5)$$

If in the definitions of efficient solution, globally efficient solution and Henig efficient solution,  $K$  is replaced by  $K \cap W$  for some neighborhood  $W$  of  $\bar{x}$ , we obtain the notions of local efficient solution, local global solution and local Henig efficient solution for (VEP), respectively.

**Remark 3.1** It follows from (1)–(4) that a Henig efficient solution is an efficient solution, and globally efficient solution is also an efficient solution.

Let  $g$  and  $h$  be mappings from  $X$  into  $\mathbb{R}^m$  and  $\mathbb{R}^\ell$ , respectively, and let  $C$  be nonempty closed subsets of  $X$ . Then  $g$  and  $h$  can be expressed as follows:  $g = (g_1, \dots, g_m)$ ,  $h = (h_1, \dots, h_\ell)$ . This paper deals with the vector equilibrium problem (VEP) in which  $K$  is described by

$$K = \left\{ x \in C : g(x) \leq 0, h(x) = 0 \right\}.$$

This constrained vector equilibrium problem is denoted by (CVEP). We set  $I := \{1, \dots, m\}$ ,  $L := \{1, \dots, \ell\}$ , and

$$\begin{aligned} I(\bar{x}) &:= \{i \in I : g_i(\bar{x}) = 0\}, \\ H &:= \{x \in C : h(x) = h(\bar{x})\}. \end{aligned}$$

Recall that a point  $\bar{x}$  is said to be a regular point in the sense of Ioffe for  $h$  relative to  $C$  if there exist numbers  $K > 0$  and  $\delta > 0$  such that for all  $x \in C \cap B(\bar{x}; \delta)$ ,

$$d_H(x) \leq K \|h(x) - h(\bar{x})\|,$$

where  $d_H(x)$  denotes the distance from  $x$  to  $H$ ,  $B(\bar{x}; \delta)$  stands for the open ball of radius  $\delta$  around  $\bar{x}$  (see, e.g., [14]).

The following assumptions are posed on Problem (CVEP).

**Assumption 3.1**

(a)  $\bar{x} \in H$ ;  $C$  is convex;  $Q$  has a base  $B$ ;  $F_{\bar{x}}, h_j$  ( $j \in L$ ) are locally Lipschitz at  $\bar{x}$ ,  $g_i$  ( $i \in I(\bar{x})$ ) are continuous in a neighborhood of  $\bar{x}$ .

(b) The functions  $F_{k,\bar{x}}$  and  $h_j$  admit upper convexificators  $\partial^* F_{k,\bar{x}}(x)$  ( $k \in J$ ) and  $\partial^* h_j(x)$  ( $j \in L$ ) at  $x$  near  $\bar{x}$ , respectively;  $g_i$  ( $i \in I(\bar{x})$ ) admit upper convexificators  $\partial^* g(\bar{x})$  at  $\bar{x}$ ; the functions  $|h_j|$  ( $j \in L$ ) are regular in the sense of Clarke at  $\bar{x}$ , that is for every  $v \in X$  there exists  $f'(\bar{x}; v)$  and  $f^0(\bar{x}; v) = f^0(\bar{x}; v)$ .

(c)  $\partial^* F_{1,\bar{x}}(\bar{x}), \dots, \partial^* F_{r,\bar{x}}(\bar{x}), \partial^* h_1(\bar{x}), \dots, \partial^* h_\ell(\bar{x})$  are bounded; the convexificator maps  $\partial^* F_{1,\bar{x}}, \dots, \partial^* F_{r,\bar{x}}, \partial^* h_1, \dots, \partial^* h_\ell$  are upper semicontinuous at  $\bar{x}$ .

We shall begin with establishing a Fritz John necessary optimality condition for local Henig efficient solution of (CVEP).

**Theorem 3.1.** *Let  $\bar{x}$  be a local Henig efficient solution of (CVEP). Assume that  $F_{\bar{x}}(\bar{x}) = 0$ , Assumption 3.1 hold. Then there exist  $\bar{\tau} \geq 0, \bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{v} := (\bar{v}_1, \dots, \bar{v}_\ell) \in \mathbb{R}^\ell$  with  $(\bar{\tau}, \bar{\mu}_1, \dots, \bar{\mu}_{|I(\bar{x})|}) \neq 0$ , and a continuous positively homogeneous function  $\Lambda$  on  $Y$  satisfying*

- (i) *If  $y_2 - y_1 \in Q \setminus \{0\}$ , then  $\Lambda(y_1) < \Lambda(y_2)$ ,*
  - (ii)  *$\exists \beta_0 > 0$  such that  $\Lambda(-b) \leq -\beta_0$  ( $\forall b \in B$ ),*
- such that

$$0 \in \text{cl} \left( \bar{\tau} \text{conv} \partial^* (\Lambda \circ F_{\bar{x}})(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{v}_j \text{conv} \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right). \quad (6)$$

where  $|I(\bar{x})|$  denotes the capacity of  $I(\bar{x})$ .

*Proof* Since  $\bar{x}$  is a local Henig efficient solution of (CVEP), there are a neighborhood  $W$  of  $\bar{x}$  and an absolutely convex neighborhood  $U$  of 0,  $U \subseteq V_B$  such that

$$\text{cone} F_{\bar{x}}(K \cap W) \cap (-\text{int} Q_U(B)) = \emptyset.$$

Applying Theorem 3.2 in [5] yields the existence a continuous positively homogeneous subadditive function  $\Lambda$  on  $Y$  such that (i), (ii) hold, and

$$(\Lambda \circ F_{\bar{x}})(x) \geq 0 \quad (\forall x \in K \cap W). \quad (7)$$

Since  $F_{\bar{x}}(\bar{x}) = 0$  and  $\Lambda$  is positively homogeneous, we have  $(\Lambda \circ F_{\bar{x}})(\bar{x}) = 0$ . In view of (7), we deduce that  $\bar{x}$  is a local minimum of the following scalar

optimization problem:

$$(P) \quad \begin{aligned} & \min(\Lambda \circ F_{\bar{x}})(x), \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad (i \in I), \\ & h_j(x) = 0 \quad (j \in J), \\ & x \in C \cap W. \end{aligned}$$

Since the function  $\Lambda$  is continuous and convex, we can apply Proposition 2.2.6 [1] to deduce that it is locally Lipschitz. Observe that in the scalar case, a local Henig solution is a local minimum. Taking account of Theorem 3.2 [14] to the scalar problem (P) yields the existence of  $\bar{\tau} \geq 0, \bar{\mu}_i \geq 0$  ( $\forall i \in I(\bar{x})$ ) with  $(\bar{\tau}, \bar{\mu}_1, \dots, \bar{\mu}_{|I(\bar{x})|}) \neq 0, \bar{\nu}_j \in \mathbb{R}$  ( $\forall j \in L$ ) such that

$$(8) \quad 0 \in \text{cl} \left( \bar{\tau} \text{conv} \partial^*(\Lambda \circ F_{\bar{x}})(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \text{conv} \partial^* h_j(\bar{x}) + N_{C \cap W}(\bar{x}) \right).$$

On the other hand,

$$N_{C \cap W}(\bar{x}) = N_C(\bar{x}).$$

Hence, (8) implies (6).  $\square$

A Fritz John necessary optimality condition for local global efficient solution of (CVEP) can be stated as follows.

**Theorem 3.2.** *Let  $\bar{x}$  be a local global efficient solution of (CVEP). Assume that  $F_{\bar{x}}(\bar{x}) = 0$ , Assumption 3.1 hold. Then there exist  $\bar{\tau} \geq 0, \bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu} := (\bar{\nu}_1, \dots, \bar{\nu}_\ell) \in \mathbb{R}^\ell$  with  $(\bar{\tau}, \bar{\mu}_1, \dots, \bar{\mu}_{|I(\bar{x})|}) \neq 0$ , and a continuous positively homogeneous function  $\Lambda$  on  $Y$  satisfying that if  $y_2 - y_1 \in Q \setminus \{0\}$ , then  $\Lambda(y_1) < \Lambda(y_2)$ , such that*

$$(9) \quad 0 \in \text{cl} \left( \bar{\tau} \text{conv} \partial^*(\Lambda \circ F_{\bar{x}})(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \text{conv} \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right).$$

*Proof* Since  $\bar{x}$  is a local global efficient solution of (CVEP), there are a neighborhood  $W$  of  $\bar{x}$  and a pointed convex cone  $H$  satisfying  $Q \setminus \{0\} \subset \text{int}H$  such that

$$F_{\bar{x}}(K) \cap (-H) \setminus \{0\} = \emptyset. \quad (10)$$

Taking  $a \in Q \setminus \{0\}$ , it follows that  $a \in \text{int}H$ . We invoke Theorem 3.3 [5] to deduce that the function defined by  $\Lambda(y) = \inf\{t \in \mathbb{R} : y \in ta - H\}$  satisfies the following conditions:

- (a) if  $y_2 - y_1 \in Q \setminus \{0\}$ , then  $\Lambda(y_1) < \Lambda(y_2)$ ;
- (b)  $\Lambda(F_{\bar{x}}(y)) \geq 0$  (for all  $y \in K$ ).

Since  $F_{\bar{x}}(\bar{x}) = 0$  and  $\Lambda$  is positively homogeneous, we have  $(\Lambda \circ F_{\bar{x}})(\bar{x}) = 0$ . In view of (b), we deduce that  $\bar{x}$  is a local minimum of the following scalar optimization problem:

$$(P1) \quad \begin{aligned} & \min(\Lambda \circ F_{\bar{x}})(x), \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad (i \in I), \\ & h_j(x) = 0 \quad (j \in J), \\ & x \in C \cap W. \end{aligned}$$

Observe that in the scalar case, a local global solution is a local minimum. Hence, we can apply Theorem 3.2 [14] to the scalar problem (P1). In the same way as in the proof of Theorem 3.1 we deduce that there exist  $\bar{\tau} \geq 0, \bar{\mu}_i \geq 0$  ( $\forall i \in I(\bar{x})$ ) with  $(\bar{\tau}, \bar{\mu}_1, \dots, \bar{\mu}_{|I(\bar{x})|}) \neq 0, \bar{\nu}_j \in \mathbb{R}$  ( $\forall j \in L$ ) such that

$$(11) \quad 0 \in \text{cl} \left( \bar{\tau} \text{conv } \partial^*(\Lambda \circ F_{\bar{x}})(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv } \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \text{conv } \partial^* h_j(\bar{x}) + N_{C \cap W}(\bar{x}) \right),$$

which together with the fact that  $N_{C \cap W}(\bar{x}) = N_C(\bar{x})$  implies (9). □

#### 4. Kuhn–Tucker necessary conditions for efficiency

To derive Kuhn–Tucker necessary conditions for Henig efficiency, we introduce the following constraint qualification, which is called (CQ1): There exist  $d_0 \in T(C; \bar{x})$  and numbers  $b_i > 0$  ( $i \in I(\bar{x})$ ) such that

- (i)  $\langle \eta_i, d_0 \rangle \leq -b_i$  ( $\forall \eta_i \in \partial^* g_i(\bar{x}), \forall i \in I(\bar{x})$ );
- (ii)  $\langle \zeta_j, d_0 \rangle = 0$  ( $\forall \zeta_j \in \partial^* h_j(\bar{x}), \forall j \in L$ ).

We also introduce another constraint qualification (CQ2): For every  $\mu_i \geq 0$  ( $\forall i \in I(\bar{x})$ ), not all zero, and  $\gamma_j \in \mathbb{R}$  ( $\forall j \in L$ ),

$$0 \notin \text{cl} \left( \sum_{i \in I(\bar{x})} \mu_i \text{conv } \partial^* g_i(\bar{x}) + \sum_{j \in L} \gamma_j \text{conv } \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right).$$

**Remark 4.1** By an argument analogous to that for the proof of Proposition 4.1 [12], we deduce that (CQ1) implies (CQ2).

A Kuhn-Tucker necessary condition for Henig efficiency can be stated as follows.

**Theorem 4.1.** *Let  $\bar{x}$  be a local Henig efficient solution of (CVEP). Assume that  $F_{\bar{x}}(\bar{x}) = 0$ ; Assumption 3.1 is fulfilled; (CQ1) or (CQ2) holds. Then there exist  $\bar{\lambda} \in Q^\Delta(B)$ ,  $\bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu} \in \mathbb{R}^\ell$ , such that*

$$0 \in \text{cl} \left( \sum_{k=1}^r \bar{\lambda}_k \text{conv} \partial^* F_{k,\bar{x}}(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \text{conv} \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right). \quad (12)$$

*Proof* Applying Theorem 3.1 yields the existence of  $\bar{\tau} \geq 0$ ,  $\bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu} := (\bar{\nu}_1, \dots, \bar{\nu}_\ell) \in \mathbb{R}^\ell$  with  $(\bar{\tau}, \bar{\mu}_1, \dots, \bar{\mu}_{|I(\bar{x})|}) \neq 0$ , and a continuous positively homogeneous function  $\Lambda$  on  $Y$  satisfying  $(\alpha)$  and  $(\beta)$  such that (6) holds. Since (CQ1) or (CQ2) holds, one gets that  $\bar{\tau} > 0$ .

Due to Assumption 3.1(c),  $\partial^* F_{1,\bar{x}}(\bar{x}), \dots, \partial^* F_{r,\bar{x}}(\bar{x})$  are bounded convexificators of  $F_{1,\bar{x}}, \dots, F_{r,\bar{x}}$  at  $\bar{x}$ , respectively, and the set-valued mappings  $\partial^* F_{1,\bar{x}}, \dots, \partial^* F_{r,\bar{x}}$  are upper semicontinuous at  $\bar{x}$ . Therefore, due to Proposition 5.1 on a chain rule in [7],  $\partial\Lambda(F_{\bar{x}}(\bar{x}))(\partial^* F_{1,\bar{x}}(\bar{x}), \dots, \partial^* F_{r,\bar{x}}(\bar{x}))$  is a convexicator of  $\Lambda \circ F_{\bar{x}}$  at  $\bar{x}$ . Observing that  $F_{\bar{x}}(\bar{x}) = 0$ , it follows from (6) that there exists a sequence

$$\begin{aligned} z_n \in & \bar{\tau} \partial\Lambda(0)(\text{conv} \partial^* F_{1,\bar{x}}(\bar{x}), \dots, \text{conv} \partial^* F_{r,\bar{x}}(\bar{x})) \\ & + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\gamma}_j \text{conv} \partial^* h_j(\bar{x}) + N(C; \bar{x}), \end{aligned} \quad (13)$$

such that  $\lim_{n \rightarrow \infty} z_n = 0$ . By (13), there exists a sequence  $\{\chi_n\} \subset \partial\Lambda(0) \subset \mathbb{R}^r$  such that

$$\begin{aligned} z_n \in & \bar{\tau} \chi_n (\text{conv} \partial^* F_{1,\bar{x}}(\bar{x}), \dots, \text{conv} \partial^* F_{r,\bar{x}}(\bar{x})) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) \\ & + \sum_{j \in L} \bar{\gamma}_j \text{conv} \partial^* h_j(\bar{x}) + N(C; \bar{x}). \end{aligned} \quad (14)$$

Since  $\partial\Lambda(F_{\bar{x}}(\bar{x}))$  is a compact set in  $\mathbb{R}^r$ , without loss of generality, we can assume that  $\chi_n \rightarrow \bar{\chi} \in \partial\Lambda(F_{\bar{x}}(\bar{x}))$ . Putting  $\bar{\lambda} = \bar{\tau} \bar{\chi}$ , one has  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_r) \in \mathbb{R}^r$ . By virtue of (14), it holds that

$$\begin{aligned} 0 \in & \text{cl} \left( \bar{\lambda} (\text{conv} \partial^* F_{1,\bar{x}}(\bar{x}), \dots, \text{conv} \partial^* F_{r,\bar{x}}(\bar{x})) \right. \\ & \left. + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\gamma}_j \text{conv} \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right), \end{aligned}$$

which implies (12).

Let us see that  $\bar{\lambda} \in Q^\Delta(B)$ . Indeed, observing that  $\Lambda$  is a convex function and  $\bar{\chi} \in \partial(\Lambda(F_{\bar{x}}(\bar{x})))$ , according to Theorem 3.1, there exists  $\beta_0 > 0$  such that for every  $y \in B$ ,

$$\begin{aligned} \langle \bar{\chi}, -y \rangle &\leq \Lambda(F_{\bar{x}}(\bar{x}) - y) - \Lambda(F_{\bar{x}}(\bar{x})) \\ &= \Lambda(-y) < -\beta_0. \end{aligned}$$

Consequently,  $\langle \bar{\chi}, y \rangle \geq \beta_0$  ( $\forall y \in B$ ). Hence,  $\bar{\chi} \in Q^\Delta(B)$ , and so  $\bar{\lambda} \in Q^\Delta(B)$ , which completes the proof.  $\square$

A Kuhn–Tucker necessary condition via the Clarke subdifferentials can be stated as follows.

**Corollary 4.1.** *Let  $\bar{x}$  be a local Henig efficient solution of (CVEP). Assume that all the hypotheses of Theorem 3.1(a), (b) hold. Suppose, in addition, that (CQ1) or (CQ2) is fulfilled with  $\partial F_{k,\bar{x}}, \partial g_i$  instead of  $\partial^* F_{k,\bar{x}}, \partial^* g_i$ , respectively. Then, there exist  $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_r) \in Q^\Delta(B)$ ,  $\bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu}_j \in \mathbb{R}$  ( $j \in J$ ), such that*

$$0 \in \sum_{k=1}^r \bar{\lambda}_k \partial F_{k,\bar{x}}(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \partial g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \text{conv } \partial^* h_j(\bar{x}) + N_C(\bar{x}), \quad (12)$$

where  $\partial F_{k,\bar{x}}(\bar{x}), \partial g_i(\bar{x})$  indicate the Clarke subdifferential of  $F_{k,\bar{x}}, g_i$  at  $\bar{x}$ , respectively.

Moreover, if the base  $B$  of  $Q$  is bounded and closed, then  $\bar{\lambda} \in \text{int}Q^*$ .

*Proof* Making use of Corollary 5.2 [1], it follows that  $F_{1,\bar{x}}, \dots, F_{r,\bar{x}}, g_i$  ( $i \in I(\bar{x})$ ) admit convexifiers  $\partial F_{1,\bar{x}}(\bar{x}), \dots, \partial F_{r,\bar{x}}(\bar{x}), \partial g_i(\bar{x})$  ( $i \in I(\bar{x})$ ) at  $\bar{x}$ ;  $\partial F_{1,\bar{x}}(\bar{x}), \dots, \partial F_{r,\bar{x}}(\bar{x})$  are bounded;  $\partial F_{1,\bar{x}}, \dots, \partial F_{r,\bar{x}}$  are upper semicontinuous at  $\bar{x}$ . Hence, Assumption 3.1 is fulfilled. We invoke Theorem 4.1 to deduce that there exist  $\bar{\lambda} \in Q^\Delta(B)$ ,  $\bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu} \in \mathbb{R}^\ell$  such that

$$0 \in \text{cl} \left( \sum_{k=1}^r \bar{\lambda}_k \text{conv } \partial F_{k,\bar{x}}(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv } \partial g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \nabla h_j(\bar{x}) + N(C; \bar{x}) \right). \quad (13)$$

Since  $\partial F_{k,\bar{x}}(\bar{x}), \partial g_i(\bar{x})$  are compact and  $N_C(\bar{x})$  is closed, it follows that the right hand side of (12) is closed, and so, the closure in (13) can be removed. Hence, (13) implies (12).

If the base  $B$  of  $Q$  is bounded and closed, by Lemma 4.5(iii) [5], we have  $Q^\Delta(B) = \text{int}Q^*$ . Hence, we get the desired conclusion.  $\square$

In case the mapping  $F_{\bar{x}}$  is Gâteaux differentiable at  $\bar{x}$  we get the following Kuhn–Tucker necessary condition.

**Corollary 4.2.** *Let  $\bar{x}$  be a local Henig efficient solution of (CVEP). Assume that all the hypotheses of Theorem 4.1 hold. Suppose, in addition, that  $F_{\bar{x}}$  is Gâteaux differentiable with the Gâteaux derivative  $\nabla_G F_{\bar{x}}(\bar{x})$ . Then, there exist  $\bar{\lambda} \in Q^\Delta(B)$ ,  $\bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu}_j \in \mathbb{R}$  ( $j \in J$ ), such that*

$$0 \in [\nabla_G F_{\bar{x}}(\bar{x})]^* \bar{\lambda} + \sum_{i \in I(\bar{x})} \bar{\mu}_i \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \text{conv } \partial^* h_j(\bar{x}) + N_C(\bar{x}), \quad (16)$$

where  $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_r)$ .

*Proof* Since  $F_{\bar{x}}$  is Gâteaux differentiable at  $\bar{x}$ , the set  $\{\nabla_G F_{\bar{x}}(\bar{x})\}$  is a convexifier of  $f$  at  $\bar{x}$ . Applying Theorem 4.1 yields the existence of  $\bar{\lambda} \in Q^\Delta(B)$ ,  $\bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu} \in \mathbb{R}^\ell$ , such that (9) holds. Taking  $\partial^* F_{\bar{x}}(\bar{x}) = \{\nabla_G F_{\bar{x}}(\bar{x})\}$ , we obtain the desired conclusion.  $\square$

Theorem 4.1 is illustrated by the following example.

