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Monique Chicourrat

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Monique Chicourrat. RE-proximities as fixed points of an operator on pseudo-proximities. *Topology and its Applications*, 2000, 104 (1-3), pp.39-51. hal-04090554

**HAL Id: hal-04090554**

**<https://uca.hal.science/hal-04090554>**

Submitted on 10 May 2023

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# RE-PROXIMITIES AS FIXED POINTS OF AN OPERATOR ON PSEUDO PROXIMITIES

*Monique CHICOURRAT*

**Abstract.** We show that the RE-proximities of [7] can be obtained as images of some natural operator defined on pseudo proximities. Considering the one-to-one correspondence between closed graph relations on the ultrafilter space (the so-called "nasses" of [10]) and pseudo proximities, this operator is indeed the correspondent of an idempotent operator defined on nasses and based upon the concepts of equivalence kernel and domain of a nasse.

## 0 - INTRODUCTION.

The results of this paper were presented in [2] and announced in [3] without proofs. Some proofs are different from those in [2].

The aim of this paper is to show that RE-proximities can be constructed as fixed points of some natural operator on the set of *pseudo proximities*.

We denote by  $\mathcal{P}(X)$  the power set of a set  $X$ .

**Definition.** Given any relation  $\delta$  on  $\mathcal{P}(X)$ , we call a *Riesz extension* of  $\delta$  any topological space  $Y$  containing a copy of  $X$  and satisfying the two following conditions :

- (E1)  $A\delta B \iff \overline{A} \cap \overline{B} \neq \emptyset$  for all  $A, B \in \mathcal{P}(X)$
- (E2)  $\{x\}\delta A \iff x \in \overline{A}$  for all  $x \in X, A \in \mathcal{P}(X)$ .

Of course, any relation  $\delta$  admitting a Riesz extension is a *proximity on  $X$* , i.e.  $\delta$  is a symmetric relation on  $\mathcal{P}(X)$  satisfying (P1)-(P3).

- (P1)  $\emptyset (non-\delta) A$
- (P2)  $A\delta(B \cup C) \iff A\delta B \text{ or } A\delta C$
- (P3)  $A \cap B \neq \emptyset \implies A\delta B$

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*1991 Mathematics subject classification.* primary: 54E05, 54D35, secondary: 54A05, 06A15.

*Key words and phrases.* proximities, regular extensions of spaces, Riesz problem, ultrafilters and round and compressed filters.

Furthermore if  $\delta$  admits a  $T_1$ -extension, then  $\delta$  is *separated* i.e.

$$(P4) \quad \{x\} \delta \{y\} \implies x = y$$

Proximities admitting  $T_1$ -extensions are known. First, it is obvious that such a proximity is a separated *LO-proximity*, i.e. satisfying the following condition :

$$(LO) \quad \text{adh}_\delta(A) \delta \text{adh}_\delta(B) \implies A\delta B$$

where  $\text{adh}_\delta(A) = \{x \in X / \{x\}\delta A\}$ .

Conversely, Gagrut, Nainpally [8] and finally Thron [15] proved that any separated LO-proximity admits a compact  $T_1$ -extension to the set  $\Gamma_\delta$  of maximal " $\delta$ -bunches" with the "absorption topology" :  $\varphi : x \in X \mapsto \delta(\{x\}) \in \Gamma_\delta$  is an embedding of  $X$  into  $\Gamma_\delta$ .

In fact, for any LO-proximity  $\delta$  on  $X$ , define  $\hat{\Gamma} = X \cup (\Gamma_\delta \setminus \Gamma_X)$  where  $\Gamma_X = \{\delta(\{x\}) : x \in X\}$ . Consider the following modification of the absorption topology (similar to constructions of Császár) : the family  $\{\hat{\Gamma}(A) : A \subset X\}$  where

$$\hat{\Gamma}(A) = \{x \in X : A \in \delta(\{x\})\} \cup \{\mathcal{G} \in (\Gamma_\delta \setminus \Gamma_X) : A \in \mathcal{G}\}$$

is a closed basis for a topology on  $\hat{\Gamma}$  and this topological space provides a compact Riesz extension for the LO-proximity  $\delta$ .

This proves that proximities admitting Riesz extensions are exactly LO-proximities and that they actually admit a compact extension.

On the other hand, Smirnov [14] proved in 1952 that *Efremovič proximities* are those proximities admitting  $T_2$ -compact extensions. These are separated *EF-proximities* (in the sense of [8]), i.e. satisfying the following

$$(EF) \quad A (non-\delta) B \iff \exists C \subset X : A (non-\delta) C \text{ and } (X \setminus C) (non-\delta) B$$

Constructions given by Smirnov and then by Leader are also absorption topologies on some sets of filters (*ends* [14]) or grills (*clusters* [11]).

In the following sections we will be interested in the RE-proximities :

**Definition.** [7] An *RE-proximity* is a proximity admitting some regular Riesz extension.

These proximities were characterized by Császár in 1986 ; in particular, he constructed regular Riesz extensions to the set of some filters (*round and compressed filters*).

In 1990, Fougères (\*) described a new method (fairly similar to Samuel's method [13]) to construct compact extensions for Efremovič proximities, using ultrafilters and based upon the so-called *nasses* of [10].

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(\*) *Personal communication*

This method is of real interest in the study of RE-proximities [5] (also in the study of "RI-proximities" [4]). It naturally yields new characterizations for these proximities and also a generalization of known results for ends or clusters and EF-proximities [12] to round and compressed filters and RE-proximities.

Moreover, we will prove here that this method also shows clearly that RE-proximities are fixed points (or images) of some natural idempotent operator on pseudo proximities.

