GLOBAL SENSITIVITY ANALYSIS IN THE SIHR EPIDEMIOLOGICAL MODEL WITH APPLICATION TO COVID19

Hacène Djellout, Cédric Chauvière, Liban Ismail

GLOBAL SENSITIVITY ANALYSIS IN THE SIHR EPIDEMIOLOGICAL MODEL WITH APPLICATION TO COVID19

March 7, 2022

LIBAN ISMAIL[1] ; HACÈNE DJELLOUT[2]; CÉDRIC CHAUVIÈRE [3]

Laboratoire de Mathématiques Blaise Pascal (LMBP), CNRS UMR 6620, Université Clermont Auvergne, Campus Universitaire des Cézeaux, 3 place Varsarely, TSA 60026, CS 60026, 63178 Aubière Cedex France.

Abstract

We study the SIHR epidemiological model to describe the evolution of the COVID19 disease and help the authorities to put in place effective strategies and controls. In this paper, we have developped a numerical approach based on Lagrange polynomials to analyze the sensitivity of input parameters (basic reproduction number, cure rate, hospitalization rate) on the evolution of the system. The sensitivity analysis has been undertaken to identify the key model parameters. Here, the quantity of interest is the endemic stationary state of the model which is a function of uncertain parameters. Using Lagrange polynomials is a natural framework for computing Sobol indices. Furthermore, we consider a sinusoidal and logistic infection rate and give simulations of the model SIHR with deterministic parameters and when the parameters follow the uniform law.

Keywords: COVID19; SIHR model; Lagrange polynomial; Stochastic collocation method; Sobol indices; Simulations.
2010 Mathematics Subject Classification:

1 Introduction

The corona virus disease (COVID-19) is a worldwide infectious disease [1, 2], caused by a new strain of coronavirus. CO stands for corona, VI for virus, and D for disease. Formerly, this disease was referred to as 2019 novel coronavirus or 2019-nCoV. The COVID-19 virus is a new virus linked to the same family of viruses as Severe Acute Respiratory Syndrome (SARS) and some types of common cold [12]. Corona-viruses are a large family of viruses that can be pathogenic in humans and animals. We know that, in humans, several corona-viruses can cause respiratory

[1] liban.ismail_abdillahi@etu.uca.fr
[2] hacene.djellout@uca.fr
[3] cedric.chauviere@uca.fr

infections whose manifestations range from a simple cold to more serious illnesses such as Middle East respiratory syndrome (MERS) where dromedary camels were thought to be the intermediate source for the transmission of the virus [22, 35, 3]. The novel corona virus was first identified by health authority of Wuhan province of China [19, 20]. Symptoms can include fever, cough and shortness of breath. In more severe cases, infection can cause pneumonia or breathing difficulties.

The simplest way to model epidemic spread in populations is to classify people into different population groups or compartments. Compartmental models are governed by a system of differential equations that follow the population over time, dividing them into different groups based on risk or infection status. They are able to predict the various properties of pathogen spread, can estimate the duration of epidemics, and can be used to understand how different situations or interventions can impact the outcome of pathogen spread. The Kermack-Mckendric SIR model is a very well established model and is widely used for various epidemics [34]. To do this, the SIR, SIRS and SEIR models have been developed which highlight the crucial role played by the $\mathcal{R}_0$ parameter, describing the average number of new infections due to a sick individual [30, 21]. The course of a disease depends on several parameters (infection rate, cure rate, hospitalization rate, vaccination, etc.). The aim of this research is to study the influence of these parameters on the spread of the epidemic. For this, we study the sensitivity analysis applied to the epidemiological model.

The sensitivity analysis studies how disturbances on the input variables of the model generate disturbances on the output variable. The author interested in a reference work can refer to [4, 15, 7]. Indeed, by studying how the response of the model reacts to variations in its input variables, the sensitivity analysis makes it possible to answer a number of questions: What are the variables that most contribute to the variability of the model response? What are the least influential variables? Which variables, or which groups of variables, interact with which others? To answer these questions, we use the Sobol indices. These indices make it possible to predict the consequences on the evolution of a disease according to the variation of the parameters of the model in order to put in place strategies to prevent disease (vaccination, prevention campaign, etc.).

To calculate the Sobol indices, applied to the numerical approximation of differential equations, we use the stochastic collocation method. The solution of the differential equation is represented by means of Lagrange polynomials. The coefficients of the polynomial basis are functions of time and can be calculated by solving a system of deterministic ordinary differential equations. Numerical examples are presented to illustrate the accuracy and efficiency of the proposed method.

Our main goal of this work is to extend the classical deterministic susceptible-infectious-removed (SIR) epidemic model by adding a hospitalized compartment and develop the stochastic collocation method to study the sensitivity analysis in the input parameters of the epidemiological model (cure rate, basic reproduction number, hospitalization rate, ... etc) on the course of the disease with an infection rate depending on time. The SIHR model appears in several research works, in particular the article [16], [31] and [28]. We assume that these parameters are not precisely known and that they can therefore be modeled as random variables with known laws. We calculate the Sobol indices of order 1 and 2 for each parameter in the endemic stationary state and we consider that the parameters follow the uniform law. The organization of the paper is as follows: in section

2, we introduce the model, describe the parameter estimation procedure, study the aymptomatic stability of disease-free and endemic equilibrium points and predict the evolution of the disease with a sinusoidal infection and a logistic infection. In section 3 and 4, we introduce the theory on sensitivity analysis and the stochastic collocation method. In section 5, we calculate the Sobol indices of the endemic stationary state to know the influences of the input random variables on the output variable and we simulate the model with two different infection rate and two different basic reproduction number when the parameters follow the uniform law. Details of calculations of stationnary Sobol indices are given in the appendix.

2 The SIHR epidemiological model

Disease models play an important role in understanding and managing the transmission dynamics of various pathogens. We can use them to describe the spatial and temporal patterns of disease prevalence, as well as to explore or better understand the factors that influence infection incidence. Modeling is a key step in understanding what treatments and interventions can be most effective and what specific factors need to be considered when trying to eradicate disease. Some recent studies provided different guidelines by introducing basic reproduction number, education and socio-economic index and lock-down strategies [17, 9]. To understand the complex dynamics underlying disease transmission, epidemiologists often use a set of models called compartmental models. Developed in the early 20th century, these models stratify a population into groups, generally based on their risk or infection status. Underlying these models is a system of differential equations that track the number of people in each category over time.

We note $\tilde{S}_t$ (respectively, $\tilde{I}_t$, $\tilde{H}_t$ and $\tilde{R}_t$) the number of the susceptible in the population (respectively, the number of the infected, hospitalized, recovered in the population N), and

$$N_t = \tilde{S}_t + \tilde{I}_t + \tilde{H}_t + \tilde{R}_t$$

where N_t is the population at time t.

We consider that the mortality from the covid of infected people and hospitalized people is the same.

The SIHR model is represented by the proportion of each compartment in the population in the following system of differential equations:

$$\begin{cases} \dfrac{dS_t}{dt} = \tau - \mu S_t - \beta(t) S_t I_t \\ \dfrac{dI_t}{dt} = \beta(t) S_t I_t - (\nu + \mu + \gamma + \alpha) I_t \\ \dfrac{dH_t}{dt} = \alpha I_t - (\nu + \mu + \lambda) H_t \\ \dfrac{dR_t}{dt} = \gamma I_t + \lambda H_t - \mu R_t. \end{cases} \tag{1}$$

with $S_t = \frac{\tilde{S}_t}{N_t}, I_t = \frac{\tilde{I}_t}{N_t}, H_t = \frac{\tilde{H}_t}{N_t}$ and $R_t = \frac{\tilde{R}_t}{N_t}$ and the initial condition $S_0 > 0, I_0 > 0, H_0 \geq 0, R_0 \geq 0$ where the interpretation of parameters is presented in Table 1.

Notations	Interpretations
τ	Birth rate of class S
μ	Natural death rate
ν	Death rate of infection
$\beta(t)$	Transmission infection rate
γ	Recovery rate of class I
λ	Recovery rate of class H
α	Hospitalized rate

Table 1: Parameters and their descriptions.

2.1 The dynamics of SIHR model

System dynamics is a methodology and a mathematical modeling technique to frame, understand and discuss complex issues and problems. System dynamics is an aspect of systems theory as a method to understand the dynamic behavior of complex systems. The model with multiple compartments is a useful tool to predict the nature of recent most dangerous disease. In this section, we study a model of four compartments SIHR. The key objectives of this section are as follows: firstly, we establish the basic reproduction rate using the method next generation matrix and analyze the stability of the disease-free and the endemic equilibrium points of the model using the basic reproduction number to understand the severity. Secondly, we perform simulations of the course of the disease with an infection rate that depending on time.

The model (1) is continuous and Lipschizien in $\mathbb{R}^4_+$. From the existence and uniqueness of the solution to the ordinary differential equation, the model admits a unique solution for any initial value $(S_0, I_0, H_0, R_0) \in \mathbb{R}^4_+$. From the equation of I in (1), we have:

$$I_t = I_0 \exp\left(\int_0^t \left[\beta(u)S_u - (\nu + \mu + \gamma + \alpha)\right] du\right),$$

therefore $I_t \geqslant 0$ for all $t \geqslant 0$. Then we have $\frac{dH_t}{dt} \geqslant -(\nu + \lambda)H_t$, so $H_t \geqslant 0$ for all $t \geqslant 0$ and $H_0 > 0$.
In the same way, we have $R_t \geqslant 0$ and $S_t \geqslant 0$ for all $t \geqslant 0$. The system (1) is positive that means that the solution remains positive for any trajectory initialized at positive conditions.