**Example 4.1.** Let  $X = \mathbb{R}^2, Y = \mathbb{R}^2, C = [0, 1] \times [0, 1], \bar{x} = (0, 0), Q = \mathbb{R}_+^2$ . Let  $F : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}^2, g : \mathbb{R}^2 \rightarrow \mathbb{R}^2, h : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined as

$$F(x, y) = \begin{cases} (\sin \frac{\pi}{y_1} + y_1 + 3y_2)(1 - x_2), \frac{1}{2}|y_1| - \frac{1}{4}y_1 + 3y_2^2 + x_2^2 y_1, & \text{if } y_1 \neq 0, \\ (0, -3y_2^2), & \text{if } y_1 = 0, \end{cases}$$

$$g = (g_1, g_2),$$

$$g_1(y) = \begin{cases} -\frac{y_1}{1+e^{\frac{1}{y_1}}} - y_2, & \text{if } y_1 \neq 0, \\ -y_2, & \text{if } y_1 = 0, \end{cases}$$

$$g_2(y) = y_2^2 - \frac{5}{2}y_2 + 1,$$

$$h(y) = y_1 - \frac{1}{2}y_2$$

( $x = (x_1, x_2) \in \mathbb{R}^2, y = (y_1, y_2) \in \mathbb{R}^2$ ). Then

$$F_{\bar{x}}(y) = \begin{cases} (\sin \frac{\pi}{y_1} + y_1 + 3y_2, \frac{1}{2}|y_1| - \frac{1}{4}y_1 + 3y_2^2), & \text{if } y_1 \neq 0, \\ (0, -3y_2^2), & \text{if } y_1 = 0, \end{cases}$$

We have  $K = \{(y_1, y_2) \in [0, 1] \times [\frac{1}{2}, 1] : y_1 = \frac{1}{2}y_2\}, T(C; \bar{x}) = \mathbb{R}_+^2$ , and  $N(C; \bar{x}) = \mathbb{R}_-^2$ , where  $\mathbb{R}_- = -\mathbb{R}_+$ . The pointed closed cone  $Q$  has the following bounded closed convex base  $B = \{(y_1, y_2) \in \mathbb{R}_+^2 : y_1 + y_2 = 1\}$ . It is easy to check that  $\text{dist}(0, B) = \frac{\sqrt{2}}{2}$ , where  $\text{dist}(0, B)$  indicates the distance from 0 to  $B$ . Taking  $U$  being the open ball of radius  $\delta = \frac{1}{2}$  around 0, then  $U$  is an absolutely convex neighborhood of 0 in  $\mathbb{R}^2$ , and

$$\text{cone} F_{\bar{x}}(K) \cap (U - B) = \emptyset.$$



Hence,  $\bar{x} = 0$  is a Henig efficient solution of the following vector equilibrium problem: Finding  $x \in K$  such that

$$F(x, y) \notin -Q \setminus \{0\} \quad (\forall y \in K).$$

It can be seen that  $\partial^* g_1(0) = \{(-1, -1), (0, -1)\}$ ,  $\partial^* g_2(0) = \{(0, -\frac{5}{2})\}$ ,  $\partial^* h(0) = \{(1, -\frac{1}{2})\}$ . Hence, taking  $\alpha > 0$ , one has  $(\alpha, 2\alpha) \in \mathbb{R}_+^2$ , and (CQ1) holds with  $b_1 = b_2 = 2\alpha$ . Thus all the hypotheses of Theorem 4.1 are fulfilled. Since the base  $B$  of  $Q$  is bounded and closed, then  $\bar{\lambda} \in \text{int}Q^*$ , and  $Q^\Delta(B) = \text{int}Q^* = \mathbb{R}_{++}^2$ . We have  $\partial^* F_{1, \bar{x}}(0) = \{(1, 3), (-1, 3)\}$ ,  $\partial_{2, \bar{x}}^*(0) = \{(-\frac{3}{4}, 0), (\frac{1}{4}, 0)\}$ . For  $\bar{\lambda} = (2, 2)$ ,  $\bar{\mu} = (\frac{1}{2}, \frac{1}{2})$ ,  $\bar{\nu} = 7$ , the optimality condition (9) in Theorem 3.2 holds at  $\bar{x} = (0, 0)$ :

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \in 2 \begin{pmatrix} \xi \\ 3 \end{pmatrix} + 2 \begin{pmatrix} \eta \\ 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} \zeta \\ -1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 \\ -\frac{5}{2} \end{pmatrix} + 7 \begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix} + \mathbb{R}_-^2$$

for  $-1 \leq \xi \leq 1$ ,  $-\frac{3}{4} \leq \eta \leq \frac{1}{4}$ ,  $-1 \leq \zeta \leq 0$ .

In what follows we give a Kuhn-Tucker necessary condition for global efficiency of (VEP).

**Theorem 4.2.** *Let  $\bar{x}$  be a local global efficient solution of (CVEP). Assume that  $F_{\bar{x}}(\bar{x}) = 0$ ; Assumption 3.1 is fulfilled; (CQ1) or (CQ2) holds. Then there exist  $\bar{\lambda} \in Q^\#$ ,  $\bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu} \in \mathbb{R}^\ell$ , such that*

$$0 \in \text{cl} \left( \sum_{k=1}^r \bar{\lambda}_k \text{conv} \partial^* F_{k, \bar{x}}(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \text{conv} \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right).$$

*Proof* As also in the proof of Theorem 4.1, we invoke Theorem 3.2 to deduce that there exist  $\bar{\tau} \geq 0$ ,  $\bar{\mu}_i \geq 0$  ( $i \in I(\bar{x})$ ),  $\bar{\nu} := (\bar{\nu}_1, \dots, \bar{\nu}_\ell) \in \mathbb{R}^\ell$  with  $(\bar{\tau}, \bar{\mu}_1, \dots, \bar{\mu}_{|I(\bar{x})|}) \neq 0$ , and a continuous positively homogeneous function  $\Lambda$  on  $Y$  satisfying that if  $y_2 - y_1 \in Q \setminus \{0\}$ , then  $\Lambda(y_1) < \Lambda(y_2)$  and (9) holds. Since (CQ1) or (CQ2) holds, one gets that  $\bar{\tau} > 0$ . By the same way as in the proof of Theorem 4.1, we arrive at

$$0 \in \text{cl} \left( \bar{\tau} \bar{\chi} (\text{conv} \partial^* F_{1, \bar{x}}(\bar{x}), \dots, \text{conv} \partial^* F_{r, \bar{x}}(\bar{x})) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv} \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\nu}_j \text{conv} \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right),$$

where  $\bar{\chi} \in \partial \Lambda(F_{\bar{x}}(\bar{x}))$ .

Let us see that  $\bar{\lambda} \in Q^\#$ . Indeed, observing that  $\Lambda$  is a convex function and  $\bar{\chi} \in \partial(\Lambda(F_{\bar{x}}(\bar{x})))$ . Hence, by Theorem 3.2, for any  $y \in Q \setminus \{0\}$ , one has

$$\begin{aligned} \langle \bar{\chi}, -y \rangle &\leq \Lambda(F_{\bar{x}}(\bar{x}) - y) - \Lambda(F_{\bar{x}}(\bar{x})) \\ &= \Lambda(-y) < \Lambda(0) = 0, \end{aligned}$$

as it can be rewritten as  $y = 0 - (-y) \in Q \setminus \{0\}$ . Hence,  $\bar{\chi} \in Q^\#$ . Since  $\bar{\tau} > 0$ , we obtain  $\bar{\lambda} := \bar{\tau}\bar{\chi} \in Q^\#$ .  $\square$

## 5. Sufficient conditions for efficiency

To derive sufficient conditions for Henig and global efficient solutions of (CVEP), we introduce some notions of generalized convexity. Let  $f$  be a function defined on  $X$  which admits an upper convexificator  $\partial^* f(\bar{x})$ . Adapting to the definition of asymptotic pseudoconvex functions in [14] and  $L$ -invex-infine on  $C$  at  $\bar{x} \in C$  in [2], we introduce the following definition.

**Definition 5.1** For  $\bar{\lambda} \in Q^\Delta(B)$ , the triple functions  $(\bar{\lambda}F_{\bar{x}}, g, h)$  is called asymptotic semiinvex-infine of type I at  $\bar{x}$  on  $C$  iff for any  $x \in C$ ,  $\chi_k^{(n)} \in \text{conv } \partial^* F_{k, \bar{x}}(\bar{x})$  ( $k \in J$ ),  $\xi_i^{(n)} \in \text{conv } \partial^* g_i(\bar{x})$  ( $i \in I(\bar{x})$ ),  $\eta_j^{(n)} \in \text{conv } \partial^* h_j(\bar{x})$  ( $j \in L$ ), there exists  $v \in N(C; \bar{x})^\circ$  satisfying

(a) Asymptotic pseudoinvexity-infine to  $\bar{\lambda}F_{\bar{x}}$  :

$$\lim_{n \rightarrow \infty} \bar{\lambda}_k \langle \chi_k^{(n)}, v \rangle \geq 0 \implies \bar{\lambda}F_{\bar{x}}(x) \geq \bar{\lambda}F_{\bar{x}}(\bar{x}),$$

(b) Asymptotic quasiinvexity-infine to  $g$  :

$$g_i(x) \leq g_i(\bar{x}) \implies \lim_{n \rightarrow \infty} \langle \xi_i^{(n)}, v \rangle \leq 0 \quad (\forall i \in I(\bar{x})),$$

(c) Asymptotic linearinvexity-infine to  $h$  :

$$h_j(x) = h_j(\bar{x}) \implies \lim_{n \rightarrow \infty} \langle \eta_j^{(n)}, v \rangle = 0 \quad (\forall j \in L).$$

### Remark 5.1

(a) In case that  $C$  is convex, it can be taken  $v := x - \bar{x}$ , as  $T(C; \bar{x}) = \mathbb{R}_+(C - \bar{x})$ .

(b) If  $C$  is convex,  $\bar{\lambda}F_{\bar{x}}$  is pseudoconvex at  $\bar{x}$  on  $C$ ,  $g_i$  ( $\forall i \in I(\bar{x})$ ) are quasiconvex at  $\bar{x}$  on  $C$ ,  $\pm h_j$  ( $\forall j \in L$ ) are quasiconvex at  $\bar{x}$  on  $C$ , then  $(\bar{\lambda}F_{\bar{x}}, g, h)$  is asymptotic semiinvex-infine of type I at  $\bar{x}$  on  $C$ .

(c) If  $C$  is convex,  $\bar{\lambda}F_{\bar{x}}$  is asymptotic pseudoconvex at  $\bar{x}$  on  $C$ ,  $g_i$  ( $i \in I(\bar{x})$ ) are asymptotic at  $\bar{x}$  on  $C$ ,  $h_j$  ( $j \in L$ ) are asymptotic quasilinear at  $\bar{x}$  (see [14]), then  $(\bar{\lambda}F_{\bar{x}}, g, h)$  is asymptotic semiinvex-infine of type II at  $\bar{x}$  on  $C$ .

**Definition 5.2** Let  $M \subset \mathbb{R}^r$  be a pointed convex cone such that  $Q \setminus \{0\} \subset \text{int}M$ . For  $\bar{\lambda} \in M^\#$ , the triple functions  $(\bar{\lambda}F_{\bar{x}}, g, h)$  is called asymptotic semiinvex-infine of type II at  $\bar{x}$  on  $C$  iff for any  $x \in C$ ,  $\chi_k^{(n)} \in \text{conv } \partial^* F_{k, \bar{x}}(\bar{x})$  ( $k \in J$ ),  $\xi_i^{(n)} \in \text{conv } \partial^* g_i(\bar{x})$  ( $i \in I(\bar{x})$ ),  $\eta_j^{(n)} \in \text{conv } \partial^* h_j(\bar{x})$  ( $j \in L$ ), there exists  $v \in N(C; \bar{x})^\circ$  satisfying (a)–(c).

In what follows we shall give a sufficient condition for Henig efficient solutions of (CVEP).

**Theorem 5.1.** Let  $\bar{x} \in K$ . Assume that  $F_{\bar{x}}(\bar{x}) = 0$ , and

(i) There exist  $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_r) \in Q^\Delta(B)$ ,  $\bar{\mu}_i \geq 0$  ( $\forall i \in I(\bar{x})$ ),  $\bar{\gamma}_j \in \mathbb{R}$  ( $\forall j \in L$ ) such that

$$0 \in \text{cl} \left( \sum_{k \in J} \bar{\lambda}_k \text{conv } \partial^* F_{k, \bar{x}}(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv } \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\gamma}_j \text{conv } \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right). \quad (17)$$

(ii) All but at most one of the upper convexifiers  $\partial^* F_{k, \bar{x}}(\bar{x})$  ( $k \in J$ ) are upper regular at  $\bar{x}$ . Assume that the functions  $(\bar{\lambda}F_{\bar{x}}, g, h)$  is asymptotic semiinvex-infine of type I at  $\bar{x}$  on  $C$ .

Then  $\bar{x}$  is a Henig efficient solution of (CVEP).

*Proof* By (17), there exist  $\chi_k^{(n)} \in \text{conv } \partial^* F_{k, \bar{x}}(\bar{x})$  ( $k \in J$ ),  $\xi_i^{(n)} \in \text{conv } \partial^* g_i(\bar{x})$  ( $i \in I(\bar{x})$ ),  $\eta_j^{(n)} \in \text{conv } \partial^* h_j(\bar{x})$  ( $j \in L$ ),  $\zeta^{(n)} \in N(C; \bar{x})$  such that

$$0 = \lim_{n \rightarrow \infty} \left[ \sum_{k \in J} \bar{\lambda}_k \chi_k^{(n)} + \sum_{i \in I(\bar{x})} \bar{\mu}_i \xi_i^{(n)} + \sum_{j \in L} \bar{\gamma}_j \eta_j^{(n)} + \zeta^{(n)} \right].$$

Then, by the asymptotic semiinvexity-infine of  $(\bar{\lambda}F_{\bar{x}}, g, h)$  at  $\bar{x}$  on  $C$ , there exists  $v \in N(C; \bar{x})^\circ$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \left[ \sum_{k \in J} \bar{\lambda}_k \langle \chi_k^{(n)}, v \rangle + \sum_{i \in I(\bar{x})} \bar{\mu}_i \langle \xi_i^{(n)}, v \rangle \right. \\ \left. + \sum_{j \in L} \bar{\gamma}_j \langle \eta_j^{(n)}, v \rangle + \langle \zeta^{(n)}, v \rangle \right] = 0. \end{aligned} \quad (18)$$

Observe that for all  $x \in K$ ,  $g_i(x) \leq 0 = g_i(\bar{x})$  ( $\forall i \in I(\bar{x})$ ). In view of the asymptotic quasiinvexity-infine of  $g_i$  at  $\bar{x}$  on  $C$ , for all  $x \in K$ , we have

$$\lim_{n \rightarrow \infty} \langle \xi_i^{(n)}, v \rangle \leq 0. \quad (19)$$

Since  $h_j(x) = 0 = h_j(\bar{x})$  ( $\forall x \in K$ ), by virtue of the asymptotic linearinvexity-infine of  $h_j$  ( $\forall j \in L$ ) at  $\bar{x}$  on  $C$ , it follows that for all  $x \in K$ ,

$$\lim_{n \rightarrow \infty} \langle \eta_j^{(n)}, v \rangle = 0. \quad (20)$$

Due to  $v \in N(C; \bar{x})^\circ$ , one has

$$\lim_{n \rightarrow \infty} \langle \zeta^{(n)}, v \rangle \leq 0. \quad (21)$$

Since all but at most one of the upper convexificators  $\partial^* F_{k, \bar{x}}(\bar{x})$  ( $k \in J$ ) are upper regular, by Rule 4.2 [7],  $\sum_{k \in J} \bar{\lambda}_k \partial^* F_{k, \bar{x}}(\bar{x})$  is an upper convexificator for the function  $\sum_{k \in J} \bar{\lambda}_k F_{k, \bar{x}}$  at  $\bar{x}$ . Combining (18)–(21) yields that for all  $x \in K$ ,

$$\lim_{n \rightarrow \infty} \left\langle \sum_{k \in J} \bar{\lambda}_k \chi_k^{(n)}, x - \bar{x} \right\rangle \geq 0.$$

In view of the asymptotic pseudoconvexity-infine of  $\bar{\lambda} F_{\bar{x}}(\cdot)$  at  $\bar{x}$ , we claim that for all  $x \in K$ ,

$$\bar{\lambda} F_{\bar{x}}(x) \geq \bar{\lambda} F_{\bar{x}}(\bar{x}) = 0. \quad (22)$$

Let us see that  $\bar{x}$  is a Henig efficient solution of (CVEP). If this were not so, for every open absolutely convex neighborhood  $U \subseteq V_B$  of 0, by (3), there would be

$$F_{\bar{x}}(K) \cap (-\text{int} Q_U(B)) \neq \emptyset. \quad (23)$$

Lemma 4.5 [5] shows that for  $\bar{\lambda} \in Q^\Delta(B)$ , there exists an open absolutely convex neighborhood  $U_0 \subseteq V_B$  of 0 such that  $\bar{\lambda} \in (Q_{U_0}(B))^* \setminus \{0\} \subseteq Q^\Delta(B)$ . Hence, there exists  $y_1 \in K$  such that

$$F_{\bar{x}}(y_1) \in -\text{int} Q_{U_0}(B).$$

Therefore,  $\bar{\lambda} F_{\bar{x}}(y_1) < 0 = F_{\bar{x}}(\bar{x})$ , which conflicts with (22). Consequently,  $\bar{x}$  is a Henig efficient solution of (CVEP).  $\square$

In the sequel we give a sufficient condition for globally efficient solutions of (CVEP).

**Theorem 5.2.** *Let  $\bar{x} \in K$ . Assume that  $F_{\bar{x}}(\bar{x}) = 0$ , and  $M \subset \mathbb{R}^r$  is a pointed convex cone such that  $Q \setminus \{0\} \subset \text{int} M$ , and*

(i) *There exist  $\bar{\lambda} := (\bar{\lambda}_1, \dots, \bar{\lambda}_r) \in M^\#$ ,  $\bar{\mu}_i \geq 0$  ( $\forall i \in I(\bar{x})$ ),  $\bar{\gamma}_j \in \mathbb{R}$  ( $\forall j \in L$ ) such that*

$$0 \in \text{cl} \left( \sum_{k \in J} \bar{\lambda}_k \text{conv } \partial^* F_{k, \bar{x}}(\bar{x}) + \sum_{i \in I(\bar{x})} \bar{\mu}_i \text{conv } \partial^* g_i(\bar{x}) + \sum_{j \in L} \bar{\gamma}_j \text{conv } \partial^* h_j(\bar{x}) + N(C; \bar{x}) \right).$$

(ii) All but at most one of the upper convexifiers  $\partial^* F_{k,\bar{x}}(\bar{x})$  ( $k \in J$ ) are upper regular at  $\bar{x}$ . Assume that the functions  $(\bar{\lambda}F_{\bar{x}}, g, h)$  is asymptotic semiinconvex-infine of type II at  $\bar{x}$  on  $C$ .

Then  $\bar{x}$  is a globally efficient solution of (CVEP).

*Proof* By an argument analogous to that use for the proof of Theorem 5.1, we get that for all  $x \in K$ ,

$$\bar{\lambda}F_{\bar{x}}(x) \geq \bar{\lambda}F_{\bar{x}}(\bar{x}) = 0. \tag{24}$$

Let us see that  $\bar{x}$  is a globally efficient solution of (CVEP). If  $\bar{x}$  is not a global efficient solution of (CVEP), then

$$F_{\bar{x}}(K) \cap (-M \setminus \{0\}) \neq \emptyset.$$

Hence, there exists  $y_1 \in K$  such that

$$F_{\bar{x}}(y_1) \in -M \setminus \{0\}.$$

Since  $\bar{\lambda} \in M^\#$ , it follows that  $\bar{\lambda}F_{\bar{x}}(y_1) < 0 = F_{\bar{x}}(\bar{x})$ , which conflicts with (24). Consequently,  $\bar{x}$  is a globally efficient solution of (CVEP).  $\square$

## 6. Conclusions

We derive Fritz John and Kuhn-Tucker necessary conditions for local Henig and global efficient solutions of vector equilibrium problems involving nonsmooth equality, inequality and set constraints via convexifiers. The Kuhn-Tucker necessary conditions obtained here via convexifiers can be sharper than those expressed in terms of the Clarke subdifferentials. Under assumptions on asymptotic semiinconvexity-infine of type I or type II of the triple  $(\bar{\lambda}F_{\bar{x}}, g, h)$ , sufficient conditions for Henig and globally efficient solutions are established. The results obtained in this paper are more general than those obtained by Gong [5] for vector equilibrium problems with only a set constraint, and those obtained by Long et al. [8] for vector equilibrium problems with subconvexlike functions.

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# INFORMATION-LEAKAGE-FREE QUANTUM DIALOGUE VIA GREENBERGER-HORNE-ZEILINGER STATES

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**Abstract.** Secure communication in general and secure dialogue in particular are highly demanded, especially in the current information exploding era. Here we are concerned with secure dialogue. Because any dialogue conducted merely by classical means is fully eavesdropped without traces left behind, quantum version of dialogue, the so-called quantum dialogue, offers a promising solution to the security problem. The security desired does not simply focus on the exchanged information but also on their classical correlations, i.e., a quantum dialogue protocol should be protected from both information theft and information leakage. Such a secure quantum dialogue protocol is proposed in this paper employing Greenberger–Horne–Zeilinger states as the quantum channel. The above-mentioned requirement for security is achieved in message rounds by using extra random bits for the encoding/decoding processes combined with two kinds of control rounds which are designed to detect eavesdropping, if any.

## 1. Introduction

Quantum mechanics (QM) was invented to make sense of physical phenomena occurring in the microscopic world. Theoretically, it is a set of postulates built to explore the invisible quantum universe. It is highly counter-intuitive exhibiting bizarre traits such as uncertainty relation, wave-particle duality, no-cloning theorem, impossibility to gain information without measurement, measurement yields probabilistic outcomes and collapses the measured object, etc. which are not encountered in the everyday macroscopic life.

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*Key words and phrases:* Quantum dialogue, GHZ states, information leakage.

At the very heart of QM are the state superposition and quantum entanglement. If a quantum system can exist in either one of a number of different quantum states, then it can also exist in a state that is linearly superposed of those states. More surprisingly, if two quantum subsystems are entangled with each other, then they behave as a whole entity losing their individuality and from a probabilistic outcome of measurement on one subsystem the state of the other untouched subsystem can be deterministically predicted, regardless of the distance between the two subsystems. This constitutes what was commonly referred to as Einstein-Podolsky-Rosen (ERP) paradox, which, in Einstein's words, implies "*spooky action at a distance*" [1]. Based on the state superposition and quantum entanglement many intriguing protocols are possible such as quantum teleportation [2], quantum superdense coding [3], quantum secret sharing [4], remote state preparation [5], joint remote state preparation [6] and so on, all of which find no counterparts in the classical world.

Of importance is the topic of secure communication. Because classical communication is totally insecure, informative messages should not be directly transferred via public media means. Instead, encrypted messages are sent which will be decrypted upon receipt. Absolute security was proved in the private key system using an encryption technique that cannot be cracked (see, e.g., [7]): the communicating partners share in advance a secret key to encode/decode the real message. However, there is a big inconvenience because the partners must meet in person for key sharing and each key must be used only one time (so the name "one-time-pad" encryption). At present, widely used is the public key system [8] in which each partner has two keys, one is put in the public key directory accessible to everyone and the other kept secret. Either key can be used to encrypt the message but decryption requires both the keys. The two keys are created using a mathematical recipe in such a way that it is extremely hard to obtain the secret key from the known public key. Thus, any sender is able to use the public key of a wanted receiver to encrypt a message but only the relevant receiver could decrypt it. The public key system is very convenient because there is no need of a prior secret key sharing as in the private key system. Nevertheless, its security is not unconditional: whenever quantum computer (a future device that can, by performing a proper quantum algorithm, easily calculate the secret key given the public one) comes to birth the public key system will be entirely broken.

Because genuine quantum computer will sooner or later be produced, one may bypass the public key system and try to more creatively exploit the proved absolute security of the "one-time-pad" encryption. It would be nice if the secret key sharing process could be done remotely under the nose of an outsider who attempts to gain content of the key. In this connection, QM enters the game showing its power through quantum key distribution (QKD) protocols which rely on superposition principle [9] or quantum entanglement [10]. The

unconditional security of QKD protocols is guaranteed by the foundations of QM, in contrast to the public key system, which relies on the computational difficulty of reversing a certain one-way mathematical function. Although QKD can be made remotely, the real message cannot be read before QKD. Hence, a reasonable problem arises as for how can ones communicate securely in urgent situations when there is no time to perform any QKD protocol? To cope with such an issue, schemes of quantum secure direct communication (QSDC) [11, 12, 13, 14, 15] have been developed allowing sending secret messages from one partner to another without a prior QKD. Yet, QSDC is just a unidirectional protocol by which the partners cannot at the same time exchange their messages. In 2004 a new kind of protocol was, for the first time, devised which is bidirectional favoring two partners to communicate with each other without doing QKD in advance, i.e., in a fashion much like in a dialogue, so the terminology "quantum dialogue" (QD) [16] (see also [17]). Since then a great deal of QD protocols (see, e.g., [18, 19, 20, 21, 22, 23, 24, 25, 26, 27]) has been put forward under different angles and via various quantum resources.