In the first section, we will recall some preliminary facts and in particular the one-to-one correspondence between nasses and pseudo proximities.

Then (section 2), using the concept of equivalence kernel of a nasse, we naturally introduce and study some operator  $\mathcal{E}$  on the set of nasses and also the correspondent operator  $\mathcal{R}$  defined on the set of pseudo proximities.

Next, we will prove in section 3 that RE-proximities are exactly those proximities which are fixed points of the operator  $\mathcal{R}$ .

Finally, in section 4, we link round and compressed filters with ultrafilters. Then one gets a direct construction of the operator  $\mathcal{R}$  on pseudo proximities in terms of round and compressed filters.

## 1 - PRELIMINARIES.

A *filter* on a set  $X$  is a non empty subset  $\mathcal{F}$  of  $\mathcal{P}(X)$  such that

$$A \cap B \in \mathcal{F} \iff A \in \mathcal{F} \text{ and } B \in \mathcal{F}$$

and a *grill* on  $X$  is a subset  $\mathcal{G}$  of  $\mathcal{P}(X)$  such that

$$\emptyset \notin \mathcal{G} \text{ and } (A \cup B \in \mathcal{G} \iff A \in \mathcal{G} \text{ or } B \in \mathcal{G}).$$

A *proper filter* will be any filter different from  $\mathcal{P}(X)$ .

Recall first that the map  $g : \mathcal{A} \subset \mathcal{P}(X) \mapsto \mathcal{A}^\# \subset \mathcal{P}(X)$  where

$$\mathcal{A}^\# = \{B \subset X : \forall A \in \mathcal{A}, A \cap B \neq \emptyset\}$$

is one-to-one between filters and grills on  $X$ , the inverse map is again  $g$  ([6], [16]).

We denote by  $\Omega(X)$  the set of ultrafilters on  $X$ . Then recall also that for any filter  $\mathcal{F}$  and grill  $\mathcal{G}$  on  $X$ ,

$$\mathcal{F} \subset \mathcal{G} \iff \exists \mathcal{U} \in \Omega(X) : \mathcal{F} \subset \mathcal{U} \subset \mathcal{G}$$

and for  $A \in \mathcal{P}(X)$ ,

$$A \in \mathcal{G} \iff \exists \mathcal{U} \in \Omega(X) : A \in \mathcal{U} \subset \mathcal{G}.$$

We consider the usual separated compact topology on  $\Omega(X)$ . The family  $\{\Omega(A) : A \subset X\}$  where  $\Omega(A)$  is the set of ultrafilters on  $X$  containing  $A$  is an open and closed basis for this topology, it is closed under finite intersection and finite union.

Thus, if  $\mathbb{F}_1$  and  $\mathbb{F}_2$  are two disjoint closed sets of  $\Omega(X)$  then there exists  $A \subset X$  such that  $\mathbb{F}_1 \subset \Omega(A)$  and  $\mathbb{F}_2 \subset \Omega(X) \setminus \Omega(A) = \Omega(X \setminus A)$ .

For  $\mathbb{H} \subset \Omega(X)$ ,  $\overline{\mathbb{H}}$  denotes the closure of  $\mathbb{H}$  in  $\Omega(X)$ , and if  $\Theta \subset \Omega(X)^2$ , we write  $\overline{\Theta}$  for the closure of  $\Theta$  for the product topology on  $\Omega(X)^2$ .

It is known that there is a bijection between filters (or grills) on  $X$  and closed subsets of  $\Omega(X)$ . Any filter  $\mathcal{F}$  (or any grill  $\mathcal{F}^\#$ ) on  $X$  and the closed subset

$$\mathbb{F} = \{\mathcal{U} \in \Omega(X) : \mathcal{F} \subset \mathcal{U}\} = \{\mathcal{U} \in \Omega(X) : \mathcal{U} \subset \mathcal{F}^\#\}$$

will be said to be *associated*. And we have

$$\mathcal{F} = \cap \{\mathcal{U} : \mathcal{U} \in \mathbb{F}\} \quad \text{and} \quad \mathcal{F}^\# = \cup \{\mathcal{U} : \mathcal{U} \in \mathbb{F}\}.$$

Furthermore, for any  $\mathbb{H} \subset \Omega(X)$ , we have  $\overline{\mathbb{H}} = \{\mathcal{U} \in \Omega(X) : \mathcal{F} \subset \mathcal{U}\}$  where  $\mathcal{F} = \cap \{\mathcal{U} : \mathcal{U} \in \mathbb{H}\}$ .

From now on, we identify any point  $x \in X$  with the ultrafilter centered at  $x$ . Thus we can write  $X \subset \Omega(X)$  and for  $A \subset X$ , the closure  $\overline{A}$  of  $A$  in  $\Omega(X)$  is  $\Omega(A)$ . In particular,  $X$  is a dense subset of  $\Omega(X)$ .

**Definition 1.1.**

1. Following Haddad's terminology, we call *nasse on  $X$*  any closed graph relation on  $\Omega(X)$ .
2. Any relation  $\delta$  on  $\mathcal{P}(X)$  such that  $\delta$  and  $\delta^-$  satisfy (P1) and (P2) will be called *pseudo proximity on  $X$* .

In the following theorem, we summarize preliminary results. These were given by Fougères and they are a small generalization of results due to Haddad [10]. One can consult [1] for basic facts on Galois connections. Proofs can be found in [4] and [5].