We consider $K_t = S_t + I_t + H_t + R_t$ and adding equations in (1), we have

$$\frac{dK_t}{dt} = \tau - \mu K_t - \nu(I_t + H_t).$$

The dynamics of the model (1) should be studied in the closed region:

$$\Lambda_t = \left\{ (S_t, I_t, H_t, R_t) \in \mathbb{R}^4 : K_t \leqslant \frac{\tau}{\mu} \text{ with } (S_0, I_0, H_0, R_0) \in \mathbb{R}^4_+ \right\}. \tag{2}$$

Lemma 2.1. *The closed region Λ_t is a positively invariant set for the COVID-19 model* (1)

Proof. We have

$$\frac{dK_t}{dt} = \tau - \mu K_t - \nu(I_t + H_t) \leqslant \tau - \mu K_t \tag{3}$$

By solving (3) and using the initial condition, we get

$$K_t \leqslant K_0 e^{-\mu t} + \frac{\tau}{\mu}\left(1 - e^{-\mu t}\right).$$

We conclude that $K_t \leqslant \frac{\tau}{\mu}$ if $K_0 \leqslant \frac{\tau}{\mu}$. Thus, Λ_t is a positively invariant set under the inputs and outputs presented in the COVID-19 model (1). □

We note that in reality the rate of infection varies with the control measures that are put in place. The article [32] considers the infection rate as a function that depends on several parameters

$$\beta(t) = \frac{\beta_0}{1 + \exp\left(\lambda_m(t - d - m/2)\right)}, \tag{4}$$

where d is the time when the control measures start to be effective, β_0 is the initial decline, m represents the duration of a process where the epdemic is near to vanish, λ_m is chosen as

$$\frac{2\log((1-\epsilon)/\epsilon)}{m}$$

and ϵ is fixed to be 0,01.

The logistics function (4) can be written in another simpler way:

$$\beta(t) = \frac{\beta_0}{1 + \exp\big(-k(t - t_0)\big)} \tag{5}$$

where k is the intensity of beta convergence towards 0 and t_0 is the day of implementation of the control measures.
The article [26] considers the following function for the infection rate:

$$\beta(t) = \beta_0 \left[1 + \epsilon \sin\left(\frac{2\pi}{T}t + \phi_0\right)\right], \tag{6}$$

where ϵ is the amplitude, T the period and ϕ_0 the initial phase.

In reality, the infection rate β probably never jumps from one value to another. Rather, it continuously changes and might go up and down several times, e.g. if social distancing measures are loosened and then tightened again. One can choose various function for β.

In this study we consider (5) and (6) for the evolution of the model (1).

2.1.1 The basic reproduction number

The basic reproduction number $\mathcal{R}_0$, also called the basic reproduction rate or rate, is an epidemiological metric used to describe the contagiousness or transmissibility of infectious agents. This number is the average number of susceptible an infected person will infect at the start of the epidemic.

This parameter describing the average number of new infections due to a sick individual, plays a crucial role. If this number is less than 1 then the epidemic will tend to die out. In this case, the disease-free equilibrium (DFE) will be locally asymptotically stable and the disease cannot persist in the population. On the other hand, it may persist or even spread to the whole population if $\mathcal{R}_0 > 1$. This implies that the disease-free equilibrium (DFE) is unstable. Using next generation matrix [23, 8] and [25], the basic reproduction of (1) can be found. Since the DFE is $E_0 = \left(\frac{\tau}{\mu}, 0, 0, 0\right)$, the basic reproduction number can be computed using the analytical approach.

We now consider the following model equivalent to the model (1)

$$\begin{cases} \dfrac{dK_t}{dt} = \tau - \mu K_t - \nu I_t - \nu H_t \\ \dfrac{dI_t}{dt} = \beta(t)(K_t - I_t - H_t - R_t)I_t - (\nu + \mu + \gamma + \alpha)I_t \\ \dfrac{dH_t}{dt} = \alpha I_t - (\nu + \mu + \lambda)H_t \\ \dfrac{dR_t}{dt} = \gamma I_t + \lambda H_t - \mu R_t. \end{cases} \tag{7}$$

to calculated the reproduction number and study the stability.

Let $\mathcal{F} = \begin{pmatrix} \tau \\ \beta(t)K_tI_t \end{pmatrix}$ represents the rate of new infection matrix and

$$\mathcal{V} = \begin{pmatrix} \mu\, K_t + \nu(I_t + H_t) \\ \beta(t)(I_t + H_t + R_t)I + (\nu + \gamma + \alpha)I_t \end{pmatrix}$$

denotes the transfert rate matrix of the individuals. Let us define $F = \frac{\partial \mathcal{F}}{\partial x_j}(E_0)$ and $V = \frac{\partial \mathcal{V}}{\partial x_j}(E_0)$ with $x_1 = K_t$ and $x_2 = I_t$. The reproduction number for the COVID-19 model given by (7) can be calculated from the relation $\mathcal{R}_0(t) = \rho(FV^{-1})$ with ρ the spectral radius of FV^{-1}, and is found to be

$$\mathcal{R}_0(t) = \frac{\beta(t)\tau}{\mu(\nu + \mu + \gamma + \alpha)} \tag{8}$$

To find the endemic equilibrium state of the model, we set

$$\frac{dK_t}{dt} = 0, \quad \frac{dI_t}{dt} = 0, \quad \frac{dH_t}{dt} = 0, \quad \frac{dR_t}{dt} = 0$$

Solving the above system, we get the endemic equilibrium state $E^* = (K^*, I^*, H^*, R^*)$, where

$$K^* = \frac{\tau}{\mu} - \frac{\nu(\mu+\nu+\gamma+\alpha)}{\mu(\mu+\nu+\lambda)}I^*, \quad H^* = \frac{\alpha}{\mu+\nu+\lambda}I^*, \qquad R^* = \frac{\gamma(\mu+\nu+\lambda)+\lambda\alpha}{\mu(\nu+\mu+\lambda)}I^*$$

$$I^* = \frac{\mu+\nu+\gamma+\alpha}{\beta(t)C_{\alpha,\gamma,\lambda,\mu,\nu}}\Big(\mathcal{R}_0(t)-1\Big),$$

$$C_{\alpha,\gamma,\lambda,\mu,\nu} = 1 + \frac{\nu(\nu+\mu+\lambda+\alpha)}{\mu(\mu+\nu+\lambda)} + \frac{\alpha}{\nu+\mu+\lambda} + \frac{\gamma(\mu+\nu+\lambda)+\lambda\alpha}{\mu(\mu+\nu+\lambda)}.$$

2.1.2 Stability of the disease-free equilibrium state and the endemic equilibrium state

In this section, we shall establish the stability of the equilibrium states. First, we give the Jacobian matrix for any point of the model (7)

$$J(E) = \begin{pmatrix} -\mu & -\nu & -\nu & 0 \\ \beta(t)I_t & \beta(t)(K_t - I_t - H_t - R_t) - \beta(t)I_t - (\nu+\mu+\gamma+\alpha) & -\beta(t)I_t & -\beta(t)I_t \\ 0 & \alpha & -(\nu+\mu+\lambda) & 0 \\ 0 & \gamma & \lambda & -\mu \end{pmatrix}.$$

Theorem 2.1. *The DFE will be locally asymptotically stable if* $\mathcal{R}_0(t) < 1$.

Proof. The Jacobian matrix corresponding to the system (7) at DFE point $E_0 = \left(\frac{\tau}{\mu}, 0, 0, 0\right)$

$$J(E_0) = \begin{pmatrix} -\mu & -\nu & -\nu & 0 \\ 0 & \frac{\beta(t)\tau}{\mu} - (\nu+\mu+\gamma+\alpha) & 0 & 0 \\ 0 & \alpha & -(\nu+\mu+\lambda) & 0 \\ 0 & \gamma & \lambda & -\mu \end{pmatrix}.$$

The characteristic polynomial of the matrix $J(E_0)$ is

$$P(X) = (-\mu - X)^2\Big(-(\nu+\mu+\lambda) - X\Big)\Big((\nu+\mu+\gamma+\alpha)(\mathcal{R}_0(t)-1) - X\Big)$$

The roots of the characteristic polynomial are $X_1 = -\mu < 0$, $X_2 = -(\mu+\nu+\lambda) < 0$ and $X_3 = (\nu+\gamma+\alpha)(\mathcal{R}_0(t)-1) < 0$ because $\mathcal{R}_0(t) < 1$. This concludes the disease free equilibrium is locally asymptotically stable if $\mathcal{R}_0(t) < 1$. □

Theorem 2.2. *The disease free equilibrium is globally asymptotically stable if* $\mathcal{R}_0(t) \leqslant 1$.

Proof. For $\mathcal{R}_0 < 1$ see the article [27].

