Convergences, pretopologies and proximities induced by Cauchy and Riesz screens

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CONVERGENCES, PRETOPOLOGIES AND PROXIMITIES INDUCED BY CAUCHY AND RIESZ SCREENS

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ABSTRACT. We answer a question of Császár [12]: under which conditions a given pretopological closure or proximity can be induced by a Cauchy structure? We give a characterization for these closures and proximities using properties of convergences and nasses [14] induced by Cauchy structures. We prove also that the set of Cauchy screens inducing a given reciprocal convergence structure is a non empty interval of the set of Cauchy screens equipped with the usual inclusion order.

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1. Introduction

The starting point of this paper is a question asked and investigated by Császár in [12]: under which conditions a given pretopological closure or proximity can be induced by a Cauchy screen (or Cauchy structure)?

Recall [1] that a pretopological space (X, c) is given by a set X and a **pretopological closure** c on X, that is, a mapping $c : \mathcal{P}(X) \to \mathcal{P}(X)$ from the power set of X to itself satisfying the following conditions:

$(CL1)$ $c(\emptyset) = \emptyset$;
$(CL2)$ $\forall A \in \mathcal{P}(X),\ A \subset c(A)$;
$(CL3)$ $\forall A, B \in \mathcal{P}(X),\ c(A \cup B) = c(A) \cup c(B)$.

It is well known that a pretopological structure on a set X can be given equivalently by a pretopological closure or a pretopological convergence or also by a family $(\mathcal{V}(x))_{x \in X}$ of filters on X, each $\mathcal{V}(x)$ being fixed at x.

Recall that a **proximity** (in the sense of [1]) on a set X is a binary symmetric relation δ on $\mathcal{P}(X)$ such that:

$(P0)$ $\forall\ A \in \mathcal{P}(X),\ \emptyset(\text{non-}\delta)A$ [1]
$(P1)$ $\forall\ A, B \in \mathcal{P}(X) : A \cap B \neq \emptyset \Rightarrow A\delta B$
$(P2)$ $\forall\ A, B, C \in \mathcal{P}(X) : A\delta(B \cup C) \Leftrightarrow A\delta B$ *or* $A\delta C$.

Such a relation induces a pretopological closure c on X defined by :

$$x \in c(A) \Leftrightarrow \{x\}\delta A.$$

[1] We write $A\delta B$ for $(A, B) \in \delta$ and $A(\text{non-}\delta)B$ for $(A, B) \notin \delta$

According to the terminology of [11] a **screen** on a set X is a family $\mathcal{C}$ of proper filters (that is, not containing $\emptyset$) on X such that:

($SC0$) $\forall x \in X$, $\dot{x} \in \mathcal{C}$ [2]
($SC1$) for any filters $\mathcal{F}$ and $\mathcal{G}$, if $\mathcal{F} \in \mathcal{C}$ and $\mathcal{F} \subset \mathcal{G}$ then $\mathcal{G} \in \mathcal{C}$.

In the terminology of [15] the pair $(X, \mathcal{C})$ is a **filterspace**.

A screen $\mathcal{C}$ is a **Cauchy screen** (or a **Cauchy structure** in [16] and [7]) if it satisfies the following condition:

($SC2$) If $\mathcal{F}$ and $\mathcal{G}$ are members of $\mathcal{C}$ such that $\mathcal{F} \subset \mathcal{G}^{\#}$ then $\mathcal{F} \cap \mathcal{G}$ is also in $\mathcal{C}$

where $\mathcal{G}^{\#}$ denotes the set $\{A \in \mathcal{P}(X) : \forall\ B \in \mathcal{G}, A \cap B \neq \emptyset\}$.

The set of screens on X is equipped with the following order: a screen $\mathcal{C}$ is **coarser than** a screen $\mathcal{C}'$, or $\mathcal{C}'$ is **finer than** $\mathcal{C}$, if $\mathcal{C}' \subset \mathcal{C}$.

Any screen $\mathcal{C}$ induces a convergence structure $\to_{\mathcal{C}}$ given by:

$$\mathcal{F} \to_{\mathcal{C}} x \quad \text{ if and only if } \quad \mathcal{F} \cap \dot{x} \in \mathcal{C}$$

It also induces a proximity δ and a pretopological closure c as follows:

$$A \delta B \Leftrightarrow \exists\ \mathcal{F} \in \mathcal{C},\ A, B \in \mathcal{F}^{\#}$$

$$x \in c(A) \Leftrightarrow \exists\ \mathcal{F} \in \mathcal{C},\ (\{x\}, A) \in \mathcal{F}^{\#} \times \mathcal{F}^{\#}$$

where $\mathcal{F}^{\#}$ is defined as above.

This closure c is also the pretopological closure induced by the convergence $\to_{\mathcal{C}}$.

In section 4 we give an answer to Császár's question. We use the one-to-one correspondence between proximities and *nasses* of Haddad (reflexive and closed graph relations on the set of ultrafilters) and also properties concerning convergences induced by Cauchy screens.

It is known that the convergences induced by a screen are the *Kent convergences* [15] and those induced by a Cauchy screen are the *reciprocal limit structures* [7]. In section 3 we study some order properties of screens and convergences: we prove for example that those (Cauchy) screens inducing a given convergence form a non empty interval of the ordered set of (Cauchy) screens, in particular there exist a coarsest and a finest one. We study also the relationship between properties of convergences and properties of some screens inducing them.

Section 4 deals with those nasses, proximities and closures induced by (Cauchy) screens and uses results from section 3.

Section 2 is about notations and definitions.

[2] $\dot{x}$ denotes the ultrafilter on X containing $\{x\}$.

2. Notations and basic definitions.

2.1. **Filters and grills.** We give here some terminology, notation and basic useful facts about filters, grills and ultrafilters. For more details, the reader may consult [19] and [20].

As in [13] a subset $\mathcal{F}$ of $\mathcal{P}(X)$ such that : $(A \in \mathcal{F},\ A \subset B \Rightarrow B \in \mathcal{F})$, will be called a **semifilter** (it is a *stack* in [20],) and for any semifilter, we write

$$\mathcal{F}^{\#} = \{A \in \mathcal{P}(X) : \forall F \in \mathcal{F}, F \cap A \neq \emptyset\}$$

(it is denoted by $\sec \mathcal{F}$ in [7]); $\mathcal{F}^{\#}$ is again a semifilter and $\mathcal{F} \mapsto \mathcal{F}^{\#}$ is an idempotent map such that: $\mathcal{F} \subset \mathcal{G} \Longleftrightarrow \mathcal{G}^{\#} \subset \mathcal{F}^{\#}$.
Furthermore we have for any semifilter $\mathcal{F}$: $A \in \mathcal{F}^{\#} \Leftrightarrow X \setminus A \notin \mathcal{F}$.

We call a **filter on** X any nonempty semifilter $\mathcal{F}$ such that $A \cap B \in \mathcal{F}$ whenever A and B are in $\mathcal{F}$. A filter is **proper** if it is not equal to $\mathcal{P}(X)$ (which is to say that $\emptyset \notin \mathcal{F}$).

We denote by $\phi(X)$ the set of proper filters on X. A filter $\mathcal{F}$ on X is said to be **centered** (or **fixed**) if its center $\cap\{F : F \in \mathcal{F}\}$ is not empty otherwise it is **free**, and if x is in $\cap\{F : F \in \mathcal{F}\}$, we say that $\mathcal{F}$ is **fixed at** x. An **ultrafilter** is a maximal filter.

If $\mathcal{F}$ and $\mathcal{G}$ are filters on X such that $\mathcal{F} \subset \mathcal{G}^{\#}$, then $\mathcal{F} \vee \mathcal{G}$ denotes the filter generated by $\mathcal{F} \cup \mathcal{G}$. We say that $\mathcal{F}$ **mesh** $\mathcal{G}$ (or $\mathcal{F}$ **and** $\mathcal{G}$ **mesh**) whenever $\mathcal{F} \subset \mathcal{G}^{\#}$.

A **filter grill** (or a **grill** for short) is a semifilter $\mathcal{G}$ not equal to $\mathcal{P}(X)$ and for which $A \cup B \in \mathcal{G}$ if and only if A or B is in $\mathcal{G}$. These were introduced by Choquet in [6]. Recall that the map $\mathcal{F} \mapsto \mathcal{F}^{\#}$ is a bijection between filters and grills, fixed points of which are exactly the ultrafilters on X and we have for any filter $\mathcal{F}$ and any grill $\mathcal{G}$: $\mathcal{F} \subset \mathcal{G}$ if and only if $\mathcal{F} \subset \mathcal{U} \subset \mathcal{G}$ for some ultrafilter $\mathcal{U}$ on X.

Let βX be the set of ultrafilters on X and for any $A \subset X$ let βA be the set of ultrafilters containing A. βX is equipped with the usual compact topology for which $\{\beta A : A \subset X\}$ is a base of open sets.
We write $\dot{x}$ for the ultrafilter containing $\{x\}$. Recall that $\dot{x}$ is an isolated point of βX and that a subset $\mathbb{D}$ of βX is dense in βX if and only if $\mathbb{D}$ contains all of the centered ultrafilters.

