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Number of zeros of exponential polynomials in zero residue characteristic

by Alain Escassut

Abstract

Let \mathbb{L} be a complete ultrametric field of residue characteristic 0 and let $F(x) = \sum_{i=1}^k f_i(x) \exp(\omega_i x)$, with $\omega_i \in \mathbb{L}$, $|\omega_i| < 1$. The number of zeros of F in the unit disk is bounded by $n - 1$.

Notations: We denote by \mathbb{L} a complete field with respect to an ultrametric absolute value, with residue characteristic 0 and we denote by v the valuation linked to \mathbb{L} as $v(x) = -\log(|x|)$ [1]. Next, we denote by \mathbb{K} an algebraically closed field \mathbb{K} complete with respect to an ultrametric absolute value with zero residue characteristic. Given $r > 0$, we denote by $d_{\mathbb{L}}(0, r)$ the disk $\{x \in \mathbb{L} \mid |x| \leq r\}$ and by $d_{\mathbb{K}}(0, r)$ the disk $\{x \in \mathbb{K} \mid |x| \leq r\}$.

In an algebraically closed field \mathbb{K} complete with respect to an ultrametric absolute value with residue characteristic $p > 0$, the number of zeros of a sum of exponential polynomials was examined in several papers. Following [6] and [8], the best estimation was given in [7] and this was also given in [4]. Here we will follow the method of [6] in order to examine a similar problem on the field \mathbb{L} which might apply to fields such as Levi-Civita fields [5].

Given $r > 0$, we denote by $H(d(0, r))$ the \mathbb{K} -algebra of analytic elements in $d_{\mathbb{K}}(0, r)$, i.e. the set of power series $\sum_{j=0}^{\infty} \lambda_j x^j$ converging in $d_{\mathbb{K}}(0, r)$ [3].

The radius of convergence of the exponential function $\exp(x) = \sum_{j=0}^{\infty} \frac{x^j}{j!}$ is 1 in the field \mathbb{K} and the function converges in all $d_{\mathbb{K}}(0, 1)$ [1], [2].

Theorem M: *Let \mathbb{L} be an ultrametric complete field of residue characteristic 0 and let $F(x) = \sum_{i=1}^k f_i(x) \exp(\omega_i x)$ where each $\omega_i \in \mathbb{L}$ satisfies $v(\omega_i) > 0$, each f_i lies in $\mathbb{L}[x]$ and $\deg(f_i) = m_i - 1$. Let $n = \sum_{i=1}^k m_i$.*

Then the number N of zeros of F in the disk $d_{\mathbb{L}}(0, 1)$, taking multiplicity into account, satisfies $N \leq n - 1$.

Before proving Theorem M, we have to recall the following Theorem A that is Theorem B.13.7 in [2] together with some notations and classical properties:

Notation: Let $r > 0$ and let $f(x) = \sum_{j=0}^{\infty} a_j x^j \in H(d_{\mathbb{K}}(0, r))$ [3]. Then

$\lim_{j \rightarrow +\infty} |a_j| r^j = 0$ and we denote by $\nu^+(r, f)$ the unique integer t such that $|a_t| r^t = \sup_{j \in \mathbb{N}} |a_j| r^j > |a_j| r^j \forall j > t$ [1], [2].

Theorem A: Let $r > 0$ and let $f \in H(d_{\mathbb{K}}(0, r))$. The number of zeros of f in $d_{\mathbb{K}}(0, r)$ is equal to $\nu^+(f, \log r)$, (taking multiplicity into account).

Proof of Theorem M: Let us first suppose that the field \mathbb{L} is algebraically closed and hence is \mathbb{K} . Denoting by D the derivation $\frac{d}{dx}$, then F satisfies a relation of the form

$$(1) \quad D^n F = c_1 D^{n-1} F + c_2 D^{n-2} F + \dots + c_n F$$

where

$$(2) \quad \prod_{i=1}^k (x - \omega_i)^{m_i} = x^n - c_1 x^{n-1} - c_2 x^{n-2} - \dots - c_n.$$

Expanding F as a power series, we have

$$F(x) = \sum_{h=0}^{\infty} a_h x^h = \sum_{h=0}^{\infty} \frac{b_h}{h!} x^h, \quad |x| < 1.$$

From the differential equation (1), we have the recurrence

$$(3) \quad b_{h+n} = c_1 b_{h+n-1} + \dots + c_n b_h, \quad h = 0, 1, 2, \dots$$

Let $\epsilon = \min_{1 \leq i \leq n} v(\omega_i)$. Since the radius of convergence of the exponential $\exp(x)$ is 1 in a field of residue characteristic 0 [1], [2], the series F converges in $d_{\mathbb{K}}(0, 1)$, hence it lies in $H(d_{\mathbb{K}}(0, 1))$, and therefore we can put $t = \nu^+(F, 1)$. Without loss of generality, we can suppose that $|a_t| = 1$. Thus we have $|a_t| = 1$ and $|a_h| < 1 \forall h > t$. Consequently by Theorem M the number of zeros of F in $d_{\mathbb{K}}(0, 1)$ is t . From (2), we can see that $v(c_j) \geq j\epsilon$ ($j = 1, 2, \dots, n$).

Now, by assumption we have $v(a_h) \geq 0$ and $v(a_h) = v(b_h) - v(h!) = v(b_h)$ because $v(h!) = 0 \forall h$, hence $v(b_h) \geq 0$ ($h = 0, 1, \dots$). By (3), we have

$$v(b_n) \geq \min_{1 \leq j \leq n} (v(c_j) + v(b_{n-j})) > \min_{1 \leq j \leq n} (\epsilon j + v(b_{n-j})) \geq \epsilon.$$

Then, by (3) with $h = 1, 2, \dots$ we have $v(b_{h+n-1}) > h\epsilon$, hence

$$v(a_{h+n-1}) > h\epsilon - v((n-1+h)!) = h\epsilon \forall h \geq 1.$$

Thus for every $h \geq 1$, we have $v(a_{h+n-1}) > 0$, hence $t \leq n-1$. Consequently, the number of zeros of F in $d(0, 1)$ is bounded by $(n-1)$, which ends the proof when the field \mathbb{L} is algebraically closed.

Consider now the general case, when \mathbb{L} is not supposed to be algebraically closed. The field \mathbb{L} then admits a unique algebraically closed extension complete with respect to an ultrametric absolute value that continues this of \mathbb{L} (see for example Theorem A.5.14 in [2]). Then $F(x)$ has continuation in the disk $d_{\mathbb{K}}(0, 1)$ and has at most $n-1$ zeros in $d_{\mathbb{K}}(0, 1)$ taking multiplicity into account, hence of course it has at most $n-1$ zeros in the disk $d_{\mathbb{L}}(0, 1)$ taking multiplicity into account.

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