If we consider $\mathcal{R}_0(t) = 1$, we have $\frac{\beta(t)\tau}{\mu} = (\nu + \mu + \gamma + \alpha)$ and

$$\frac{dV_t}{dt} = \frac{dI_t}{dt} = \big(\beta(t)(K_t - I_t - H_t - R_t) - (\nu + \mu + \gamma + \alpha)\big)I_t = \big(\beta(t)S_t - (\nu + \mu + \gamma + \alpha)\big)I_t$$

For $\frac{dV_t}{dt} = 0$, we have $S_t = \frac{\tau}{\mu}$. By Lasalle invariance principle [18], the DFE point is globally asymptotically stable. □

Theorem 2.3. *The endemic equilibrium state $E^* = (K^*, I^*, H^*, R^*)$ is locally asymptotically stable if $\mathcal{R}_0(t) > 1$.*

Proof. For the prove, see the article [27]. □

2.2 Parameter estimation and simulations

This epidemic is deadlier than the 21st century seasonal flu epidemics in France. You have to go back to 1957-1958 and 1968-1969 to find epidemics of seasonal that claimed more lives in France than Covid-19. As of 19 November 2020, the reports related to Covid-19 in medico-social establishments published daily by the National Public Health Agency show a total of 47,127 deaths and 2,086,288 confirmed positive cases by polymerase chain reaction (PCR).

The total population in France is $N = 64821957$. According to the report cases about COVID-19 infection at october 1, we can see that:

$S_0 = 64806884/64821957$, $I_0 = 13970/64821957$, $H_0 = 626/64821957$, $R_0 = 474/64821957$.

We have estimated the important model parameters using the France infection cases from 01th october to 15th november, 2020 which are given in Table 2.

Parameter	**value:01 October-15 November**
τ	714000/64821954=0.011
μ	599000/64821954=0.00924
ν	0,0002
γ	0.423254705
λ	0.05732
α	0.0336454691

Table 2: Parameters estimations.

The $\mathcal{R}_0$ (initial virus reproduction rate) is calculated from a population that is fully susceptible to be infected (that has not yet been vaccinated or immunized against an infectious agent). It corresponds to the product of three factors: $\mathcal{R}_0 = A \times B \times C$, the risk of contracting the virus during contact, the number of contacts in a unit of time and the number of days an infected person is contagious (up to 14 days for coronavirus).

In the case of the coronavirus, which is a very contagious virus, this $\mathcal{R}_0$ was before confinement at 3 or more. For the same virus, the $\mathcal{R}_0$ can vary from one population to another depending on population density, susceptibility and other factors. With $\mathcal{R}_0$ given, we have

$$\beta_0 = \frac{\mu(\mu+\nu+\gamma+\alpha)}{\tau}\mathcal{R}_0.$$

In Figure (1) and (2), we consider the transmission rate with sinusoidale function (6) for two different basic reproduction rate $\mathcal{R}_0 = 3$ in Figure (1) and $\mathcal{R}_0 = 0.95$ in Figure (2) .

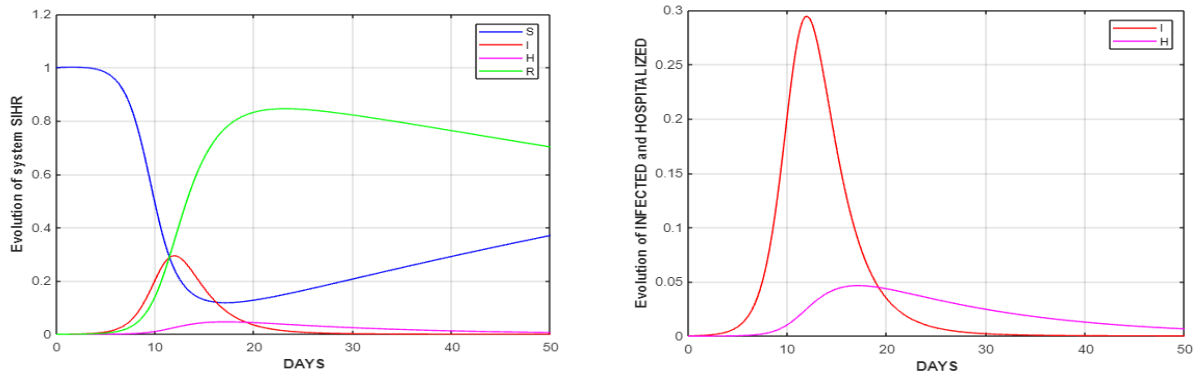


Figure 1: Evolution of the system with sinusoidal transmission rate: $\mathcal{R}_0 = 3,\ \epsilon = 0.1,\ T = 50$(days), $\phi_0 = \pi/3$.

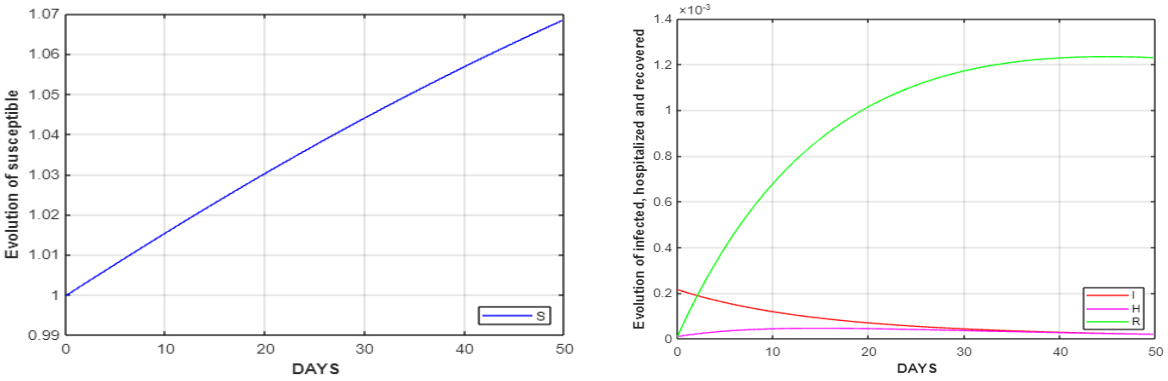


Figure 2: Evolution of the system with sinusoidal transmission rate: $\mathcal{R}_0 = 0.95,\ \epsilon = 0.1,\ T = 50$(days), $\phi_0 = \pi/3$.

In Figure (3) and (4), we consider the transmission rate with logistic function (5) for two different basic reproduction rate $\mathcal{R}_0 = 3$ in Figure (3) and $\mathcal{R}_0 = 0.95$ in Figure (4) .
In Figure (1) and (3), we see exactly the same evolution of the system with two different transmission rates. We notice that there will be one wave after 15 days.
We observe in Figure (1) and (3), that the susceptible population is decreasing with time and more people are getting exposed. Since infected population is increasing, there is a large number of

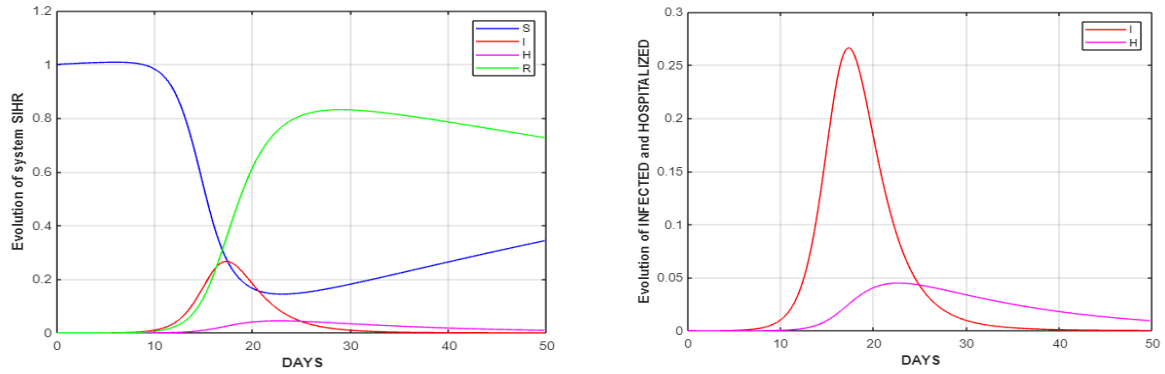


Figure 3: Evolution of the system with logistic transmission rate: $\mathcal{R}_0 = 3,\ k = 1.5,\ t_0 = 10.$

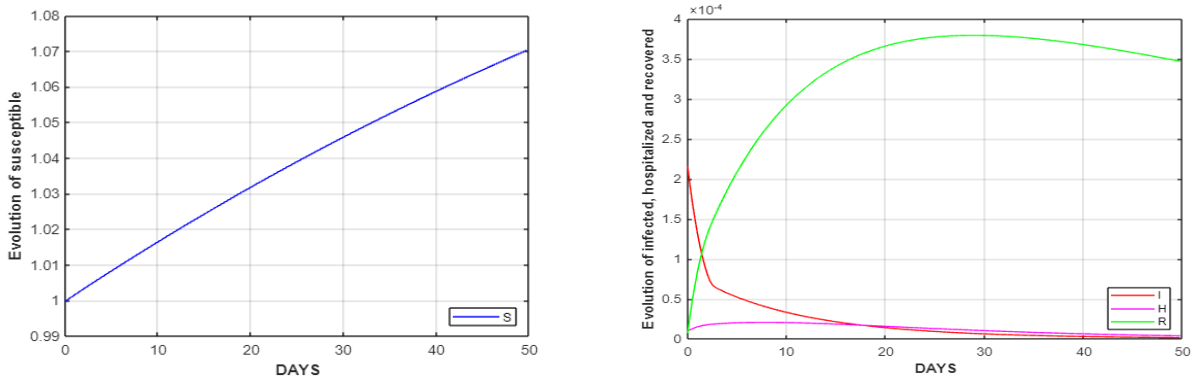


Figure 4: Evolution of the system with logistic transmission rate: $\mathcal{R}_0 = 0.95,\ k = 1.5,\ t_0 = 10.$

infected individuals over time which can lead to an outbreak in a very short time. If we analyze the rest of the population dynamics, we see that the infected population grew faster in the first 20 days, which shows the spread of the epidemic over time.