Let Alice and Bob be two remote partners enjoying a QD protocol. In all the above-cited QD protocols, although Alice and Bob safely obtain each other's information, there exists a security loophole that any third party is able to obtain some classical correlation between the partners' information simply by listening to their public announcements [28, 29]. In information theory this kind of security loophole bears the name "information leakage". To get rid of the information leakage problem, a number of interesting protocols have been devised. Those protocols utilize different quantum resources/technologies such as EPR pairs [30], two-qutrit entangled states [31], W states [32], single quantum entangled states [33], single-photon states [34], auxiliary quantum operations [35], hyperentanglement [36], entanglement swapping [37], quantum authentication [38], single photons in both polarization and spatial-mode degrees of freedom [39] and reference frame independence combined with measurement device independence [40] and so on. However, the above proposals employ ordered batches of quantum states, so the full message can be read only at the end of a protocol, losing the taste of a dialogue.

In this paper we suggest an information-leakage-free quantum dialogue protocol using three-qubit Greenberger–Horne–Zeilinger (GHZ) states [41] of which two qubits travel forth and back between Alice and Bob. The desired security is ensured by random checking possible eavesdropping in both directions from Alice to Bob and vice versa. Classical correlations between Alice's and Bob's information are not leaked out to any unauthorized outsider thanks to a judicious fashion of encryption/decryption. In Section 2 we outline the GHZ states. Section 3 describes the encoding and decoding processes. Section 4 presents typical kinds of eavesdropping. Section 5 introduces methods to detect eavesdropping attacks. Section 6 is the quantum dialogue protocol.

Finally, Section 7 is the conclusion.

## 2. GHZ states

There are two non-equivalent classes of genuine tripartite entangled states, the GHZ class and the W one [42]. Here we are concerned with the GHZ class. In terms of the single-qubit Pauli operators

$$(2.1) \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$(2.2) \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the complete set of the GHZ class consists of 8 orthonormal states  $\{|G_{ijk}\rangle_{ABC}; i, j, k \in \{0, 1\}\}$  that exhibit entanglement among three qubits  $A, B$  and  $C$  in the following form

$$(2.3) \quad |G_{i,j,k}\rangle_{ABC} = Z_A^i X_A^j \otimes X_B^k |G_{0,0,0}\rangle_{ABC},$$

where  $Z_A^i X_A^j$  act on qubit  $A$ ,  $X_B^k$  on qubit  $B$  and

$$(2.4) \quad |G_{0,0,0}\rangle_{ABC} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{ABC},$$

with  $|mnl\rangle_{ABC} \equiv |m\rangle_A \otimes |n\rangle_B \otimes |l\rangle_C \equiv |m\rangle_A |n\rangle_B |l\rangle_C$  for any  $m, n, l \in \{0, 1\}$ . The entangled state  $|G_{0,0,0}\rangle_{ABC}$  can be generated from the product state  $|000\rangle_{ABC}$  by application of the unitary operators  $CNOT_{AC}CNOT_{BC}H_C$ , with  $H_C$  the single-qubit Hadarmard gate,

$$(2.5) \quad H_A |m\rangle_A = \frac{1}{\sqrt{2}}[(-1)^m |m\rangle_A + |m \oplus 1\rangle_A],$$

and  $CNOT_{TC}$  the two-qubit controlled-NOT ( $CNOT$ ) gate with  $C$  the control qubit and  $T$  the target one,

$$(2.6) \quad CNOT_{TC} |m\rangle_T |n\rangle_C = |m \oplus n\rangle_T |n\rangle_C,$$

where  $\oplus$  represents the XOR operation. Indeed,

$$\begin{aligned}
& CNOT_{AC}CNOT_{BC}H_C |000\rangle_{ABC} \\
&= CNOT_{AC}CNOT_{BC} |0\rangle_A |0\rangle_B \frac{(|0\rangle + |1\rangle)_C}{\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}CNOT_{AC} |0\rangle_A (|00\rangle + |11\rangle)_{BC} \\
(2.7) \quad &= \frac{1}{\sqrt{2}} |000\rangle + |111\rangle)_{ABC} \equiv |G_{0,0,0}\rangle_{ABC}.
\end{aligned}$$

Putting (2.4) into (2.3) and resorting to the action rule of the Pauli operators,

$$(2.8) \quad X^p |m\rangle = |m \oplus p\rangle,$$

$$(2.9) \quad Z^q |m\rangle = (-1)^{qm} |m\rangle,$$

where  $|m\rangle$  is a Fock state with  $m \in \{0, 1\}$  and  $p, q$  are any nonnegative integers, we have another more explicit expression of  $|G_{i,j,k}\rangle_{ABC}$  :

$$(2.10) \quad |G_{i,j,k}\rangle_{ABC} = \frac{1}{\sqrt{2}} [|j\rangle_A |k\rangle_B |0\rangle_C + (-1)^i |j \oplus 1\rangle_A |k \oplus 1\rangle_B |1\rangle_C].$$

Furthermore, in terms of two-qubit maximally entangled states (the Bell-states),

$$(2.11) \quad |\mathcal{B}_{mn}\rangle_{AB} = \frac{1}{\sqrt{2}} \sum_{s=0}^1 (-1)^{ms} |s\rangle_A |s \oplus n\rangle_B$$

and the Hadamard-states

$$(2.12) \quad |\pm\rangle_C = \frac{1}{\sqrt{2}} (|0\rangle_C \pm |1\rangle_C),$$

$|G_{i,j,k}\rangle_{ABC}$  can also be re-expressed as

$$(2.13) \quad |G_{i,j,k}\rangle_{ABC} = \frac{1}{\sqrt{2}} [|\mathcal{B}_{i,j \oplus k}\rangle_{AB} |+\rangle_C + (-1)^j |\mathcal{B}_{i \oplus 1, j \oplus k}\rangle_{AB} |-\rangle_C].$$

The expressions (2.3), (2.10) and (2.13) for  $|G_{i,j,k}\rangle_{ABC}$  are helpful for later consideration.

### 3. Exchanging secret bits

Suppose that Alice has two secret bits  $a_1, a_2$  while Bob has one secret bit  $b_1$ . How can they securely exchange their secret bits? As a reminder, the

security here is meant in the sense that no third party is able to learn *any* information about Alice's and Bob's bits, i.e., not only  $a_1, a_2, b_1$  themselves but also their classical correlations (i.e., their XOR values)  $a_1 \oplus a_2, a_1 \oplus b_1, a_2 \oplus b_1, a_1 \oplus a_2 \oplus b_1$  must be kept confidential from the outsider. To achieve such level of security Bob prepares a GHZ state  $|G_{i,j,k}\rangle_{ABC}$  with certain  $i, j, k \in \{0, 1\}$  which are picked up at his own choice (that is, except Bob noone knows the values of  $i, j, k$ ). Bob keeps qubit  $C$  with himself but sends qubits  $A, B$  to Alice. Alice applies  $Z_A^{a_1} X_A^{a_2}$  on qubit  $A$  and  $X_B^r$  on qubit  $B$ , with  $r$  a random bit, transforming  $|G_{i,j,k}\rangle_{ABC}$  to  $Z_A^{a_1} X_A^{a_2} \otimes X_B^r |G_{i,j,k}\rangle_{ABC}$ . On one hand, using the relationships

$$(3.1) \quad X^a X^b = X^{a \oplus b},$$

$$(3.2) \quad Z^a Z^b = Z^{a \oplus b},$$

$$(3.3) \quad Z^a X^b = (-1)^{ab} X^b Z^a$$

and (2.3) we have (up to a common sign)

$$(3.4) \quad \begin{aligned} Z_A^{a_1} X_A^{a_2} \otimes X_B^r |G_{i,j,k}\rangle_{ABC} &= Z_A^{i \oplus a_1} X_A^{j \oplus a_2} \otimes X_B^{k \oplus r} |G_{0,0,0}\rangle_{ABC} \\ &= |G_{i \oplus a_1, j \oplus a_2, k \oplus r}\rangle_{ABC}. \end{aligned}$$

On the other hand, using (2.10) with the action rules (2.8) and (2.9) yields

$$(3.5) \quad \begin{aligned} &Z_A^{a_1} X_A^{a_2} \otimes X_B^r |G_{i,j,k}\rangle_{ABC} \\ &= \frac{1}{\sqrt{2}} Z_A^{a_1} X_A^{a_2} \otimes X_B^r [|j\rangle_A |k\rangle_B |0\rangle_C \\ &\quad + (-1)^i |j \oplus 1\rangle_A |k \oplus 1\rangle_B |1\rangle_C] \\ &= \frac{1}{\sqrt{2}} [|j \oplus a_2\rangle_A |k \oplus r\rangle_B |0\rangle_C \\ &\quad + (-1)^{i+a_1} |j \oplus a_2 \oplus 1\rangle_A |k \oplus r \oplus 1\rangle_B |1\rangle_C]. \end{aligned}$$

After application of  $Z_A^{a_1} X_A^{a_2} \otimes X_B^r$  on  $|G_{i,j,k}\rangle_{ABC}$ , Alice sends qubits  $A, B$  back to Bob, who is able to extract Alice's bits  $a_1, a_2, r$  by performing a suitable joint measurement on the three qubits  $A, B, C$ . Namely, Bob's measurement proceeds as follows. First, he applies  $H_C CNOT_{BC} CNOT_{AC}$  on  $Z_A^{a_1} X_A^{a_2} \otimes$

$X_B^r |G_{i,j,k}\rangle_{ABC}$ . By virtue of (2.5), (2.6) and (3.5) Bob gets

$$\begin{aligned}
& H_C CNOT_{BC} CNOT_{AC} Z_A^{a_1} X_A^{a_2} \otimes X_B^r |G_{i,j,k}\rangle_{ABC} \\
= & \frac{1}{\sqrt{2}} H_C CNOT_{BC} CNOT_{AC} [|j \oplus a_2\rangle_A |k \oplus r\rangle_B |0\rangle_C \\
& + (-1)^{(i \oplus a_1)} |j \oplus a_2 \oplus 1\rangle_A |k \oplus r \oplus 1\rangle_B |1\rangle_C] \\
= & \frac{1}{\sqrt{2}} |j \oplus a_2\rangle_A H_C CNOT_{BC} [|k \oplus r\rangle_B |0\rangle_C \\
& + (-1)^{(i \oplus a_1)} |k \oplus r \oplus 1\rangle_B |1\rangle_C] \\
= & \frac{1}{\sqrt{2}} |j \oplus a_2\rangle_A |k \oplus r\rangle_B H_C [|0\rangle_C + (-1)^{(i \oplus a_1)} |1\rangle_C] \\
(3.6) \quad = & |j \oplus a_2\rangle_A |k \oplus r\rangle_B |i \oplus a_1\rangle_C.
\end{aligned}$$

Next, since the qubits  $A, B, C$  have become disentangled, Bob independently measures each of them in their computational bases. If the outcomes of measurement on qubits  $A, B$  and  $C$  are respectively  $a, b$  and  $c$ , then, as seen from (3.6),  $a = j \oplus a_2$ ,  $b = k \oplus r$  and  $c = i \oplus a_1$ . Because Bob knew  $i, j$  and  $k$ , it is straightforward for him and only him to decode Alice's bits as  $a_1 = c \oplus i$ ,  $a_2 = a \oplus j$  and  $r = b \oplus k$ . Because the bit  $r$  is random, Bob can exploit it to hide his secret bit  $b_1$  in  $d = b_1 \oplus r$  and publicly announces  $d$  via a reliable (classical) channel to enable Alice to decode Bob's secret bit  $b_1$  as  $d \oplus r$ . Only Alice is able to do the correct decoding because  $r$  was set by herself. Of course, a third party can hear  $d$  from Bob's public announcement but he/she can by no means infer  $b_1$  from  $d$  because  $d$  itself is random thanks to the randomness of  $r$ . Note that  $d$  does not express any classical correlations between  $a_1, a_2$  and  $b_1$  so no information leakage occurs, in contrast to Refs. [18, 19, 20, 21, 22, 23, 24, 25, 26, 27]. All the above-described actions constitute a round called message round. In the message round the secret bits are exchanged securely between Alice and Bob without any information leakage, if there are no attacks from the outsider.

#### 4. Outsider's attacks

In practice there is often an outside enemy intending to eavesdrop others' communication. Name such eavesdropper Eve who is supposed to be capable of doing anything allowed by the laws of QM. As qubit  $C$  remains always with Bob, Eve can physically attack only qubits  $A$  and  $B$  when they travel forth and back between Alice and Bob. Eve is aware that these qubits are members of a GHZ state, which is maximally entangled, so their reduced density matrix

is  $I/2$  ( $I$  is the  $2 \times 2$  identity matrix) and no information can be extracted from them. Hence, one of Eve's strategies is to disturb the secret bits' exchanging.

Eve can undertake the so-called measure-resend attack: when qubits  $A, B$  are traveling Eve measures them and then let them go on along their route. Eve may utilize either the computational basis  $\{|0\rangle_A, |1\rangle_A\}$  to measure one qubit (say, qubit  $A$ ) or the Bell basis  $\{|\mathcal{B}_{mn}\rangle_{AB}; m, n \in \{0, 1\}\}$  to measure two qubits  $A$  and  $B$  jointly. In case of the computational basis  $|G_{i,j,k}\rangle_{ABC}$  collapses to  $|j\rangle_A |k\rangle_B |0\rangle_C$  or  $|j \oplus 1\rangle_A |k \oplus 1\rangle_B |1\rangle_C$ , while in case of the Bell basis  $|G_{i,j,k}\rangle_{ABC} \rightarrow |\mathcal{B}_{i,j \oplus k}\rangle_{AB} |+\rangle_C$  or  $|\mathcal{B}_{i \oplus 1, j \oplus k}\rangle_{AB} |-\rangle_C$ . In both cases the genuine tripartite entanglement is demolished and Bob's joint measurement on the three qubits  $A, B, C$  would generally yields the outcomes  $a \neq j \oplus a_2$ ,  $b \neq k \oplus r$  and  $c \neq i \oplus a_1$ . As a consequence, Bob's decoding is wrong and so is Alice's.

An easier way for Eve to disturb is application of the Pauli  $X$  operator on either qubit  $A$  or  $B$ . This does not destroy the tripartite entanglement but changes  $|G_{i,j,k}\rangle_{ABC}$  to another GHZ state, i.e., modifies the quantum correlation on which Alice and Bob rely for securely exchanging their secret bits. Such type of disturbance is sometimes referred to as denial-of-service attack.

Also, Eve can implement another kind of attack in which she captures qubits  $A, B$  while they travel from Bob to Alice and replace them by two ancillary qubits  $A', B'$  which are of course not entangled with qubit  $C$ . By Eve's doing so, the bits that Alice and Bob decrypt would differ from those they expect to obtain. Because of the manner this kind of attack is implemented, it gets its own name too: the capture-replace attack. A disadvantage of the capture-replace attack is the cost to pay for ancillary qubits  $A', B'$ .

Interestingly, there is a delicate kind of attack under the name intercept-replace attack. This kind of attack allows Eve to gain full information at a cost of consuming additional quantum entanglement resource together with quantum memory. It is pretty wise and proceeds as follows. Eve prepares ahead a GHZ state  $|G_{i',j',k'}\rangle_{A'B'C'}$  of her three qubits  $A', B', C'$  with certain  $i', j', k'$  chosen at her will and ambushes *en route* between Bob and Alice. When Bob sends qubits  $A, B$  of the state  $|G_{ijk}\rangle_{ABC}$  to Alice, Eve intercepts them and stores them in her quantum memory. After that she keeps qubit  $C'$  with herself and sends qubits  $A', B'$  to Alice. Alice (being unaware of the qubits' substitution:  $A, B \rightarrow A', B'$ ) encodes her secret bits  $a_1, a_2$  on  $A'$  and a random bit  $r$  on  $B'$  then sends  $A', B'$  back to Bob. This time Eve intercepts the qubits  $A', B'$  and performs a suitable joint measurement on  $A', B', C'$  to learn the values of  $a_1, a_2$  and  $r$ . Having known  $a_1, a_2, r$  Eve takes out the qubits  $A, B$  which she has previously stored in the quantum memory and encodes  $a_1, a_2$  on qubit  $A$  while  $r$  on qubit  $B$ , followed by sending  $A, B$  back to Bob. Bob, with all the three qubits  $A, B, C$  at hand, is in the position to carry out a suitable joint measurement on the trio to readily infer the bits  $a_1, a_2, r$ . Finally, Bob



publicly discloses the bit  $d = b_1 \oplus r$  from which not only Alice but also Eve can deduce Bob's secret bit  $b_1 = d \oplus r$ . In other words, by the intercept-replace attack Eve is able to eavesdrop all the secret bits that Alice and Bob have exchanged.

## 5. Unmasking eavesdropper

To detect presence of the eavesdropper Eve, Alice and Bob must deploy appropriate checking methods. One of the checking methods is like this. Alice and Bob switch the mode of actions from exchanging secret bits to detecting Eve's possible interference. Then, instead of the encoding procedure as in the message round mentioned in Section 3, Alice measures qubits  $A$  and  $B$ , while Bob measures qubit  $C$ , with their measurement outcomes to be compared. They have two options for their measurement bases. In the first option both Alice and Bob use the computational bases  $\{|0\rangle_{A(B,C)}, |1\rangle_{A(B,C)}\}$ . In the second option Alice uses the Bell basis  $\{|\mathcal{B}_{mn}\rangle_{AB}; m, n \in \{0, 1\}\}$  but Bob the Hadamard basis  $\{|\pm\rangle_C\}$ . If it is the first option and the measurement outcomes are  $x, y, z \in \{0, 1\}$  (corresponding to the event when Alice finds states  $|x\rangle_A |y\rangle_B$  and Bob finds state  $|z\rangle_C$ ) then, from (2.10), the outcomes must satisfy the following constraint: either  $\{x = j, y = k, z = 0\}$  or  $\{x = j \oplus 1, y = k \oplus 1, z = 1\}$ . If it is the second option, then, from (2.13), the measurement outcomes must satisfy the constraint that Alice finds  $|\mathcal{B}_{i,j \oplus k}\rangle_{AB}$  and Bob finds  $|+\rangle_C$  or Alice finds  $|\mathcal{B}_{i \oplus 1, j \oplus k}\rangle_{AB}$  and Bob finds  $|-\rangle_C$ . It can be verified that the above-specified constraints are generally violated by the denial-of-service, capture-replace and intercept-replace attacks. For the measure-resend attack, if Eve uses the computational basis when Alice and Bob choose the first option or if Eve uses the Bell basis when Alice and Bob choose the second option, then Eve safely passes the test. This feature helps to correctly design the checking method. In detail, Bob should decide to switch to the checking mode after Alice's confirmation of her receipt of qubits and only then Alice and Bob discuss with each other on the choice of measurement option. Because Eve undertakes the measure-resend attack earlier she does not know the option chosen by Alice and Bob. It is this fact that could unmask presence of Eve in the Bob-to-Alice route. However, Eve can attack on the Alice-to-Bob route as well. So, that route must also be 'guarded'. This time Alice is the person who decides to switch to the checking mode in which Alice encodes three random bits  $r_1, r_2, r_3$  and sends the encoded qubits back to Bob. Upon Bob's receipt of the qubits Alice requests Bob to cooperate as follows. First, Bob performs the joint measurement on the three qubits  $A, B, C$  as described in Section 3 to obtain the outcomes  $a, b, c$ . After

that Alice tells Bob the values of  $r_1, r_2, r_3$ . Since Bob knows  $i, j, k$  he is able to check whether all the equalities  $a = j \oplus r_2$ ,  $b = k \oplus r_3$ ,  $c = i \oplus r_1$  hold or not. Transparently, if Eve attacks along the Alice-to-Bob route then the above equalities will not be always held. Therefore, if both routes from Bob to Alice and from Alice to Bob are often put under checking processes the probability for Eve to survive is vanishing after a large enough number of checking process. To not confuse between the two kinds of checking processes, the round of checking in the Bob-to-Alice route is called forward checking round, while the round of checking in the Alice-to-Bob route is called backward checking round.

## 6. Quantum dialogue protocol

So far three kinds of rounds of action have been designed, which are message round, forward checking round and backward checking round. The message round allows Alice and Bob to exchange their bits, the forward checking round tests Eve's interference during the time when Bob sends his two qubits to Alice and the backward checking round controls the time when Alice returns the two qubits back to Bob.

Quantum dialogue protocol consists of a number of consecutive rounds of actions each of which takes place impromptu with a probability  $p_m, p_f$  and  $p_b$  ( $p_m + p_f + p_b = 1$ ) for message round, forward checking round and backward checking round, respectively. If in a checking (either forward or backward) round Eve is unmasked, Alice and Bob abort the protocol. Otherwise, in each message round Alice "asks" by two bits and Bob "answers" by one bit. Therefore, message round by message round, it resembles that Alice and Bob "talk" one to another akin in a dialogue which is here quantum. Similarly to the calculations in Refs. [16, 26], for the present protocol the probability that Eve survives (i.e., remains masked) can also be evaluated which approaches zero for a long enough dialogue, i.e., the protocol is asymptotically secure against Eve's attacks.

A primitive version of the present quantum dialogue protocol that seems to boost capacity of the quantum channel can be thought of. Namely, in a message round, instead of encoding  $a_1, a_2, r$ , Alice may encode three secret bits  $a_1, a_2, a_3$ . Then, after decoding Alice's bits, Bob can also hide his three secret bits  $b_1, b_2, b_3$  respectively into the encrypted bits  $d_1 = a_1 \oplus b_1$ ,  $d_2 = a_2 \oplus b_2$ ,  $d_3 = a_3 \oplus b_3$ , which are to be revealed openly. Since Alice knows  $a_1, a_2, a_3$  she is able to easily decrypt Bob's bits as  $b_1 = a_1 \oplus d_1$ ,  $b_2 = a_2 \oplus d_2$ ,  $b_3 = a_3 \oplus d_3$ . In this way each partner can exchange three secret bits per GHZ state, a considerable increase in the quantum channel capacity. Unfortunately,

although Eve is unable to exploit the publicly announced bits  $d_1, d_2, d_3$  to deduce Alice's and Bob's secret bits  $a_1, a_2, a_3, b_1, b_2, b_3$  with certainty [27], she knows classical correlations between Alice's and Bob's bits through  $d_1, d_2, d_3$ . From the cryptography perspective, a protocol is secure if Eve cannot have any information about the secret communicated bits both before and after classical announcement. The just outlined primitive quantum dialogue protocol thus suffers a weakness under the name "information leakage". That is why in the present quantum dialogue protocol among the bits Alice encodes there is one random bit  $r$ . This random bit guarantees security of Bob's secret bit  $b_1$  because from the published bit  $d = b_1 \oplus r$  nobody except Alice is able to deduce  $b_1$ . It is this random bit that serves to prevent the present quantum dialogue protocol from information leakage.