**Theorem 1.1.** *For any relation  $\delta$  on  $\mathcal{P}(X)$ , consider*

$$\varphi(\delta) = \{(\mathcal{U}, \mathcal{V}) \in \Omega(X)^2 : \mathcal{U} \times \mathcal{V} \subset \delta\}$$

*and for any relation  $\Theta$  on  $\Omega(X)$ , define*

$$\psi(\Theta) = \cup \{\mathcal{U} \times \mathcal{V} : (\mathcal{U}, \mathcal{V}) \in \Theta\}$$

*i.e.  $A \psi(\Theta) B$  iff  $\Theta(\overline{A}) \cap \overline{B} \neq \emptyset$  (or  $(\overline{A} \times \overline{B}) \cap \Theta \neq \emptyset$ ).*

1.  $(\varphi, \psi)$  is a Galois connection i.e.  $\varphi$  and  $\psi$  are monotone and

$$\Theta \subset \varphi(\delta) \iff \psi(\Theta) \subset \delta.$$

Thus the restriction  $\varphi : \text{Im}\psi \rightarrow \text{Im}\varphi$  is the inverse of  $\psi : \text{Im}\varphi \rightarrow \text{Im}\psi$ .

2. For a relation  $\Theta$  on  $\Omega(X)$ , we have  $\varphi(\psi(\Theta)) = \overline{\Theta}$ .

3.  $\Theta \in \text{Im}\varphi$  iff  $\Theta$  is a nasse on  $X$ .  
 $\delta \in \text{Im}\psi$  iff  $\delta$  is a pseudo proximity on  $X$ .

4. For any pseudo proximity  $\delta$  on  $X$ ,

$\delta$  is symmetric iff  $\varphi(\delta)$  is symmetric,

$\delta$  satisfies (P3) iff  $\varphi(\delta)$  is reflexive.

It follows in particular from this theorem that there is a one-to-one correspondence between proximities and reflexive and symmetric closed relations on  $\Omega(X)$ .

**Definition 1.2.** For any pseudo proximity  $\delta$ , the relation

$$\Delta = \{(\mathcal{U}, \mathcal{V}) \in \Omega(X)^2 : \mathcal{U} \times \mathcal{V} \subset \delta\}$$

is said to be *the nasse of  $\delta$* .

**Remark 1.1.** The nasse of  $\delta$  is a closed subset  $\Delta$  of  $\Omega(X)^2$  and for any closed subset  $\mathbb{F}$  of  $\Omega(X)$ ,  $\Delta(\mathbb{F})$  is closed in  $\Omega(X)$ .

**Remark 1.2.** For any pseudo proximity  $\delta$  with associated nasse  $\Delta$ , we have

$$A\delta B \iff \Delta(\overline{A}) \cap \overline{B} \neq \emptyset.$$

Finally, we recall the following nice characterization given by Haddad ([9], [10]) for EF-proximities. This characterization is the main fact leading to a simple compact extension for EF-proximities to the set of ultrafilters [5].

**Theorem 1.2.** A (pseudo) proximity  $\delta$  with nasse  $\Delta$  satisfies condition (EF) iff  $\Delta$  is an equivalence relation on  $\Omega(X)$ .

## 2 - THE OPERATORS $\mathcal{E}$ AND $\mathcal{R}$

The following lemmas introduce the concepts of equivalence kernel and domain of a symmetric relation. ( The proofs are left to the reader ).

Firstly, we will show that these notions generate idempotent operators on symmetric nasses and on symmetric pseudo proximities.

**Lemma 2.1.** *Let  $R$  be a symmetric relation on a set  $E$  and define*

$$\mathbb{D}(R) = \{x \in E : R(x) \neq \emptyset \text{ and for any } y \in R(x), R(y) = R(x)\}.$$

*Then  $R(\mathbb{D}(R)) = \mathbb{D}(R)$  and  $R$  is an equivalence relation on  $\mathbb{D}(R)$ . Furthermore,  $\mathbb{D}(R)$  is the largest subset of  $E$  with these two properties.*

**Definition 2.1.** (With the above notations).

The set  $\mathbb{D}(R)$  is called *the equivalence domain of  $R$*  and  $R' = R \cap \mathbb{D}(R)^2$  is its *equivalence kernel*. An equivalence class of  $R'$  will be said *equivalence class of  $R$* .

**Lemma 2.2.** *For any symmetric relation  $R$  on  $E$ , a non void subset  $A$  of  $E$  is an equivalence class of  $R$  iff  $A \times A \subset R$  and  $R(A) = A$  (or  $R(A) \subset A$ ).*

**Lemma 2.3.** *If  $R$  is a symmetric relation on  $E$  and  $F \subset E$ , consider the relation  $R_F = R \cap F^2$  on  $F$ . Then  $F \cap \mathbb{D}(R) \subset \mathbb{D}(R_F)$  and  $R' \cap F^2 \subset R'_F$ .*

**Remark 2.1.** For any  $x \in E$ , we have :

$$x \in \mathbb{D}(R) \iff (R(x) \neq \emptyset, R(x) \times R(x) \subset R, \text{ and } R(R(x)) = R(x)).$$

**Remark 2.2.** The equivalence kernel  $R'$  of  $R$  is not necessarily a maximal equivalence relation contained in  $R$ . It can even be empty. For example, if  $E = \{1, 2, 3\}$ , and

$$x R y \iff (x = y) \text{ or } (x = 1 \text{ and } y \in \{2, 3\}) \text{ or } (y = 1 \text{ and } x \in \{2, 3\}).$$

Then  $R' = \emptyset$ .

**Definition 2.2.**

1. For any symmetric nasse  $\Delta$ , let  $\Delta'$  be the equivalence kernel of  $\Delta$  and define  $\mathcal{E}(\Delta)$  to be the (symmetric) nasse  $\overline{\Delta'}$ .