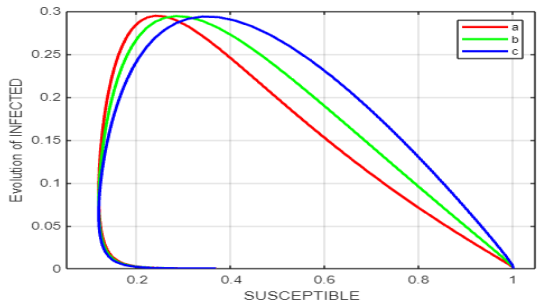
In Figure (2) and (4), for $\mathcal{R}_0 = 0.95 < 1$, we note that the population increases linearly and that the number of infected and hospitalized decreases until approaching to zero. We see exactly the same evolution of the system with two different transmission rates.

We note that with a reproduction rate $\mathcal{R}_0$ equal to 0.95, the epidemic will not last over time.

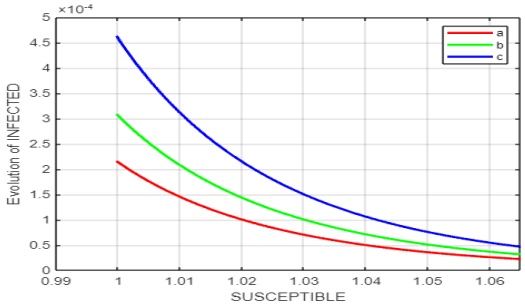
To compare the two evolutions with the two different rates, for an $\mathcal{R}_0 = 3$ we have a peak of infectivity in the population after 15 days, see figures (1) and (3). In contrary, there is not a peak of infectivity in the population for an $\mathcal{R}_0 = 0.95$ and an epidemic that will not last over time, see figures (2) and (4).

We have theoretically shown that if $\mathcal{R}_0 > 1$ then the epidemic will spread over time and with $\mathcal{R}_0 < 1$ there will be no propagation and the epidemic will quickly disappear.

Now we are going to show, with the two different basic reproduction rates and with different initial conditions, that there is an existence of a periodic global attraction orbit.
We observe in the figures (5) and (6), with two different $\mathcal{R}_0$ (3 or 0.95), that for different initial conditions the corresponding orbits converge towards a limit cycle.



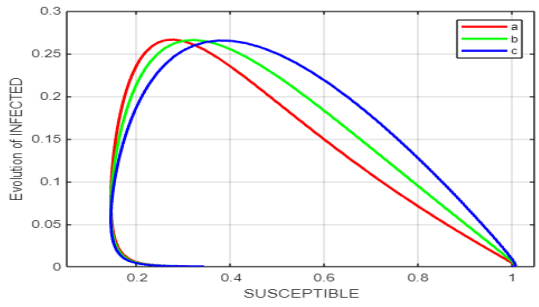
$\mathcal{R}_0 = 3$



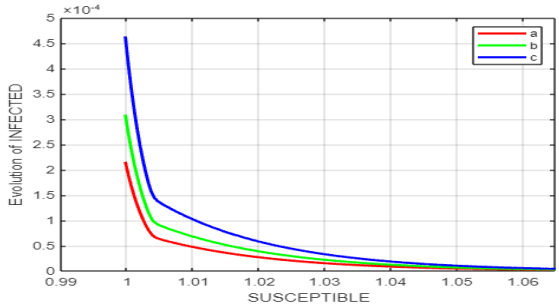
$\mathcal{R}_0^f = 0.95$

Figure 5: T=50 days. Periodic infection parameter $\beta(t)$ as in (6). $N = 64821957$, $S0 = 64806884/N = 0.9997$; $\epsilon = 0.1$; and $\phi_0 = \pi/3$. Orbits with initial conditions : $I0 = 13970/N = 0.00021; H0 = 626/N = 9.6 * 10^{-6}; R0 = 474/N = 7.3 * 10^{-6}$; -(a) , $I0 = 20000/N = 0.0003$; $H0 = 2000/N = 3*10^{-5}$; $R0 = 1000/N = 1.5*10^{-5}$; -(b), $I0 = 30000/N = 0.00046; H0 = 4000/N = 6*10^{-5}; R0 = 3000/N = 4.5 * 10^{-5}$; -(c).

In the figures (5) and (6), for an $\mathcal{R}_0 < 1$, we notice that the total number of the population begins with 1 and increases over time while the number of infected decreases until it approaches to zero. For an $\mathcal{R}_0 > 1$, we notice that the total number of the population starts with 1 and decreases over time while the number of infected increases drastically and decreases rapidly until it approaches to zero.



$\mathcal{R}_0 = 3$



$\mathcal{R}_0^f = 0.95$

Figure 6: Logistic infection parameter $\beta(t)$ as in (5). $N = 64821957$, $S0 = 64806884/N = 0.9997$; $k = 1.5$; and $t_0 = 10$. Orbits with initial conditions : $I0 = 13970/N = 0.00021; H0 = 626/N = 9.6 * 10^{-6}; R0 = 474/N = 7.3 * 10^{-6}$; -(a) , $I0 = 20000/N = 0.0003$; $H0 = 2000/N = 3 * 10^{-5}$; $R0 = 1000/N = 1.5 * 10^{-5}$; -(b), $I0 = 30000/N = 0.00046$; $H0 = 4000/N = 6 * 10^{-5}$; $R0 = 3000/N = 4.5 * 10^{-5}$; -(c).

We conclude that with two different transmission rates (sinusoidale or logistic), with different initial starting conditions, with two different values of $\mathcal{R}_0$, the evolution of the disease will stabilize and converge towards the same points of equilibrium.

3 Sensitivity analysis

During the development, construction or use of a mathematical model, sensitivity analysis can prove to be a valuable tool. It is possible to group the sensitivity analysis methods into three classes: screening methods, which consist of a qualitative analysis sensitivity of the output variable to the input variables, local analysis methods [33] , which quantitatively assess the impact of a small variation around a given value of the inputs and finally the global sensitivity analysis methods which are interested in the variability of the output of the model in the whole of its range of variation. Global sensitivity analysis studies how the variability of inputs affects that of output, by determining how much of the variance of output is due to a given input or set of inputs. While the local sensitivity analysis is more concerned with the value of the response variable, the global sensitivity analysis is concerned with its variability. The author interested in a reference work can refer to [4, 15, 7]. In this work, we use the global sensitivity namely the Sobol indices.

Sobol's sensitivity indices are used when considering the following model:

$$Y = g(X) = g(X_1, ..., X_p) \tag{9}$$

with $X \in I_X \subset \mathbb{R}^p$ the input random vector and $Y \in I_Y \subset \mathbb{R}$ the output variable. The input variables X_i, with $i = 1, ..., p$, are independent and the function g of this model may not be known explicitly.

For such a model, there are 2^{p-1} indices, one for each variable and each interaction of variables. These indices belong to the category of global indices, since they represent the proportion of the variance explained by the input variable concerned. They allow us to measure the influence of the uncertainties of the input variables on the output variable Y. More specifically, they can give us information about: the input variables that generate the most variability on the output variable Y, variables which have little influence on the Y output, the interactions between input variables which have an impact on the output Y.

For the calculation of the Sobol indices, we use the following conditional formulas for the continuous case:

$$\mathbb{E}(Y|X = x) = \int_{I_Y} y f_{Y|X=x}(y)dy = \int_{I_Y} y \frac{f_{X,Y}(x, y)}{f_X(x)} dy \tag{10}$$

$$\mathbb{V}(Y|X = x) = \mathbb{E}(Y^2|X = x) - \big(\mathbb{E}(Y|X = x)\big)^2 \tag{11}$$

and we need the following theorem.

Theorem 3.1. *Let X and Y be two random variables, continuous or discrete, such that $\mathbb{E}(|Y|) < \infty$, then the expectation of Y satisfies:*

$$\mathbb{E}(Y) = \mathbb{E}\big(\mathbb{E}(Y|X)\big).$$

For all random variables X and Y, the variance of Y satifies:

$$\mathbb{V}(Y) = \mathbb{E}\big(\mathbb{V}(Y|X)\big) + \mathbb{V}\big(\mathbb{E}(Y|X)\big). \tag{12}$$

In formula (12), we notice that the variance $\mathbb{V}(\mathbb{E}(Y|X))$ increases with the influence of X. Indeed, if X has a strong influence on Y, it also has one on its variance . By fixing X, we then have the variance of $Y|X$ which is on average smaller than Y. And the more this influence increases, the more the expectation $\mathbb{E}(\mathbb{V}(Y|X))$ decreases. According to formula (12), we thus have the variance $\mathbb{V}(\mathbb{E}(Y|X))$ which increases with the decrease in $\mathbb{E}(\mathbb{V}(Y|X))$. It is precisely this variance which is used in the calculation of the so-called first order Sobol indices defined in the following definition:

Definition 3.1. *We denote by S_i the part of the variance of Y to the variable X_i:*

$$S_i = \frac{\mathbb{V}\big(\mathbb{E}(Y|X_i)\big)}{\mathbb{V}(Y)} \qquad i = 1, ..., p. \tag{13}$$

This first-order indice therefore gives us an evaluation of the influence of an input variable X_i on the output variable Y. The S_i indices, between 0 and 1, is high when the influence of X_i is large.