Variations of the present protocol are possible. In the present protocol in a message round Alice can "ask" two bits  $a_1, a_2$  but Bob can "answer" just one bit  $b_1$ . In case Alice needs to "ask" just one bit  $a_1$  but Bob wishes to "answer" two bits  $b_1, b_2$ , they let each other know their intention. Upon their agreement, Alice now encodes  $a_1, r_1, r_2$  with  $r_1, r_2$  two random bits. Later, after deducing Alice's bits, Bob broadcasts two encrypted bits  $d_1 = r_1 \oplus b_1, d_2 = r_2 \oplus b_2$ . Obviously, only Alice is able to decode Bob's two secret bits  $b_1, b_2$  from  $d_1, d_2$  thanks to her knowledge of the random bits  $r_1, r_2$ . Such a modified quantum dialogue protocol is also free of information leakage.

Furthermore, the present protocol can be extended to be symmetric with respect to Alice and Bob in the sense that in each message round each of the two can communicate two secret bits with the other. Such a symmetric quantum dialogue protocol requires Bob to prepare fourpartite GHZ states of the form  $|G_{ijkl}\rangle_{ABCD} = [|j\rangle_A |k\rangle_B |l\rangle_C |0\rangle_D + (-1)^i |j \oplus 1\rangle_A |k \oplus 1\rangle_B |l \oplus 1\rangle_C |1\rangle_D] / \sqrt{2}$  of which qubit  $D$  is kept at home but qubits  $A, B, C$  are sent to Alice. In this extended version of quantum dialogue Alice encodes on the qubits  $A, B, C$  four bits  $a_1, a_2, r_1, r_2$  with  $a_1, a_2$  being the two secret bits and  $r_1, r_2$  two random ones. This provides room for Bob to encrypt his two secret bits  $b_1, b_2$  into  $d_1 = r_1 \oplus b_1, d_2 = r_2 \oplus b_2$  which will be safely decrypted only by Alice. The mathematical formulation of such a symmetric quantum dialogue protocol is cumbersome but straightforward, so we will not represent it in detail here.

## 7. Conclusion

In summary, we have proposed a quantum dialogue protocol which uses GHZ states as the working quantum channel. Two of the three qubits in the GHZ state travel like a shuttle between two remote partners carrying secret

and random bits: the secret bits can be exchanged safely while the random bits are to protect the quantum dialogue from information leakage. As in most bidirectional communication protocols, three kinds of rounds of actions, named message round, forward checking round and backward checking round, are designed in the present protocol. The message round serves as an exchanger of meaningful information between the communicators. The forward and backward checking rounds are to detect the eavesdropper's attacks in both routes between the communicators. The figure of merit in the present quantum dialogue protocol is its high level of security featured by keeping confidential not only the communicators' informative bits themselves but also any their classical correlations. In other words, the proposed quantum dialogue protocol is both safe and free of information leakage. In a message round of the present protocol Alice can communicate two bits with Bob, while Bob can communicate only one bit with Alice. Yet, we also outline possible variations of the present quantum dialogue protocol towards those in which Bob can communicate two bits with Alice and Alice communicates only one bit with Bob or Alice can communicate two bits with Bob and Bob can also communicate two bits with Alice. In theory, constructions of information-leakage-free quantum dialogue protocols for Alice and Bob to exchange any numbers of bits are possible, but the bigger number of exchangeable bits the higher scale of multipartite entanglement and the more complicated the manipulation of qubits.

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## A NEW CLASS OF UNIQUE RANGE SETS FOR MEROMORPHIC FUNCTIONS IGNORING MULTIPLICITY WITH 15 ELEMENTS

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**Abstract.** In this paper, we give a new class of unique range sets for meromorphic functions ignoring multiplicity with 15 elements.

### 1. Introduction. Main results

In this paper, by a meromorphic function we mean a meromorphic function on the complex plane  $\mathbb{C}$ .

Let  $f$  be a non-constant meromorphic function on  $\mathbb{C}$ . For every  $a \in \mathbb{C}$ , we define the function  $\nu_f^a : \mathbb{C} \rightarrow \mathbb{N}$  by

$$\nu_f^a(z) = \begin{cases} 0 & \text{if } f(z) \neq a \\ d & \text{if } f(z) = a \text{ with multiplicity } d, \end{cases}$$

and set  $\nu_f^\infty = \nu_{\frac{1}{f}}^0$ , and define the function  $\bar{\nu}_f^a : \mathbb{C} \rightarrow \mathbb{N}$  by  $\bar{\nu}_f^a(z) = \min \{ \nu_f^a(z), 1 \}$ , and set  $\bar{\nu}_f^\infty = \bar{\nu}_{\frac{1}{f}}^0$ . For  $f \in \mathcal{M}(\mathbb{C})$  and a non-empty set  $S \subset \mathbb{C} \cup \{\infty\}$ , we define

$$E_f(S) = \bigcup_{a \in S} \{ (z, \nu_f^a(z)) : z \in \mathbb{C} \}, \quad \bar{E}_f(S) = \bigcup_{a \in S} \{ (z, \bar{\nu}_f^a(z)) : z \in \mathbb{C} \}.$$

Let  $\mathcal{F}$  be a nonempty subset of  $\mathcal{M}(\mathbb{C})$ . Two functions  $f, g$  of  $\mathcal{F}$  are said to *share  $S$ , counting multiplicity* (share  $S$  CM) if  $E_f(S) = E_g(S)$ , and to *share  $S$ , ignoring multiplicity* (share  $S$  IM) if  $\bar{E}_f(S) = \bar{E}_g(S)$ .

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If the condition  $E_f(S) = E_g(S)$  implies  $f = g$  for any two non-constant meromorphic (entire) functions  $f, g$ , then  $S$  is called a unique range set for meromorphic (entire) functions counting multiplicity, or in brief, URSM (URSE). A set  $S \subset \mathbb{C} \cup \{\infty\}$  is called a unique range set for meromorphic (entire) functions ignoring multiplicity, or in brief, URSM-IM (URSE-IM), if the condition  $\overline{E}_f(S) = \overline{E}_g(S)$  implies  $f = g$  for any pair of non-constant meromorphic (entire) functions.

In 1976 Gross ([8]) proved that there exist three finite sets  $S_j$  ( $j = 1, 2, 3$ ) such that any two entire functions  $f$  and  $g$  satisfying  $E_f(S_j) = E_g(S_j)$ ,  $j = 1, 2, 3$  must be identical. In the same paper Gross posed the following question:

**Question A.** *Can one find two (or possible even one) finite set  $S_j$  ( $j = 1, 2$ ) such that any two entire functions  $f$  and  $g$  satisfying  $E_f(S_j) = E_g(S_j)$  ( $j = 1, 2$ ) must be identical.*

Yi [16]-[18], [20] first gave an affirmative answer to Question A. Since then, many results have been obtained for this and related topics (see [1]-[13], [15]-[21]).

Concerning to Question A, a natural question is the following.

**Question B.** *What is the smallest cardinality for such a finite set  $S$  such that any two meromorphic functions  $f$  and  $g$  satisfying either  $E_f(S) = E_g(S)$  or  $\overline{E}_f(S) = \overline{E}_g(S)$  must be identical.*

So far, the best answer to Question B for case URSM was obtained by Frank and Reinders [5]. They proved the following result.

**Theorem C.** *The set  $\{z \in \mathbb{C} \mid P_{FR}(z) = \frac{(n-1)(n-2)}{2}z^n + n(n-2)z^{n-1} + \frac{(n-1)n}{2}z^{n-2} - c = 0\}$ , where  $n \geq 11$  and  $c \neq 0, 1$ , is a unique range set for meromorphic functions counting multiplicity.*

In 1997, H. X. Yi [19] first gave an answer to question B for the case URSM-IM with 19 elements. Since then, many results have been obtained for this topic (see [2], [3]). So far, the best answer to Question B for the case URSM-IM was obtained by B. Chakraborty [3]. He proved the following result.

**Theorem D.** *Let  $S = \{z \in \mathbb{C} \mid P_{FR}(z) = 0\}$ . If  $n \geq 15$ , then  $S$  is a URSM-IM.*

In this paper, we give a new class of unique range sets for meromorphic functions ignoring multiplicity with 15 elements. Note that this class is different from B. Chakraborty's in [3].

Now let us describe main results of the paper.

Let  $n \in \mathbb{N}^*$ ,  $n \geq 3$ .

Consider polynomial  $P(z)$  as follows:

$$P(z) = z^n - \frac{2na}{n-1}z^{n-1} + \frac{na^2}{n-2}z^{n-2} + 1 = Q(z) + 1, \quad (1.1)$$

where  $a \in \mathbb{C}$ ,  $a \neq 0$ . Suppose that

$$Q(1) \neq -1, \quad (1.2)$$

$$Q(1) \neq -2. \quad (1.3)$$

Note that  $P(z)$ , defined by (1.1) with condition (1.2), has no multiple zeros. Clearly,  $P'(z) = nz^{n-3}(z-a)^2$ , and  $P(z)$  is different from  $P_{FR}$ . Moreover  $P'(z)$  has a zero at 0 of order  $n-3$ , and a zero at  $a$  of order 2.

The polynomials of the form (1.1) were investigated in [1] and [11].

We shall prove the following theorem.

**Theorem 1.** *Let  $P(z)$  be defined by (1.1) with conditions (1.2) and (1.3), and let  $S = \{z \in \mathbb{C} \mid P(z) = 0\}$ . If  $n \geq 15$ , then  $S$  is a URSM-IM.*

## 2. Lemmas, Definitions

We assume that the reader is familiar with the notations of Nevanlinna theory (see, for example, [4], [14]).

We need some lemmas.

**Lemma 2.1.** [14] *Let  $f$  be a non-constant meromorphic function on  $\mathbb{C}$  and let  $a_1, a_2, \dots, a_q$  be distinct points of  $\mathbb{C} \cup \{\infty\}$ . Then*

$$(q-2)T(r, f) \leq \sum_{i=1}^q \bar{N}\left(r, \frac{1}{f-a_i}\right) + S(r, f),$$

where  $S(r, f) = o(T(r, f))$  for all  $r$ , except for a set of finite Lebesgue measure.

**Lemma 2.2.** [4]

*For any non-constant meromorphic function  $f$ ,*

$$N\left(r, \frac{1}{f'}\right) \leq N\left(r, \frac{1}{f}\right) + \bar{N}(r, f) + S(r, f).$$

**Definition.** Let  $f$  be a non-constant meromorphic function, and  $k$  be a positive integer. We denote by  $\bar{N}_{(k)}(r, f)$  the counting function of the poles of order  $\geq k$  of  $f$ , where each pole is counted only once, and by  $N_{(1)}(r, f)$  the counting function of the simple poles of  $f$ .

**Lemma 2.3.** *Let  $f, g$  be two non-constant meromorphic functions and let  $f^{-1}(0) = g^{-1}(0)$ . Set*

$$F = \frac{1}{f}, \quad G = \frac{1}{g}, \quad L = \frac{F''}{F'} - \frac{G''}{G'}.$$

Suppose that  $L \neq 0$ . Then

$$\begin{aligned} 1) [11 - 13] \quad N(r, L) &\leq \bar{N}_{(2)}(r, f) + \bar{N}_{(2)}(r, g) + \bar{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1) + \\ &\quad \bar{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1) + \bar{N}(r, \frac{1}{f'}; f \neq 0) + \bar{N}(r, \frac{1}{g'}; g \neq 0). \end{aligned}$$

Moreover, if  $a$  is a common simple zero of  $f$  and  $g$ , then  $L(a) = 0$ .

$$\begin{aligned} 2) \quad &\bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1) + \bar{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1) \\ &\leq N(r, L) + \frac{1}{2}(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) + N(r, \frac{1}{f}; \nu_1 \geq 2) + N(r, \frac{1}{g}; \nu_2 \geq 2). \end{aligned}$$

**Proof.** 2) By using properties of the Stieltjes integral (see [4, p. 5, p. 14]), we get:

$$N(r, \frac{1}{f}) - n(0, \frac{1}{f}) = \sum_{0 < |a_m| < r} \ln \frac{r}{|a_m|},$$

where  $a_m$  are zeros of  $f$ , counting multiplicity; and

$$\bar{N}(r, \frac{1}{f}) - \bar{n}(0, \frac{1}{f})$$

is the same sum, where each zero  $a_m$  is counted only once.

Similarly, we obtain equalities for  $N(r, \frac{1}{f}; \nu_1 \geq 2)$ ,  $\bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1)$ ,  $N(r, \frac{1}{g}; \nu_2 \geq 2)$ ,  $\bar{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1)$ .

We are going to prove 2) by using these inequalities and the arguments in (Lemma 2.2 [12]), (Lemma 2.4 [13]) and (Lemma 2.6 [3]).

Set

$$\begin{aligned} M &= \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{g}) + \bar{N}(r, \frac{1}{f}; \nu_1 > \nu_2 \geq 1) + \bar{N}(r, \frac{1}{g}; \nu_2 > \nu_1 \geq 1), \\ T &= N(r, L) + \frac{1}{2}(N(r, \frac{1}{f}) + N(r, \frac{1}{g})) + N(r, \frac{1}{f}; \nu_1 \geq 2) + N(r, \frac{1}{g}; \nu_2 \geq 2). \end{aligned}$$

Let  $a$  be a zero of  $f$  with multiplicity  $p$ . From  $f^{-1}(0) = g^{-1}(0)$  it follows that  $a$  is a zero of  $f$  with multiplicity  $q$ . We consider the following cases:

**Case 1.** Assume that  $p = q$ .

If  $p = q = 1$ , then  $a$  is counted with  $1 + 1 + 0 + 0 = 2$  times in  $M$ . From this and the proof of Part 1) (Lemma 2.2 [12]), (Lemma 2.4 [13]) it follows that  $a$  is a zero of  $L$ . Then it is counted with  $1 + \frac{1}{2}(1 + 1) = 2$  times in  $T$ .

If  $p = q \geq 2$ , then  $a$  is counted with  $1 + 1 + 0 + 0 = 2$  times on  $M$ . By the proof of Part 1) (Lemma 2.2 [12]), (Lemma 2.4 [13]) it follows that  $a$  is not a pole of  $L$ . Then it is counted with  $0 + \frac{1}{2}(p + p) + p + p = 3p$  times in  $T$ .

**Case 2.** Assume that  $p > q$ .

If  $p > q$  and  $q = 1$ , then  $p \geq 2$  and  $a$  is counted with  $1 + 1 + 1 + 0 = 3$  times in  $M$ . By the proof of Part 1) (Lemma 2.2[12]), (Lemma 2.4 [13]) it follows that  $a$  is a pole of  $L$ , and by  $p \geq 2$  we see that  $a$  is counted with  $1 + \frac{1}{2}(p + 1) + p + 0 = p + 1 + \frac{p+1}{2} > 3$  times in  $T$ .

If  $p > q$  and  $q \geq 2$ , then  $p \geq 2$  and  $a$  is counted with  $1 + 1 + 1 + 0 = 3$  times in  $M$ . From this and the proof of Part 1) (Lemma 2.2 [12]), (Lemma 2.4 [13]) it follows that  $a$  is a pole of  $L$ , and by  $p \geq 2$ ,  $q \geq 2$  we see that  $a$  is counted with  $1 + \frac{1}{2}(p + q) + p + q = 1 + \frac{3(p+q)}{2} > 3$  times in  $T$ .

**Case 3.** Assume that  $q > p$ .

The proof is completed by using the arguments similar to ones in Case 2.

A polynomial  $R(z)$  is called a *strong uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions  $f$  and  $g$ , and a nonzero constant  $c$ , the condition  $R(f) = cR(g)$  implies  $f = g$  (see [1], [7], [11]). In this case we say that,  $R(z)$  is a SUPM (SUPE). A polynomial  $R(z)$  is called a *uniqueness polynomial for meromorphic (entire) functions* if for arbitrary two non-constant meromorphic (entire) functions  $f$  and  $g$ , the condition  $R(f) = R(g)$  implies  $f = g$  (see[1], [7], [11]). In this case we say  $R(z)$  is a UPM (UPE). Let  $R(z)$  be a polynomial of the degree  $q$ . Assume that the derivative of  $R(z)$  has mutually distinct  $k$  zeros  $d_1, d_2, \dots, d_k$  with multiplicities  $q_1, q_2, \dots, q_k$ , respectively. We often consider polynomials satisfying the following condition introduced by Fujimoto ([6]):

$$R(d_i) \neq R(d_j), 1 \leq i < j \leq k. \quad (2.1)$$

The number  $k$  is called the *derivative index* of  $R$ .

**Lemma 2.4.** (Fujimoto [7]).

Let  $R(z)$  be a polynomial of the degree  $q$  satisfying the condition (2.1). Then  $R(z)$  is a uniqueness polynomial if and only if

$$\sum_{1 \leq l < m \leq k} q_l q_m > \sum_{i=1}^k q_i.$$

In particular, the above inequality is always satisfied whenever  $k \geq 4$ . When  $k = 3$  and  $\max\{q_1, q_2, q_3\} \geq 2$  or when  $k = 2$ ,  $\min\{q_1, q_2\} \geq 2$  and  $q_1 + q_2 \geq 5$ , then also the above inequality holds.

H. Fujimoto [6] proved the following:

**Lemma 2.5.** *Let  $R(z)$  be a polynomial of the degree  $q$  satisfying the condition (2.1), we assume furthermore that  $q \geq 5$  and there are two non-constant meromorphic functions  $f$  and  $g$  such that*

$$\frac{1}{R(f)} = \frac{c_0}{R(g)} + c_1$$

for two constants  $c_0 (\neq 0)$  and  $c_1$ . If  $k \geq 3$  or if  $k = 2$ ,  $\min\{q_1, q_2\} \geq 2$ , then  $c_1 = 0$ .

**Lemma 2.6.** [1]  $\sum_{i=0}^m \binom{m}{i} \frac{(-1)^i}{n+m+1-i}$  is not an integer, where  $n, m \geq 1$  are integers.

In [1], Banerjee proved the Lemma for  $n, m \geq 3$ , but it is clear that the Lemma is valid for  $n, m \geq 1$ .

**Lemma 2.7.** *Let  $P(z)$  be defined by (1.1) with conditions (1.2) and (1.3), and let  $n \geq 6$ . Then  $P(z)$  is a strong uniqueness polynomial for meromorphic functions*

**Proof.**

By Lemma 2.6, we see that  $\frac{1}{n} - \frac{2}{n-1} + \frac{1}{n-2}$  is not an integer. Set  $A = \frac{1}{n} - \frac{2}{n-1} + \frac{1}{n-2}$ . Then  $A \neq 0$ . We have  $P(0) = Q(0) + 1 = 1$ ,  $P(a) = Q(a) + 1 = nAa^n + 1$ . From this and  $a \neq 0$ , we get  $P(a) \neq P(0)$ . Set  $F = P(f), G = P(g)$ . From  $P(f) = cP(g)$ ,  $c \neq 0$ , it implies

$$F = cG, T(r, f) + S(r, f) = T(r, g) + S(r, g), S(r, f) = S(r, g). \quad (2.2)$$

Now we consider the following possible cases:

*Case 1.*  $c \neq 1$ .

If  $c = P(a)$ , from (2.2) and  $P(0) = 1$  we have

$$F - 1 = P(a)\left(G - \frac{1}{P(a)}\right). \quad (2.3)$$

We consider  $P(z) - \frac{1}{P(a)}$ . By  $P(0) = 1$  and  $P(a) = c \neq 1$  we obtain  $P(0) - \frac{1}{P(a)} \neq 0$ . Moreover, since  $P(a) = nAa^n + 1 \neq -1$  and  $P(a) = c \neq 1$  we obtain

$P(a) - \frac{1}{P(a)} \neq 0$ . Therefore,  $P(z) - \frac{1}{P(a)}$  has only simple zeros, let they be given by  $b'_i, i = 1, 2, \dots, n$ . Note that  $P(z) - 1$  has a zero at 0 of order  $n - 2$ , and two distinct simple zeros. Let  $c'_i, i = 1, 2$ , be distinct simple zeros of  $P(z) - 1$ . Applying Lemma 2.1 to the function  $g$  and the values  $b'_1, b'_2, \dots, b'_n, \infty$ , and by (2.2), (2.3) we get

$$\begin{aligned} (n-1)T(r, g) &\leq \bar{N}(r, g) + \sum_{i=1}^n \bar{N}(r, \frac{1}{g-b'_i}) + S(r, g) \\ &\leq T(r, g) + \bar{N}(r, \frac{1}{f}) + \sum_{i=1}^2 \bar{N}(r, \frac{1}{f-c'_i}) + S(r, g) \\ &\leq T(r, g) + T(r, f) + 2T(r, f) + S(r, g) \\ &= 4T(r, g) + S(r, g) \\ (n-5)T(r, g) &\leq S(r, g). \end{aligned}$$

This is a contradiction to the assumption that  $n \geq 6$ .

If  $c \neq P(a)$ , then from (2.2) we have

$$F - c = c(G - 1). \quad (2.4)$$

We consider  $P(z) - c$ . By  $P(0) = 1$  and  $c \neq 1$  we have  $P(0) - c = 1 - c \neq 0$ . Moreover  $c \neq P(a)$ . So  $P(a) - c \neq 0, P(0) - c \neq 0$ . Therefore  $P(z) - c$  has only simple zeros, let they be given by  $e_i, i = 1, 2, \dots, n$ . Now we consider  $P(z) - 1$ . We see that  $P(0) = 1, P(z) - P(0) = P(z) - 1$  has a zero at 0 of order  $n - 2$ , and 2 distinct simple zeros. Let  $t_i, i = 1, 2$ , be distinct simple zeros of  $P(z) - 1$ . Applying Lemma 2.1 to the function  $f$  and the values  $e_1, e_2, \dots, e_n, \infty$ , and by (2.4) we get

$$\begin{aligned} (n-1)T(r, f) &\leq \bar{N}(r, f) + \sum_{i=1}^n \bar{N}(r, \frac{1}{g-e_i}) + S(r, f) \\ &\leq T(r, f) + \bar{N}(r, \frac{1}{g}) + \sum_{i=1}^2 \bar{N}(r, \frac{1}{f-t_i}) + S(r, f) \\ &\leq T(r, f) + T(r, g) + 2T(r, g) + S(r, f) \\ &= 4T(r, f) + S(r, f) \\ (n-5)T(r, f) &\leq S(r, f). \end{aligned}$$

This is a contradiction to the assumption that  $n \geq 6$ .

*Case 2.*  $c = 1$ . Then

$$P(f) = P(g) \quad (2.5)$$

Applying Lemma 2.4 to (2.5) we obtain  $f = g$ .

### 3. Proof of Theorem 1

Now we use the above Lemmas to prove the main result of the paper.

Recall that  $P(z) = (z - a_1)\dots(z - a_n)$ ,  $P'(z) = nz^{n-3}(z - a)^2$ .