For any filter $\mathcal{F}$, we denote by $\beta(\mathcal{F})$ the closed set of all ultrafilters containing $\mathcal{F}$ (or contained in $\mathcal{F}^{\#}$). For any filters $\mathcal{F}$ and $\mathcal{G}$, we have: $\mathcal{F} \subset \mathcal{G}^{\#}$ if and only if $\beta(\mathcal{F}) \cap \beta(\mathcal{G}) \neq \emptyset$, in particular $A \in \mathcal{F}^{\#}$ if and only if $\beta A \cap \beta(\mathcal{F}) \neq \emptyset$.
For all filters $\mathcal{F}$ and $\mathcal{G}$, $\beta(\mathcal{F} \cap \mathcal{G}) = \beta(\mathcal{F}) \cup \beta(\mathcal{G})$, that is, any ultrafilter containing $\mathcal{F} \cap \mathcal{G}$ contains $\mathcal{F}$ or $\mathcal{G}$.
Recall also that there exists a bijection between closed subsets of βX and filters on X. Actually for any subset $\mathbb{F}$ of βX and any filter $\mathcal{F}$, we have: $\mathcal{F} = \cap\{\mathcal{U} : \mathcal{U} \in \mathbb{F}\}$ if and only if $\beta(\mathcal{F}) = \overline{\mathbb{F}}$.

Finally, an **elementary filter** $\mathcal{F}$ [11] is a filter for which $\beta(\mathcal{F})$ is finite, i.e. $\mathcal{F}$ is a finite intersection of ultrafilters. If $\mathcal{U}, \mathcal{V} \in \beta X$, we call the filter

$\mathcal{U} \cap \mathcal{V}$ a **bi-ultrafilter**, notice that $\beta(\mathcal{U} \cap \mathcal{V}) = \{\mathcal{U}, \mathcal{V}\}$. Any ultrafilter is a bi-ultrafilter.

2.2. **Convergences.** A **convergence structure** (or a **convergence**) on a set X is a relation ξ from $\phi(X)$ to X such that:

- $(C0)$ $(\dot{x}, x) \in \xi$ for all $x \in X$,
- $(C1)$ For all filters $\mathcal{F}$ and $\mathcal{G}$, $(\mathcal{G}, x) \in \xi$ whenever $(\mathcal{F}, x) \in \xi$ and $\mathcal{F} \subset \mathcal{G}$.

We will also write $\mathcal{F} \to_\xi x$ or $\mathcal{F} \to x$ for $(\mathcal{F}, x) \in \xi$ and we will say that $\mathcal{F}$ **converges to** x or that x is a **limit point** of $\mathcal{F}$. A filter $\mathcal{F}$ is said to be **convergent** if it admits a limit point.

A convergence ξ is **coarser than** a convergence τ (or τ is **finer than** ξ) if:

$$\mathcal{F} \to_\tau x \Longrightarrow \mathcal{F} \to_\xi x$$

and we write : $\tau \leq \xi$.

Following [15] a convergence ξ is a **Kent convergence** provided that the following is satisfied:

- $(C2)$ If $\mathcal{F} \to x$ then $\mathcal{F} \cap \dot{x} \to x$.

A convergence ξ is said to be a **limit structure** if the following is satisfied:

- $(C3)$ if x is a limit point of the filters $\mathcal{F}_1$ and $\mathcal{F}_2$, then $\mathcal{F}_1 \cap \mathcal{F}_2$ converges to x.

A convergence ξ is **pseudotopological** [5] provided that:

- $(C4)$ $\mathcal{F} \to x \Longleftrightarrow \forall\, \mathcal{U} \in \beta(\mathcal{F}),\ \mathcal{U} \to x$.

Given a convergence ξ one assigns to any point x of X the **neighborhood filter** $\mathcal{V}_\xi(x)$ (or $\mathcal{V}(x)$) of x defined by:

$$\mathcal{V}_\xi(x) = \cap\{\mathcal{F} : \mathcal{F} \to_\xi x\}.$$

We have also $\mathcal{V}_\xi(x) = \cap\{\mathcal{U} \in \beta X : \mathcal{U} \to_\xi x\}$.
The pretopology associated to the system of neighborhood filters $(\mathcal{V}_\xi(x))_{x \in X}$ is the **pretopology associated to** ξ.

A convergence ξ is **pretopological** [5] provided that for all x, $\mathcal{V}_\xi(x) \to_\xi x$, or equivalently if:

- $(C5)$ $\mathcal{F} \to_\xi x \Longleftrightarrow \mathcal{V}_\xi(x) \subset \mathcal{F}$

that is to say that ξ is the convergence related to its associated pretopological structure.
We have : $(C5) \Rightarrow (C4) \Rightarrow (C3) \Rightarrow (C2)$.

Recall (with the terminology of [18]) that for any convergence ξ there exists a **pseudotopological modification** $Ps\xi$ which is the finest pseudotopological convergence coarser than ξ and a **pretopological modification** $Pr\xi$ which is the finest pretopological convergence coarser than ξ. They are respectively defined by:

$$\mathcal{F} \to_{Ps\xi} x \Longleftrightarrow \forall\, \mathcal{U} \in \beta(\mathcal{F}),\ \mathcal{U} \to_\xi x \Longleftrightarrow \beta(\mathcal{F}) \subset \mathbb{F}_x$$

where $\mathbb{F}_x = \{\mathcal{U} \in \beta X : \mathcal{U} \to_\xi x\}$, and:

$$\mathcal{F} \to_{Pr\xi} x \Longleftrightarrow \mathcal{V}_\xi(x) \subset \mathcal{F}.$$

In other words, $Pr\xi$ is the convergence associated to the pretopology with neighborhood filters $(\mathcal{V}_\xi(x))_{x\in X}$. We have $\xi \leq Ps\xi \leq Pr\xi$.

3. **Convergences and screens.**

To every screen $\mathcal{C}$ one assigns its **induced convergence** which is defined by:

$$\mathcal{F} \to_{\mathcal{C}} x \quad \text{if and only if} \quad \mathcal{F} \cap \dot{x} \in \mathcal{C}$$

Notice that any convergent filter is member of $\mathcal{C}$.

For any screen $\mathcal{C}$ the neighborhood filter of a point x (with respect to the induced convergence) is given by:

$$\mathcal{V}(x) = \cap\{\mathcal{F} : \mathcal{F} \to_{\mathcal{C}} x\} = \cap\{\mathcal{F} : \mathcal{F} \cap \dot{x} \in \mathcal{C}\} = \cap\{\mathcal{F} \in \mathcal{C} : \mathcal{F} \subset \dot{x}\}$$

According to [11] a screen $\mathcal{C}$ is said to be **Riesz** if for all $x \in X$, $\mathcal{V}(x) \in \mathcal{C}$.

It is known that to a Cauchy screen $\mathcal{C}$ on X one can associate an **equivalence relation $\sim$ on $\mathcal{C}$**, which we call **associated to $\mathcal{C}$**, defined as follows:

$$\mathcal{F} \sim \mathcal{G} \Longleftrightarrow \mathcal{F} \cap \mathcal{G} \in \mathcal{C}.$$

Then we have:

$$\mathcal{F} \to_{\mathcal{C}} x \Longleftrightarrow \mathcal{F} \in \mathcal{C} \text{ and } \mathcal{F} \sim \dot{x}.$$

3.1. **Properties of convergences induced by particular screens.** Recall first the following definitions and results of [15] and [7] (propositions 3.1.1 and 3.1.2).

Definition 3.1.1. *A convergence ξ is called* **symmetric** *if the following is satisfied:*

($S0$) *If $\mathcal{F}$ is a convergent filter and $\mathcal{F} \subset \dot{y}$ then $\mathcal{F}$ converges to y.*

The convergence ξ is said to be **reciprocal** *if provided some filter converges to both x and y, then a filter converges to x if and only if it converges to y.*

It is easy to see that a reciprocal convergence is symmetric.

Proposition 3.1.1. [15] *A convergence ξ is induced by a screen if and only if it is a symmetric Kent convergence. Any symmetric Kent convergence is induced by the screen of its convergent filters.*

Proposition 3.1.2. [7] *A convergence on a set X is induced by a Cauchy structure if and only if it is a reciprocal limit structure.*
Furthermore for any reciprocal limit structure ξ the screen $Conv(\xi)$ of its convergent filters is a Cauchy screen inducing ξ.

From these results we get corollaries 3.1.1 and 3.1.2 below.

Corollary 3.1.1. *A symmetric Kent convergence ξ is induced by a Cauchy screen if and only if the screen $Conv(\xi)$ of its convergent filters is a Cauchy screen.*

For a pretopological convergence it is easy to see that (S0) is equivalent to each one of properties (S1), (S1'), (S1") given below. Closures satisfying (S1) are called **weakly separated** in [10]. Thus proposition 3.1.1 gives the following corollary.

Corollary 3.1.2. *A pretopological convergence with associated neighborhood filters $(\mathcal{V}(x))_{x\in X}$ and closure c is induced by a screen if and only if it satisfies (S0) which in turn is equivalent to each of the following properties:*
(S1) $c(\{x\}) \cap c(A) \neq \emptyset \Longleftrightarrow x \in c(A)$
(S1') $\mathcal{V}(x) \neq \mathcal{V}(y) \Longrightarrow \exists V \in \mathcal{V}(x) :\ y \notin V$ *i.e.* $y \notin c(\{x\})$
(S1") $y \in c(\{x\})$ *if and only if for all* $A \subset X :\ x \in c(A) \Longleftrightarrow y \in c(A)$.
In particular c is symmetric in the sense of [10]*, i.e.*

$$y \in c(\{x\}) \quad \text{if and only if} \quad x \in c(\{y\}).$$

Remark and example. For any limit structure ξ, if $Conv(\xi)$ is a Cauchy screen, then ξ is not necessarily induced by a Cauchy screen. For example take a set X, x_0 and y_0 distinct points of X, and consider the convergence ξ for which each $\dot{x}$ converges to x and $\dot{x}_0 \cap \dot{y}_0$ and $\dot{x}_0$ converge to y_0, ξ is a limit structure but it is not reciprocal not even symmetric and it is easy to see that $Conv(\xi)$ is a Cauchy screen.

Remark about property $(S0)$**.**
For any limit structure, property (S0) is equivalent to the following symmetry axiom of [7] (denoted by (S0) in [16]):

$$(S0') \qquad \text{if } \dot{x} \to y \text{ then: } \mathcal{F} \to x \Longleftrightarrow \mathcal{F} \to y.$$

Concerning Riesz screens we can prove the following result.