Then, higher order Sobol indices, assess the importance of the combined effects of several input variables on the output variable. We find their expression from a decomposition of the variance, in the same way as for the first order indices. This variance decomposition is based on the Hoeffding-Sobol decomposition of a function [14]. This is generally presented with input variables of law $\mathcal{U}([0, 1])$, but it is also true in the general case [6, 5].

Definition 3.2. ***Hoeffding-Sobol decomposition***
Let $g : I = I_1 \times ... \times I_p \subset \mathbb{R}^p \to \mathbb{R}$ a squared integrable function with respect to the joint density function $f : I \to \mathbb{R}^+$ of p variables $(X_1, ..., X_p)$. It is assumed that these variables are independent. Then we have g in (9) *which admits a unique decomposition of the form:*

$$Y = g(x_1, ..., x_p) = h_0 + \sum_{i=1}^{p} h_i(x_i) + \sum_{1 \leqslant i \leqslant j \leqslant p} h_{i,j}(x_i, x_j) + ... + h_{1,...,p}(x_1, ..., x_p) \tag{14}$$

where

$$\begin{cases} h_0 = \mathbb{E}(Y) \\ h_i(x_i) = \mathbb{E}(Y|X_i = x_i) - h_0 \\ h_{i,j}(x_i, x_j) = \mathbb{E}(Y|X_i = x_i, X_j = x_j) - h_i(x_i) - h_j(x_j) - h_0 \\ .... \end{cases} \tag{15}$$

for all $k = 1, ..., s$

$$\int_{I_X} h_{i_1,...,i_s}(x_{i_1}, ..., x_{i_s}) f_{X_{i_k}}(x_{i_k}) dx_{i_k} = 0, \tag{16}$$

$f_{X_{i_k}}$ *is density of* X_{i_k}.

We deduce from the conditions (16) and the independence of the variables $\big(X_i\big)_{1 \leqslant i \leqslant p}$ that the functions h_U (with $U \subseteq \{1, ..., p\}$) are orthogonal:

$$\int_{I_X} h_U(x_u) h_V(x_V) f(x) dx = 0,$$

if $U \neq V$, with $U, V \subseteq \{1, ..., p\}$.

To obtain sensitivity indices for several variables of order greater than 1, we use the decomposition of the variance:

$$\mathbb{V}(Y) = \Big(\sum_{i=1}^{n} V_i \Big) + \sum_{1 \leqslant i < j \leqslant n} V_{i,j} + ... + V_{1,...,n}, \tag{17}$$

where

$$\begin{cases} V_i = \mathbb{V}\big(\mathbb{E}(Y|X_i)\big) \\ V_{i,j} = \mathbb{V}\big(\mathbb{E}(Y|X_i, X_j)\big) - V_i - V_j \\ ..... \\ V_{1,...,n} = \mathbb{V}(Y) - \sum_{i=1}^{n} V_i - \sum_{1 \leqslant i < j \leqslant n} V_{i,j} - ... - \displaystyle\sum_{1 \leqslant i_1 < i_2 < ... < i_{n_1} \leqslant n} V_{i_1,...,i_{n-1}} \end{cases}$$

Definition 3.3. *By equations* (13)*,* (16) *and* (17) *we obtain the formulas of the various Sobol indices of orders greater than 1:*

$$\begin{cases} S_{i,j} = \dfrac{\mathbb{V}(h_{i,j}(X_i, X_j))}{\mathbb{V}(Y)} = \dfrac{\mathbb{V}\big(\mathbb{E}(Y|X_i, X_j)\big)}{\mathbb{V}(Y)} - S_i - S_j, \\ S_{i,j,k} = \dfrac{\mathbb{V}(h_{i,j,k}(X_i, X_j, X_k))}{\mathbb{V}(Y)} = \dfrac{\mathbb{V}\big(\mathbb{E}(Y|X_i, X_j, X_k)\big)}{\mathbb{V}(Y)} - S_i - S_j - S_k - S_{i,j} - S_{i,k} - S_{j,k}, \\ .... \\ S_{1,...,p} = \dfrac{\mathbb{V}(h_{i,j,k}(X_1, ..., X_p))}{\mathbb{V}(Y)} = \dfrac{\mathbb{V}\big(\mathbb{E}(Y|X_1, ..., X_p)\big)}{\mathbb{V}(Y)} - \displaystyle\sum_{U \subseteq \{1,...,p\}} S_U. \end{cases} \tag{18}$$

Corollary 3.1. *By decomposing the variance, equation* (17)*, we have the sum of the sensitivity indices equal to 1.*

These sensitivity indices can easily be interpreted: the more indice S_i (respectively $S_{i,j,k}$) will be close to 1, the greater the influence of variable X_i (respectively of variables X_i, X_j, X_k).

We can also study the total sensitivity indice for the variable X_i. This represents the sum of the sensitivity indices involving the variable X_i. We note $S_{T_i} = \sum_{j \in I_i} S_j$, where I_i represents all the sets of indices containing the indice i.

Sobol indices are easier to compute from a meta-model. This is why we introduce Lagrange's polynomial in the next section.

4 The Stochastic collocation method (SCM)

We denote by $(\Omega, \mathcal{F}, \mathbb{P})$ the probability space, where as usual Ω is the set of possible outcomes, $\mathcal{F}$ is a σ-algebra over Ω and $\mathbb{P}$ is a function $\mathcal{F} \to [0; 1]$ that gives a probability measure on $\mathcal{F}$. Consider an $\mathbb{R}$-valued random variable X of support $I \subset \mathbb{R}$ that describes input uncertainties. We assume that the probability law of X is known and that it is defined by a probability density function $p(x)$, $x \in I$.
We describe in this section, a method called the stochastic collocation method (SCM). This method makes it possible to represent a random variable Y of unknown distribution as a function of a random variable X of known distribution in the form.

$$Y = \sum_{i=1}^{n} y_i L_i(X) \tag{19}$$

where $\{L_i(x)\}_{i=1}^n$ are Lagrange polynomial basis and $\{y_i\}_{i=1}^n$ the real numbers. If a relation $Y = f(X)$ exists then $y_i = f(x_i)$ and relation (19) becomes the projection of f on the basis of Lagrange polynomials.

Before describing the method, we will recall two concepts that are important. These two notions are the interpolation polynomials and the quadrature rule.

Let $\mathcal{P}_n$ denote the linear space of polynomials of degree less or equal to n. Let $(n+1)$ distinct points $x_0, x_1, ..., x_n$ and corresponding values $y_0, y_1, ..., y_n$, then there exists a unique polynomial $P \in \mathcal{P}_n$ such that $P(x_i) = y_i$. Lagrange establishes a representation of such polynomials under the form

$$P(x) = \sum_{i=0}^{n} y_i L_i(x),$$

where

$$L_i(x) = \prod_{j=0 | j \neq i}^{n} \frac{x - x_j}{x_i - x_j}, \qquad i = 0, ..., n. \tag{20}$$

This equality shows that the quality of the approximation depends only on the choice of the points x_i, for more details see [11].

Designing a set of points is not easy and for numerical simulations, it is convenient to choose points x_i which correspond to quadrature rules because they present good approximation properties. We recall here that a quadrature rule is written as a weighted sum as follows

$$\int_I f(x)p(x)dx \approx \sum_{i=0}^{n} \omega_i f(x_i) \tag{21}$$

where ω_i are the quadrature weight and x_i are the quadrature points and $p(x)$ is a positive function that corresponds to a probability density of support I. We choose x_i and ω_i so that the relation (21) is exact when f is a polynomial of degree less than or equal to $2n-1$, for more details see [13, 24, 29].
For more information on the stochastic collocation method see [10] which applies the method with random variables and stochastic processes.
The method is illustrated on the following example:

Example 4.1. *We consider a function with four variables which depends on time:*

$$S_t(\beta,\alpha,\gamma,\lambda) = \sum_{i,j,k,l=0}^{N} S_{i,j,k,l}(t)L_i(\beta)L_j(\alpha)L_k(\gamma)L_l(\lambda) \tag{22}$$

assume (22) *satisfies the differential equation* (1)*, we obtain :*

$$\begin{aligned}\sum_{i,j,k,l=0}^{N} \frac{dS_{i,j,k,l}(t)}{dt}L_i(\beta)L_j(\alpha)L_k(\gamma)L_l(\lambda) =& \tau - \beta\Big(\sum_{i,j,k,l=0}^{N} S_{i,j,k,l}(t)L_i(\beta)L_j(\alpha)L_k(\gamma)L_l(\lambda)\Big)\times \\ & \Big(\sum_{i^*,j^*,k^*,l^*=0}^{N} I_{i^*,j^*,k^*,l^*}(t)L_i^*(\beta)L_j^*(\alpha)L_k^*(\gamma)L_l^*(\lambda)\Big) \\ & - \mu_1 \sum_{i,j,k,l=0}^{N} S_{i,j,k,l}(t)L_i(\beta)L_j(\alpha)L_k(\gamma)L_l(\lambda)\end{aligned} \tag{23}$$

We set $\beta=\beta_m$, $\alpha=\alpha_n$, $\gamma=\gamma_p$ *and* $\lambda=\lambda_q$ *for these particular values we have*

$$L_i(\beta_m)=\delta_{im},\ L_j(\alpha_n)=\delta_{jn},\ L_k(\gamma_p)=\delta_{kp}\ ,\ L_l(\lambda_q)=\delta_{lq}$$

where δ_{im} *is the Kronecker symbol.*

The equation (23) *becomes*

$$\frac{dS_{mnpq}}{dt} = \tau - \beta_m S_{mnpq} I_{mnpq} - \mu_1 S_{mnpq} \tag{24}$$

Similar equations can be written for the other components of the differential equation.