2. For any symmetric pseudo proximity  $\delta$  with symmetric nasse  $\Delta$ , we define  $\mathcal{R}(\delta)$  to be the symmetric pseudo proximity on  $X$  whose nasse is  $\mathcal{E}(\Delta) = \overline{\Delta'}$ .

**Proposition 2.4.**

1. For any symmetric nasse  $\Delta$ ,  $\mathcal{E}(\Delta)$  is also symmetric and :  $\mathcal{E}(\Delta) \subset \Delta$  and  $\mathcal{E}(\mathcal{E}(\Delta)) = \mathcal{E}(\Delta)$ .

2. For any symmetric pseudo proximity  $\delta$  :  $\mathcal{R}(\delta) \subset \delta$  and  $\mathcal{R}(\mathcal{R}(\delta)) = \mathcal{R}(\delta)$ .

**Proof.** First observe that if  $\Delta'$  is the equivalence kernel of  $\Delta$  then  $\mathcal{E}(\Delta) = \overline{\Delta'}$  is symmetric since  $\Delta'$  is.

Of course,  $\mathcal{E}(\Delta) \subset \Delta$  since  $\Delta' \subset \Delta$ . Thus, to prove that  $\mathcal{E}$  is idempotent we show that  $\mathcal{E}(\Delta) \subset \mathcal{E}(\mathcal{E}(\Delta))$ .

Actually, we have  $\mathbb{D}(\Delta) \subset \mathbb{D}(\mathcal{E}(\Delta))$  since for  $\mathcal{U} \in \mathbb{D}(\Delta)$ ,  $\Delta(\mathcal{U}) = \Delta'(\mathcal{U}) = \overline{\Delta'}(\mathcal{U})$  and then

$$\Delta(\mathcal{U}) \times \Delta(\mathcal{U}) \subset \Delta' \subset \overline{\Delta'} \quad \text{and} \quad \overline{\Delta'}(\overline{\Delta'}(\mathcal{U})) \subset \Delta(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U}) = \overline{\Delta'}(\mathcal{U}).$$

Hence,  $\Delta' = \Delta' \cap \mathbb{D}(\Delta)^2 \subset \overline{\Delta'} \cap \mathbb{D}(\Delta)^2 \subset \overline{\Delta'} \cap \mathbb{D}(\mathcal{E}(\Delta))^2 = (\mathcal{E}(\Delta))'$  and thus  $\mathcal{E}(\Delta) \subset \mathcal{E}(\mathcal{E}(\Delta))$ .

Finally, assertion 2. of the proposition is true since the operators  $\varphi$  and  $\psi$  of theorem 1.1 are monotone.

**Remark 2.3.** If  $\mathcal{E}(\Delta)$  is reflexive so is  $\Delta$  and thus if  $\mathcal{R}(\delta)$  is a proximity, so is  $\delta$ . The converse is not true. Furthermore, the operators  $\mathcal{E}$  and  $\mathcal{R}$  are neither increasing nor decreasing operators.

Actually, assume that  $X$  contains at least three points  $x, y, z \in X$ . Let  $\Delta_1$  be the diagonal of  $\Omega(X)^2$  and consider  $\Delta_2 = \Delta_1 \cup \{(x, y), (y, x), (x, z), (z, x)\}$  and  $\Delta_3 = \Delta_2 \cup \{(y, z), (z, y)\}$ . These relations are symmetric nasses on  $X$ , we have  $\mathcal{E}(\Delta_1) = \Delta_1$  and  $\mathcal{E}(\Delta_3) = \Delta_3$ . And since any singleton of  $X^2$  is open in  $\Omega(X)^2$ , we have  $\mathcal{E}(\Delta_2) = \Delta'_2 = \{(\mathcal{U}, \mathcal{U}) : \mathcal{U} \in \Omega(X) \setminus \{x, y, z\}\}$ .

Now observe that  $\Delta_2$  is reflexive, but  $\mathcal{E}(\Delta_2)$  is not. And we have  $\Delta_1 \subset \Delta_2 \subset \Delta_3$  but  $\mathcal{E}(\Delta_1) \not\subset \mathcal{E}(\Delta_2)$  and  $\mathcal{E}(\Delta_3) \not\subset \mathcal{E}(\Delta_2)$ .

**Proposition 2.5.** For any proximity  $\delta$  with nasse  $\Delta$ , the pseudo proximity  $\mathcal{R}(\delta)$  is a proximity iff  $X \subset \mathbb{D}(\Delta)$ .

**Proof.** We prove that for any symmetric nasse  $\Delta$ ,  $\mathcal{E}(\Delta)$  is reflexive iff  $X \subset \mathbb{D}(\Delta)$ .

Actually, for any  $x \in X$ , since  $\{(x, x)\}$  is open in  $\Omega(X)^2$ , we have the following :

$$x \in \mathbb{D}(\Delta) \iff (x, x) \in \Delta' \iff (x, x) \in \overline{\Delta'}.$$

Furthermore,  $\{(x, x) : x \in X\}$  is dense in the diagonal of  $\Omega(X)^2$ . Thus,  $X \subset \mathbb{D}(\Delta)$  iff  $\mathcal{E}(\Delta) = \overline{\Delta'}$  is reflexive.

### 3. RE-PROXIMITIES AS IMAGES OF THE OPERATOR $\mathcal{R}$

In this section, we use results and constructions presented also in [5].