Proposition 3.1.3. *A convergence ξ is induced by a Riesz screen if and only if ξ is pretopological and symmetric. In this case $Conv(\xi)$ is a Riesz screen inducing ξ.*

Proof. If ξ is pretopological with property (S0) then by proposition 3.1.1, ξ is induced by the screen $Conv(\xi)$ of its convergent filters. Therefore the neighborhood filter at a point x for the convergence $\to_{Conv(\xi)}$ coincides with the neighborhood filter for the convergence ξ. Finally $Conv(\xi)$ is a Riesz screen since ξ is pretopological and every neighborhood filter is convergent.

Conversely if ξ is induced by a Riesz screen $\mathcal{C}$, then the neighborhood filter at a point x is member of $\mathcal{C}$ and so converges to x since it is fixed at x. This proves that ξ is pretopological. □

It follows from propositions 3.1.2 and 3.1.3:

Corollary 3.1.3. *A convergence ξ is induced by a Riesz Cauchy screen if and only if it is pretopological and reciprocal.*

According to [11] a pretopological space with associated neighborhood filters $(\mathcal{V}(x))_{x\in X}$ is an **(S2)-pretopological space** if the following separation condition is satisfied:

$$(S2) \qquad \forall x, y \in X, \quad \mathcal{V}(x) \subset \mathcal{V}(y)^{\#} \Longrightarrow \mathcal{V}(x) = \mathcal{V}(y).$$

Notice that the condition $[\mathcal{V}(x) \subset \mathcal{V}(y)^{\#}]$ is equivalent to the existence of a filter converging to both x and y.

It is then easy to see that a pretopological convergence ξ, with associated neighborhood filters $(\mathcal{V}(x))_{x\in X}$, is reciprocal if and only if condition $(S2)$ is satisfied.

Corollary 3.1.4. *A convergence ξ is induced by a Riesz Cauchy screen if and only if it is pretopological and reciprocal if and only if ξ is the convergence associated to an $(S2)$ pretopological space.*

Finally we will characterize reciprocal pseudotopological convergences using the following definition taken from [17].

Definition 3.1.2. *A Cauchy screen $\mathcal{C}$ is* **full** *if a filter $\mathcal{F}$ is in $\mathcal{C}$ whenever $\beta(\mathcal{F})$ is contained in an equivalence class of $\mathcal{C}$ (for the equivalence relation $\sim$).*

Proposition 3.1.4. *A convergence ξ is induced by a full Cauchy screen if and only if ξ is pseudotopological and reciprocal. In this case $Conv(\xi)$ is a full Cauchy screen inducing ξ.*

Proof. Let ξ be induced by some full Cauchy screen $\mathcal{C}$. The convergence ξ is reciprocal. We prove that it is pseudotopological. If $\mathcal{F}$ is a filter such that every ultrafilter of $\beta(\mathcal{F})$ converges to some point x, then $\beta(\mathcal{F})$ is contained in the equivalence class of $\dot{x}$ in $\mathcal{C}$ and then $\mathcal{F}$ is in $\mathcal{C}$ and is also equivalent to $\dot{x}$, therefore $\mathcal{F}$ converges to x.

Conversely if ξ is pseudotopological and reciprocal then it is induced by the Cauchy screen $Conv(\xi)$. Let us see that this screen is full. Assume that for some filter $\mathcal{F}$, $\beta(\mathcal{F})$ is contained in an equivalence class of $Conv(\xi)$. There exists a filter $\mathcal{G}$ converging to a point x such that $\beta(\mathcal{F})$ is contained in the equivalence class of $\mathcal{G}$ which is also the equivalence class of $\dot{x}$. Then every member of $\beta(\mathcal{F})$ converges to x and since ξ is pseudotopological we conclude that $\mathcal{F}$ converges also to x and $\mathcal{F}$ is in $Conv(\xi)$. □

3.2. **Order properties for the set of screens inducing a given convergence.** Recall that x is a **cluster point** for a given filter $\mathcal{F}$ with respect to a given convergence if and only if there exists a filter $\mathcal{G}$ containing $\mathcal{F}$ converging to x. We introduce now the following definitions generalizing notions given by Császár in the pretopological setting (cf. [10], [12]).

Definition 3.2.1. *Let ξ be a convergence on a set X. A filter $\mathcal{F}$ is called:*

(i) **ξ-compressed** *if: $\mathcal{F} \subset \dot{x} \Rightarrow \mathcal{F} \to x$;*

(ii) **strongly ξ-compressed** *whenever any cluster point for $\mathcal{F}$ is a limit point for $\mathcal{F}$.*

We denote by $Comp(\xi)$ the set of all ξ-compressed filters and by $SComp(\xi)$ the set of all strongly ξ-compressed filters.

Remarks.

1. If ξ is the convergence induced by a pretopological closure c on X, then the *c-compressed* and *strongly c-compressed filters* of Császár are precisely the ξ-compressed and strongly ξ-compressed filters respectively.

2. A convergent filter is compressed if and only if the convergence is symmetric. Generally a convergent filter is not necessarily compressed even for a Kent convergence or a limit structure as the example given in the preceding subsection shows.

Lemma 3.2.1. *The sets $Comp(\xi)$ and $SComp(\xi)$ are screens on X and we have: $SComp(\xi) \subset Comp(\xi)$.*

The proof is straightforward.

Theorem 3.2.1. *Let ξ be a symmetric Kent convergence on X. Denote by $Conv(\xi)$ the screen of convergent filters.*
For any screen $\mathcal{C}$ inducing a symmetric Kent convergence ξ', we have :

(i) $\xi \leq \xi' \Leftrightarrow Conv(\xi) \subset \mathcal{C}$

(ii) $\xi' \leq \xi \Leftrightarrow \mathcal{C} \subset Comp(\xi)$

In particular ξ is induced by any screen between $Conv(\xi)$ and $Comp(\xi)$, $Conv(\xi)$ is the finest, $Comp(\xi)$ is the coarsest.

Proof.

(i) Clearly, if $\xi \leq \xi'$ where ξ' is induced by $\mathcal{C}$ then $Conv(\xi) \subset \mathcal{C}$. Conversely assume that $Conv(\xi) \subset \mathcal{C}$, if $\mathcal{F}$ converges to x for ξ then $\mathcal{F} \cap \dot{x}$ converges to x since ξ satisfies $(C2)$, so $\mathcal{F} \cap \dot{x}$ is in $\mathcal{C}$.

(ii) Assume that $\xi' \leq \xi$ and let $\mathcal{F} \in \mathcal{C}$. If $\mathcal{F} \subset \dot{x}$ then $\mathcal{F} \cap \dot{x} \in \mathcal{C}$, therefore $\mathcal{F} \to_{\xi'} x$ and $\mathcal{F} \to_{\xi} x$, which proves that $\mathcal{F}$ is ξ-compressed and that $\mathcal{C} \subset Comp(\xi)$.

Conversely assume that $\mathcal{C} \subset Comp(\xi)$. If $\mathcal{F} \to_{\xi'} x$ then $\mathcal{F} \cap \dot{x} \in \mathcal{C}$ and $\mathcal{F} \cap \dot{x} \in Comp(\xi)$ so $\mathcal{F} \cap \dot{x} \to_{\xi} x$ and therefore $\mathcal{F} \to_{\xi} x$. □

Corollary 3.2.1. *Every screen inducing some pretopological symmetric convergence is a Riesz screen,*
$Conv(\xi)$ is the finest, $Comp(\xi)$ is the coarsest.

The proof is clear since $Conv(\xi)$ is a Riesz screen contained in every screen inducing ξ.

Theorem 3.2.2. *For all limit structure ξ, the screen $SComp(\xi)$ is a Cauchy screen.*
Furthermore if ξ is reciprocal (i.e. induced by some Cauchy screen) then:

(a) *$Conv(\xi)$ and $SComp(\xi)$ are Cauchy screens such that $Conv(\xi) \subset SComp(\xi)$.*
(b) *For all Cauchy screen $\mathcal{C}$, we have :*
$\mathcal{C}$ induces ξ if and only if $Conv(\xi) \subset \mathcal{C} \subset SComp(\xi)$.

Proof. We check that $SComp(\xi)$ fulfills condition $(SC2)$ of Cauchy screens.
Let $\mathcal{F}$ and $\mathcal{G}$ be members of $SComp(\xi)$ such that $\mathcal{F} \subset \mathcal{G}^{\#}$ and assume that x is a cluster point of the filter $\mathcal{F} \cap \mathcal{G}$. Then there exists a filter and therefore an ultrafilter $\mathcal{U}$ containing $\mathcal{F} \cap \mathcal{G}$ and converging to x. Since this ultrafilter contains $\mathcal{F}$ or $\mathcal{G}$, x is a cluster point for $\mathcal{F}$ or $\mathcal{G}$ and so a limit point for this filter. Therefore the filter $\mathcal{F} \vee \mathcal{G}$ converges also to x, finally x is a cluster point and so a limit point for both $\mathcal{F}$ and $\mathcal{G}$ and by property (C3), $\mathcal{F} \cap \mathcal{G}$ converges to x. We conclude that $\mathcal{F} \cap \mathcal{G}$ is in $SComp(\xi)$. Thus $SComp(\xi)$ is a Cauchy screen.

Now assume that ξ is reciprocal.
(a) $Conv(\xi) \subset SComp(\xi)$: let $\mathcal{F}$ be a filter converging to a point x, if y is a cluster point for $\mathcal{F}$, then there is a filter $\mathcal{G}$ containing $\mathcal{F}$ and converging to y, so $\mathcal{G}$ converges to both x and y, and since ξ is reciprocal $\mathcal{F}$ also converges to y.