Furthermore, it can be shown that

$$\begin{aligned}\mathbb{E}[S(t,\beta,\alpha,\gamma,\lambda)] &= \sum_{i=0}^{N}\sum_{j=0}^{N} S_{ijkl}(t)\int\int L_i(\beta)L_j(\alpha)L_k(\gamma)L_l(\lambda)p(\beta)p(\alpha)p(\gamma)p(\lambda)d\beta d\alpha d\gamma d\lambda \\ &= \sum_{i,j,k,l=0}^{N} \omega_i\omega_j\omega_k\omega_l S_{ijkl}(t) \end{aligned} \tag{25}$$

and

$$\big(S(t,\beta,\alpha,\gamma,\lambda)\big)^2 = \sum_{i,j,k,l,i^*,j^*,k^*,l^*=0}^{N} S_{ijkl}(t)S_{i^*j^*k^*l^*}(t)L_i(\beta)L_j(\alpha)L_k(\gamma)L_l(\lambda)L_{i^*}(\beta)L_{j^*}(\alpha)L_{k^*}(\gamma)L_{l^*}(\lambda)$$

becomes after simplification:

$$\mathbb{E}\big[\big(S(t,\beta,\alpha,\gamma,\lambda)\big)^2\big] = \sum_{i,j,k,l=0}^{N} \omega_i\omega_j\omega_k\omega_l\bigg(S_{ijkl}(t)\bigg)^2 \tag{26}$$

Then, the variance writes :

$$\mathbb{V}(S) = \mathbb{V}\big[(S(t,\beta,\alpha,\gamma,\lambda)\big] = \mathbb{E}\big[\big(S(t,\beta,\alpha,\gamma,\lambda)\big)^2\big] - \big[\mathbb{E}[S(t,\beta,\alpha,\gamma,\lambda)]\big]^2 \tag{27}$$

Now we will calculate the Sobol indices for S. We recall the Sobol indice S_β:

$$S_\beta = \frac{\mathbb{V}\big(\mathbb{E}(S|\beta)\big)}{\mathbb{V}(S)} = \frac{\mathbb{E}\big[\big(\mathbb{E}(S|\beta)\big)^2\big] - \big[\mathbb{E}\big(\mathbb{E}(S|\beta)\big)\big]^2}{\mathbb{V}(S)}$$

We have

$$\begin{aligned}\mathbb{E}(S|\beta) &= \sum_{i,j,k,l} S_{ijkl}(t)L_i(\beta)\int\int\int L_j(\alpha)L_k(\gamma)L_l(\lambda)p(\alpha)p(\gamma)p(\lambda)d\alpha d\gamma d\lambda \\ &= \sum_{i,j,k,l} S_{ijkl}(t)L_i(\beta)\omega_j\omega_k\omega_l\end{aligned}$$

and its square

$$\big(\mathbb{E}(S|\beta)\big)^2 = \sum_{i,j,k,l,i^*,j^*,k^*,l^*} S_{ijkl}(t)S_{i^*j^*k^*l^*}(t)\omega_j\omega_k\omega_l L_i(\beta)\omega_{j^*}\omega_{k^*}\omega_{l^*}L_{i^*}(\beta)$$

Then, the mean of the squared term is

$$\begin{aligned}\mathbb{E}\big[\big(\mathbb{E}(S|\beta)\big)^2\big] &= \sum_{i,j,k,l,i^*,j^*,k^*,l^*} S_{ijkl}(t)S_{i^*j^*k^*l^*}(t)\omega_j\omega_k\omega_l\omega_{j^*}\omega_{k^*}\omega_{l^*}\int L_i(\beta)L_{i^*}(\beta)p(\beta)d\beta \\ &= \sum_{i,j,k,l,j^*,k^*,l^*} S_{ijkl}(t)S_{ij^*k^*l^*}(t)\omega_i\omega_j\omega_k\omega_l\omega_{j^*}\omega_{k^*}\omega_{l^*}\end{aligned}$$

And the last term $\mathbb{E}\big(\mathbb{E}(S|\beta)\big) = \mathbb{E}(S)$ *is given by* (25).

For the other Sobol indices of order 1, we do exactly the same.
Now we are going to calculate the Sobol indice of order 2. As for the indices of order 1, we will just calculate a single indice of order 2 and the other indices are done in exactly the same way.
We recall the Sobol indice $S_{\beta,\alpha}$*:*

$$S_{\beta,\alpha} = \frac{\mathbb{V}\big(\mathbb{E}(S|\beta,\alpha)\big)}{\mathbb{V}(S)} = \frac{\mathbb{E}\big[\big(\mathbb{E}(S|\beta,\alpha)\big)^2\big] - \big[\mathbb{E}\big(\mathbb{E}(S|\beta,\alpha)\big)\big]^2}{\mathbb{V}(S)} - S_\beta - S_\alpha$$

With

$$\begin{aligned}\mathbb{E}(S|\beta,\alpha) &= \sum_{i,j,k,l} S_{ijkl}(t) L_i(\beta) L_j(\alpha) \int\int L_k(\gamma) L_l(\lambda) p(\gamma) p(\lambda) d\gamma d\lambda \\ &= \sum_{i,j,k,l} S_{ijkl}(t) L_i(\beta) L_j(\alpha) \omega_k \omega_l\end{aligned}$$

For the square, we have:

$$\big(\mathbb{E}(S|\beta,\alpha)\big)^2 = \sum_{i,j,k,l,i^*,j^*,k^*,l^*} S_{ijkl}(t) S_{i^*j^*k^*l^*}(t) L_i(\beta) L_j(\alpha) L_{i^*}(\beta) L_{j^*}(\alpha) \omega_k \omega_l \omega_{k^*} \omega_{l^*}$$

The expectation of the squared term:

$$\begin{aligned}\mathbb{E}\big[\big(\mathbb{E}(S|\beta,\alpha)\big)^2\big] &= \sum_{i,j,k,l,i^*,j^*,k^*,l^*} S_{ijkl}(t) S_{i^*j^*k^*l^*}(t) \omega_k \omega_l \omega_{k^*} \omega_{l^*} \int L_i(\beta) L_{i^*}(\beta) p(\beta) d\beta \int L_j(\alpha) L_{j^*}(\alpha) p(\alpha) d\alpha \\ &= \sum_{i,j,k,l,k^*,l^*} S_{ijkl}(t) S_{ijk^*l^*}(t) \omega_k \omega_l \omega_{k^*} \omega_{l^*} \omega_i \omega_j.\end{aligned}$$

And finally the last term: $\mathbb{E}\big(\mathbb{E}(S|\beta,\alpha)\big) = \mathbb{E}(S)$ *is given by* (25)

For the Sobol indices of order 1 and 2 for S*,* I*,* H *and* R *we do exactly the same.*

5 Numerical application

In this section, we simulate the compartments (S, I, H, R) of the model (1) with two different infection rate and two different reproduction number when the parameters follows the uniform law and we give the Sobol indices for the endemic stationary state defined by the following system:

$$\begin{cases} S^* = S^*(\mathcal{R}_0, \alpha, \gamma, \lambda) = \dfrac{\tau}{\mu \times \mathcal{R}_0} \\ I^* = I^*(\mathcal{R}_0, \alpha, \gamma, \lambda) = \dfrac{\tau}{\mu + \nu + \gamma + \alpha}\left(1 - \dfrac{1}{\mathcal{R}_0}\right) \\ H^* = H^*(\mathcal{R}_0, \alpha, \gamma, \lambda) = \dfrac{\alpha\tau}{(\mu + \nu + \lambda)(\mu + \nu + \gamma + \alpha)}\left(1 - \dfrac{1}{\mathcal{R}_0}\right) \\ R^* = R^*(\mathcal{R}_0, \alpha, \gamma, \lambda) = \dfrac{\gamma\tau(\mu + \nu + \lambda) + \lambda\alpha\tau}{\mu(\mu + \nu + \lambda)(\mu + \nu + \gamma + \alpha)}\left(1 - \dfrac{1}{\mathcal{R}_0}\right), \end{cases} \tag{28}$$

We set $\mu_2 = \mu + \nu$ and note $X = (X_1, X_2, X_3, X_4) = (\mathcal{R}_0, \alpha, \gamma, \lambda)$ the random input variables of the system (28).
For the rest, we consider that $(\mathcal{R}_0, \alpha, \gamma, \lambda)$ follow the uniform law $\mathcal{U}[a_{X_i}, b_{X_i}]$.