The aim of this section is to prove that RE-proximities are exactly the proximities which are the fixed points (or images) of  $\mathcal{R}$  :



**Theorem 3.1.** *If  $\delta$  is a proximity on  $X$  with associated nasse  $\Delta$ , let  $\Delta'$  be the equivalence kernel of  $\Delta$  and  $\mathbb{D}(\Delta)$  be the equivalence domain of  $\Delta$ .*

*Then,  $\delta$  is an RE-proximity iff one of the following conditions is satisfied. (By theorem 1.1., these conditions are equivalent.)*

$$(RE1) \quad \mathcal{E}(\Delta) = \Delta$$

$$(RE2) \quad \mathcal{R}(\delta) = \delta \quad , \quad \text{or equivalently} \quad : \quad (A\delta B \iff \Delta'(\overline{A}) \cap \overline{B} \neq \emptyset).$$

*Furthermore, if these conditions are satisfied, then  $\delta$  admits a regular Riesz extension to  $\mathbb{D}(\Delta)$ .*

**Theorem 3.2.** *Let  $\delta$  be any proximity with nasse  $\Delta$ .*

1. *Considering  $\Delta'$  as a relation on  $\Omega(X)$ , define the following operator on  $\mathbb{D}(\Delta)$  :*

$$c : H \subset \mathbb{D}(\Delta) \mapsto \Delta'(\overline{H}) \subset \mathbb{D}(\Delta).$$

*This operator is a regular topological closure on  $\mathbb{D}(\Delta)$ .*

2. *If  $\mathcal{R}(\delta)$  is a proximity then  $\mathcal{R}(\delta)$  is an RE-proximity and  $(\mathbb{D}(\Delta), c)$  is a regular extension of  $\mathcal{R}(\delta)$  .*

**Lemma 3.3.** *Suppose that  $E$  is a topological space and  $R$  is a symmetric relation on  $E$  such that for all closed subsets  $F$  of  $E$ ,  $R(F)$  is closed in  $E$ . Let  $R'$  be the equivalence kernel of  $R$ .*

*Then the operator  $c : H \subset \mathbb{D}(R) \mapsto R'(\overline{H}) \subset \mathbb{D}(R)$  defines a topology on  $\mathbb{D}(R)$  and if  $E$  is a separated normal space, then this topology is regular.*

**Proof.** Remark that for  $H \subset E$ , we have  $R'(H) = R'(H \cap \mathbb{D}(R)) = R(H) \cap \mathbb{D}(R)$  and if  $H \subset \mathbb{D}(R)$ ,  $R'(H) = R(H)$ .

That  $c$  is a topological closure on  $\mathbb{D}(R)$  is now based on the following observation : for  $H \subset \mathbb{D}(R)$ , we have  $R'(\overline{H}) = R'(\overline{H} \cap \mathbb{D}(R)) = R(\overline{H}) \cap \mathbb{D}(R)$ , then  $c(H)$  is finally the image of the closure of  $H$  in the subspace  $\mathbb{D}(R)$  by the equivalence relation  $R'$  and it is closed in  $\mathbb{D}(R)$ .

To prove the regularity of  $(\mathbb{D}(R), c)$ , take  $x \in \mathbb{D}(R)$ ,  $H \subset \mathbb{D}(R)$  such that  $x \notin R'(\overline{H})$ . Then,  $R'(x) \cap R'(\overline{H}) = \emptyset$ , and hence,  $R(x)$  and  $R(\overline{H})$  are disjoint closed subsets of  $E$  (recall that  $R'(x) = R(x) \subset \mathbb{D}(R)$ ).

Take closed sets  $F_1$  and  $F_2$  in  $E$  such that  $F_1 \cup F_2 = E$  and  $R(x) \cap F_1 = \emptyset$ ,  $R(\overline{H}) \cap F_2 = \emptyset$ . Then  $R'(F_1) \cup R'(F_2) = \mathbb{D}(R)$ ,  $x \notin R'(F_1)$ , and  $R'(\overline{H}) \cap R'(F_2) = \emptyset$ .

Remark now that  $R'(F_i) = R'(\overline{F_i \cap \mathbb{D}(R)})$  and the proof is complete.

**Proof of theorem 3.2.** From lemma 3.3, the operator  $c$  defines a regular topology on  $\mathbb{D}(\Delta)$ . To prove the assertion 2., first, let us notice that  $\Delta'$  is an equivalence relation on  $\mathbb{D}(\Delta)$  and thus by theorem 1.1, we have

$$A \mathcal{R}(\delta) B \iff (\overline{A} \times \overline{B}) \cap \Delta' \neq \emptyset \iff \Delta'(\overline{A}) \cap \Delta'(\overline{B}) \neq \emptyset.$$

Hence, if  $\mathcal{R}(\delta)$  is a proximity, then  $X \subset \mathbb{D}(\Delta)$  (by proposition 2.5.), and we have

$$(E1) \quad : \quad A \mathcal{R}(\delta) B \iff c(A) \cap c(B) \neq \emptyset$$

$$(E2) \quad : \quad \{x\} \mathcal{R}(\delta) A \iff (\{x\} \times \overline{A}) \cap \Delta' \neq \emptyset \iff x \in \Delta'(\overline{A}) \iff x \in c(A)$$

and  $(\mathbb{D}(\Delta), c)$  is a regular Riesz extension of  $\mathcal{R}(\delta)$ .

Theorem 3.2. proves that if the conditions (RE) are satisfied then  $\delta$  is an RE-proximity. The two next lemmas prove that the converse is true.

**Lemma 3.4.** *If  $Y$  is a topological space, let  $\pi$  be the proximity defined on  $Y$  by  $(A\pi B \iff c(A) \cap c(B) \neq \emptyset)$  where  $c(A)$  is the closure of  $A$  in  $Y$ . Denote by  $\Delta$  the nasse of  $\pi$ . If  $Y$  is regular, then  $Y \subset \mathbb{D}(\Delta)$  and  $\pi$  satisfies (RE2) (i.e.  $\mathcal{R}(\pi) = \pi$ ).*

**Proof.** To start with, remark that if  $Y$  is regular, then  $y \in c(A)$  iff  $\{y\}\pi A$ .

Fix  $y \in Y$ . To prove that  $y \in \mathbb{D}(\Delta)$ , we use remark 2.1.