(b) By proposition 3.2.1 a Cauchy screen $\mathcal{C}$ induces ξ if and only if $Conv(\xi) \subset \mathcal{C} \subset Comp(\xi)$.
Since $SComp(\xi) \subset Comp(\xi)$, the condition $Conv(\xi) \subset \mathcal{C} \subset SComp(\xi)$ is sufficient for $\mathcal{C}$ to induce ξ.
Conversely, assume that a Cauchy screen $\mathcal{C}$ induces ξ. We have to check that $\mathcal{C} \subset SComp(\xi)$. If $\mathcal{F} \in \mathcal{C}$ and x is a cluster point for $\mathcal{F}$ then there is a filter $\mathcal{G}$ containing $\mathcal{F}$ and converging to x. We then have: $\mathcal{F} \sim \mathcal{G}$ and $\mathcal{G} \sim \dot{x}$ in $\mathcal{C}$, and so $\mathcal{F} \sim \dot{x}$ in $\mathcal{C}$ which proves that $\mathcal{F}$ converges to x. Finally $\mathcal{F} \in SComp(\xi)$. □

If ξ is the convergence ξ induced by a pretopological closure c the preceding result implies the following which was proved by Császár in [12]:

Corollary 3.2.2. *If c is a pretopological closure on X then the strongly c-compressed filters constitute a Cauchy screen.*

Finally we have also:

Corollary 3.2.3. *For any reciprocal limit structure ξ, $SComp(\xi)$ is the coarsest Cauchy screen contained in $Comp(\xi)$.*

Remark. Let ξ be a convergence induced by a Cauchy screen; $Conv(\xi)$ is the finest Cauchy screen inducing ξ and $SComp(\xi)$ is the coarsest.
Let $\mathcal{C}$ be a Cauchy screen inducing a convergence ξ'.
We have: $\xi \leq \xi' \Leftrightarrow Conv(\xi) \subset \mathcal{C}$, but it is false that: $\xi' \leq \xi \Leftrightarrow \mathcal{C} \subset SComp(\xi)$.

In the example below, we construct reciprocal (actually separated) pretopological convergences ξ and ξ', which are induced by the Cauchy screens of their strongly compressed filters and such that $\xi' \leq \xi$ and $SComp(\xi') \not\subset SComp(\xi)$.

Example. Consider the usual convergence ξ' on the set of reals $\mathbb{R}$, and then take non convergent (distinct) ultrafilters $\mathcal{U}_0$ and $\mathcal{U}_1$ containing respectively the non compact subsets $]-\infty, 0]$ and $[0, +\infty[$. In particular, these ultrafilters do not contain any neighborhood filter for ξ'.

Now take the pretopological convergence ξ for which $\mathcal{V}_\xi(x) = \mathcal{V}_{\xi'}(x)$ if x is neither 0 nor 1, and $\mathcal{V}_\xi(0) = \mathcal{V}_{\xi'}(0) \cap \mathcal{U}_0$, $\mathcal{V}_\xi(1) = \mathcal{V}_{\xi'}(1) \cap \mathcal{U}_1$.

Each of these pretopological convergences is a limit structure on $\mathbb{R}$, ξ' is reciprocal and actually separated, i.e. $x \neq y \Longrightarrow \mathcal{V}_{\xi'}(x) \not\subset \mathcal{V}_{\xi'}(y)^{\#}$.
The convergence ξ is also separated and thus reciprocal. To prove this, we use the following equivalence for filters $\mathcal{F}$ and $\mathcal{G}$: $\mathcal{F} \subset \mathcal{G}^{\#}$ if and only if $\beta(\mathcal{F}) \cap \beta(\mathcal{G}) \neq \emptyset$. Since any ultrafilter containing the intersection of two filters contains necessarily one of them (property recalled in section 2) we have: $\beta(\mathcal{V}_\xi(0)) = \beta(\mathcal{V}_{\xi'}(0)) \cup \{\mathcal{U}_0\}$ and $\beta(\mathcal{V}_\xi(1)) = \beta(\mathcal{V}_{\xi'}(1)) \cup \{\mathcal{U}_1\}$. Of course if x is in $\mathbb{R} \setminus \{0, 1\}$, $\beta(\mathcal{V}_\xi(x)) = \beta(\mathcal{V}_{\xi'}(x))$. For any distinct points $x, y \in \mathbb{R}$, $\beta(\mathcal{V}_{\xi'}(x)) \cap \beta(\mathcal{V}_{\xi'}(y)) = \emptyset$ and each one of $\mathcal{U}_0$ and $\mathcal{U}_1$ does not contain any neighborhood filter for ξ'. Finally with $\mathcal{U}_0 \neq \mathcal{U}_1$, we check that if x, y are distinct points, then $\beta(\mathcal{V}_\xi(x)) \cap \beta(\mathcal{V}_\xi(y)) = \emptyset$, which proves that ξ is separated.
Therefore each one of ξ' and ξ is induced by the Cauchy screen of its strongly compressed filters.

Let $\mathcal{F}_0$ be the bi-ultrafilter $\mathcal{U}_0 \cap \mathcal{U}_1$, with respect to ξ', $\mathcal{F}_0$ is not convergent and has no cluster point since the only ultrafilters containing $\mathcal{F}_0$ are $\mathcal{U}_0$ and $\mathcal{U}_1$ and they do not converge for ξ'. Therefore $\mathcal{F}_0$ is a strongly ξ'-compressed filter.
With respect to ξ, 0 and 1 are cluster points for $\mathcal{F}_0$ since $\mathcal{U}_0$ and $\mathcal{U}_1$ converge to 0 and 1 respectively. Furthermore $\mathcal{U}_i \notin \beta(\mathcal{V}_\xi(j))$ for $\{i, j\} = \{0, 1\}$, therefore $\mathcal{F}_0$ does not contain $\mathcal{V}_\xi(j)$ either and does not converge to j for ξ. So $\mathcal{F}_0$ is not strongly ξ-compressed. Finally $SComp(\xi')$ is not contained in $SComp(\xi)$.

We complete our study with the following result.

Proposition 3.2.1. *For all reciprocal pseudotopological convergence ξ, the Cauchy screens $Conv(\xi)$ and $SComp(\xi)$ are full.*

Proof. It has already been shown that $Conv(\xi)$ is full. We check now that $SComp(\xi)$ is full.

Assume that for some filter $\mathcal{F}$, $\beta(\mathcal{F})$ is contained in an equivalence class of $SComp(\xi)$. If x is a cluster point for $\mathcal{F}$ then some member $\mathcal{U}$ of $\beta(\mathcal{F})$ converges to x. Therefore $\mathcal{U}$ and so every member of $\beta(\mathcal{F})$ is in the equivalence class of $\dot{x}$, that is, every ultrafilter containing $\mathcal{F}$ converges to x and

then $\mathcal{F}$ converges also to x since ξ is pseudotopological. We conclude that $\mathcal{F}$ is in $SComp(\xi)$. □

4. Pretopological closures and proximities induced by Cauchy screens

A proximity δ on a set X induces a pretopological closure c on X defined by :

$$x \in c(A) \Leftrightarrow \{x\}\delta A$$

and the neighborhood filter of a point x is given by:

$$\mathcal{V}_\delta(x) = \delta(\{x\})^\# \quad \text{where} \quad \delta(\{x\}) = \{A \subset X : \{x\}\delta A\}.$$

A screen $\mathcal{C}$ induces a proximity δ and a pretopological closure c as defined in the introduction.

This closure c is also the associated closure of the pretopological structure induced by the convergence $\to_{\mathcal{C}}$ since for all $x \in X$ the neighborhood filters at x coincide. Actually the neighborhood filter of x related to $\to_{\mathcal{C}}$ is:

$$\mathcal{V}(x) = \cap\{\mathcal{F} : \mathcal{F} \in \mathcal{C}, \mathcal{F} \subset \dot{x}\}$$

and we have:

$$\delta(\{x\}) = \cup\{\mathcal{F}^\# : \mathcal{F} \in \mathcal{C}, \mathcal{F} \subset \dot{x}\} = \mathcal{V}(x)^\#$$

therefore $\mathcal{V}(x) = \delta(\{x\})^\#$ and then $\mathcal{V}(x) = \mathcal{V}_\delta(x)$.

In [12], Császár asked for a characterization of those pretopological closures and those proximities induced by Cauchy screens. We give an answer to both questions.

4.1. **Proximities and screens.** If a screen $\mathcal{C}$ induces a proximity δ then every member $\mathcal{F}$ of $\mathcal{C}$ is such that $\mathcal{F}^\# \times \mathcal{F}^\# \subset \delta$, that is, $\mathcal{F}^\#$ is a **δ-clan** in the sense of Thron [20], or equivalently, $\mathcal{F}$ is a **δ-compressed filter** in the sense of [10].

For every proximity δ the set $\mathcal{C}_\delta$ of δ-compressed filters is a screen and it is known that δ is induced by $\mathcal{C}_\delta$ (indeed it is the coarsest one): for all subsets A and B of X, $A\delta B$ if and only if there exists a δ-clan containing A and B. Thron [20] gave a short proof for this result using basic properties of grills, filters and ultrafilters. Thus every proximity is induced by a screen.

We call a **Riesz proximity** (or **RI-proximity**, [3]) on X any proximity δ for which there exists a pretopological closure space (Y, φ) containing X as a dense subset and such that we have: $A\delta B$ if and only if $\varphi(A) \cap \varphi(B) \neq \emptyset$. These proximities have been studied and characterized in [2], [3], [10].