Suppose  $n \geq 15$  and  $\overline{E}_f(S) = \overline{E}_g(S)$ , where  $S = \{z \in \mathbb{C} \mid P(z) = 0\}$ .

Set

$$F = \frac{1}{P(f)}, G = \frac{1}{P(g)}, L = \frac{F''}{F'} - \frac{G''}{G'},$$

$$T(r) = T(r, f) + T(r, g), S(r) = S(r, f) + S(r, g).$$

Then  $T(r, P(f)) = nT(r, f) + S(r, f)$  and  $T(r, P(g)) = nT(r, g) + S(r, g)$ , and hence  $S(r, P(f)) = S(r, f)$  and  $S(r, P(g)) = S(r, g)$ , since  $P(f)$  and  $f$ , and  $P(g)$  and  $g$  have the same growth estimates, respectively.

We consider two following cases:

**Case 1**  $L \equiv 0$ . Then, we have  $\frac{1}{P(f)} = \frac{c}{P(g)} + c_1$  for some constants  $c (\neq 0)$  and  $c_1$ . By Lemma 2.5 we obtain  $c_1 = 0$ .

Therefore, there is a constant  $C \neq 0$  such that  $P(f) = CP(g)$ . Then, applying Lemma 2.7 we obtain  $f = g$ .

**Case 2**  $L \not\equiv 0$ .

**Claim 1** We have

$$(n-2)T(r) \leq \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) - N_0(r, \frac{1}{f'}) - N_0(r, \frac{1}{g'}) + S(r), \quad (3.1),$$

where  $N_0(r, \frac{1}{f'})$  ( $N_0(r, \frac{1}{g'})$ ) is the counting function of those zeros of  $f'$ , which are not zeros of function  $(f - a_1)\dots(f - a_n)f(f - a)((g - a_1)\dots(g - a_n)g(g - a))$ .

Then, applying the Lemma 2.1 to the functions  $f, g$  and the values  $a_1, a_2, \dots, a_n, 0, a, \infty$ , and noting that

$$\sum_{i=1}^q \overline{N}(r, \frac{1}{f - a_i}) = \overline{N}(r, \frac{1}{P(f)}), \sum_{i=1}^q \overline{N}(r, \frac{1}{g - a_i}) = \overline{N}(r, \frac{1}{P(g)}),$$

we obtain

$$(n+1)T(r) \leq \overline{N}(r, f) + \overline{N}(r, g) + \overline{N}(r, \frac{1}{P(f)}) + \overline{N}(r, \frac{1}{P(g)}) + \overline{N}(r, \frac{1}{f}) + \overline{N}(r, \frac{1}{g}) +$$



$$\overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) - N_0\left(r, \frac{1}{f'}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r). \quad (3.2)$$

On the other hand,

$$\begin{aligned} \overline{N}(r, f) + \overline{N}(r, g) &\leq (T(r, f) + T(r, g)) + S(r) = T(r) + S(r), \\ \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) &\leq (T(r, f) + T(r, g)) + S(r) = T(r) + S(r), \\ \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}\left(r, \frac{1}{g-a}\right) &\leq (T(r, f) + T(r, g)) + S(r) = T(r) + S(r). \end{aligned}$$

From this and (3.2) we obtain (3.1)

**Claim 2** We have

$$\begin{aligned} \overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right) &\leq \\ \left(\frac{n}{2} + 3\right)T(r) + \overline{N}\left(r, \frac{1}{[P(f)]'}; P(f) \neq 0\right) + \overline{N}\left(r, \frac{1}{[P(g)]'}; P(g) \neq 0\right) + S(r). \end{aligned}$$

By  $\overline{E}_f(S) = \overline{E}_g(S)$  we get  $(P(f))^{-1}(0) = (P(g))^{-1}(0)$ , and note that

$$\overline{N}_{(2)}(r, P(f)) = \overline{N}(r, f), \quad \overline{N}_{(2)}(r, P(g)) = \overline{N}(r, g).$$

Then applying the Lemma 2.3 to the functions  $P(f), P(g)$  we obtain

$$\begin{aligned} N(r, L) &\leq \overline{N}(r, f) + \overline{N}(r, g) + \\ &\quad \overline{N}\left(r, \frac{1}{P(f)}; \nu_1 > \nu_2 \geq 1\right) + \overline{N}\left(r, \frac{1}{P(g)}; \nu_2 > \nu_1 \geq 1\right) + \\ ((3.3)) \quad &\quad \overline{N}\left(r, \frac{1}{[P(f)]'}; P(f) \neq 0\right) + \overline{N}\left(r, \frac{1}{[P(g)]'}; P(g) \neq 0\right); \end{aligned}$$

$$\begin{aligned} &\overline{N}\left(r, \frac{1}{P(f)}\right) + \overline{N}\left(r, \frac{1}{P(g)}\right) + \overline{N}\left(r, \frac{1}{P(f)}; \nu_1 > \nu_2 \geq 1\right) + \overline{N}\left(r, \frac{1}{P(g)}; \nu_2 > \nu_1 \geq 1\right) \leq \\ N(r, L) + \frac{1}{2} &\left(N\left(r, \frac{1}{P(f)}\right) + N\left(r, \frac{1}{P(g)}\right)\right) + N\left(r, \frac{1}{P(f)}; \nu_1 \geq 2\right) + N\left(r, \frac{1}{P(g)}; \nu_2 \geq 2\right). \end{aligned} \quad (3.4)$$

Moreover,

$$\overline{N}(r, f) + \overline{N}(r, g) \leq T(r) + S(r). \quad (3.5)$$

Obviously,

$$N\left(r, \frac{1}{P(f)}\right) \leq nT(r, f) + S(r, f); \quad N\left(r, \frac{1}{P(g)}\right) \leq nT(r, g) + S(r, g),$$

$$N(r, \frac{1}{P(f)}) + N(r, \frac{1}{P(g)}) \leq nT(r) + S(r). \quad (3.6)$$

On the other hand, from  $P(f) = (f - a_1)\dots(f - a_n)$  it follows that every zero with multiplicity  $\geq 2$  of  $P(f)$  is a zero of  $f - a_i$  with multiplicity  $\geq 2$ ,  $i = 1, 2, \dots, n$ , and therefore, it is a zero of  $f'$ , so we have

$$N_{(2)}(r, \frac{1}{P(f)}) \leq N(r, f').$$

From this and Lemma 2.2 we obtain

$$N_{(2)}(r, \frac{1}{P(f)}) \leq N(r, \frac{1}{f}) + \bar{N}(r, f) \leq 2T(r, f) + S(r, f).$$

Similarly, we have

$$\bar{N}_{(2)}(r, \frac{1}{P(g)}) \leq N(r, \frac{1}{g}) + \bar{N}(r, g) \leq 2T(r, g) + S(r, g).$$

Therefore,

$$N_{(2)}(r, \frac{1}{P(f)}) + N_{(2)}(r, \frac{1}{P(g)}) \leq 2T(r) + S(r). \quad (3.7)$$

Combining (3.1)-(3.7) we get

$$\begin{aligned} & \bar{N}(r, \frac{1}{P(f)}) + \bar{N}(r, \frac{1}{P(g)}) \leq \\ & (\frac{n}{2} + 3)T(r) + \bar{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \bar{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) + S(r). \end{aligned}$$

Claim 2 is proved.

**Claim 3** We have

$$\begin{aligned} & \bar{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) + \bar{N}(r, \frac{1}{[P(g)]'}; P(g) \neq 0) \leq \\ & 2T(r) + N_0(r, \frac{1}{f'}) + N_0(r, \frac{1}{g'}) + S(r). \end{aligned}$$

We have

$$\begin{aligned} & \bar{N}(r, \frac{1}{[P(f)]'}; P(f) \neq 0) \\ & = \bar{N}(r, \frac{1}{f^{n-3}(f-a)^2 f'}; P(f) \neq 0) \\ & \leq \bar{N}(r, \frac{1}{f}) + \bar{N}(r, \frac{1}{f-a}) + \bar{N}_o(r, \frac{1}{f'}) \\ (3.8) \quad & \leq 2T(r, f) + \bar{N}_o(r, \frac{1}{f'}) + S(r, f). \end{aligned}$$

Similarly,

$$\overline{N}\left(r, \frac{1}{[P(g)]^r}; P(g) \neq 0\right) \leq 2T(r, g) + \overline{N}_o\left(r, \frac{1}{g}\right) + S(r, g). \quad (3.9)$$

Inequalities (3.8) and (3.9) give us

$$(3.10) \quad \overline{N}\left(r, \frac{1}{[P(f)]^r}; P(f) \neq 0\right) + \overline{N}\left(r, \frac{1}{[P(g)]^r}; P(g) \neq 0\right) \leq 2T(r) + \overline{N}_o\left(r, \frac{1}{f}\right) + \overline{N}_o\left(r, \frac{1}{g}\right) + S(r).$$

Claim 3 is proved.

Claim 1, 2, 3 give us:

$$(n-2)T(r) \leq \left(\frac{n}{2} + 5\right)T(r) + S(r), \quad (n-14)T(r) \leq S(r)$$

This is a contradiction to the assumption that  $n \geq 15$ . So  $L \equiv 0$ . Therefore  $f = g$ . Theorem 1 is proved.

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# NONAUTONOMOUS ATTRACTORS FOR YOUNG DIFFERENTIAL EQUATIONS DRIVEN BY UNBOUNDED VARIATION PATHS

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**Abstract.** We prove the existence of the pullback attractor of the generated flow by a dissipative nonautonomous differential equations driven by unbounded variation paths under the condition of smallness of nonlinear term. In case perturbed term is linear we prove that the attractor is singleton and also is forward one.

## 1. Introduction

This work extends the study on the long term behavior of the solution of the dissipative Young equations driven by Hölder paths in [10], [11] to the general case where coefficient functions now depend on time. Namely, we consider system

$$(1.1) \quad dx_t = [A(t)x + f(t, x_t)]dt + g(t, x_t)d\omega_t,$$

in which  $A, f, g$  are continuous functions, the driving path  $\omega$  is of bounded  $p$ -variation for some  $p \in (1, 2)$ . This equation is understood in the form

$$(1.2) \quad x_t = x_0 + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)d\omega_s,$$

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where the first integral is of Riemann type, meanwhile the second one is defined in the Young sense.

The existence and uniqueness of the solution of (1.1) is proved in e.g. [17], [2] and the references there in. It is proved in [2] that (1.1) generates a two-parameter flow of homeomorphism on  $\mathbb{R}^d$ . This allow to study the asymptotic behaviour of the solution of system in the frame work of dynamical system theory. One interesting topic is pull back and forward attractors. They are invariant sets those attract all the trajectories of the system. In this paper we develop techniques from [11] which deal with autonomous equation, to prove the existence of a nonautonomous attractor for the generated flow from (1.1). In case  $g$  is linear the attractor is singleton and is also forward one. Since the notation of nonautonomous attractor is understood as  $\omega$ -wise we keep the presentation simple by deal with the problem for a deterministic system. The results in this paper can be applied to a stochastic equation with Hölder noises where a random attractor is established.

## 2. Preliminaries and main results

### Young integral

Let us first briefly make a survey on Young integrals. Let  $\mathcal{C}([a, b], \mathbb{R}^r)$ ,  $r \geq 1$ , denote the space of all continuous paths  $x : [a, b] \rightarrow \mathbb{R}^r$  equipped with supremum norm  $\|\cdot\|_{\infty, [a, b]}$  given by  $\|x\|_{\infty, [a, b]} = \sup_{t \in [a, b]} |x_t|$ , where  $|\cdot|$  is the Euclidean norm of a vector in  $\mathbb{R}^r$ . For  $p \geq 1$  and  $[a, b] \subset \mathbb{R}$ ,  $\mathcal{C}^p([a, b], \mathbb{R}^r) \subset \mathcal{C}([a, b], \mathbb{R}^r)$  denotes the space of all continuous paths  $x : [a, b] \rightarrow \mathbb{R}^r$  which is of finite  $p$ -variation, i.e.

$$(2.1) \quad \|x\|_{p, [a, b]} := \left( \sup_{\Pi(a, b)} \sum_{i=1}^n \|x_{t_{i+1}} - x_{t_i}\|^p \right)^{1/p} < \infty,$$

where the supremum is taken over the whole class of finite partitions of  $[a, b]$  (see e.g. [12]).  $\mathcal{C}^p([a, b], \mathbb{R}^r)$  with the  $p$ -var norm

$$\|x\|_{p, [a, b]} := |x_a| + \|x\|_{p, [a, b]},$$

is a nonseparable Banach space [12, Theorem 5.25, p. 92]. Also for each  $0 < \alpha < 1$ , we denote by  $\mathcal{C}^{\alpha\text{-Hol}}([a, b], \mathbb{R}^r)$  the space of Hölder continuous functions with exponent  $\alpha$  on  $[a, b]$  equipped with the norm

$$\|x\|_{\alpha\text{-Hol}, [a, b]} := \|x_a\| + \sup_{a \leq s < t \leq b} \frac{|x_t - x_s|}{(t - s)^\alpha}.$$



It is known that  $\mathcal{C}^{\alpha-\text{Hol}}([a, b], \mathbb{R}^r) \subset \mathcal{C}^{\frac{1}{p}}([a, b], \mathbb{R}^r)$ .

Now, consider  $y \in \mathcal{C}^q([a, b], \mathbb{R}^{d \times m})$  and  $x \in \mathcal{C}^p([a, b], \mathbb{R}^m)$  with  $\frac{1}{p} + \frac{1}{q} > 1$ , the Young integral  $\int_a^b y_t dx_t$  can be defined as

$$\int_a^b y_s dx_s := \lim_{|\Pi| \rightarrow 0} \sum_{[u, v] \in \Pi} y_u(x_v - x_u),$$

where the limit is taken over all the finite partitions  $\Pi$  of  $[a, b]$  with  $|\Pi| := \max_{[u, v] \in \Pi} |v - u|$  (see [19]). This integral satisfies the additive property by construction, and the so-called *Young-Loeve estimate* [12, Theorem 6.8, p. 116]

$$\left| \int_s^t y_u dx_u - y_s[x_t - x_s] \right| \leq K \|y\|_{q, [s, t]} \|x\|_{p, [s, t]}, \quad \forall [s, t] \subset [a, b],$$

where

$$(2.2) \quad K := (1 - 2^{1 - \frac{1}{p} - \frac{1}{q}})^{-1}.$$

This implies

$$\left| \int_s^t y_u dx_u \right| \leq \|x\|_{p, [s, t]} \left( |y_s| + (K + 1) \|y\|_{q, [s, t]} \right).$$

### Assumptions

Now we introduce conditions on driving path  $\omega$  and coefficient functions  $A, f, g$ .

(**H**<sub>0</sub>) For  $p \in (1, 2)$

$$\overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=-n}^{n-1} \|\omega\|_{p\text{-var}, [k, k+1]}^p < +\infty$$

(**H**<sub>1</sub>)  $A$  is continuous and bounded on  $\mathbb{R}$  by  $\|A\|$ . Moreover  $A$  satisfies the *uniform contraction condition*, i.e. there exists  $C_A \geq 1, \lambda_A > 0$  such that for all  $s < t$

$$(2.3) \quad \|\Phi(t, s)\| \leq C_A e^{-\lambda_A(t-s)},$$

where  $\Phi(t, s)$  is the Cauchy matrix of the equation  $dz_t = A(t)z_t dt$ .

(**H**<sub>2</sub>)  $f(t, x)$  is continuous and locally Lipchitz continuity w.r.t.  $x$  uniformly on  $t$  and there exists  $C_f > 0$  and  $b \in L^1(\Delta, \mathbb{R}^d)$ , for all closed interval  $\Delta \subset \mathbb{R}$  such that the following properties hold:

$$\begin{cases} (i) & |f(t, x)| \leq C_f |x| + b(t), \quad \forall x \in \mathbb{R}^d, \forall t \in \mathbb{R}, \\ (ii) & \sup_{k \in \mathbb{Z}} \|b\|_{L^1(k, k+1)} < \infty. \end{cases}$$

(**H<sub>3</sub>**)  $g(t, x)$  is differentiable in  $x$  with  $\partial_x g$  is locally  $\delta$ -Holder continuous w.r.t.  $x$  uniformly in  $t$  for some  $\delta > p - 1$ , and there exist some constants  $0 < C_g$ ,  $1 - \frac{1}{p} < \beta \leq 1$ , an increasing convex function  $k : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  vanish at 0

$$\begin{cases} (i) & |g(t, x) - g(t, y)| \leq C_g |x - y|, \quad \forall x, y \in \mathbb{R}^d, \quad \forall t \in \mathbb{R}, \\ (ii) & |g(t, x) - g(s, x)| + \|\partial_x g(t, x) - \partial_x g(s, x)\| \leq k(|t - s|)^\beta =: h^*(|t - s|) \\ & \quad \forall x \in \mathbb{R}^d, \quad \forall s, t \in \mathbb{R}. \\ (iii) & \lim_{t \rightarrow \infty} \frac{\log h^*(t)}{|t|} = 0 \end{cases}$$

**Remark 2.1.** (i) (**H<sub>0</sub>**) is satisfied for almost all realizations of fractional Brownian motion with Hurst index  $H > 1/2$  ([15]). We introduce the notation

$$(2.4) \quad \Gamma_p := \max \left\{ \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \|\omega\|_{p\text{-var}, [k, k+1]}^p, \overline{\lim}_{n \rightarrow \infty} \frac{1}{n} \sum_{k=-n}^{-1} \|\omega\|_{p\text{-var}, [k, k+1]}^p \right\}$$

which is finite under assumption (**H<sub>0</sub>**).

(ii) Assumption (**H<sub>1</sub>**) ensures that the semigroup  $\Phi(t) = e^{At}$ ,  $t \in \mathbb{R}$  generated by  $A$  satisfies the following properties: for all  $a < b \leq t$

$$(2.5) \quad \|\Phi(t, \cdot)\|_{\infty, [a, b]} \leq C_A e^{-\lambda_A(t-b)},$$

$$(2.6) \quad \|\Phi(t, \cdot)\|_{p, [a, b]} \leq C_A^2 \|A\| e^{-\lambda_A(t-b)} (b - a).$$

We recall here the theorem on existence and uniqueness of solution from [2]. Under these conditions, system (1.1) possesses a unique solution on whole  $\mathbb{R}$  which starts at an arbitrary time  $t_0$  and generates a stochastic two-parameter flow of homeomorphism  $\Psi(t, s)$ ,  $t \geq s$  in which  $\Psi(t, s)x_0$  is the solution to (1.1) at time  $t$  with initial value  $x_0$  at time  $s$ . Moreover, we have the following estimate for the growth of the solution.

**Proposition 2.1.** *The solution  $x$  of (1.1) is of bounded  $p$ -variation on each  $[u, v] \subset \mathbb{R}$  and satisfies*

$$\begin{aligned} \|x\|_{p, [u, v]} &\leq \left[ |x_u| + D[1 + h^*(|u| \vee |v| \vee |u - v|)](1 + \|\omega\|_{p, [u, v]})N_{[u, v]} \right] \times \\ &\quad \times e^{2L(v-u) + \kappa N_{[u, v]} N_{[u, v]}^{\frac{p-1}{p}}}, \end{aligned}$$

where  $\kappa = \log \frac{K+2}{K+1}$ ,  $L = \|A\| + C_f$ , and  $D$  is a generic constant and  $N_{[u, v]}$  is estimated as

$$(2.7) \quad N_{[u, v]} \leq 1 + [2(K+1)C_g]^p \|\omega\|_{p, [u, v]}^p.$$

**Proof.** By computation we have for  $s, t \in [u, v]$ ,  $s < t$

$$|x_t - x_s| \leq \hat{A}_{s,t}^{1/p} + C_f \int_s^t |x_r| dr + C_g \|\omega\|_{p,[s,t]} \left( |x_s| + K \|x\|_{p,[s,t]} \right),$$

where

$$\hat{A}_{s,t} := \left[ \int_s^t b(r) dr + \|\omega\|_{p,[s,t]} \left( |g(0, 0)| + h^*(|u| \vee |v|) + Kh^*(|u - v|) \right) \right]^p.$$

The rest follows step by step in [11, Theorem 2.4].

From now on, we always denote by  $D$  a generic constant.

### 3. Nonautonomous attractors

In what follows we recall the notion of the (global) pullback attractor of a two-parameter flow (see more for instance in [8], [14], [9]).

**Definition 3.1.** ([7]) For a given two-parameter flow  $\Psi(t, s)$ , a family of sets  $\mathcal{A}_t$  of  $\mathbb{R}^d$ ,  $t \in \mathbb{R}$  is called the pullback (forward) attractor of  $\Psi$  if

- (i) is compact set for  $t \in \mathbb{R}$ ,
- (ii) is invariant, i.e  $\Psi(t, s)\mathcal{A}_s = \mathcal{A}_t$  for all  $s \leq t$  in  $\mathbb{R}$ ,
- (iii) globally pullback (forward) attracting, i.e for every  $t \in \mathbb{R}$  and every  $\tilde{D}$  bounded

$$\lim_{s \rightarrow -\infty} d(\Psi(t, s)\tilde{D}|\mathcal{A}_t) = 0, \quad \left( \lim_{t \rightarrow +\infty} d(\Psi(t, s)\tilde{D}|\mathcal{A}_t) = 0 \right),$$

in which  $d$  is Hausdorff semi-distance between nonempty closed subsets  $E, F$  of  $\mathbb{R}^d$  is defined as  $d(E|F) = \sup\{\inf\{d(x, y)|y \in F\}|x \in E\}$ .

In general, one may consider the attracting on a family of nonempty sets  $(\tilde{D}_t)$  instead of a single set as in Definition 3.1. Below, we consider the family  $\tilde{D}$  of tempered set  $\tilde{D}_t$ , i.e.  $\tilde{D}_t$  is a subset of the closed ball  $\bar{B}(0, r_t)$  where the radius  $r_t$  is tempered, i.e.

$$\lim_{t \rightarrow \pm\infty} \frac{1}{t} \max\{\log r_t, 0\} = 0.$$

The pullback attracting property now can be written as

$$\lim_{s \rightarrow -\infty} d(\Psi(t, s)\tilde{D}_s|\mathcal{A}_t) = 0,$$

It is known that the existence of a nonautonomous pullback attractor is ensured by the existence of the pullback absorbing set. A family of set  $\mathcal{B}_t$  is said to be nonautonomous pullback absorbing if for almost all  $\omega$ , for each  $t$  there exists  $T = T(t, \tilde{\mathcal{D}})$  such that

$$\Psi(t, s)\tilde{\mathcal{D}}_s \subset \mathcal{B}_t$$

for all  $s < t - T(t, \tilde{\mathcal{D}})$ . Assume that there exists a family of compact pullback absorbing sets  $\mathcal{B}_t$ . Then there is a pullback attractor  $\mathcal{A}_t$  given by ([7])

$$(3.1) \quad \mathcal{A}_t = \bigcap_{\tau \leq t} \overline{\bigcup_{s \leq \tau} \Psi(t, s)\mathcal{B}_s}.$$

### 3.1. Existence

We consider

$$x_t = x_0 + \int_0^t f(s, x_s)ds + \int_0^t g(s, x_s)d\omega_s,$$

with  $(\mathbf{H}_0) - (\mathbf{H}_4)$ . Thanks to the "variation of constants" formula for Young differential equations (see e.g. [20] or [10]),  $x_t$  satisfies

$$(3.2) \quad x_t = \Phi(t, t_0)x_{t_0} + \int_{t_0}^t \Phi(t, s)f(s, x_s)ds + \int_{t_0}^t \Phi(t, s)g(s, x_s)d\omega_s, \quad t \geq t_0.$$

In the Lemma below we are going to estimate the solution base on (3.2).