If  $(\mathcal{U}, \mathcal{V}) \in \Delta(y) \times \Delta(y)$ , for any  $(A, B) \in \mathcal{U} \times \mathcal{V}$ , we have  $\{y\}\pi A$  and  $\{y\}\pi B$ , and then  $A\pi B$ . So  $(\mathcal{U}, \mathcal{V}) \in \Delta$  and  $\Delta(y) \times \Delta(y) \subset \Delta$ .

Furthermore, fix  $\mathcal{U} \in \Delta(y)$  and  $\mathcal{V} \in \Delta(\mathcal{U})$ . For any  $A \in \mathcal{U}$ , we have  $y \in c(A)$  and for any  $B \in \mathcal{V}$ ,  $c(A) \cap c(B) \neq \emptyset$ . Now set  $B \in \mathcal{V}$ , if  $y \notin c(B)$ , take an open set  $O$  such that  $y \in O \subset c(O) \subset Y \setminus c(B)$ . Then  $y \notin Y \setminus O$ , so  $Y \setminus O \notin \mathcal{U}$ , and also  $c(B) \cap c(O) = \emptyset$ , and thus  $O \notin \mathcal{U}$ . This contradicts that  $\mathcal{U}$  is a ultrafilter. Therefore,  $\mathcal{V} \in \Delta(y)$  and finally,  $\Delta(\Delta(y)) = \Delta(y)$ .

Hence  $Y \subset \mathbb{D}(\Delta)$  and :

$$\begin{aligned} A\pi B &\iff \exists y \in Y & : & \{y\}\pi A \text{ and } \{y\}\pi B \\ &\iff \exists y \in Y & : & \overline{A} \cap \Delta(y) \neq \emptyset \text{ and } \overline{B} \cap \Delta(y) \neq \emptyset \\ &\implies \Delta'(\overline{A}) \cap \overline{B} \neq \emptyset & & (\text{because } \Delta(y) \times \Delta(y) \subset \Delta'). \\ &\iff A \mathcal{R}(\pi) B \end{aligned}$$

Finally, since  $\mathcal{R}(\pi) \subset \pi$ , we have  $\mathcal{R}(\pi) = \pi$ , and condition (RE2) is satisfied.

**Lemma 3.5.** *If  $\pi$  is a proximity on a set  $Y$  and  $X \subset Y$ , let  $\delta$  be the proximity induced by  $\pi$  on  $X$ , i.e.*

$$\delta = \pi \cap \mathcal{P}(X)^2.$$

*If  $\mathcal{R}(\pi) = \pi$  then  $\mathcal{R}(\delta) = \delta$ .*

**Proof.** Let  $\Delta_Y$  be the nasse of  $\pi$ , and  $\Delta_X$  that of  $\delta$ . We have to prove that if  $\mathcal{E}(\Delta_Y) = \Delta_Y$  then  $\mathcal{E}(\Delta_X) = \Delta_X$ .

Since  $X \subset Y$ ,  $\Omega(X)$  can be identified with the open-closed subset  $\{\mathcal{U} \in \Omega(Y) : X \in \mathcal{U}\}$  of  $\Omega(Y)$ . Now one can easily verify that  $\Delta_X = \Delta_Y \cap \Omega(X)^2$ .

So, by Lemma 2.3,  $\Delta'_Y \cap \Omega(X)^2 \subset \Delta'_X$ . But  $\Omega(X)^2$  is open in  $\Omega(Y)^2$ ; for this reason, the density of  $\Delta'_Y$  in  $\Delta_Y$  implies that of  $\Delta'_Y \cap \Omega(X)^2$  in  $\Delta_Y \cap \Omega(X)^2$ , and thus, that of  $\Delta'_X$  in  $\Delta_X$ .

**Remark 3.1.** It can be shown that any separated RE-proximity actually admits a separated regular Riesz extension [5] : if  $\delta$  satisfies one of the "(RE)-conditions" and if it is separated, then the quotient space  $\mathbb{D}(\Delta)/\Delta'$  provides a separated and regular Riesz extension of  $\delta$ .

**Remark 3.2.** We finally remark that of course the constructions and results of Theorem 3.2 and of the preceding remark are available for EF-proximities and in fact, we obtain as a corollary known results for these proximities [5] : if  $\delta$  is an EF-proximity with nasse  $\Delta$ , then  $\mathbb{D}(\Delta) = \Omega(X)$  and  $\Delta' = \Delta$ ; the operator  $c : \mathbb{H} \subset \Omega(X) \mapsto \Delta(\overline{\mathbb{H}}) \subset \Omega(X)$  is a compact topological closure on  $\Omega(X)$  and  $(\Omega(X), c)$  is a Riesz extension of  $\delta$ ; furthermore if  $\delta$  is separated, then the separated quotient space  $\Omega(X)/\Delta$  provides a separated compact Riesz extension of  $\delta$ .

#### 4. THE OPERATOR $\mathcal{R}$ AND ROUND AND COMPRESSED FILTERS

In this section we will prove that the pseudo proximity  $\mathcal{R}(\delta)$  can be directly defined from  $\delta$ , using the link between round and compressed filters and ultrafilters.

We need some preliminary notations.

**Let  $\delta$  be any pseudo proximity on  $X$  and  $\Delta$  its associated nasse.**

**Notations.** Let  $\gamma$  be the relation defined on  $\mathcal{P}(X)$  by

$$A\gamma B \iff A \text{ (non-}\delta\text{)} (X \setminus B).$$

Again we denote by  $\gamma(A)$  the set  $\{B \subset X : A\gamma B\}$ .

