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Number of zeros of exponential polynomials in zero residue characteristic

by Alain Escassut

Abstract

Let $\mathbb{L}$ be a complete ultrametric field of residue characteristic 0 and let $F(x) = \sum_{i=1}^{k} f_i(x)exp(\omega_i x)$, with $\omega_i \in \mathbb{L}$, $|\omega_i| < 1$. The number of zeros of F in the unit disk is bounded by $n-1$.

Notations: We denote by $\mathbb{L}$ a complete field wit respect to an ultrametric absolute value, with residue characteristic 0 and we denote by v the valuation linked to $\mathbb{L}$ as $v(x) = -\log(|x|)$ [1]. Next, we denote by $\mathbb{K}$ an algebraically closed field $\mathbb{K}$ complete with respect to an ultrametric absolute value with zero residue characteristic. Given $r > 0$, we denote by $d_{\mathbb{L}}(0,r)$ the disk $\{x \in \mathbb{L} \mid |x| \leq r\}$ and by $d_{\mathbb{K}}(0,r)$ the disk $\{x \in \mathbb{K} \mid |x| \leq r\}$.

In an algebraically closed field $\mathbb{K}$ complete with respect to an ultrametric absolute value with residue characteristic $p > 0$, the number of zeros of a sum of exponential polynomials was examined in several papers. Following [6] and [8], the best estimation was given in [7] and this was also given in [4]. Here we will follow the method of [6] in order to examine a similar problem on the field $\mathbb{L}$ which might apply to fields such as Levi-Civita fields [5].

Given $r > 0$, we denote by by $H(d(0,r))$ the $\mathbb{K}$-algebra of anaytic elements in $d_{\mathbb{K}}(0,r)$, i.e. the set of power series $\sum_{j=0}^{\infty} \lambda_j x^j$ converging in $d_{\mathbb{K}}(0,r)$ [3].

The radius of convergence of the exponential function $exp(x) = \displaystyle\sum_{j=0}^{\infty} \frac{x^j}{j!}$ is 1 in the field $\mathbb{K}$ and the fuction converges in all $d_{\mathbb{K}}(0,1)$ [1], [2].

Theorem M: *Let $\mathbb{L}$ be an ultrametric complete field of residue characteristic 0 and let $F(x) = \sum_{i=1}^{k} f_i(x)exp(\omega_i x)$ where each $\omega_i \in \mathbb{L}$ satisfies $v(\omega_i) > 0$, each f_i lies in $\mathbb{L}[x]$ and $\deg(f_i) = m_i - 1$. Let $n = \sum_{i=1}^{k} m_i$.*

Then the number N of zeros of F in the disk $d_{\mathbb{L}}(0,1)$, taking multiplicity into account, satisfies $N \leq n-1$.

Before proving Theorem M, we have to recall the following Theorem A that is Theorem B.13.7 in [2] together with some notations and classical properties:

Notation: Let $r > 0$ and let $f(x) = \displaystyle\sum_{j=0}^{\infty} a_j x^j \in H(d_{\mathbb{K}}(0,r))$ [3]. Then $\lim\limits_{j \to +\infty} |a_j| r^j = 0$ and we denote by $\nu^+(r,f)$ the unique integer t such that $|a_t| r^t = \sup_{j \in \mathbb{N}} |a_j| r^j > |a_j| r^j \ \forall j > t$ [1], [2].

Theorem A: *Let $r > 0$ and let $f \in H(d_{\mathbb{K}}(0,r))$. The number of zeros of f in $d_{\mathbb{K}}(0,r)$ is equal to $\nu^+(f, \log r)$, (taking multiplicity into account).*

Proof of Theorem M: Let us first suppose that the field $\mathbb{L}$ is algebraically closed and hence is $\mathbb{K}$. Denoting by D the derivation $\dfrac{d}{dx}$, then F satisfies a relation of the form

$$D^n F = c_1 D^{n-1} F + c_2 D^{n-2} F + ... + c_n F \tag{1}$$

where

$$\prod_{i=1}^{k} (x - \omega_i)^{m_i} = x^n - c_1 x^{n-1} - c_2 x^{n-2} - ... - c_n. \tag{2}$$

Expanding F as a power series, we have

$$F(x) = \sum_{h=0}^{\infty} a_h x^h = \sum_{h=0}^{\infty} \frac{b_h}{h!} x^h, \ |x| < 1.$$

From the differential equation (1), we have the recurrence

$$b_{h+n} = c_1 b_{h+n-1} + ... + c_n b_h, \ h = 0, \ 1, \ 2, \ .... \tag{3}$$

Let $\epsilon = \min_{1 \le i \le n} v(\omega_i)$. Since the radius of convergence of the exponential $exp(x)$ is 1 in a field of residue characteristic 0 [1], [2], the series F converges in $d_{\mathbb{K}}(0,1)$, hence it lies in $H(d_{\mathbb{K}}(0,1))$, and therefore we can put $t = \nu^+(F,1)$. Without loss of generality, we can suppose that $|a_t| = 1$. Thus we have $|a_t| = 1$ and $|a_h| < 1 \ \forall h > t$. Consequently by Theorem M the number of zeros of F in $d_{\mathbb{K}}(0,1)$ is t. From (2), we can see that $v(c_j) \ge j\epsilon$ $(j = 1, \ 2, \ ..., n)$.

Now, by assumption we have $v(a_h) \ge 0$ and $v(a_h) = v(b_h) - v(h!) = v(b_h)$ because $v(h!) = 0 \ \forall h$, hence $v(b_h) \ge 0$ $(h = 0, \ 1, \ ...)$. By (3), we have

$$v(b_n) \ge \min_{1 \le j \le n} (v(c_j) + v(b_{n-j})) > \min_{1 \le j \le n} (\epsilon j + v(b_{n-j})) \ge \epsilon.$$

Then, by (3) with $h = 1, \ 2, \ ...$ we have $v(b_{h+n-1}) > h\epsilon$, hence

$$v(a_{h+n-1}) > h\epsilon - v((n-1+h)!) = h\epsilon \ \forall h \ge 1.$$

Thus for every $h \ge 1$, we have $v(a_{h+n-1}) > 0$, hence $t \le n-1$. Consequently, the number of zeros of F in $d(0,1)$ is bounded by $(n-1)$, which ends the proof when the field $\mathbb{L}$ is algebraically closed.

Consider now the general case, when $\mathbb{L}$ is not supposed to be algebraically closed. The field $\mathbb{L}$ then admits a unique algebraically closed extension complete with respect to an ultrametric absolute value that continues this of $\mathbb{L}$ (see for example Theorem A.5.14 !n [2]). Then $F(x)$ has continuation in the disk $d_{\mathbb{K}}(0,1)$ and has at most $n-1$ zeros in $d_{\mathbb{K}}(0,1)$ taking multiplicity into account, hence of course it has at most $n-1$ zeros in the disk $d_{\mathbb{L}}(0,1)$ taking multiplicity into account.

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