**Lemma 3.1.** *The following estimate holds for any  $a < b \leq t$*

$$\begin{aligned} \left| \int_a^b \Phi(t, s)g(s, x_s)d\omega_s \right| &\leq KC_A \left(1 + C_A \|A\| (b - a)\right) \|\omega\|_{p, [a, b]} e^{-\lambda_A(t-b)} \times \\ &\times \left[ C_g \|x\|_{p, [a, b]} + (h^*(|a| \vee |b|) + h^*(|b - a|)) + |g(0, 0)| \right]. \end{aligned}$$

**Proof.** Firstly by assumption, we choose  $2 > q \leq p$  such that  $q\beta \leq 1$ , then

$$\begin{aligned} |g(t, x)| &\leq C_g |x| + h^*(|t|) + |g(0, 0)|, \quad \forall t, \\ \|g(\cdot, x)\|_{q, [s, t]} &\leq C_g \|x\|_{p, [s, t]} + h^*(|t - s|), \quad \forall s \leq t. \end{aligned}$$

Using (2.5) and (2.6) we have

$$\begin{aligned}
& \left| \int_a^b \Phi(t, s)g(s, x_s)d\omega_s \right| \\
& \leq \|\omega\|_{p\text{-var}, [a, b]} \left( |\Phi(t, a)g(a, x_a)| + K \|\Phi(t, \cdot)g(\cdot, x)\|_{q, [a, b]} \right) \\
& \leq \|\omega\|_{p, [a, b]} \left[ \|\Phi(t, a)\| \cdot |g(a, x_a)| + K \|\Phi(t, \cdot)\|_{p, [a, b]} \|g(\cdot, x)\|_{\infty, [a, b]} \right. \\
& \quad \left. + K \|\Phi(t, \cdot)\|_{\infty, [a, b]} \|g(\cdot, x)\|_{q, [a, b]} \right] \\
& \leq \|\omega\|_{p, [a, b]} \left[ C_A e^{-\lambda_A(t-a)} (C_g |x_a| + h^*(|a|) + |g(0, 0)|) \right. \\
& \quad \left. + KC_A^2 \|A\| e^{-\lambda_A(t-b)} (b-a) \left( C_g \|x\|_{\infty, [a, b]} + (h^*(|a| \vee h^*(|b|)) + |g(0, 0)|) \right) \right. \\
& \quad \left. + KC_A e^{-\lambda_A(t-b)} \left( C_g \|x\|_{p, [a, b]} + h^*(|b-a|) \right) \right] \\
& \leq KC_A \left[ 1 + C_A \|A\| (b-a) \right] \|\omega\|_{p, [a, b]} e^{-\lambda_A(t-b)} \times \\
& \quad \times \left[ C_g \|x\|_{p, [a, b]} + (h^*(|a| \vee |b|) + h^*(|b-a|) + |g(0, 0)|) \right].
\end{aligned}$$

■

Next we denote by  $\Delta_k$  the interval  $[k, k+1]$ ,  $k \in \mathbb{Z}$  and prove that the solution at time  $t$  can be estimates via its norm on consecutive  $\Delta_k$  that cover  $[t_0, t]$ .

**Lemma 3.2.** *Assume that  $\lambda := \lambda_A - C_A C_f > 0$ . The following estimate hold*

$$\begin{aligned}
(3.3) \quad e^{\lambda(t-t_0)} |x_t| & \leq C_A |x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)} b(s) ds \\
& + M \sum_{k=\lfloor t_0 \rfloor}^n \|\omega\|_{p, \Delta_k} e^{\lambda(k-t_0)} \left[ C_g \|x\|_{p, \Delta_k} + Dh^*(|k| + 1) \right], \forall t \in \Delta_n,
\end{aligned}$$

where  $M := KC_A e^{\lambda_A} (1 + C_A \|A\|)$ .

**Proof.** Firstly note that  $\lambda > 0$  by  $(\mathbf{H}_4)$ . Using  $(\mathbf{H}_1)$  and  $(\mathbf{H}_2)$  we have

$$\begin{aligned}
|x_t| & \leq |\Phi(t, t_0)x_{t_0}| + \int_{t_0}^t |\Phi(t, s)f(s, x_s)| ds + \left| \int_{t_0}^t \Phi(t, s)g(s, x_s)d\omega_s \right| \\
& \leq C_A e^{-\lambda_A(t-t_0)} |x_{t_0}| + \int_{t_0}^t C_A e^{-\lambda_A(t-s)} \left( C_f |x_s| + b(s) \right) ds + \\
& \quad \left| \int_{t_0}^t \Phi(t, s)g(s, y_s)d\omega_s \right|
\end{aligned}$$

$$\leq C_A e^{-\lambda_A(t-t_0)} |x_{t_0}| + \int_{t_0}^t C_A e^{-\lambda_A(t-s)} b(s) ds + \beta_t + L_f \int_{t_0}^t e^{-\lambda_A(t-s)} |x_s| ds$$

where  $\beta_t := \left| \int_{t_0}^t \Phi(t, s) g(s, x_s) d\omega_s \right|$ ,  $L_f := C_A C_f$ . This implies,

$$e^{\lambda_A(t-t_0)} |x_t| \leq C_A |x_{t_0}| + \int_{t_0}^t C_A e^{\lambda_A(s-t_0)} b(s) ds + e^{\lambda_A(t-t_0)} \beta_t + L_f \int_{t_0}^t e^{\lambda_A(s-t_0)} |x_s| ds.$$

By applying the continuous Gronwall Lemma we obtain

$$\begin{aligned} e^{\lambda_A(t-t_0)} |x_t| &\leq C_A |x_{t_0}| + \int_{t_0}^t C_A e^{\lambda_A(s-t_0)} b(s) ds + e^{\lambda_A(t-t_0)} \beta_t + \\ &\int_{t_0}^t L_f e^{L_f(t-s)} \left[ C_A |x_{t_0}| + \int_{t_0}^s C_A e^{\lambda_A(u-t_0)} b(u) du + e^{\lambda_A(s-t_0)} \beta_s \right] ds \\ &\leq C_A e^{L_f(t-t_0)} |x_{t_0}| + \int_{t_0}^t C_A e^{L_f(t-s) + \lambda_A(s-t_0)} b(s) ds + e^{\lambda_A(t-t_0)} \beta_t + \\ &+ \int_{t_0}^t L_f e^{L_f(t-s) + \lambda_A(s-t_0)} \beta_s ds \end{aligned}$$

and then

(3.4)

$$e^{\lambda(t-t_0)} |x_t| \leq C_A |x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)} b(s) ds + e^{\lambda(t-t_0)} \beta_t + \int_{t_0}^t L_f e^{\lambda(s-t_0)} \beta_s ds.$$

Now we use Lemma 3.1 to estimate  $\beta_s$ . Assume  $t_0 = n_0 \in \mathbb{Z}$ ,

$$\begin{aligned} &e^{\lambda(s-t_0)} \beta_s \\ &= e^{\lambda(s-t_0)} \left| \int_{n_0}^s \Phi(s, u) g(u, x_u) d\omega_u \right| \\ &\leq e^{\lambda(s-t_0)} \sum_{k=n_0}^{\lfloor s \rfloor - 1} \left| \int_{\Delta_k} \Phi(s, u) g(u, x_u) d\omega_u \right| + \left| \int_{\lfloor s \rfloor}^s \Phi(s, u) g(u, x_u) d\omega_u \right| \\ &\leq e^{\lambda(s-t_0)} \sum_{k=n_0}^{\lfloor s \rfloor} K C_A (1 + C_A \|A\|) \|\omega\|_{p, \Delta_k} e^{-\lambda_A(s-k-1)} \times \\ &\quad \times \left[ C_g \|x\|_{p, \Delta_k} + h^*(|k| + 1) + |g(0, 0)| \right] \\ (3.5) &\leq M \sum_{k=n_0}^{\lfloor s \rfloor} \|\omega\|_{p, \Delta_k} e^{\lambda(k-n_0)} e^{-L_f(s-k)} \left[ C_g \|x\|_{p, \Delta_k} + D h^*(|k| + 1) \right]. \end{aligned}$$

Replacing (3.5) into (3.4) and considering  $t \in [n, n + 1]$  we obtain

$$\begin{aligned}
 & e^{\lambda(t-t_0)}|x_t| \\
 & \leq C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)}b(s)ds \\
 & + M \sum_{k=n_0}^n \|\omega\|_{p,\Delta_k} e^{\lambda(k-n_0)-L_f(t-k)} \left[ C_g\|x\|_{p,\Delta_k} + Dh^*(|k| + 1) \right] \\
 & + L_f M \int_{t_0}^t \sum_{k=n_0}^{\lfloor s \rfloor} \|\omega\|_{p,\Delta_k} e^{\lambda(k-n_0)-L_f(s-k)} \times \\
 & \hspace{15em} \times \left[ C_g\|x\|_{p,\Delta_k} + Dh^*(|k| + 1) \right] ds \\
 & \leq C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)}b(s)ds + M \sum_{k=n_0}^n \|\omega\|_{p,\Delta_k} e^{\lambda(k-n_0)} \times \\
 (3.6) \hspace{15em} & \hspace{15em} \times \left[ C_g\|x\|_{p,\Delta_k} + Dh^*(|k| + 1) \right].
 \end{aligned}$$

The continuity of  $x$  at  $t = (n + 1)$  implies that (3.6) holds for all  $t \in [n, n + 1]$ . Now for  $t_0 \in (n_0 - 1, n_0)$ , similar to (3.5) and (3.6) we have

$$e^{\lambda(s-t_0)}\beta_s \leq M \sum_{k=n_0-1}^{\lfloor s \rfloor} \|\omega\|_{p,\Delta_k} e^{\lambda(k-t_0)} e^{-L_f(s-k)} \left[ C_g\|x\|_{p,\Delta_k} + Dh^*(|k| + 1) \right],$$

and by replacing  $t_0$  in the final term of (3.4) by  $(n_0 - 1)$  then

$$\begin{aligned}
 e^{\lambda(t-t_0)}|x_t| & \leq C_A|x_{t_0}| + \int_{t_0}^t C_A e^{\lambda(s-t_0)}b(s)ds \\
 & + M \sum_{k=n_0-1}^n \|\omega\|_{p,\Delta_k} e^{\lambda(k-t_0)} \left[ C_g\|x\|_{p,\Delta_k} + Dh^*(|k| + 1) \right].
 \end{aligned}$$

This proves (3.3). ■

**Proposition 3.1.** *Define*

$$(3.7) \quad \Lambda_k := [2(K + 1)C_g] \|\omega\|_{p,\Delta_k},$$

$$(3.8) \quad G_k := \|\omega\|_{p,\Delta_k} \left( 1 + \Lambda_k^{p-1} \right) e^{\kappa(1+\Lambda_k^p)+2L},$$

$$(3.9) \quad H_k := (1 + \|\omega\|_{p,\Delta_k})^2 \left( 1 + \Lambda_k^{2p-1} \right) e^{\kappa(1+\Lambda_k^p)+2L},$$

where  $\kappa, L$  in Proposition 2.1 and

$$(3.10) \quad \zeta := \sum_{k=1}^{\infty} e^{-\lambda k} h^*(|k|+1) H_{-k} \prod_{j=1}^k \left(1 + MC_g G_{-j}\right)$$

(which can be infinity), where  $\lambda, M$  is defined in Lemma 3.2.

Assume further that

$$(3.11) \quad \lambda > \hat{G} := C_A e^{\lambda_A + 2L} (1 + C_A \|A\|) \left\{ \left[ 2(K+1)C_g \Gamma_p \right]^p + \left[ 2(K+1)C_g \Gamma_p \right] \right\},$$

where  $\Gamma_p$  is defined in (2.4). Then  $\zeta(\omega)$  is finite.

**Proof.**

Due to the inequality  $\log(1 + ae^b) \leq a + b$  for  $a, b \geq 0$ , we have

$$\begin{aligned} \log \left( 1 + MC_g G_k \right) &\leq \left[ M e^{2L+\kappa} + 2 \right] \left[ 2(K+1) \right]^{p-1} C_g^p \|\omega\|_{p, \Delta_k}^p + \\ &\quad M e^{2L+\kappa} C_g \|\omega\|_{p, \Delta_k}. \end{aligned}$$

It follows that

$$\begin{aligned} \overline{\lim}_{m \rightarrow \infty} \frac{1}{m} \log \prod_{k=1}^m \left( 1 + MC_g G_{-k} \right) &= \overline{\lim}_{m \rightarrow \infty} \frac{1}{m} \sum_{k=1}^m \log \left( 1 + MC_g G_{-k} \right) \\ &\leq \left[ M e^{2L} \frac{K+2}{2(K+1)^2} + \frac{1}{K+1} \right] \left\{ \left[ 2(K+1)C_g \Gamma_p \right]^p + \left[ 2(K+1)C_g \Gamma_p \right] \right\} \\ &\leq C_A e^{\lambda_A + 2L} (1 + C_A \|A\|) \left\{ \left[ 2(K+1)C_g \Gamma_p \right]^p + \left[ 2(K+1)C_g \Gamma_p \right] \right\} = \hat{G}. \end{aligned}$$

Meanwhile, (3.7) and (3.9) yield

$$\log H_k \leq D \left[ 1 + \|\omega\|_{p, \Delta_k} + \|\omega\|_{p, \Delta_k}^p \right],$$

where we use the inequalities  $\log(1 + a + b) \leq \log(1 + a) + \log(1 + b)$ ,  $\forall a, b \geq 0$  and  $\log(1 + ab) \leq \log(1 + a) + \log b$ ,  $\forall a \geq 0, b \geq 1$ . As a result,

$$\overline{\lim}_{m \rightarrow \infty} \frac{\log H_{-m}}{m} = 0$$

and then by assumption on  $h^*$

$$\overline{\lim}_{m \rightarrow \infty} \frac{\log h^*(|m|+1) H_{-m}}{m} = 0.$$



Hence, there exists for each  $0 < 2\delta < \lambda - \hat{G}$  an  $m_0 = m_0(\delta, \omega)$  such that for all  $m \geq m_0$ ,

$$e^{(-\delta + \hat{G})m} \leq \prod_{k=1}^m [1 + G_{-k}] \leq e^{(\delta + \hat{G})m}$$

and

$$e^{-\delta m} \leq h^*(|m| + 1)H_{-m} \leq e^{\delta m}.$$

Consequently,

$$\zeta \leq \sum_{k=1}^{m_0-1} e^{-\lambda k} h^*(|k| + 1)H_{-k} \prod_{j=1}^k (1 + MC_g G_{-j}) + \sum_{k=n_0}^{\infty} e^{-(\lambda - 2\delta - \hat{G})k}$$

which is finite. ■

We are now in position to state the first main result of this paper.

**Theorem 3.2.** *Assume that  $(\mathbf{H}_0) - (\mathbf{H}_4)$  are satisfied. Then under the condition (3.11) the flow generated by system (1.1) possesses a pullback attractor  $\mathcal{A}_t$ .*

**Proof.** We first consider  $t_0 = n_0 \in \mathbb{Z}^-, n_0 \leq t = n \in \mathbb{Z}$ . From Lemma 3.2 we have

$$\begin{aligned} |x_n|e^{\lambda(n-n_0)} &\leq C_A|x_{n_0}| + \int_{n_0}^n C_A e^{\lambda(s-t_0)}|b(s)|ds + M \sum_{k=n_0}^{n-1} e^{\lambda(k-n_0)} \|\omega\|_{p, \Delta_k} \times \\ &\quad \times [C_g \|x\|_{p, \Delta_k} + Dh^*(|k| + 1)] \end{aligned}$$

Using  $(\mathbf{H}_2)(iii)$ ,  $\int_{n_0}^n C_A e^{\lambda(s-t_0)}|b(s)|ds \leq D \sum_{k=n_0}^{n-1} e^{\lambda(k-n_0)}$ . Then dominating each  $\|x\|_{p, \Delta_k}$  by estimation in Proposition 2.1 with the observation that

$$N_{\Delta_k}^{\frac{p-1}{p}} \leq 1 + \Lambda_k^{p-1}, \quad N_{\Delta_k}^{\frac{2p-1}{p}} \leq 2[1 + \Lambda_k^{2p-1}]$$

we obtain

$$|x_n|e^{\lambda(n-n_0)} \leq C_A|x_{n_0}| + MC_g \sum_{k=n_0}^{n-1} e^{\lambda(k-n_0)} G_k |x_k| + D \sum_{k=n_0}^{n-1} e^{\lambda(k-n_0)} h^*(|k| + 1)H_k.$$

Fix  $n_0$ , put  $m = n - n_0$  and  $z_k = e^{\lambda k}|x_{k+n_0}|$ . We have

$$z_m \leq C_A z_0 + MC_g \sum_{k=0}^{m-1} G_{k+n_0} z_k + D \sum_{k=0}^{m-1} e^{\lambda k} h^*(|k + n_0| + 1)H_{k+n_0}$$

holds for all  $m \in \mathbb{Z}^+$ . Thank to Lemma 4.1

$$z_m \leq C_A z_0 \prod_{k=0}^{m-1} (1 + MC_g G_{k+n_0}) + D \sum_{k=0}^{m-1} e^{\lambda k} h^*(|k+n_0|+1) H_{k+n_0} \times \\ \times \prod_{j=k+1}^{m-1} (1 + MC_g G_{j+n_0})$$

or

$$|x(n, n_0, x_{n_0})| \\ \leq C_A |x_{n_0}| e^{-\lambda(n-n_0)} \prod_{k=0}^{n-n_0-1} (1 + MC_g G_{k+n_0}) \\ + D \sum_{k=0}^{n-n_0-1} e^{-\lambda(n-n_0-k)} h^*(|k+n_0|+1) H_{k+n_0} \prod_{j=k+1}^{n-n_0-1} (1 + MC_g G_{j+n_0}) \\ \leq C_A |x_{n_0}| e^{-\lambda(n-n_0)} \prod_{k=1-n}^{-n_0} (1 + MC_g G_{-k}) + \\ + D e^{-\lambda n} \sum_{k=1-n}^{-n_0} e^{-\lambda k} h^*(|k|+1) H_{-k} \prod_{j=1-n}^k (1 + MC_g G_{-j}).$$

We consider two cases:

If  $n \geq 1$ ,

$$|x(n, n_0, x_{n_0})| \leq \\ C_A e^{-\lambda n} \prod_{k=1-n}^0 (1 + MC_g G_{-k}) |x_{n_0}| e^{\lambda n_0} \prod_{k=1}^{-n_0} (1 + MC_g G_{-k}) + \\ D e^{-\lambda n} \sum_{k=1-n}^0 e^{-\lambda k} h^*(|k|+1) H_{-k} \prod_{j=1-n}^k (1 + MC_g G_{-j}) + \\ D e^{-\lambda n} \prod_{j=1-n}^0 (1 + MC_g G_{-j}) \sum_{k=1}^{-n_0} e^{-\lambda k} h^*(|k|+1) H_{-k} \prod_{j=1}^k (1 + MC_g G_{-j}).$$

If  $n \leq 0$ ,

$$|x(n, n_0, x_{n_0})| \leq \frac{C_A e^{-\lambda n}}{\prod_{k=1}^{-n_0} (1 + MC_g G_{-k})} |x_{n_0}| e^{\lambda n_0} \prod_{k=1}^{-n_0} (1 + MC_g G_{-k}) +$$

$$\frac{De^{-\lambda n}}{\prod_{k=1}^{1-n} (1 + MC_g G_{-k})} \sum_{k=1}^{-n_0} e^{-\lambda k} h^*(|k| + 1) H_{-k} \prod_{j=1}^k (1 + MC_g G_{-j}).$$

Hence in both cases,

$$\begin{aligned} |x(n, n_0, x_{n_0})| &\leq a_n^1 |x_{n_0}| e^{\lambda n_0} \prod_{k=1}^{-n_0} (1 + MC_g G_{-k}) \\ &\quad + a_n^1 \sum_{k=1}^{-n_0} e^{-\lambda k} h^*(|k| + 1) H_{-k} \prod_{j=1}^k (1 + MC_g G_{-j}) + a_n^2 \\ &\leq a_n^1 |x_{n_0}| e^{\lambda n_0} \prod_{k=1}^{-n_0} (1 + MC_g G_{-k}) \\ &\quad + a_n^1 \sum_{k=1}^{\infty} e^{-\lambda k} h^*(|k| + 1) H_{-k} \prod_{j=1}^k (1 + MC_g G_{-j}) + a_n^2 \\ &\leq a_n^1 |x_{n_0}| e^{\lambda n_0} \prod_{k=1}^{-n_0} [1 + MC_g G_{-k}] + a_n^1 \zeta + a_n^2 \end{aligned}$$

where  $\zeta(\omega)$  is given in (3.10)

$$a_n^1 := De^{-\lambda n} \times \begin{cases} \prod_{k=1-n}^0 [1 + MC_g G_{-k}], & \text{if } n \geq 1 \\ \left( \prod_{k=1}^{1-n} [1 + MC_g G_{-k}] \right)^{-1}, & \text{if } n \leq 0, \end{cases}$$

and

$$a_n^2 := \begin{cases} D_2 \sum_{k=1-n}^0 e^{-\lambda(n+k)} h^*(|k| + 1) H_{-k} \prod_{j=1-n}^k [1 + MC_g G_{-j}], & \text{if } n \geq 1, \\ 0, & \text{if } n \leq 0. \end{cases}$$

Hence, for fixed  $n$ , if  $x_{n_0}$  lies in a tempered set,

$$|x(n, n_0, x_{n_0})| \leq 1 + a_n^1 \zeta + a_n^2 =: \rho_n$$

when  $-n_0$  large enough. It is easy to see that  $\lim_{n \rightarrow \infty} \frac{\log \rho_n}{n} = 0$ .

Using Proposition 2.1 again to estimate  $|x(t, t_0, x_{t_0})|$  with arbitrary  $t, t_0 \in \mathbb{R}$ , by computation we have a tempered function  $\hat{\rho}_t$  is tempered such that

$$(3.12) \quad |x(t, t_0, x_{t_0})| \leq \hat{\rho}_t$$

when  $-t_0$  large enough.