The calculation of $\mathcal{R}_0$ is important to evaluate the epidemiological situation in a department or a region. Estimates of the weekly reproduction number are based on the numbers of positive PCR tests for SARS-COV-2, on emergency room visits for suspected COVID-19 and from hospitalizations. The value of $\mathcal{R}_0$ was above 3 before the November 2020 lockdown . The goal of control efforts is to reduce $\mathcal{R}_0$ below the cutoff value of 1 and as close to 0 as possible, thereby bringing an outbreak under control. On November 19, three weeks after the start of lockdown in France, the $\mathcal{R}_0$ rate was less than 1 in metropolitan France in the three data sources: from the SI-DEP virological data $\mathcal{R}_0 = 0.65$, from the passages to emergencies $\mathcal{R}_0 = 0.87$ and from hospitalization data $\mathcal{R}_0 = 0.95$. All three reproduction rates are down from pre-containment estimates.

In this study, the sections (5.1) and (5.2), we consider the initial value of $\mathcal{R}_0 = 3$ before lockdown and the sections (5.3) and (5.4) consider the final value of $\mathcal{R}_0^f = 0.95$ after lockdown to see the impact of the input variables when they follow the uniform law on the evolution of the system over time.
In Table (3), we give the values of the different parameters with their confidence intervals. We have chosen 95% for $\mathcal{R}_0$ before the lockdown and 75% for $\mathcal{R}_0$ after the lockdown so that we have exactly the same values obtained theoretically. For $\mathcal{R}_0 = 3$, the smaller the confidence interval, the more we have a good approximation of reality and for $\mathcal{R}_0^f = 0.95$, the larger the interval, the better we approximate the theoretical calculations.

Parameter	Values	confidence interval $[a_{X_i}, b_{X_i}]$
$\mathcal{R}_0$	3	[2.85;3.15] at 95%
$\mathcal{R}_0^f$	0.95	[0.7;1.2] at 75%
α	0.0336454691	[0.030 ;0.037] at 90%
γ	0.423254705	[0.38;0.46] at 90%
λ	0.05732	[0.0515;0.0630] at 90%

Table 3: Parameters estimations.

In Table (4) we give the Sobol indices of the stationary endemic state with $\mathcal{R}_0 = 3$ that we calculated in an analytical way (see formulae in the Appendix).

Sobol Indices	$X^* = S^*$	$X^* = I^*$	$X^* = H^*$	$X^* = R^*$
$X^*_{\mathcal{R}_0}$	1	0.069946	0.025106	0.98471
X^*_{α}	0	0.005864	0.34415	0.0012
X^*_{γ}	0	0.92393	0.33147	0.010134
X^*_{λ}	0	0	0.29659	0.0013
$X^*_{\mathcal{R}_0,\alpha}$	0	$1.22 * 10^{-6}$	$7.22 * 10^{-5}$	0.012711
$X^*_{\mathcal{R}_0,\gamma}$	0	$1.933 * 10^{-5}$	$6.936 * 10^{-5}$	0.0027712
$X^*_{\mathcal{R}_0,\lambda}$	0	0	$6.2 * 10^{-5}$	$2.7205 * 10^{-5}$
$X^*_{\alpha,\gamma}$	0	$6.49 * 10^{-5}$	0.0008	0.005536
$X^*_{\alpha,\lambda}$	0	0	0.0008	0.012712
$X^*_{\gamma,\lambda}$	0	0	0.0008	0.0027721

Table 4: Sobol indices with $\mathcal{R}_0 = 3$ calculated analytically .

In section (5.1), we compute the Sobol indices with a sinusoidal transmission rate (6) and in the section (5.2) with a logistic transmission rate (5). And we compare with the indices which are obtained analytically in table (4).

5.1 Sobol indice with sinusoidal transmission rate before lockdown

In this subsection, first we give in the table (5) the steady state Sobol indices using the stochatic collocation method defined in section (4) and the influences of all variable on each compartment with $\mathcal{R}_0$ before the lockdown and sinusoidal transmission rate.

Sobol Indices	$X^* = S^*$	$X^* = I^*$	$X^* = H^*$	$X^* = R^*$
$X^*_{\mathcal{R}_0}$	0.9970	0.0700	0.0252	0.9882
X^*_{α}	0.0000	0.0059	0.3407	0.0012
X^*_{γ}	0.0012	0.9237	0.3347	0.0074
X^*_{λ}	0.0000	0.0000	0.2967	0.0014
$X^*_{\mathcal{R}_0,\alpha}$	0.0000	0.0000	0.0001	0.0000
$X^*_{\mathcal{R}_0,\gamma}$	0.0001	0.0002	0.0001	0.0001
$X^*_{\mathcal{R}_0,\lambda}$	0.0000	0.0000	0.0001	0.0000
$X^*_{\alpha,\gamma}$	0.0001	0.0001	0.0008	0.0001
$X^*_{\alpha,\lambda}$	0.0000	0.0000	0.0009	0.0000
$X^*_{\gamma,\lambda}$	0.0000	0.0000	0.0008	0.0000

Table 5: Sobol indices calculated numerically with $\mathcal{R}_0 = 3$ and sinusoidale transmission rate.

In Figures (7), (8), (9) and (10), we have the compartment simulations for each parameter. The Sobol indices S_α, S_λ, I_α, I_λ, R_α and R_λ that remain close to zero over time, the have not been plotted.

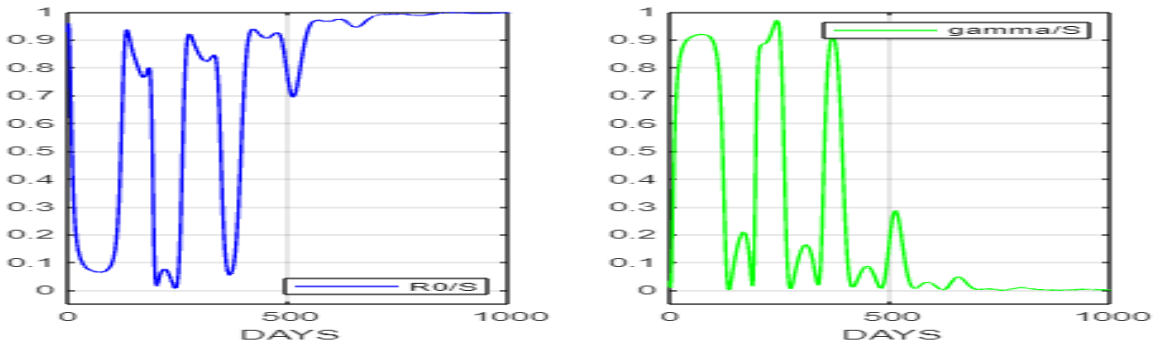


Figure 7: Evolution of compartment SUSCEPTIBLE to the variability of input variables.

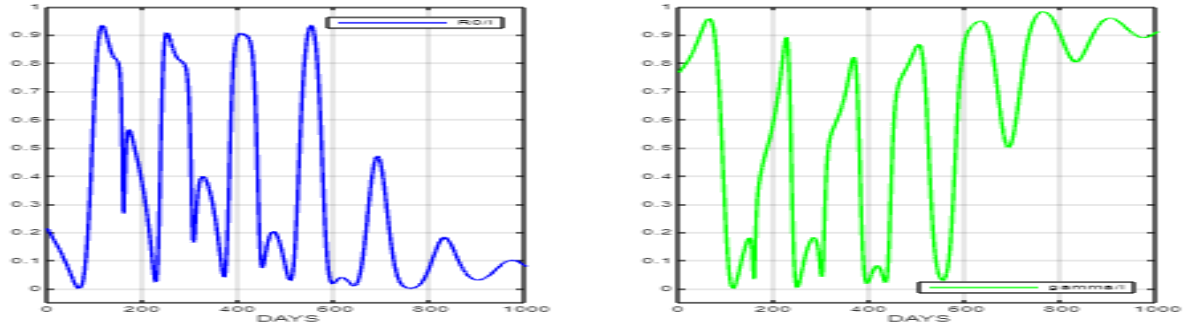


Figure 8: Evolution of compartment INFECTED to the variability of input variables

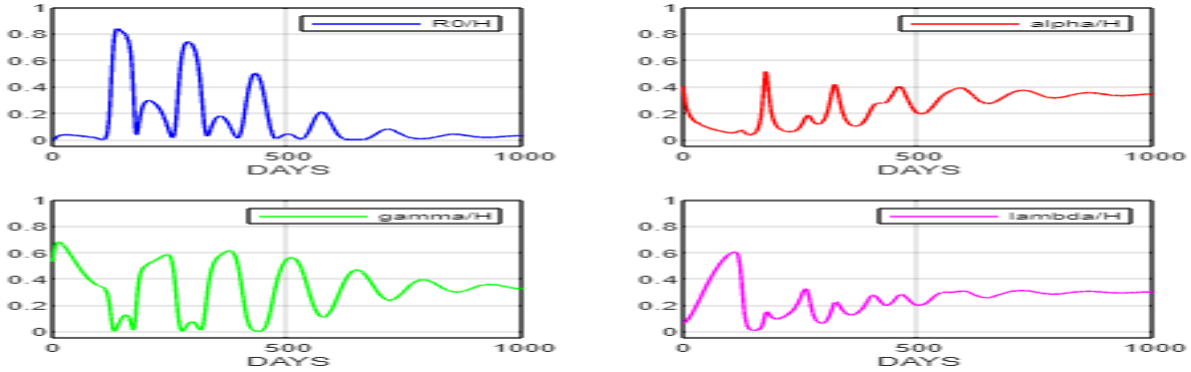


Figure 9: Evolution of compartment HOSPITALIZED to the variability of input variables.