Theorem 4.1.1. *The proximities induced by a Riesz screen are precisely the Riesz proximities.*

Proof. It is known (see [2], [10], [3]) that a proximity δ is a Riesz proximity if and only if one of the equivalent following conditions is true (c denotes the pretopological closure induced by δ):

(RI) $\quad \forall\ A, B \in \mathcal{P}(X),\ c(A) \cap c(B) \neq \emptyset \Rightarrow A\delta B$

(RI') $\quad \forall\ x \in X,\ \delta(\{x\}) \times \delta(\{x\}) \subset \delta,$

which means that for all $x \in X$, $\mathcal{V}_\delta(x)$ is δ-compressed. Then if δ is a Riesz proximity the screen $\mathcal{C}_\delta$ of δ-compressed filters is a Riesz screen since it induces δ and $\mathcal{V}_\delta(x)$ is also the neighborhood filter at x related to $\to_{\mathcal{C}_\delta}$. Conversely let δ be a proximity induced by a Riesz screen $\mathcal{C}$. Then for all $x \in X$, the neighborhood filter $\mathcal{V}(x)$ is in $\mathcal{C}$ and so is compressed for the proximity δ. Thus δ is a Riesz proximity since $\mathcal{V}(x) = \mathcal{V}_\delta(x)$. □

We obtain a characterization for those proximities induced by some Cauchy screen using the following notion of [14].

Definition 4.1.1. *A* **nasse** *on X is a reflexive and closed graph relation on βX.*

The following facts were proved in [14]:

1. there is a bijection between proximities (or dually the **topogeneous orders** of Császár [8]) on X and the so-called **nasses** of Haddad. For any proximity δ with associated nasse Δ, we have

$$(\mathcal{U}, \mathcal{V}) \in \Delta \Leftrightarrow \mathcal{U} \times \mathcal{V} \subset \delta \qquad \text{and} \qquad A\delta B \Leftrightarrow (\beta A \times \beta B) \cap \Delta \neq \emptyset.$$

   We recover here Thron's result which says that the screen of δ-compressed filters induces δ. Actually if $A\delta B$ then there is a pair $(\mathcal{U}, \mathcal{V}) \in (\beta A \times \beta B) \cap \Delta$. It follows that A and B are members of $\mathcal{U} \cup \mathcal{V}$ and from $(\mathcal{U}, \mathcal{V}) \in \Delta$ we get that $\mathcal{U} \cup \mathcal{V}$ is a δ-clan (i.e. $\mathcal{U} \cap \mathcal{V}$ is δ-compressed).
2. Recall that an **Efrĕmovič-proximity** (or **EF-proximity** for short) is a proximity δ on a set X with the following separation property: for all A, B in $\mathcal{P}(X)$,

$$A(\text{non-}\delta)B \Longrightarrow \exists\ C \in \mathcal{P}(X) : A(\text{non-}\delta)C \text{ and } (X \setminus C)(\text{non-}\delta)B.$$

   For any proximity the converse implication is true, as one can easily see.
   A proximity δ is an EF-proximity if and only if its nasse Δ is an equivalence relation.

We will prove the following theorem.

Theorem 4.1.2. *Let δ be a proximity on X.*

1. *δ is induced by a screen $\mathcal{C}$ if and only if its associated nasse is the closure in $(\beta X)^2$ of the relation $\Theta_{\mathcal{C}} = \bigcup_{\mathcal{F} \in \mathcal{C}} \beta(\mathcal{F}) \times \beta(\mathcal{F})$ on βX induced by $\mathcal{C}$.*

2. *δ is induced by a Cauchy screen if and only if its associated nasse Δ is the closure in $(\beta X)^2$ of a transitive and symmetric relation on βX if and only if Δ is the closure in $(\beta X)^2$ of an equivalence relation on βX.*

Notations. Any family $\mathcal{B}$ of filters on X induces a relation $\Theta_{\mathcal{B}}$ on βX defined by:

$$\Theta_{\mathcal{B}} = \bigcup_{\mathcal{F}\in\mathcal{B}} \beta(\mathcal{F}) \times \beta(\mathcal{F}).$$

If $\mathcal{C}$ is the set of all filters containing a member of $\mathcal{B}$, we then have $\Theta_{\mathcal{B}} = \Theta_{\mathcal{C}}$ and $(\mathcal{U}, \mathcal{V}) \in \Theta_{\mathcal{C}}$ if and only if $\mathcal{U} \cap \mathcal{V} \in \mathcal{C}$.

This defines a relation on βX which is clearly symmetric and reflexive with domain

$$\mathbb{D} = \mathcal{C} \cap \beta X = \bigcup_{\mathcal{F}\in\mathcal{C}} \beta(\mathcal{F})$$

Since $\mathcal{C}$ and $\mathbb{D}$ contain exactly the same ultrafilters, we get:
$\mathcal{C}$ is a screen if and only if $\mathbb{D}$ contains all of the centered ultrafilters, that is, if and only if $\mathbb{D}$ is dense in βX.

Note that **if $\mathcal{C}$ is a Cauchy screen then** the relation $\Theta_{\mathcal{C}}$ is the restriction on $\mathbb{D} = \mathcal{C} \cap \beta X$ of the equivalence relation $\sim$ associated to $\mathcal{C}$ (recalled in the beginning of section 3), and therefore **$\Theta_{\mathcal{C}}$ is an equivalence relation on** $\mathbb{D}$.

Now to an arbitrary relation Θ on βX let us associate the family of filters $\mathcal{C}_\Theta$ on X defined as follows:

$$\mathcal{C}_\Theta = \{\mathcal{F} \in \phi(X) : \beta(\mathcal{F}) \times \beta(\mathcal{F}) \subset \Theta\}$$

and let $\mathcal{C}^b_\Theta$ be the screen of the bi-ultrafilters of $\mathcal{C}_\Theta$.

Theorem 4.1.3. *A relation Θ is induced by some screen $\mathcal{C}$ (in the sense that $\Theta = \Theta_{\mathcal{C}}$) if and only if Θ is a symmetric and reflexive relation on a dense subset of βX.*
Furthermore, if Θ is such a relation with dense domain $\mathbb{D}$ then Θ is induced by any screen between $\mathcal{C}^b_\Theta$ and $\mathcal{C}_\Theta$, $\mathcal{C}_\Theta$ is the coarsest one inducing Θ, $\mathcal{C}^b_\Theta$ is the finest one.

Proof. For any symmetric and reflexive relation Θ on a dense subset $\mathbb{D}$ of βX, $\mathbb{D}$ contains all centered ultrafilters and it is straightforward that $\mathcal{C}_\Theta$ and $\mathcal{C}^b_\Theta$ are screens on X inducing Θ. Furthermore any screen inducing Θ is clearly contained in $\mathcal{C}_\Theta$, and it contains $\mathcal{C}^b_\Theta$ since if $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters such that $\mathcal{U} \cap \mathcal{V}$ is in $\mathcal{C}^b_\Theta$ then $(\mathcal{U}, \mathcal{V})$ is in Θ and there exists $\mathcal{F} \in \mathcal{C}$ such that $\mathcal{U}, \mathcal{V}$ are both in $\beta(\mathcal{F})$, so $\mathcal{F} \subset \mathcal{U} \cap \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V}$ is also in $\mathcal{C}$. □

Theorem 4.1.4. *For all relations Θ on βX the following are equivalent:*

1. *Θ is induced by a Cauchy screen.*
2. *Θ is an equivalence relation on its domain and its domain is a dense subset of βX.*

3. $\mathcal{C}_\Theta$ *is a Cauchy screen.*
4. *The screen* $\mathcal{C}^e_\Theta$ *of elementary filters of* $\mathcal{C}_\Theta$ *is a Cauchy screen.*

In case one of the above conditions holds then, among the Cauchy screens inducing Θ, $\mathcal{C}_\Theta$ *is the coarsest and* $\mathcal{C}^e_\Theta$ *is the finest.*

Proof. 1. $\Rightarrow$ 2. has already been observed.

Now we prove: 2. $\Rightarrow$ 3. Assume that Θ is an equivalence relation on a dense domain $\mathbb{D}$ of βX.

The screen $\mathcal{C}_\Theta$ is the coarsest screen inducing Θ, we check that it is a Cauchy screen.

Let $\mathcal{F}, \mathcal{G} \in \mathcal{C}_\Theta$ such that $\mathcal{F} \subset \mathcal{G}^\#$. The sets $\beta(\mathcal{F})^2$ and $\beta(\mathcal{G})^2$ are subsets of Θ and $\beta(\mathcal{F}) \cap \beta(\mathcal{G}) \neq \emptyset$. Since Θ is transitive, it follows that $\beta(\mathcal{F}) \times \beta(\mathcal{G}) \subset \Theta$ and $\beta(\mathcal{G}) \times \beta(\mathcal{F}) \subset \Theta$, and finally $\beta(\mathcal{F} \cap \mathcal{G})^2 = (\beta(\mathcal{F}) \cup \beta(\mathcal{G}))^2 \subset \Theta$, and $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}_\Theta$.

Next we prove: 3. $\Rightarrow$ 4. Assume that $\mathcal{C}_\Theta$ is a Cauchy screen. Let $\mathcal{F}, \mathcal{G} \in \mathcal{C}^e_\Theta$ such that $\mathcal{F} \subset \mathcal{G}^\#$, then $\mathcal{F}, \mathcal{G}$ are elementary and $\mathcal{F}, \mathcal{G} \in \mathcal{C}_\Theta$. Therefore, $\mathcal{F} \cap \mathcal{G}$ is in $\mathcal{C}_\Theta$ and it is again elementary, i.e. $\mathcal{F} \cap \mathcal{G} \in \mathcal{C}^e_\Theta$. This proves that the screen $\mathcal{C}^e_\Theta$ is Cauchy.

Finally 4. $\Rightarrow$ 1. is clear since the screen $\mathcal{C}^e_\Theta$ is between $\mathcal{C}_\Theta$ and $\mathcal{C}^b_\Theta$, which proves that $\mathcal{C}^e_\Theta$ induces Θ. by the preceding theorem.