Therefore, there exists a family of sets  $\mathcal{B}_t = \bar{B}(0, \hat{\rho}_t)$  which absorbs  $\tilde{\mathcal{D}}$ . This ensures the existence of the pullback attractor  $\mathcal{A}_t$  for system (1.1) which is given by (3.1). ■

**Remark 3.1.** *In [11] we prove that under (3.11) the RDS generated by the autonomous equation possesses a random pullback attractor. Thus, the criteria still holds for the nonautonomous case.*

#### 4. Special case: $g$ linear

In this part, we consider (1.1) where  $g(t, x)$  is linear in  $x$ . For convenience we assume path  $x$  valued in  $\mathbb{R}$  instead of  $\mathbb{R}^m$ . Then  $g$  has the form  $g(t, x) = C(t)x$  where  $C$  is a  $\mathbb{R}^{d \times d}$ -valued, continuous functions. Then, (1.1) becomes

$$(4.1) \quad dx_t = [A(t)x_t + f(t, x_t)]dt + C(t)x_t d\omega_t, t \in \mathbb{R}, x(t_0) = x_{t_0} \in \mathbb{R}^d.$$

We need the following assumption for  $C$ .

$$(\mathbf{H}_C) \quad C \text{ is continuous and } \hat{C} := \sup_k \|C\|_{p, \Delta_k} < \infty.$$

Note that in this situation,  $(\mathbf{H}_3)$  is not fulfilled. However, as proved in [3], (4.1) possesses a unique solution  $x(\cdot, t_0, x_0)$  start at  $t_0$  from  $x_0 \in \mathbb{R}^d$  which is of bounded  $p$ -variation on any compact subset of  $\mathbb{R}$ . Using the estimate

$$(4.2) \quad \|Cx\|_{p, [s, t]} \leq \|C\|_{\infty, [s, t]} \|x\|_{p, [s, t]} + \|x\|_{\infty, [s, t]} \|C\|_{p, [s, t]},$$

we can treat (4.1) as the general equation in previous section by considering  $C_g = \hat{C}$  and omitting  $(\mathbf{H}_3)(ii), (iii)$ . Then one obtains the similar results that the equation generates a two-parameter flow of homeomorphism on  $\mathbb{R}^d$ .

This case is treated in [4] using a kind of Lyapunov function. Here we revise the problem on the existence of the random pullback attractor of the system in such a case by using semi group method as in previous section.

**Theorem 4.1.** *Assume that  $(\mathbf{H}_0), (\mathbf{H}_1), (\mathbf{H}_2), (\mathbf{H}_C)$  are satisfied. Then there exists  $\varepsilon > 0$  such that if  $\hat{C} < \varepsilon$  the flow generated by the system (4.1) possesses a pullback attractor  $\mathcal{A}_t$ .*

**Proof.** The proof is followed step by step of Theorem 3.2. ■

In what follow we impose a stronger condition on  $f$  to study the difference between two solutions of the system which facilitates the proof of singleton attractor.

(**H<sub>5</sub>**):  $f$  is global Lipchitz continuous with Lipchitz constant  $C_f$  (here we use an abuse notation for simplicity).

We use the linearity of  $g$  to obtain a further result that the pullback attractor in Theorem 4.1 is singleton and moreover forward attractor.

**Theorem 4.2.** *Under the assumption in Theorem 4.1 and (**H<sub>5</sub>**), the pullback attractor  $\mathcal{A}_t$  is singleton for each  $t$  and moreover is forward attractor.*

**Proof.** We fix  $\bar{t}$  and consider  $\mathcal{A}_{\bar{t}}$ . Take  $a^1(\bar{t}), a^2(\bar{t}) \in \mathcal{A}_{\bar{t}}$ , by the invariance of  $\mathcal{A}_{\bar{t}}$ , for each  $t_0 < \bar{t}$  there exist  $b^1 := b^1(t_0), b^2 := b^2(t_0) \in \mathcal{A}_{t_0}$  such that

$$a^i(\bar{t}) = x(\bar{t}, t_0, b^i), \quad i = 1, 2.$$

Put  $x^i := x^i(\cdot, t_0, b^i)$  and  $z := x^1 - x^2$  then

$$\begin{aligned} dz_t &= d(x_t^1 - x_t^2) \\ &= [A(t)x_t^1 + f(t, x_t^1) - A(t)x_t^2 - f(t, x_t^2)]dt + [C(t)x_t^1 - C(t)x_t^2]d\omega_t \\ &= [A(t)z_t + f(t, x_t^1) - f(t, x_t^2)]dt + C(t)z_t d\omega_t \\ &=: [A(t)z_t + F(t, z_t)]dt + C(t)z_t d\omega_t, \quad t \geq t_0, \\ z_{t_0} &= b^1 - b^2, \end{aligned}$$

in which by the definition  $|F(t, z_t)| \leq C_f |z_t|$ ,  $F(t, 0) \equiv 0$ .

Note that using the estimate in (4.2) one obtains a similar result to that in Proposition 3.1. Then repeat the arguments in Lemma 3.2 with for short  $t_0 = n_0 \in \mathbb{Z}$

$$|z_t| e^{\lambda(t-n_0)} \leq C_A |z_{n_0}| + D\hat{C} \sum_{k=n_0}^n \|\omega\|_{p, \Delta_k} e^{\lambda(k-n_0)} \|z\|_{p, \Delta_k}, \quad \forall t \in \Delta_n,$$

in which the norm  $\|z\|_{p, \Delta_k}$  can be estimated similar to  $x$  in Proposition 2.1, namely

$$(4.3) \quad \|z\|_{p, \Delta_k} \leq |z_k| e^{D(1+\hat{C}^p \|\omega\|_{p, \Delta_k}^p)}.$$

Hence

$$|z_t| e^{\lambda(t-n_0)} \leq C_A |z_{n_0}| + D\hat{C} \sum_{k=n_0}^n \|\omega\|_{p, \Delta_k} e^{D\|\omega\|_{p, \Delta_k}^p} e^{\lambda(k-t_0)} |z_k|, \quad \forall t \in \Delta_n.$$

This leads to

$$|z_n| \leq C_A |z_{n_0}| e^{-\lambda(n-n_0)} \prod_{k=n_0}^{n-1} \left[ 1 + D\hat{C} \|\omega\|_{p,\Delta_k} e^{D\|\omega\|_{p,\Delta_k}^p} \right].$$

Since  $b^1, b^2 \in \mathcal{A}_{n_0}$ ,  $|z_{n_0}| \leq 2\hat{\rho}_{n_0}$ . Note that

$$\lim_{n_0 \rightarrow -\infty} \frac{\log \hat{\rho}_{n_0}}{n_0} = 0.$$

For  $\bar{t} = \bar{n} \in \mathbb{Z}$ , follow the arguments in Theorem 3.2 for  $\hat{C}$  small enough,  $|a^1(\bar{t}) - a^2(\bar{t})| = |z_{\bar{t}}| \rightarrow 0$  as  $n_0 \rightarrow \infty$  or  $a^1(\bar{t}) = a^2(\bar{t})$ . Using (4.3) to estimate  $z_t$  via  $z_n$  which  $[n, n+1]$  contain  $t$ , this holds for arbitrary  $\bar{t} \in \mathbb{R}$ . Therefore,  $\mathcal{A}_{\bar{t}}$  is one point set.

Finally, the above arguments show that the difference of two solutions of the system tends to zero in the forward direction, the attractor is then the forward one. The proof is completed.  $\blacksquare$

## Appendix

The proof of following Lemmas can be seen in [11]

**Lemma 4.1** (Discrete Gronwall Lemma). *Let  $a$  be a non negative constant and  $u_n, \alpha_n, \beta_n$  be nonnegative sequences satisfying*

$$u_n \leq a + \sum_{k=0}^{n-1} \alpha_k u_k + \sum_{k=0}^{n-1} \beta_k, \quad \forall n \geq 1$$

then

$$u_n \leq \max\{a, u_0\} \prod_{k=0}^{n-1} (1 + \alpha_k) + \sum_{k=0}^{n-1} \beta_k \prod_{j=k+1}^{n-1} (1 + \alpha_j)$$

for all  $n \geq 1$ .

**Lemma 4.2** (Gronwall-type Lemma). *If  $y$  satisfies the following condition*

$$(4.4) \quad |y_t - y_s| \leq \hat{A}_{s,t}^{1/q} + a_1 \int_s^t |y_u| du + a_2 \|\omega\|_{p,[s,t]} (|y_s| + a_3 \|y\|_{q-\text{var},[s,t]})$$

for all  $s, t$ , where  $a_1, a_2, a_3$  are positive real constants, then

$$(4.5) \quad \|y\|_{p,[u,v]} \leq \left[ |y_u| + 2\hat{A}_{u,v}^{1/q} N_{[u,v]} \right] e^{2a_1(v-u) + \kappa N_{[u,v]}} N_{[u,v]}^{\frac{p-1}{p}}(\omega)$$

with  $\kappa = \log \frac{a_3+2}{a_3+1}$ , and

$$N_{[u,v]} \leq 1 + [2a_2(a_3 + 1)]^p \|\omega\|_{p,[s,t]}^p.$$

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## ON THE EQUATION $F(n^2 + m^2 + k) = H(n) + H(m) + K$

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**Abstract.** We give all solutions of the equation

$$F(n^2 + m^2 + k) = H(n) + H(m) + K \quad (\forall n, m \in \mathbb{N}),$$

where  $k \in \mathbb{N}$  is the sum of two fixed squares,  $K \in \mathbb{C}$  and  $F, H$  are completely multiplicative functions.

### 1. Introduction.

Let  $\mathcal{P}, \mathbb{N}_0, \mathbb{N}, \mathbb{Z}$  and  $\mathbb{C}$  be the set of primes, non-negative integers, positive integers, integers and complex numbers, respectively. Let  $\mathcal{M}$  ( $\mathcal{M}^*$ ) be the set of all multiplicative (completely multiplicative) functions, respectively. For  $D \in \mathbb{N}, D \geq 2$  we denote by  $\chi_D^*(n)$  the principal Dirichlet character and by  $\chi_D(n)$  the non-principal Dirichlet character (mod  $D$ ). For numbers  $x, y \in \mathbb{Z}$  we denote by  $(x, y)$  the greatest common divisor of  $x$  and  $y$ . For each  $n \in \mathbb{N}$  let  $\bar{n} \in \{0, 1, 2, 3\}$  be such that  $\bar{n} \equiv n \pmod{4}$ .

Furthermore, we define the sets  $\mathcal{B}$  and  $\mathcal{E}$  as follows:

$$\mathcal{B} = \{n^2 + m^2 \mid n, m \in \mathbb{N}\} = \{2, 5, 8, 10, 13, 17, 18, 20, 25, \dots\}$$

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and

$$\mathcal{E} = \{n^2 + m^2 > 0 \mid n, m \in \mathbb{N}_0\} = \{1, 2, 4, 5, 8, 9, 10, 13, 16, 17, 18, 20, 25, \dots\}.$$

It is obvious that

$$\mathcal{B} \subset \mathcal{E}.$$

In 1996 P. V. Chung [2] characterized all multiplicative functions  $f$  satisfying the equation

$$f(m^2 + n^2) = f(n^2) + f(m^2) \quad \text{for every } n, m \in \mathbb{N}.$$

P. V. Chung proved that there are only two possible categories of solutions, the first of which contains the identity function

In 2014 B. Bojan [1] determined all  $f : \mathbb{N} \rightarrow \mathbb{C}$  for which

$$f(n^2 + m^2) = f^2(n) + f^2(m) \quad \text{for every } n, m \in \mathbb{N}$$

holds.

Poo-Sung Park in [14] and [15] proved that if  $f \in \mathcal{M}$  and  $k \in \mathbb{N}$ ,  $k \geq 3$  satisfy one of following two conditions

$$f(x_1^2 + \dots + x_k^2) = f(x_1)^2 + \dots + f(x_k)^2$$

or

$$f(x_1^2 + \dots + x_k^2) = f(x_1^2) + \dots + f(x_k^2)$$

for all positive integers  $x_1, \dots, x_k$ , then  $f$  is the identity function.

I. Kátai and B. M. Phong proved in [5] that if the sets

$$\mathcal{A} = \{a_1 < a_2 < \dots\} \subseteq \mathbb{N}, \quad \mathcal{S} := \{m^2 \mid m \in \mathbb{N}\}$$

and the arithmetical functions  $f : \mathcal{A} + \mathcal{S} \rightarrow \mathbb{C}$ ,  $g : \mathcal{A} \rightarrow \mathbb{C}$  and  $h : \mathcal{S} \rightarrow \mathbb{C}$  satisfy the equation

$$f(a + n^2) = g(a) + h(n^2) \quad \text{for every } a \in \mathcal{A}, n \in \mathbb{N},$$

then the assumption  $8\mathbb{N} \subseteq \mathcal{A} - \mathcal{A}$  implies that there is a complex number  $A$  such that

$$g(a) = Aa + \bar{g}(a), \quad h(n^2) = An^2 + \bar{h}(n) \quad \text{and} \quad f(a + n^2) = A(a + n^2) + \bar{g}(a) + \bar{h}(n)$$

hold for every  $a \in \mathcal{A}, n \in \mathbb{N}$ . Furthermore

$$\begin{aligned} \bar{g}(a) &= \bar{g}(b) \quad \text{if } a \equiv b \pmod{120}, \quad (a, b \in \mathcal{A}), \\ \bar{h}(n) &= \bar{h}(m) \quad \text{if } n \equiv m \pmod{60}, \quad (n, m \in \mathbb{N}) \end{aligned}$$

are true.

B.M.M.Khanh [12] determined all solutions of the equation

$$f(n^2 + m^2 + k) = f(n)^2 + f(m)^2 + K \quad (\forall n, m \in \mathbb{N}),$$

where  $k \in \mathbb{N}_0$  and  $K \in \mathbb{C}$ . In [10] and [11] she gave all functions  $f : \mathbb{N} \rightarrow \mathbb{C}$  which satisfy the equation

$$f(n^2 + Dm^2 + k) = f(n)^2 + Df(m)^2 + k \quad \text{for every } n, m \in \mathbb{N}.$$

The conjecture of I. Kátai and B. M. Phong formulated in [6] was proved by B. M. M. Khanh in [10].

In [7] and [8] I. Kátai and B. M. Phong gave all arithmetical functions  $f, h : \mathbb{N} \rightarrow \mathbb{C}$ , which satisfy the relations

$$f(a^2 + b^2 + c^2 + d^2 + k) = h(a) + h(b) + h(c) + h(d) + K$$

for every  $a, b, c, d \in \mathbb{N}$ , where  $k \in \mathbb{N}_0$  and  $K \in \mathbb{C}$ .

Recently, in [9] I. Kátai and B. M. Phong gave all functions  $f, h : \mathbb{N} \rightarrow \mathbb{C}$  which satisfy the relation

$$f(a^2 + b^2 + c^2 + k) = h(a) + h(b) + h(c) + K$$

for every  $a, b, c \in \mathbb{N}$ , where  $k \in \mathbb{N}_0$  and  $K \in \mathbb{C}$ .

In this paper we prove the following result.

**Theorem 1.** *The numbers  $k \in \mathcal{E}, K \in \mathbb{C}$  and the functions  $F, H \in \mathcal{M}^*$  satisfy the equation*

$$(1.1) \quad F(n^2 + m^2 + k) = H(n) + H(m) + K \quad \text{for every } n, m \in \mathbb{N}$$

*if and only if one of the following assertions holds:*

- (T1)  $K = k, \quad H(m) = m^2, F(n) = n,$
- (T2)  $K = -1, \quad H(m) = 1, F(n) = 1,$
- (T3)  $K = -2, \quad H(m) = 1, \quad F(n^2 + m^2 + k) = 0,$
- (T4)  $K = -1, \quad k \equiv 2 \pmod{3}, \quad H(m) = \chi_3^*(m), \quad F(n) = \chi_3(n)$

*for every  $n, m \in \mathbb{N}$ , where  $\chi_3^*(m) \pmod{3}$  is the principal Dirichlet character and  $\chi_3(m) \pmod{3}$  is the non-principal Dirichlet character, i.e  $\chi_3^*(0) = 0, \chi_3^*(1) = 1, \chi_3^*(2) = 1, \chi_3(0) = 0, \chi_3(1) = 1, \chi_3(2) = -1.$*

By Theorem 1, we have

**Corollary 1.** *If the numbers  $k \in \mathcal{E}$ ,  $K \in \mathbb{C} \setminus \{-1, -2\}$  and the functions  $F, H \in \mathcal{M}^*$  satisfy the equation*

$$F(n^2 + m^2 + k) = H(n) + H(m) + K \quad \text{for every } n, m \in \mathbb{N},$$

then

$$K = k, \quad H(m) = m^2 \quad \text{and} \quad F(n) = n \quad \text{for every } n, m \in \mathbb{N}.$$

## 2. Lemmas.

Assume that the functions  $F, H \in \mathcal{M}^*$  and the numbers  $k \in \mathbb{N}_0, K \in \mathbb{C}$  satisfy the equation (1.1). Since  $H \in \mathcal{M}^*$ , we have  $H(1) = 1$ ,  $H(4) = H(2)^2$  and  $H(6) = H(2)H(3)$ .

We shall use the following results due to the second author:

**Lemma 1.** (B. M. M. Khanh [13], Lemma 6). *We have*

$$\begin{cases} H(7) &= 2H(5) - 1 \\ H(8) &= 2H(5) + H(2)^2 - 2 \\ H(9) &= H(2)H(3) + 2H(5) - H(2) - 1 \\ H(10) &= H(2)H(3) + 3H(5) - H(3) - 2 \\ H(11) &= H(2)H(3) + 4H(5) - H(3) - H(2) - 2 \\ H(12) &= H(2)H(3) + 4H(5) + H(2)^2 - H(2) - 4 \end{cases}$$

and

$$\begin{aligned} H(\ell + 12m) &= H(\ell + 9m) + H(\ell + 8m) + H(\ell + 7m) - \\ &\quad - H(\ell + 5m) - H(\ell + 4m) - H(\ell + 3m) + H(\ell) \end{aligned}$$

holds for every  $\ell, m \in \mathbb{N}$ .

**Lemma 2.** (B. M. M. Khanh [13], Lemma 7). *Let*

$$\begin{cases} A &= \frac{1}{120} \left( H(2)H(3) + 4H(5) - H(3) - H(2) - 3 \right), \\ \Gamma_2 &= \frac{-1}{8} \left( H(2)H(3) - 4H(5) + 4H(2)^2 - H(3) + 3H(2) - 3 \right), \\ \Gamma_3 &= \frac{-1}{3} \left( H(2)H(3) - 2H(5) + 2H(3) - H(2) \right), \\ \Gamma_4 &= \frac{1}{4} \left( H(2)H(2) - 2H(2)^2 - H(3) + H(2) + 1 \right), \\ \Gamma_5 &= \frac{1}{5} \left( H(2)H(3) - H(5) - H(3) - H(2) + 2 \right), \\ \Gamma &= \frac{1}{4} \left( H(2)H(3) - 4H(5) + 2H(2)^2 + 3H(3) + H(2) + 1 \right), \end{cases}$$

$$\mathcal{S}(\ell) = \Gamma_2\chi_2^*(\ell) + \Gamma_3\chi_3^*(\ell) + \Gamma_4\chi_4(\ell - 1) + \Gamma_5\chi_5(\ell) + \Gamma,$$

where  $\chi_2^*(\ell) \pmod{2}$ ,  $\chi_3^*(\ell) \pmod{3}$  are the principal Dirichlet characters and  $\chi_4(\ell) \pmod{4}$ ,  $\chi_5(\ell) \pmod{5}$  are the real, non-principal Dirichlet characters.

Then we have

$$(2.1) \quad H(n) = An^2 + \mathcal{S}(n) \quad \text{for every } n \in \mathbb{N}.$$

**Lemma 3.** Assume that the numbers  $k \in \mathcal{E}$  and the function  $G \in \mathcal{M}^*$  satisfy the equation

$$G(n^2 + m^2 + k) = 1 \quad \text{for every } n, m \in \mathbb{N}.$$

Then  $G(n) = 1$  for every  $n \in \mathbb{N}$ .

**Proof.** This lemma follows from Theorem 1.1 of Fehér J., K.-H. Indlekofer and N. M. Timofeev [4] (see also Fehér J. and I. Kátai [3]).

**Lemma 4.** If the numbers  $k \in \mathcal{E}$ ,  $K \in \mathbb{C}$  and the functions  $F, H \in \mathcal{M}^*$  satisfy the equations

$$H(m) = 1 \quad \text{for every } m \in \mathbb{N}$$

and

$$(2.2) \quad F(n^2 + m^2 + k) = H(n) + H(m) + K = 2 + K \quad (\forall n, m \in \mathbb{N}),$$

then one of the assertions (T2), (T3) of Theorem 1 holds.

**Proof.** Assume that  $F, H \in \mathcal{M}^*$  satisfy (2.2). Since

$$(k + 25)^2 + 25 + k = (k + 25)(k + 26) \quad \text{and} \quad 25 = 5^2 = 3^2 + 4^2, \quad 26 = 1^2 + 5^2,$$

by using the facts  $F \in \mathcal{M}^*$ , we infer from (2.2) that

$$\begin{aligned} 2 + K &= F\left((k + 25)^2 + 5^2 + k\right) = F(3^2 + 4^2 + k)F(1^2 + 5^2 + k) = \\ &= (2 + K)^2, \end{aligned}$$

which implies that  $K \in \{-1, -2\}$ .

If  $K = -1$ , then we infer from (2.2) that

$$F(n^2 + m^2 + k) = 2 + K = 1 \quad \text{for every } n, m \in \mathbb{N},$$

and so Lemma 3 with  $G = F \in \mathcal{M}^*$  proves that  $F(n) = 1$  for every  $n \in \mathbb{N}$ . Thus the case (T2) holds.

If  $K = -2$ , then we infer from (2.2) that

$$F(n^2 + m^2 + k) = 2 + K = 0 \quad \text{for every } n, m \in \mathbb{N},$$

and so the case (T3) holds.

Lemma 4 is thus proved.

**Lemma 5.** *Let  $p \in \{2, 3\}$ . Assume that the numbers  $k \in \mathcal{E}$ ,  $K \in \mathbb{C}$  and the functions  $F, H \in \mathcal{M}^*$  satisfy the equations*

$$(2.3) \quad F(n^2 + m^2 + k) = H(n) + H(m) + K \quad \text{and} \quad H(m) = \chi_p^*(m),$$

where  $\chi_p^* \pmod{p}$  is the principal Dirichlet character. Then  $p = 3$  and the assertion (T4) of Theorem 1 holds.

**Proof.** Since

$$(k+25)^2 + 25 + k = (k+25)(k+26) \quad \text{and} \quad (k+900)^2 + 900 + k = (k+900)(k+901),$$

we infer from  $F \in \mathcal{M}^*$  and (2.3) that

$$F((k+25)^2 + 5^2 + k) = F(3^2 + 4^2 + k)F(1^2 + 5^2 + k)$$

and

$$F((k+900)^2 + 30^2 + k) = F(18^2 + 24^2 + k)F(1^2 + 30^2 + k),$$

which with (2.3) imply that

$$\begin{cases} H(k+1) &= -\chi_p^*(5) - K + (\chi_p^*(3) + \chi_p^*(4) + K)(1 + \chi_p^*(5) + K) = \\ &= -1 - K + (1 + K)(2 + K) = (K + 1)^2 \\ H(k) &= -\chi_p^*(30) - K + (\chi_p^*(18) + \chi_p^*(24) + K)(1 + \chi_p^*(30) + K) = \\ &= -K + K(1 + K) = K^2. \end{cases}$$

In the above steps we use the fact  $\chi_p^*(3) + \chi_p^*(4) = 1$ ,  $\chi_p^*(5) = 1$  if  $p \in \{2, 3\}$ . Since  $H(m) = \chi_p^*(m) \in \{0, 1\}$ , we have  $H(k) = K^2 \neq (K + 1)^2 = H(k + 1)$ , consequently

$$\left( H(k), H(k+1) \right) = \left( K^2, (K + 1)^2 \right) \in \left\{ (1, 0), (0, 1) \right\}.$$

These imply that either

$$k \equiv -1 \pmod{p} \quad \text{and} \quad K = -1$$

or

$$k \equiv 0 \pmod{p} \quad \text{and} \quad K = 0.$$

We distinguish the proof for four cases according to  $p \in \{2, 3\}$ .