If  $\mathcal{A} \subset \mathcal{P}(X)$ , we define

$$\gamma(\mathcal{A}) = \cup\{\gamma(A) : A \in \mathcal{A}\} \quad \text{and} \quad \mathcal{A}^\delta = \cap\{\delta(A) : A \in \mathcal{A}\}.$$

Remark that for any filter  $\mathcal{F}$  on  $X$ ,  $\gamma(\mathcal{F})$  is a filter and  $\mathcal{F}^\delta$  is a grill on  $X$ .

The following lemma is a small generalization of a result given in [10] for proximities. The proof can be found also in [5].

**Lemma 4.1.** *For any filter  $\mathcal{F}$  on  $X$  with associated closed  $\mathbb{F} \subset \Omega(X)$ ,  $\gamma(\mathcal{F})^\# = \mathcal{F}^\delta$  and the closed subset of  $\Omega(X)$  associated with the filter  $\gamma(\mathcal{F})$  (or with the grill  $\mathcal{F}^\delta$ ) is  $\Delta(\mathbb{F})$ .*

Recall now some definitions. The definition of  $\delta$ -cluster is due to Leader [11].

**Definition 4.1.** ([7], [16]) A proper filter  $\mathcal{F}$  on  $X$  is said to be  $\delta$ -round iff  $\gamma(\mathcal{F}) = \mathcal{F}$ .

We say that  $\mathcal{F}$  is  $\delta$ -compressed or  $\mathcal{F}^\#$  is a  $\delta$ -clan iff  $\mathcal{F}^\# \times \mathcal{F}^\# \subset \delta$ .

A grill  $\mathcal{G}$  on  $X$  is called a  $\delta$ -cluster iff  $\mathcal{G} = \mathcal{G}^\delta$ .

It is easy to see that any  $\delta$ -cluster is a maximal  $\delta$ -clan (the converse is false [2]) and note that if  $\delta$  is a proximity, then for any  $\mathcal{U} \in \Omega(X)$ ,  $\mathcal{U} \subset \mathcal{U}^\delta$  and thus  $(\mathcal{U}^\delta)^\delta \subset \mathcal{U}^\delta$ . It follows that the grill  $\mathcal{U}^\delta$  is a  $\delta$ -clan iff  $\mathcal{U}^\delta$  is a  $\delta$ -cluster.

Recall that for any filter  $\mathcal{F}$ , with associated closed  $\mathbb{F} \subset \Omega(X)$ ,  $\mathcal{F}^\#$  is the union of the ultrafilters of  $\mathbb{F}$ . So, using Lemma 4.1., one gets the following result.

**Proposition 4.2.** *For any proper filter  $\mathcal{F}$  on  $X$  with associated closed  $\mathbb{F} \subset \Omega(X)$  :  $\mathcal{F}$  is  $\delta$ -round iff  $\mathbb{F} = \Delta(\mathbb{F})$  ; and  $\mathcal{F}$  is  $\delta$ -compressed iff  $\mathbb{F} \times \mathbb{F} \subset \Delta$ .*

In particular, the associated closed  $\mathbb{F} \subset \Omega(X)$  of a ultrafilter  $\mathcal{U}$  is  $\{\mathcal{U}\}$ . So by Lemma 4.1,  $\Delta(\mathcal{U})$  is the closed subset of  $\Omega(X)$  associated to the filter  $\gamma(\mathcal{U})$  and therefore with Lemma 2.2, we can deduce the following theorem.

**Theorem 4.3.** *For any symmetric pseudo proximity  $\delta$  :*

1.  $\mathcal{U} \in \mathbb{D}(\Delta) \iff \gamma(\mathcal{U})$  is  $\delta$ -round compressed.
2. *For any proper filter  $\mathcal{F}$  on  $X$ , with associated closed  $\mathbb{F} \subset \Omega(X)$ ,  $\mathcal{F}$  is  $\delta$ -round compressed iff there exists  $\mathcal{U} \in \mathbb{D}(\Delta)$  such that  $\mathbb{F} = \Delta(\mathcal{U})$  and then  $\mathcal{F} = \gamma(\mathcal{U})$  and  $\mathcal{F}^\# = \mathcal{U}^\delta$ .*

*Thus for any proximity  $\delta$ , if  $\mathcal{F}$  is a  $\delta$ -round compressed filter, then  $\mathcal{F}^\#$  is a  $\delta$ -cluster.*

This result now yields a direct construction of  $\mathcal{R}(\delta)$  from  $\delta$  and hence gives also characterizations of RE-proximities. The characterization (RE4) was given (in terms of the "order"  $\gamma$ ) by Császár [7].

For any symmetric pseudo-proximity  $\delta$  with associated nasse  $\Delta$ , denote by  $\Delta'$  the equivalence kernel of  $\Delta$ . Note that for any  $A \in \mathcal{P}(X)$ , and any  $\mathcal{U} \in \Omega(X)$ , we have :

$\mathcal{U} \in \Delta'(\overline{A})$  iff  $\mathcal{U} \in \mathbb{D}(\Delta)$  and  $A \in \mathcal{U}^\delta$ . Since  $\Delta'$  is an equivalence relation on  $\mathbb{D}(\Delta)$ , using the characterization of a pseudo proximity with its associated nasse (theorem 1.1), we deduce :