Assume now that one of the conditions 1. to 4. is true. We prove that $\mathcal{C}^e_\Theta$ is the finest Cauchy screen inducing Θ.

Let $\mathcal{C}$ be a Cauchy screen inducing also Θ, note first that $\mathbb{D} = \mathcal{C} \cap \beta X = \mathcal{C}^e_\Theta \cap \beta X$.

Let $\mathcal{F} \in \mathcal{C}^e_\Theta$, $\mathcal{F}$ is the intersection of a finite number of ultrafilters, say $\mathcal{U}_1, \ldots, \mathcal{U}_n$. Each $(\mathcal{U}_i, \mathcal{U}_j)$ is in Θ and therefore $\mathcal{U}_i \cap \mathcal{U}_j$ is a member of $\mathcal{C}$. In particular each $\mathcal{U}_i$ and $\mathcal{U}_i \cap \mathcal{U}_{i+1}$ are members of $\mathcal{C}$. Finally, since $\mathcal{C}$ is a Cauchy screen, by finite induction we prove that for all $1 \leq i \leq n$, $\mathcal{U}_1 \cap \ldots \cap \mathcal{U}_i$ is again member of $\mathcal{C}$, so $\mathcal{F} \in \mathcal{C}$. □

Now we can prove Theorem 4.1.2.

Proof of Theorem 4.1.2.

1. Let δ be a proximity and $\mathcal{C}$ be a screen on X. We have:

δ is induced by $\mathcal{C} \Leftrightarrow [\ A \delta B \Leftrightarrow \exists \mathcal{F} \in \mathcal{C} : A, B \in \mathcal{F}^\#\ ]$

$\Leftrightarrow [\ A \delta B \Leftrightarrow \exists \mathcal{F} \in \mathcal{C} : (\beta A \times \beta B) \cap (\beta(\mathcal{F}))^2 \neq \emptyset\ ] \Leftrightarrow [\ A \delta B \Leftrightarrow (\beta A \times \beta B) \cap \Theta_\mathcal{C} \neq \emptyset\ ]$

Since $\{\beta A \times \beta B : A, B \in \mathcal{P}(X)\}$ is a base for the open sets of the product topology on $(\beta X)^2$, we have finally:

$$\delta \text{ is induced by } \mathcal{C} \Leftrightarrow [\ A \delta B \Leftrightarrow (\beta A \times \beta B) \cap \overline{\Theta_\mathcal{C}} \neq \emptyset\ ]$$

which is to say that $\overline{\Theta_\mathcal{C}}$ is the nasse associated to δ.

2. Note that a nasse Δ is always reflexive and therefore if Θ is a symmetric and transitive relation on βX which is dense in Δ then

$$\Theta' = \Theta \cup \{(\mathcal{U}, \mathcal{U}) : \mathcal{U} \in \beta X\}$$

is an equivalence relation on βX which is again dense in Δ.

Now the second part of the theorem follows from the first part and theorem 4.1.4. □

Actually from theorem 4.1.2 and theorem 4.1.4 we get the following result. Recall that a Cauchy screen $\mathcal{C}$ is **totally bounded** ([7]) if every ultrafilter is a Cauchy filter.

Theorem 4.1.5. *A proximity induced by a Cauchy screen is also induced by a totally bounded and full Cauchy screen.*

Proof. According to theorem 4.1.2 let Θ be an equivalence relation on βX which is dense in the nasse Δ of a proximity δ induced by a Cauchy screen. The Cauchy screen $\mathcal{C}_\Theta = \{\mathcal{F} : \beta(\mathcal{F}) \times \beta(\mathcal{F}) \subset \Theta\}$ induces δ and is clearly totally bounded since Θ is reflexive on βX. Furthermore if $\mathcal{F}$ is a filter such that $\beta(\mathcal{F})$ is contained in an equivalence class of $\mathcal{C}_\Theta$ for $\sim$ then $\beta(\mathcal{F})$ is contained in an equivalence class for Θ since Θ is the restriction of $\sim$ to βX. Thus $\mathcal{F} \in \mathcal{C}_\Theta$, which proves finally that $\mathcal{C}_\Theta$ is a full and totally bounded Cauchy screen inducing Θ and therefore the proximity δ. □

We give now some corollaries. The first one recovers in particular Thron's result: any proximity is induced by the screen of its compressed filters.
We recall the following result.

Proposition 4.1.1. [4] *For any proximity δ with associated nasse Δ, a filter $\mathcal{F}$ is δ-compressed if and only if $\beta(\mathcal{F}) \times \beta(\mathcal{F}) \subset \Delta$.*

Corollary 4.1.1. *Any proximity δ is induced by the screen $\mathcal{C}_\delta$ of δ-compressed filters and also by the screens $\mathcal{C}_\delta^b$ and $\mathcal{C}_\delta^e$ consisting of bi-ultrafilters and elementary filters respectively of $\mathcal{C}_\delta$.*

Proof. Let Δ be the nasse associated to a proximity δ. The nasse Δ is a symmetric and reflexive relation on βX, therefore using theorem 4.1.3, $\Delta = \Theta_{\mathcal{C}_\Delta} = \Theta_{\mathcal{C}_\Delta^b} = \Theta_{\mathcal{C}_\Delta^e}$ where $\mathcal{C}_\Delta = \{\mathcal{F} : \beta(\mathcal{F}) \times \beta(\mathcal{F}) \subset \Delta\}$ and $\mathcal{C}_\Delta^b$ and $\mathcal{C}_\Delta^e$ are the screens consisting of bi-ultrafilters and elementary filters respectively of $\mathcal{C}_\delta$.
It follows from the preceding proposition that $\mathcal{C}_\Delta$ is also the screen $\mathcal{C}_\delta$ of all δ-compressed filters. Finally theorem 4.1.2 implies that the proximity δ is induced by the screens $\mathcal{C}_\delta$, $\mathcal{C}_\delta^b$ and $\mathcal{C}_\delta^e$. □

Theorem 4.1.4 and Haddad's characterization of EF-proximities imply also the following result which was proved by Császár in [12].

Corollary 4.1.2. *For any proximity δ, the screen $\mathcal{C}_\delta$ is Cauchy if and only if $\mathcal{C}_\delta^e$ is also Cauchy if and only if δ in an EF-proximity,*

Proof. Let Δ be the nasse associated to δ. The screen $\mathcal{C}_\Delta$ is again $\mathcal{C}_\delta$. Therefore since a proximity δ is an EF-proximity if and only if its nasse Δ is an equivalence relation on βX ([14]), we get from theorem 4.1.4 that the

screen $\mathcal{C}_\delta$ is Cauchy if and only if $\mathcal{C}_\delta^e$ is also Cauchy if and only if Δ is an equivalence relation and thus if and only if δ in an EF-proximity. □

Remarks.

1. For any proximity δ the screen $\mathcal{C}_\delta$ is the coarsest screen inducing δ, generally a finest one fails to exist ([12], example 3.18, p.212).

2. Of course if δ is an EF-proximity then $\mathcal{C}_\delta$ is the coarsest Cauchy screen inducing δ. Császár [12] gave examples of proximities and closures induced by Cauchy screens for which neither a coarsest nor a finest compatible Cauchy screen exists.

Actually in the next subsection we show that for any RE-proximity δ which is not an EF-proximity, there does not exist a coarsest Cauchy screen inducing δ.

3. It follows from corollaries 4.1.1 and 4.1.2 that a proximity induced by a Cauchy screen and which is not an EF-proximity is induced by screens which are not Cauchy, indeed $\mathcal{C}_\delta$ is one of them.

We conclude this section with the following result. This result can be found also in [12] where it is proved directly that a proximity induced by a Cauchy screen satisfies property (RI).

Proposition 4.1.2. *A proximity induced by a Cauchy screen is a Riesz proximity.*

Proof. It was proved in [3] that a proximity δ is a Riesz proximity if and only if its nasse is the closure of $\Theta \circ \Theta^-$ for some relation Θ on βX. From theorem 4.1.2 a proximity induced by a Cauchy screen is a Riesz proximity since its nasse is the closure of an equivalence relation Θ and we have $\Theta = \Theta \circ \Theta^-$. □

4.2. **About RE-proximities.** An **RE-proximity** on X is a proximity δ for which there exists a regular topological closure space (Y, φ), Y containing a copy of X and such that we have: $A\delta B$ if and only if $\varphi(A) \cap \varphi(B) \neq \emptyset$. For basic properties and results concerning RE-proximities, the reader may consult [9], [4].

It is known that to any proximity δ on X one can associate a *topogeneous order* (in the sense of Császár) γ on X defined by:

$$\forall A, B \in \mathcal{P}(X) \qquad A\gamma B \quad \text{if and only if} \quad A(\text{non-}\delta)(X \setminus B).$$

For any filter $\mathcal{F}$ on X the set $\gamma(\mathcal{F})$ consisting of subsets B of X such that $A\gamma B$ for some $A \in \mathcal{F}$ is a filter on X and the inclusion $\gamma(\mathcal{F}) \subset \mathcal{F}$ is always true. If $x \in X$, the filter $\gamma(\dot{x})$ is simply denoted by $\gamma(x)$.

Definition 4.2.1. *Given a proximity δ on X, a filter $\mathcal{F}$ on X for which $\mathcal{F} = \gamma(\mathcal{F})$ is a* **δ-round filter** *(or a γ-round filter).*

Round filters were introduced in [9] using a somewhat different notation.