**The case I:**  $p = 2$ ,  $K = -1$ ,  $k \equiv 1 \pmod{2}$ ,  $H(m) = \chi_2^*(m)$ .

We shall prove that this case does not occur.

In this case, we have

$$F(n^2 + m^2 + k) = \chi_2^*(n) + \chi_2^*(m) - 1 \quad \text{for every } n, m \in \mathbb{N}.$$

It is clear to check that

$$\chi_2^*(n) + \chi_2^*(m) = \overline{n^2 + m^2} \quad \text{for every } n, m \in \mathbb{N},$$

consequently

$$(2.4) \quad F(\eta + k) = \bar{\eta} - 1 \quad \text{for every } \eta \in \mathcal{B},$$

where  $\bar{\eta} \in \{0, 1, 2, 3\}$  such that  $\eta \equiv \bar{\eta} \pmod{4}$ .

Since  $k \in \mathcal{E}$  and  $k \equiv 1 \pmod{2}$ , we have  $k\eta \in \mathcal{B}$  for every  $\eta \in \mathcal{B}$ , and so it follows from (2.4) that

$$(2.5) \quad F(k)F(\eta + 1) = F(k\eta + k) = \overline{k\eta} - 1 = \bar{\eta} - 1 \quad \text{for every } \eta \in \mathcal{B}.$$

In the last relation we use  $k \equiv 1 \pmod{4}$ , because  $k \equiv 1 \pmod{2}$  and  $k \in \mathcal{E}$ . Since  $2 = 1^2 + 1^2 \in \mathcal{B}$  and  $8 = 2^2 + 2^2 \in \mathcal{B}$ , we obtain from (2.5) that

$$F(k)F(3) = \bar{2} - 1 = 1 \quad \text{and} \quad F(k)F(3)^2 = F(k)F(8 + 1) = \bar{8} - 1 = -1.$$

These imply that

$$(2.6) \quad F(3) = F(k) = -1$$

and

$$(2.7) \quad F(\eta + 1) = 1 - \bar{\eta} \quad \text{for every } \eta \in \mathcal{B}.$$

In the next part we deduce from (2.7) that

$$(2.8) \quad F(n) = \chi_4(n) \quad \text{for every } n \in \mathbb{N},$$

where  $\chi_4(n)$  denotes the non-principal Dirichlet character  $\pmod{4}$ . This relation with  $k \equiv 1 \pmod{4}$  implies that  $F(k) = 1$ , which contradicts to (2.6).

Since  $F(3) = -1$ , it follows from (2.7) that

$$\begin{aligned} F(2) &= -F(2)F(3) = -F(6) = -F(1^2 + 2^2 + 1) = -(1 - \bar{5}) = 0, \\ F(7) &= -F(3)F(7) = -F(21) = -F(2^2 + 4^2 + 1) = -(1 - \bar{20}) = -1, \\ F(5) &= -F(5)F(7) = -F(35) = -F(3^2 + 5^2 + 1) = -(1 - \bar{34}) = 1. \end{aligned}$$

Thus we have proved that  $F(n) = \chi_4(n)$  for  $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ .

Now assume that  $F(n) = \chi_4(n)$  for all  $n < P$ , where  $P \geq 11$ . We will prove that  $F(P) = \chi_4(P)$ . Since  $F \in \mathcal{M}^*$ , we may assume that  $P = p \in \mathcal{P}$ . One can check that there are  $n, m \in \mathbb{N}$  such that

$$n^2 + m^2 + 1 = pQ, \text{ where } Q < p, (Q, 2) = 1.$$

Then  $n \equiv m \pmod{2}$ ,  $\chi_4(pQ) = (-1)^n$  and

$$F(n^2 + m^2 + 1) = 1 - \overline{n^2 + m^2} = (-1)^n = \chi_4(pQ),$$

consequently we infer from our assumptions and from (2.7) that

$$F(p) = \frac{F(pQ)}{F(Q)} = \frac{F(n^2 + m^2 + 1)}{F(Q)} = \frac{\chi_4(pQ)}{\chi_4(Q)} = \chi_4(p).$$

This relation shows that  $F(n) = \chi_4(n)$  for all  $n \in \mathbb{N}$

Thus the case I does not occur.

**The case II:**  $p = 3$ ,  $K = -1$ ,  $k \equiv 2 \pmod{3}$ ,  $H(m) = \chi_3^*(m)$ .

In this case, we have

$$F(n^2 + m^2 + k) = \chi_3^*(n) + \chi_3^*(m) - 1 \quad \text{for every } n, m \in \mathbb{N}.$$

It is clear to check that

$$\chi_3^*(n) + \chi_3^*(m) - 1 = \chi_3(n^2 + m^2 + 2) \quad \text{for every } n, m \in \mathbb{N},$$

consequently

$$(2.9) \quad F(\eta + k) = \chi_3(\eta + 2) = \chi_3(\eta + k) \quad \text{for every } \eta \in \mathcal{B},$$

where  $\chi_3(n)$  is the non-principal character  $\pmod{3}$ , i. e.  $\chi_3(0) = 0, \chi_3(1) = 1, \chi_3(2) = -1$ .

Let us note that for  $\alpha \in \mathcal{B}$  and  $\eta \in \mathcal{E}$  the condition  $\alpha\eta \notin \mathcal{B}$  may hold only in the case  $\alpha = 2u^2$  and  $\eta = 2v^2$ . Thus, we have

$$(2.10) \quad \eta(n^2 + m^2) \in \mathcal{B} \quad \text{for every } \eta \in \mathcal{E} \text{ and } n, m \in \mathbb{N}, n \neq m.$$

Consequently, we have  $13k = (2^2 + 3^2)k \in \mathcal{B}$ ,  $25k = (4^2 + 5^2)k \in \mathcal{B}$ ,  $90k = (3^2 + 9^2)k \in \mathcal{B}$  and  $97k = (4^2 + 9^2)k \in \mathcal{B}$ . Thus, we infer from (2.9) that

$$\begin{aligned} F(k)F(2)F(7) &= F(k)F(14) = F(13k + k) = \chi_3(13k + k) = \\ &= \chi_3(k)\chi_3(14) = \chi_3(2)\chi_3(2)\chi_3(7) = 1, \end{aligned}$$



$$\begin{aligned} F(k)F(2)F(13) &= F(k)F(26) = F(25k + k) = \chi_3(25k + k) = \\ &= \chi_3(k)\chi_3(26) = \chi_3(2)^2\chi_3(13) = 1, \end{aligned}$$

$$\begin{aligned} F(k)F(7)F(13) &= F(k)F(91) = F(90k + k) = \chi_3(90k + k) = \\ &= \chi_3(k)\chi_3(91) = \chi_3(2)\chi_3(1) = -1 \end{aligned}$$

and

$$\begin{aligned} F(k)F(2)F(7)^2 &= F(k)F(98) = F(97k + k) = \chi_3(97k + k) = \\ &= \chi_3(k)\chi_3(98) = \chi_3(2)\chi_3(2) = 1. \end{aligned}$$

We infer from these relations that

$$F(7) = 1, F(2) = -1, F(13) = 1 \text{ and } F(k) = -1,$$

which with (2.9) and (2.10) implies

(2.11)

$$\begin{aligned} F(n^2 + m^2 + 1) &= -F(k)F(n^2 + m^2 + 1) = -F(k(n^2 + m^2) + k) = \\ &= -\chi_3(k(n^2 + m^2) + k) = -\chi_3(k)\chi_3(n^2 + m^2 + 1) = \\ &= \chi_3(n^2 + m^2 + 1) \text{ for every } n, m \in \mathbb{N}, n \neq m. \end{aligned}$$

Since  $F(2) = -1$  and  $F(7) = 1$ , it follows from (2.11) that

$$F(3) = -F(2)F(3) = -F(6) = -F(1^2 + 2^2 + 1) = -\chi_3(6) = 0$$

and

$$F(5) = F(5)F(7) = F(35) = F(3^2 + 5^2 + 1) = \chi_3(35) = -1.$$

Thus we have proved that  $F(n) = \chi_3(n)$  for  $n \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ .

Now assume that  $F(n) = \chi_3(n)$  for all  $n < P$ , where  $P \geq 11$ . We will prove that  $F(P) = \chi_3(P)$ . Since  $F \in \mathcal{M}^*$ , we may assume that  $P = p \in \mathcal{P}$ . One can check that there are  $n, m \in \mathbb{N}, n \neq m$  such that

$$n^2 + m^2 + 1 = pQ, \text{ where } Q < p, (Q, 3) = 1.$$

Then we infer from our assumptions and from (2.11) that

$$F(p) = \frac{F(pQ)}{F(Q)} = \frac{F(n^2 + m^2 + 1)}{F(Q)} = \frac{\chi_3(pQ)}{\chi_3(Q)} = \chi_3(p).$$

This relation shows that  $F(n) = \chi_3(n)$  for all  $n \in \mathbb{N}$ , therefore (T4) is thus proved.

**The case III:**  $p = 2$ ,  $K = 0$ ,  $k \equiv 0 \pmod{2}$ ,  $H(m) = \chi_2^*(m)$ .

We prove that this case does not occur. Indeed, we have

$$F(8+k) = F(2^2 + 2^2 + k) = \chi_2^*(2) + \chi_2^*(2) + K = 0,$$

which implies that  $F(2) = 0$  or  $F(Q) = 0$ , where  $Q|k+8$ ,  $(Q, 2) = 1$ . The case  $F(2) = 0$  is not occur, because

$$0 = F(2)F\left(1 + \frac{k}{2}\right) = F\left(1^2 + 1^2 + k\right) = \chi_2^*(1) + \chi_2^*(1) + K = 2.$$

If  $F(Q) = 0$ ,  $(Q, 2) = 1$ , then there are  $n, m \in \mathbb{N}$  such that

$$n^2 + m^2 + k \equiv 0 \pmod{Q} \quad \text{and} \quad (n, 2) = (m, 2) = 1.$$

Then

$$0 = F(Q)F\left(\frac{n^2 + m^2 + k}{Q}\right) = F\left(n^2 + m^2 + k\right) = \chi_2^*(n) + \chi_2^*(m) + K = 2,$$

which is impossible.

**The case IV:**  $p = 3$ ,  $K = 0$ ,  $k \equiv 0 \pmod{3}$ ,  $H(m) = \chi_3^*(m)$ .

We prove that this case does not occur.

Indeed, we have

$$(k+3)^2 + 1 + k = (k+2)(k+5),$$

which implies that

$$\begin{aligned} 0 &= \chi_3^*(k+3) + \chi_3^*(1) + K - (\chi_3^*(1) + \chi_3^*(1) + K)(\chi_3^*(1) + \chi_3^*(2) + K) = \\ &= 1 - 2^2 = -3. \end{aligned}$$

This is impossible.

Lemma 5 is proved.

**Lemma 6.** *Assume that  $k \in \mathbb{N}$ ,  $K \in \mathbb{C}$ . Then there isn't a function  $F \in \mathcal{M}^*$  such*

$$(2.12) \quad F(n^2 + m^2 + k) = \chi_5(n) + \chi_5(m) + K \quad \text{for every } n, m \in \mathbb{N},$$

where  $\chi_5(n)$  is the Dirichlet non-principal character  $\pmod{5}$ .

**Proof.** Assume in the contradiction that there is  $F \in \mathcal{M}^*$  such that (2.12) is true.

We infer from the following relations

$$\begin{aligned}(k+25)^2 + 25 + k &= (k+25)(k+26), \\ (k+169)^2 + 169 + k &= (k+169)(k+170), \\ (k+676)^2 + 676 + k &= (k+676)(k+677)\end{aligned}$$

that

$$\begin{aligned}F\left((k+25)^2 + 25 + k\right) - F(k+25)F(k+26) &= \\ &= \chi_5(k) + \chi_5(5) + K - (\chi_5(3) + \chi_5(4) + K)(\chi_5(1) + \chi_5(5) + K) = \\ &= \chi_5(k) + K - K(1+K) = \chi_5(k) - K^2 = 0, \\ F\left((k+169)^2 + 169 + k\right) - F(k+169)F(k+170) &= \\ &= \chi_5(k+4) + \chi_5(13) + K - (\chi_5(5) + \chi_5(12) + K)(\chi_5(1) + \chi_5(13) + K) = \\ &= \chi_5(k+4) - 1 + K - (-1+K)K = \chi_5(k+4) - (K-1)^2 = 0\end{aligned}$$

and

$$\begin{aligned}F\left((k+676)^2 + 676 + k\right) - F(k+676)F(k+677) &= \\ &= \chi_5(k+1) + \chi_5(26) + K - (\chi_5(10) + \chi_5(24) + K)(\chi_5(1) + \chi_5(26) + K) = \\ &= \chi_5(k+1) + 1 + K - (1+K)(2+K) = \chi_5(k+1) - (K+1)^2 = 0.\end{aligned}$$

These imply that

$$\left(\chi_5(k), \chi_5(k+1), \chi_5(k+4)\right) = \left(K^2, (K+1)^2, (K-1)^2\right).$$

Since  $\chi_5(n) \in \{1, -1, -1, 1, 0\}$  for every  $n \in \mathbb{N}$ , we have

$$\begin{aligned}\left(\chi_5(k), \chi_5(k+1), \chi_5(k+4)\right) &= \left(K^2, (K+1)^2, (K-1)^2\right) \in \\ &\in \left\{(1, -1, 0), (-1, -1, 1), (-1, 1, -1), (1, 0, -1), (0, 1, 1)\right\}.\end{aligned}$$

This relation is true in the following case:

$$(2.13) \quad k \equiv 0 \pmod{5}, \quad K = 0, \quad \chi_5(k) = 0, \quad \chi_5(k+1) = \chi_5(k+4) = 1.$$

Since

$$(k+10)^2 + 4 + k = (k+13)(k+8),$$

and so from (2.12) and (2.13) we have

$$\begin{aligned} 0 &= F\left((k+10)^2 + 2^2 + k\right) - F(2^2 + 3^2 + k)F(2^2 + 2^2 + k) = \\ &= \left(\chi_5(k+10) + \chi_5(2) + K\right) - (\chi_5(2) + \chi_5(3) + K)(2\chi_5(2) + K) = \\ &= -1 - (-2)^2 = -5, \end{aligned}$$

which is impossible.

Lemma 6 is proved.

### 3. Proof of Theorem 1.

It is easy to check that the functions defined in (T1), (T2), (T3) and (T4) satisfy the functional equation (1.1). Now we prove the "only if" part.

Assume that  $k \in \mathcal{E}$ ,  $K \in \mathbb{C}$  and  $F, H \in \mathcal{M}^*$  satisfy (1.1). Since  $H \in \mathcal{M}^*$ , using (2.1), we have

$$\begin{aligned} 0 &= H(nm) - H(n)H(m) = \\ &= A(nm)^2 + \mathcal{S}(nm) - \left((An^2 + \mathcal{S}(n))(Am^2 + \mathcal{S}(m))\right) = \\ &= (A - A^2)n^2m^2 - A\mathcal{S}(m)n^2 - A\mathcal{S}(n)m^2 + \mathcal{S}(nm) - \mathcal{S}(n)\mathcal{S}(m) \end{aligned}$$

holds for every  $n, m \in \mathbb{N}$ . Since  $\mathcal{S}(n)$  is an bounded function, the above equation shows that

$$(3.1) \quad \begin{cases} A^2 &= A \\ A\mathcal{S}(n) &= 0 \quad \text{for every } n \in \mathbb{N} \\ \mathcal{S}(nm) &= \mathcal{S}(n)\mathcal{S}(m) \quad \text{for every } n, m \in \mathbb{N}, (n, m) = 1. \end{cases}$$

The first equation implies  $A \in \{0, 1\}$ .

a) Assume that  $A = 0$ . Then (2.1) implies that  $H(n) = \mathcal{S}(n)$ . It follows from  $A = 0$  and from the definitions of  $A$ , we have

$$H(5) = \frac{1}{4}(-H(2)H(3) + H(3) + H(2) + 3).$$

Since  $H \in \mathcal{M}^*$ , we have  $H(1) = 1$ ,  $H(4) = H(2)^2$ ,  $H(6) = H(2)H(3)$ ,  $H(8) = H(2)^3$ ,  $H(9) = H(3)^2$ ,  $H(10) = H(2)H(5)$  and  $H(12) = H(2)^2H(3)$ . It can check from Lemma 1 that

$$(3.2) \quad \begin{cases} H(7) &= \frac{1}{2}(-H(2)H(3) + H(3) + H(2) + 1), \\ H(8) &= \frac{1}{2}(-H(2)H(3) + H(3) + H(2) - 1 + 2H(2)^2), \\ H(9) &= \frac{1}{2}(H(2)H(3) + H(3) - H(2) + 1), \\ H(10) &= \frac{1}{4}(H(2)H(3) - H(3) + 3H(2) + 1), \\ H(11) &= 1, \\ H(12) &= H(3) - 1 + H(2)^2. \end{cases}$$

Furthermore, we infer from Lemma 2 that

$$(3.3) \quad \begin{cases} \Gamma_2 &= \frac{1}{4}(-H(2)H(3) + H(3) - H(2) + 3 - 2H(2)^2), \\ \Gamma_3 &= \frac{1}{2}(-H(2)H(3) - H(3) + H(2) + 1), \\ \Gamma_4 &= \frac{1}{4}(H(2)H(3) - 2H(2)^2 - H(3) + H(2) + 1), \\ \Gamma_5 &= \frac{1}{4}(H(2)H(3) - H(3) - H(2) + 1), \\ \Gamma &= \frac{1}{2}(H(2)H(3) + H(3) - 1 + H(2)^2) \end{cases}$$

and

$$(3.4) \quad H(\ell) = \mathcal{S}(\ell) := \Gamma_2\chi_2^*(\ell) + \Gamma_3\chi_3^*(\ell) + \Gamma_4\chi_4(\ell - 1) + \Gamma_5\chi_5(\ell) + \Gamma.$$

These imply that

$$\begin{cases} H(8) - H(2)^3 &= -\frac{1}{2}(H(2) - 1)(2H(2)^2 + H(3) - 1) = 0 \\ H(9) - H(3)^2 &= \frac{1}{2}(H(3) - 1)(-2H(3) + H(2) - 1) = 0 \\ H(10) - H(2)F(5) &= \frac{1}{4}(H(2) - 1)(H(2) + 1)(H(3) - 1) = 0. \end{cases}$$

This system has four solutions

$$(H(2), H(3)) \in \{(1, 1), (0, 1), (1, 0), (-1, -1)\}$$

In order to prove Theorem 1, we distinguish the proof for four cases.

**Case (I):** Assume that  $H(2) = 1$ ,  $H(3) = 1$ . Then we infer from (3.2) and (3.3) that

$$A = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma_5 = 0 \quad \text{and} \quad \Gamma = 1,$$

therefore it follows from (3.4) that  $H(m) = 1$  for every  $m \in \mathbb{N}$ .

Thus, Lemma 4 implies the proof of (T2) and (T3) of Theorem 1.

**Case (II):** Assume that  $H(2) = 0, H(3) = 1$ . Then we infer from (3.2) and (3.3) that

$$A = \Gamma_3 = \Gamma_4 = \Gamma_5 = \Gamma = 0 \quad \text{and} \quad \Gamma_2 = 1,$$

therefore it follows from (3.4) that  $H(m) = \chi_2^*(m)$  for every  $m \in \mathbb{N}$ .

Lemma 5 implies that this case does not occur.

**Case (III):** Assume that  $H(2) = 1, H(3) = 0$ . Then we infer from (3.2) and (3.3) that

$$A = \Gamma_2 = \Gamma_4 = \Gamma_5 = \Gamma = 0 \quad \text{and} \quad \Gamma_3 = 1,$$

therefore it follows from (3.4) that  $H(m) = \chi_3^*(m)$  for every  $m \in \mathbb{N}$ .

Thus, Lemma 5 implies the proof of (T4) of Theorem 1.

**Case (IV):** Assume that  $H(2) = -1, H(3) = -1$ . Then we infer from (3.2) and (3.3) that

$$A = \Gamma_2 = \Gamma_3 = \Gamma_4 = \Gamma = 0 \quad \text{and} \quad \Gamma_5 = 1,$$

therefore it follows from (3.4) that  $H(m) = \chi_5(m)$  for every  $m \in \mathbb{N}$

Thus, Lemma 6 implies that this case does not occur.

b) Assume now that  $A = 1$ . Then (3.1) implies that  $\mathcal{S}(n) = 0$  for every  $n \in \mathbb{N}$  and so  $H(m) = m^2$  for every  $m \in \mathbb{N}$ . We obtain from (1.1) that

$$(3.5) \quad F(n^2 + m^2 + k) = n^2 + m^2 + K \quad \text{for every } n, m \in \mathbb{N}.$$

We shall prove that  $k = K$ . Since

$$(k + 25)^2 + 25 + k = (k + 25)(k + 26) \quad \text{and} \quad 25 = 5^2 = 3^2 + 4^3, 26 = 1^2 + 5^2,$$

we infer from (3.5), using the fact  $F \in \mathcal{M}^*$  that

$$(3.6) \quad \begin{aligned} 0 &= F\left((k + 25)^2 + 5^2 + k\right) - F(k + 25)F(k + 26) = \\ &= (k + 25)^2 + 5^2 + K - (25 + K)(26 + K) = -(K + 50 + k)(K - k). \end{aligned}$$

We also have

$$(k + 100)^2 + 100 + k = (k + 100)(k + 101) \quad \text{and} \quad 100 = 10^2 = 6^2 + 8^2, 101 = 1^2 + 10^2,$$

which with (3.5) implies

$$\begin{aligned}
(3.7) \quad 0 &= F\left((k+100)^2 + 100 + k\right) - F\left((k+100)(k+101)\right) = \\
&= (k+100)^2 + 100 + K - (100+K)(101+K) = \\
&= -(K+200+k)(K-k).
\end{aligned}$$

It is obvious from (3.6) and (3.7) that  $k = K$ .

Now let

$$G(n) := \frac{F(n)}{n}, \quad G \in \mathcal{M}^* \quad \text{for every } n \in \mathbb{N}.$$

Then we infer from (3.5) and  $k = K$  that  $G(n^2 + m^2 + 1) = 1$  for every  $n, m \in \mathbb{N}$ , therefore Lemma 3 implies that  $G(n) = 1$  and  $F(n) = n$  for every  $n \in \mathbb{N}$ . The proof of (T1) is finished.

Theorem 1 is proved.

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