**Theorem 4.4.**

1.  $AR(\delta)B \iff \exists \mathcal{U} \in \mathbb{D}(\Delta) : A, B \in \mathcal{U}^\delta$   
 $\iff$  *there exists a  $\delta$ -round compressed filter  $\mathcal{F}$  such that  $A, B \in \mathcal{F}^\#$ .*
2. *A proximity  $\delta$  is an RE-proximity iff one of the following equivalent statements is satisfied :*

$$(RE3) \quad A\delta B \iff \exists \mathcal{U} \in \mathbb{D}(\Delta) : A, B \in \mathcal{U}^\delta$$

$$(RE4) \quad A\delta B \iff \text{there exists a } \delta\text{-round compressed filter } \mathcal{F} \text{ such that } A, B \in \mathcal{F}^\#.$$

**Remark 4.1.** Note that theorem 4.3 is a generalization of results available for EF-proximities ([12], [16], [2]). As a matter of fact, the following assertions are equivalent if  $\Delta$  is an equivalence relation and  $\mathbb{F} \subset \Omega(X)$  :

1.  $\mathbb{F}$  is maximal for the property  $\mathbb{F} \times \mathbb{F} \subset \Delta$
2.  $\mathbb{F}$  is minimal for the property  $\mathbb{F} = \Delta(\mathbb{F})$
3.  $\mathbb{F} \times \mathbb{F} \subset \Delta$  and  $\mathbb{F} = \Delta(\mathbb{F})$
4. There exists  $\mathcal{U} \in \Omega(X)$  such that  $\mathbb{F} = \Delta(\mathcal{U})$ .

Now let  $\delta$  be an EF-proximity and  $\mathcal{F}$  some filter on  $X$ . Since minimality for the filter  $\mathcal{F}$  (and maximality for the associated grill) means maximality for the associated closed  $\mathbb{F}$ , and vice-versa, then the following properties are equivalent [5] :

1.  $\mathcal{F}^\#$  is a maximal  $\delta$ -clan
2.  $\mathcal{F}$  is a minimal  $\delta$ -compressed filter
3.  $\mathcal{F}$  is a maximal  $\delta$ -round filter
4.  $\mathcal{F}$  is a  $\delta$ -round compressed filter
5.  $\mathcal{F} = \gamma(\mathcal{U})$  for some  $\mathcal{U} \in \Omega(X)$  (and  $\mathcal{F}^\# = \mathcal{U}^\delta$ )
6.  $\mathcal{F}^\#$  is a  $\delta$ -cluster.

**Remark 4.2.** For any proximity  $\delta$ , let us denote by  $\Lambda_\delta$  the set of  $\delta$ -round and compressed filters. It is shown in [5] that for any separated RE-proximity  $\delta$ , one gets a (separated) regular Riesz extension of  $\delta$  to the set  $\Lambda_\delta$ . This extension is equivalent (in the same way that compactifications of a completely regular space are equivalent) to  $\mathbb{D}(\Delta)/\Delta'$ , and to the Riesz extension constructed by Császár.

But also in the general case of RE-proximities, it can be proved that the regular Riesz extension constructed by Császár is a quotient of a subspace of  $\mathbb{D}(\Delta)$ . Actually, consider

$\Lambda_X = \{\gamma(\{x\}) : x \in X\}$  ( $\Lambda_X$  is the set of *convergent filters*) and  $\hat{\Lambda}_\delta = X \cup (\Lambda_\delta \setminus \Lambda_X)$ ; the topology generated by the family  $\{\hat{\Lambda}_\delta(A) : A \subset X\}$  where  $\hat{\Lambda}_\delta(A) = \{x \in X : A \in \gamma(\{x\})\} \cup \{\mathcal{F} \in (\Lambda_\delta \setminus \Lambda_X) : A \in \mathcal{F}\}$  provides a regular Riesz extension for the RE-proximity  $\delta$  [7]. One can show that this space is also a quotient of a subspace of  $\mathbb{D}(\Delta)$  (see [2]), namely, it is a quotient of  $\hat{\mathbb{D}} = (\mathbb{D}(\Delta) \setminus \mathbb{E}_X) \cup X$  where  $\mathbb{E}_X = \bigcup_{x \in X} \Delta(x)$ .

**Remark 4.3.** Finally, let us remark that Császár's *preregular systems of filters* can be linked with ultrafilters. These are systems  $\Lambda$  of filters on  $X$  inducing a proximity  $\delta$  as follows, for which  $\Lambda \subset \Lambda_\delta$  :

$$A\delta B \iff \exists \mathcal{F} \in \Lambda : A, B \in \mathcal{F}^\#$$

Such a proximity  $\delta$  is an RE-proximity (theorem 4.4.). If  $\Delta$  is the nasse of this proximity, then preregular systems of filters  $\Lambda$  inducing  $\delta$  correspond exactly to subsets  $\mathbb{E}$  of  $\mathbb{D}(\Delta)$  such that  $\mathbb{E} = \Delta(\mathbb{E})$  and  $\Delta \cap \mathbb{E}^2$  is dense in  $\Delta$ . The correspondence is clearly the following:

$$\mathbb{E} = \{\mathcal{U} \in \mathbb{D}(\Delta) : \gamma(\mathcal{U}) \in \Lambda\} \quad \text{and} \quad \Lambda = \{\gamma(\mathcal{U}) : \mathcal{U} \in \mathbb{E}\}.$$

Then, the results obtained for the Riesz extensions  $\mathbb{D}(\Delta)$ ,  $\mathbb{D}(\Delta)/\Delta'$  and  $\Lambda_\delta$  can be generalized to  $\mathbb{E}$ ,  $\mathbb{E}/(\Delta \cap \mathbb{E}^2)$  and  $\Lambda$ .

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Université Blaise Pascal (Clermont II)  
 Laboratoire de Mathématiques Pures  
 63177 AUBIÈRE Cedex  
 e-mail : chicour@ucfma.univ-bpclermont.fr