As in [4], let δ be a proximity with associated nasse Δ and let

$$\mathbb{D}(\Delta) = \{\mathcal{U} \in \beta X : \Delta(\mathcal{U})^2 \subset \Delta \text{ and } \Delta(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})\}.$$

It is called the **equivalence domain of** Δ.
The restriction Δ' of Δ to the set $\mathbb{D}(\Delta)$ is the **equivalence kernel of** Δ. It is an equivalence relation on $\mathbb{D}(\Delta)$.

Theorem 4.2.1. *The equivalence relation Δ' on $\mathbb{D}(\Delta)$ is induced by the set $\mathcal{B}$ of δ-round and compressed filters in the sense that $\Delta' = \Theta_{\mathcal{B}} = \bigcup_{\mathcal{F}\in\mathcal{B}} \beta(\mathcal{F}) \times \beta(\mathcal{F})$.*
Furthermore the equivalence classes of Δ' are exactly the sets $\beta(\mathcal{F})$ for $\mathcal{F} \in \mathcal{B}$.
Consequently, the set $\mathcal{C}_{rd}$ of all filters containing a member of $\mathcal{B}$ satisfies the condition $(SC2)$ of Cauchy screens and we have also $\Delta' = \Theta_{\mathcal{C}_{rd}}$.

Proof. From [4] we have the following:
1. the equivalence classes of Δ' are exactly the sets $\Delta(\mathcal{U})$ for $\mathcal{U} \in \mathbb{D}(\Delta)$, which are also the sets $\beta(\gamma(\mathcal{U}))$.
2. $\mathcal{U} \in \mathbb{D}(\Delta)$ if and only if $\gamma(\mathcal{U}) \in \mathcal{B}$.
3. $\mathcal{F} \in \mathcal{B}$ if and only if $\mathcal{F} = \gamma(\mathcal{U})$, or equivalently $\beta(\mathcal{F}) = \Delta(\mathcal{U})$, for some $\mathcal{U} \in \mathbb{D}(\Delta)$.

It follows that $\mathcal{B} = \{\gamma(\mathcal{U}) : \mathcal{U} \in \mathbb{D}(\Delta)\}$ and that the equivalence classes of Δ' are the sets $\beta(\mathcal{F})$ for $\mathcal{F} \in \mathcal{B}$.

Since equivalence classes are disjoint or equal, for any members $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{B}$ such that $\mathcal{F}_1 \subset \mathcal{F}_2^{\#}$, we have $\mathcal{F}_1 = \mathcal{F}_2$.

From this it follows that the set $\mathcal{C}_{rd}$ satisfies condition $(SC2)$: for any members $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{C}_{rd}$ such that $\mathcal{F}_1 \subset \mathcal{F}_2^{\#}$ we have $\mathcal{F}_1 \cap \mathcal{F}_2 \in \mathcal{C}_{rd}$. □

Theorem 4.2.2. *Let δ be any proximity on X and $\mathcal{C}_{rd}$ the set of filters containing a δ-round and compressed filter. The set $\mathcal{C}_{rd}$ is a Cauchy screen if and only if for every x in X, $\gamma(x)$ is a δ-round and compressed filter if and only if $\mathbb{D}(\Delta)$ is dense in βX. And then $\mathcal{C}_{rd}$ is a full Cauchy screen.*

Proof. From $\Delta' = \Theta_{\mathcal{C}_{rd}}$, we have that $\mathcal{C}_{rd}$ is a screen (i.e. it contains every centered ultrafilter) if and only if the domain $\mathbb{D}(\Delta)$ of Δ' contains every centered ultrafilter, that is, if and only if $\mathbb{D}(\Delta)$ is dense in βX.

As it was recalled in the proof of the preceding proposition, an ultrafilter $\mathcal{U}$ on X is in $\mathbb{D}(\Delta)$ if and only if $\gamma(\mathcal{U})$ is in $\mathcal{B}$ [4]. Finally $\mathcal{C}_{rd}$ is a screen if and only if for every x in X, $\gamma(x)$ is a δ-round and compressed filter. And then it is a Cauchy screen. Let us prove that it is also full.

Assume that for some filter $\mathcal{F}$, $\beta(\mathcal{F})$ is contained in an equivalence class of $\mathcal{C}_{rd}$. The relation $\Theta_{\mathcal{C}_{rd}} = \Delta'$ is the restriction on $\mathbb{D}(\Delta)$ of the relation $\sim$ on $\mathcal{C}_{rd} \cap \beta X$. Therefore $\beta(\mathcal{F})$ is contained in an equivalence class of Δ', that is a set $\beta(\mathcal{G})$ for some δ-round and compressed filter $\mathcal{G}$. This inclusion gives $\mathcal{G} \subset \mathcal{F}$ and finally $\mathcal{F} \in \mathcal{C}_{rd}$. □

Since every centered ultrafilter is isolated in βX, the closure $\overline{\Delta'}$ of Δ' in βX is a nasse (that is, reflexive) on βX if and only if $\mathbb{D}(\Delta)$ is dense in βX. According to [4], under this condition, the proximity $\mathcal{R}(\delta)$ associated to $\overline{\Delta'}$ is an RE-proximity. It follows from the above discussion and theorem 4.1.2 that this proximity is induced by the Cauchy screen $\mathcal{C}_{rd}$.

We have the following characterization from [4]: δ is an RE-proximity if and only if its nasse Δ is the closure in $(\beta X)^2$ of its equivalence kernel Δ' if and only if $\mathcal{R}(\delta) = \delta$. In particular if δ is an RE-proximity $\mathbb{D}(\Delta)$ is dense in βX and δ is induced by the Cauchy screen $\mathcal{C}_{rd}$. Finally $\mathcal{C}_{rd}$ is a screen inducing the proximity δ if and only if δ is an RE-proximity.

So we have the following proposition:

Theorem 4.2.3. *An arbitrary RE-proximity δ is always induced by a full Cauchy screen, namely the screen $\mathcal{C}_{rd}$ generated by the set of δ-round and compressed filters. Consequently the same is true of arbitrary RC or EF-proximities.*

A *preregular system of filters* of Császár [9] generates a Cauchy screen inducing an RE-proximity. We give another equivalent definition.

Consider a family $\mathcal{B}$ of filters such that:

$$(PSF1) \qquad \forall x \in X, \quad \exists \mathcal{F} \in \mathcal{B}, \quad \mathcal{F} \subset \dot{x}.$$

We associate to $\mathcal{B}$ a proximity $\delta_{\mathcal{B}}$ or δ for short, on X defined by:

$$A\delta_{\mathcal{B}}B \text{ if and only if there exists } \mathcal{F} \in \mathcal{B} \text{ such that } A, B \in \mathcal{F}^{\#}.$$

Notice that if $\mathcal{C}$ is the screen generated by $\mathcal{B}$, that is the set of all filters containing some member of $\mathcal{B}$, then $\delta_{\mathcal{C}} = \delta_{\mathcal{B}}$ and $\mathcal{C}$ satisfies $(PSF1)$ if and only if $\mathcal{B}$ does, if and only if $\mathcal{C}$ contains all centered ultrafilters.
The *topogeneous order* γ associated to δ is defined by:

$$[A\gamma B] \text{ if and only if } [\forall \mathcal{F} \in \mathcal{B}, A \in \mathcal{F}^{\#} \Rightarrow B \in \mathcal{F}].$$

The family $\mathcal{B}$ is a **preregular system of filters** in the sense of [9] if and only if :

$$(PSF2) \qquad \text{every } \mathcal{F} \text{ in } \mathcal{B} \text{ is } \delta\text{-round},$$

that is, for any A in some $\mathcal{F}$ in $\mathcal{B}$, there exists a $B \in \mathcal{F}$ such that $B\gamma A$.
It was proved in [9] that a proximity δ is an RE-proximity if and only if δ is $\delta_{\mathcal{B}}$ for some preregular system of filters $\mathcal{B}$.

Now it follows easily from these definitions that if $\mathcal{B}$ is a preregular system of filters on X, then $\mathcal{B}$ is a subset of the set of δ-round and compressed filters and then again, as shown in the proof of theorem 4.2.1: for all $\mathcal{F}_1, \mathcal{F}_2 \in \mathcal{B}$ such that $\mathcal{F}_1 \subset \mathcal{F}_2^{\#}$ we have $\mathcal{F}_1 = \mathcal{F}_2$. Therefore the screen $\mathcal{C}$ generated by $\mathcal{B}$ is a Cauchy screen on X.

Finally we have the following result:

Proposition 4.2.1. *If δ is an RE-proximity which is not an EF-proximity then a coarsest Cauchy screen inducing δ does not exist.*

Proof. Let Δ be the nasse associated to δ and Δ' be its equivalence kernel which is dense in Δ. Δ is not an equivalence relation on βX and $\mathbb{D}(\Delta) \neq \beta X$. It is easy to check that there exist three different ultrafilters $\mathcal{U}, \mathcal{V}, \mathcal{W}$ in $\beta X \setminus \mathbb{D}(\Delta)$ such that $\mathcal{U}\Delta\mathcal{V}\Delta\mathcal{W}$ and $\mathcal{U}(\text{non-}\Delta)\mathcal{W}$. Therefore one can see that the relations $\Delta_1 = \Delta' \cup \beta(\mathcal{U} \cap \mathcal{V})^2 \cup \{(\mathcal{X}, \mathcal{X}), \mathcal{X} \in \beta X\}$ (recall that $\beta(\mathcal{U} \cap \mathcal{V}) = \{\mathcal{U}, \mathcal{V}\}$) and $\Delta_2 = \Delta' \cup \beta(\mathcal{W} \cap \mathcal{V})^2 \cup \{(\mathcal{X}, \mathcal{X}), \mathcal{X} \in \beta X\}$ are equivalence relations on βX which are dense again in Δ'.

Let $\mathcal{C}_{rd}$ be the screen generated by the δ-round and compressed filters and consider the following screens $\mathcal{C} = \mathcal{C}_{rd} \cup \beta X$, $\mathcal{C}_1 = \mathcal{C} \cup \{\mathcal{U} \cap \mathcal{V}\}$ and $\mathcal{C}_2 = \mathcal{C} \cup \{\mathcal{W} \cap \mathcal{V}\}$. Since $\mathcal{C}_{rd}$ is Cauchy and full it is easy to see that $\mathcal{C}$ is actually a totally bounded and full Cauchy screen as well as $\mathcal{C}_1$ and $\mathcal{C}_2$ (because $\mathcal{U}, \mathcal{V}, \mathcal{W}$ are not in $\mathcal{C}_{rd}$). These screens induce δ again since $\Theta_{\mathcal{C}_1} = \Delta_1$ and $\Theta_{\mathcal{C}_2} = \Delta_2$ which are dense in the nasse Δ.

Finally there does not exist a Cauchy screen inducing δ and containing both $\mathcal{C}_1$ and $\mathcal{C}_2$ because any Cauchy screen containing them should contain also $\mathcal{U} \cap \mathcal{W}$ also since the filters $\mathcal{U} \cap \mathcal{V}$ and $\mathcal{V} \cap \mathcal{W}$ mesh, and then $\mathcal{U} \cap \mathcal{W}$ would be δ-compressed and $(\mathcal{U}, \mathcal{W})$ would be in Δ which is not the case. □

From the preceding proof one can extract the following observation:

Corollary 4.2.1. *Among the full and/or totally bounded Cauchy screens inducing a given RE-proximity which is not EF, a coarsest one does not exist.*

4.3. **Pretopological closures induced by a screen.** The pretopological closure c on X induced by a screen $\mathcal{C}$ is defined by:

$(*)$ $x \in c(A)$ if and only if there exists $\mathcal{F} \in \mathcal{C}$ such that $(\{x\}, A) \in \mathcal{F}^{\#} \times \mathcal{F}^{\#}$.

It is the closure associated to the proximity δ induced by $\mathcal{C}$: $x \in c(A)$ if and only if $\{x\} \delta A$.
As it was observed at the beginning of this section, the pretopological closure induced by $\mathcal{C}$ is also the closure associated to the pretopological modification of the convergence $\to_{\mathcal{C}}$ induced by $\mathcal{C}$.
We will use the two points of view to study the closures induced by Cauchy screens.

Theorem 4.3.1. *For a pretopological closure c the following are equivalent:*

(i) *c is induced by a screen*
(ii) *c is induced by a proximity*
(iii) *c is symmetric, i.e.* $\forall x, y \in X,\ y \in c(\{x\}) \Leftrightarrow x \in c(\{y\})$

Proof. Since every proximity is induced by a screen, we get easily that $(i) \Leftrightarrow (ii)$.

Furthermore it is clear that a closure c induced by a proximity (or a screen) is symmetric.

Conversely assume that c is symmetric, then take the proximity π defined by:

$$A\pi B \quad \text{if} \quad A \cap c(B) \neq \emptyset \text{ or } c(A) \cap B \neq \emptyset.$$

By the symmetry axiom on c, if $c(\{x\}) \cap B \neq \emptyset$ then $x \in c(B)$. Therefore if $\{x\}\pi B$ then $x \in c(B)$ and so c is induced by the proximity π. □

Remark. If δ is a proximity inducing c, then $A\delta B$ provided $A \cap c(B) \neq \emptyset$ and it is easy to see that π is the smallest proximity inducing c when c is symmetric.

Theorem 4.3.2. *For a pretopological closure c the following are equivalent:*

(i) *c is induced by a Riesz screen*
(ii) *c is induced by a Riesz proximity*
(iii) *c satisfies condition $(S1)$ (or equivalently conditions $(S1')$, $(S1'')$ of corollary 3.1.2):*

$$(S1) \quad \forall x \in X,\ \forall A \in \mathcal{P}(X),\ c(\{x\}) \cap c(A) \neq \emptyset \Leftrightarrow x \in c(A)$$

Proof. Since a proximity is induced by a Riesz screen if and only if it is a Riesz proximity, it is clear that $(i) \Leftrightarrow (ii)$.

By condition (RI) for a Riesz proximity (in the preceding subsection) the condition $(S1)$ is clearly necessary for c to be induced by a Riesz proximity.

Conversely assume that (S1) is true, then take the proximity π defined by: $A\pi B$ if and only if $c(A) \cap c(B) \neq \emptyset$. This proximity is clearly a Riesz proximity and by condition (S1), c is induced by π. □

Remark. If δ is a Riesz proximity inducing c, we have that $c(A) \cap c(B) \neq \emptyset \Rightarrow A\delta B$ and finally when $(S1)$ is true, π is the smallest Riesz proximity inducing c.

To characterize closures induced by Cauchy screens we use the convergence point of view. We get from proposition 3.1.2 that:

Proposition 4.3.1. *A pretopological closure is induced by a Cauchy screen $\mathcal{C}$ if and only if it is the closure associated to the pretopological modification of a reciprocal limit structure (namely the convergence $\to_{\mathcal{C}}$).*

Theorem 4.3.3. *For a pretopological closure c with associated neighborhood filters $(\mathcal{V}_x)_{x\in X}$, the following are equivalent :*

(i) *c is induced by a Cauchy screen,*
(ii) *c is (the closure associated to) the pretopological modification of a reciprocal pseudotopology,*
(iii) *there exists a family $(\mathbb{F}_x)_{x\in X}$ of subsets of βX satisfying the following condition:*

$$(S) \quad \forall x, y \in X, \quad \mathcal{V}_x = \cap\{\mathcal{U} : \mathcal{U} \in \mathbb{F}_x\} \quad \textit{and} \quad [\mathbb{F}_x \cap \mathbb{F}_y \neq \emptyset \Rightarrow \mathbb{F}_x = \mathbb{F}_y].$$

We prove first the following propositions.

Proposition 4.3.2. *Let ξ be any pseudotopological convergence on X and for any $x \in X$, write $\mathbb{F}_x = \{\mathcal{U} \in \beta X : \mathcal{U} \to x\}$.*
The convergence ξ is reciprocal if and only if :

$$\forall x, y \in X, \quad \mathbb{F}_x \cap \mathbb{F}_y \neq \emptyset \Longrightarrow \mathbb{F}_x = \mathbb{F}_y.$$

In particular a pretopological convergence ξ with associated neighborhood filters $(\mathcal{V}(x))_{x\in X}$, is reciprocal if and only if the separation condition $(S2)$ of Császár is satisfied (see subsection 3.1).

Proof. The first part is straightforward since the convergence is given by: $\mathcal{F} \to x$ if and only if $\beta(\mathcal{F}) \subset \mathbb{F}_x$.

In the pretopological case, one has to note that $\mathbb{F}_x = \beta(\mathcal{V}(x))$ for every x, and therefore that $[\mathcal{V}(x) \text{ mesh } \mathcal{V}(y) \Longleftrightarrow \mathbb{F}_x \cap \mathbb{F}_y \neq \emptyset]$ and $[\mathcal{V}(x) = \mathcal{V}(y) \Longleftrightarrow \mathbb{F}_x = \mathbb{F}_y]$. □

Proposition 4.3.3. *If ξ is a reciprocal convergence then so is its pseudotopogical modification.*

Proof. The pseudotopological modification of ξ is given by:

$$\mathcal{F} \to x \text{ if and only if } \beta(\mathcal{F}) \subset \mathbb{F}_x$$

where $\mathbb{F}_x = \{\mathcal{U} \in \beta X : \mathcal{U} \to_\xi x\}$. Notice that an ultrafilter converges to a point x for ξ if and only if it converges to x for the pseudotopological modification. We then have $\mathbb{F}_x = \{\mathcal{U} \in \beta X : \mathcal{U} \to x\}$ and therefore using proposition 4.3.2, if ξ is reciprocal then so is its pseudotopological modification. □

Proof of theorem 4.3.3.

With proposition 4.3.3, (i) $\Leftrightarrow$ (ii) is clear since a convergence and its pseudotopological modification have the same pretopological modification.

(ii) $\Leftrightarrow$ (iii) :

Let ξ be a reciprocal pseudotopology inducing c and take $\mathbb{F}_x = \{\mathcal{U} : \mathcal{U} \to_\xi x\}$. We clearly have $\mathcal{V}_x = \cap\{\mathcal{F} : \mathcal{F} \to_\xi x\} = \cap\{\mathcal{U} : \mathcal{U} \in \mathbb{F}_x\}$ and since ξ is reciprocal, using proposition 4.3.2, we get condition (S).

Conversely assume that there exists a family $(\mathbb{F}_x)_{x\in X}$ satisfying condition (S) and define a convergence ξ as follows: $\mathcal{F} \to_\xi x$ if $\beta(\mathcal{F}) \subset \mathbb{F}_x$.

Let us check that it is indeed a convergence.
The condition $\mathcal{V}_x = \cap\{\mathcal{U} : \mathcal{U} \in \mathbb{F}_x\}$ means that $\beta(\mathcal{V}_x)$ is the closure in βX of $\mathbb{F}_x$. Since $\dot{x} \in \beta(\mathcal{V}_x)$ and $\{\dot{x}\}$ is open in βX, we have $\dot{x} \in \mathbb{F}_x$. Therefore ξ is a convergence and it is pseudotopological since for any ultrafilter $\mathcal{U}$, $\mathcal{U} \to_\xi x$ if and only if $\mathcal{U} \in \mathbb{F}_x$. Now from condition (S), ξ is also reciprocal and c is the closure associated to the pretopological modification of ξ. □

Finally theorem 4.3.3 with proposition 3.1.3 yield the following result from [12]:

Corollary 4.3.1. *A pretopological closure is induced by a Riesz Cauchy screen if and only if it is the closure of an $(S2)$-pretopological space.*

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