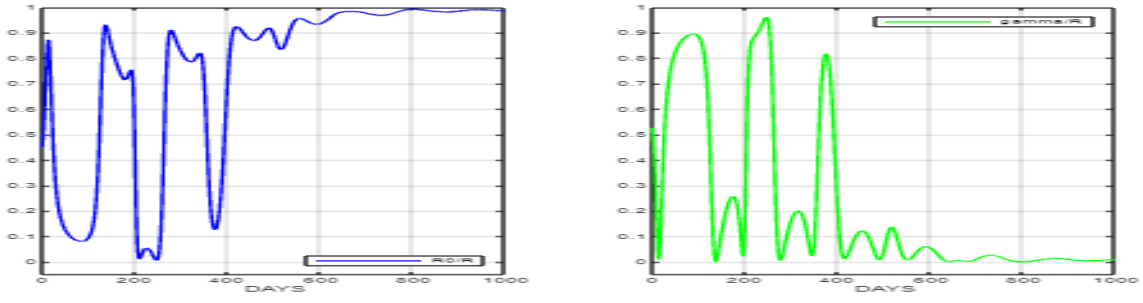


Figure 10: Evolution of compartment RECOVERED to the variability of input variables.

In Table (5), we notice the variable $\mathcal{R}_0$ strongly influences the compartments of susceptible S and recovered R at 99%, it influences the compartments of infected I at 7% and hospitalized H at 2%. We also find, in the Figure (9), that the variables α and λ only influence the compartment of hospitalized H people. The variable γ influences the compartments of infected I at more than 92% and hospitalized H people at more than 33%, see Table (5).

According to the Sobol indices, we can clearly see that the parameters that most influence the compartment of the infected are γ and $\mathcal{R}_0$ see Figure (8).

We see in the Figure (9), that all the input variables influence the compartment of hospitalized people with different intensity.

The Sobol indices of order 2 also inform us that the combined effects on the compartments remains weak. For example in the Table (5) the influence of α and γ on H is 0.08%. All the compartment simulations for the 2nd order indices are given in the Figures (11), (12), (13) and (14).

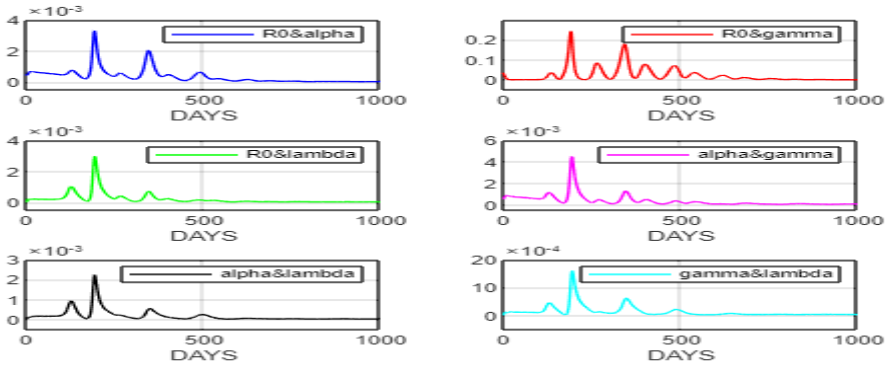


Figure 11: Evolution of the compartment SUSCEPTIBLE in relation to the variability of the mixed-variables.

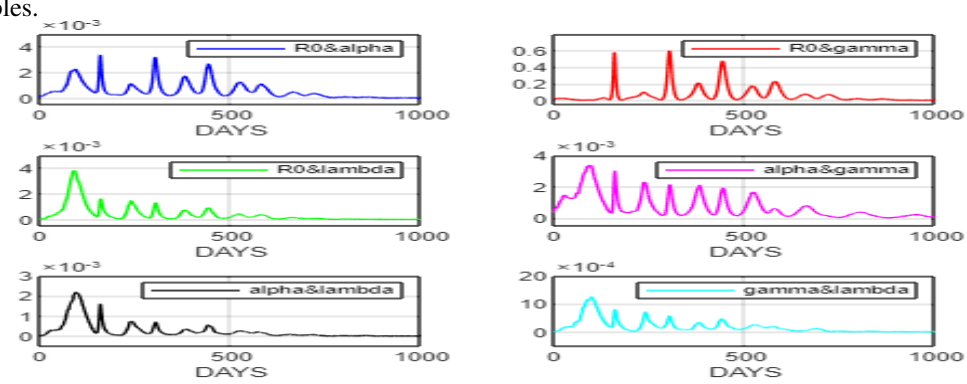


Figure 12: Evolution of the compartment INFECTED in relation to the variability of the mixed-variables.

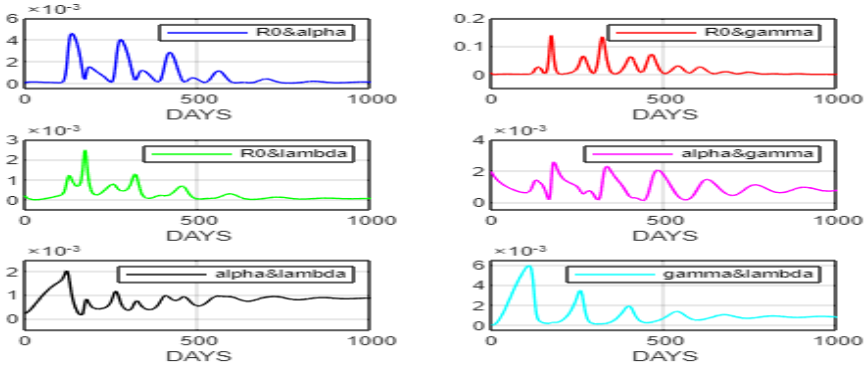


Figure 13: Evolution of the compartment HOSPITALIZED in relation to the variability of the mixed-variables.

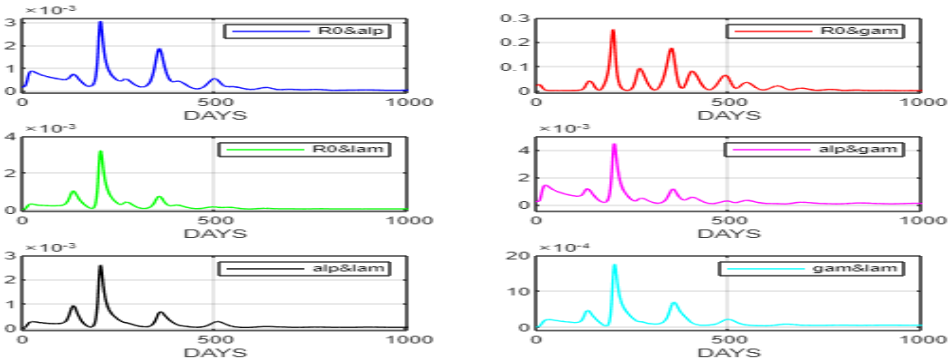


Figure 14: Evolution of the compartment RECOVERED in relation to the variability of the mixed-variables.

5.2 Sobol indice with logistic transmission rate before lockdown

In this subsection, first we give in the table (6) the steady state Sobol indices using the stochatic collocation method defined in section (4) and the influences of all variable on each compartment with $\mathcal{R}_0$ before the lockdown using a logistic transmission rate.

Sobol Indices	$X^* = S^*$	$X^* = I^*$	$X^* = H^*$	$X^* = R^*$
$X^*_{\mathcal{R}_0}$	0.9999	0.0710	0.0252	0.9846
X^*_{α}	0.0000	0.0058	0.3429	0.0011
X^*_{γ}	0.0000	0.9223	0.3306	0.0129
X^*_{λ}	0.0000	0.0000	0.2985	0.0013
$X^*_{\mathcal{R}_0,\alpha}$	0.0000	0.0000	0.0001	0.0000
$X^*_{\mathcal{R}_0,\gamma}$	0.0001	0.0008	0.0002	0.0001
$X^*_{\mathcal{R}_0,\lambda}$	0.0000	0.0000	0.0001	0.0000
$X^*_{\alpha,\gamma}$	0.0000	0.0001	0.0008	0.0000
$X^*_{\alpha,\lambda}$	0.0000	0.0000	0.0009	0.0000
$X^*_{\gamma,\lambda}$	0.0000	0.0000	0.0008	0.0000

Table 6: Sobol indices calculated numerically with $\mathcal{R}_0 = 3$ and logistic transmission rate.

Figures (15), (16), (17) and (18), show the compartment simulations for each parameter. The Sobol indices S_α, S_λ, I_α, I_λ, R_α and R_λ that remain close to zero over time, so they have not been represented.

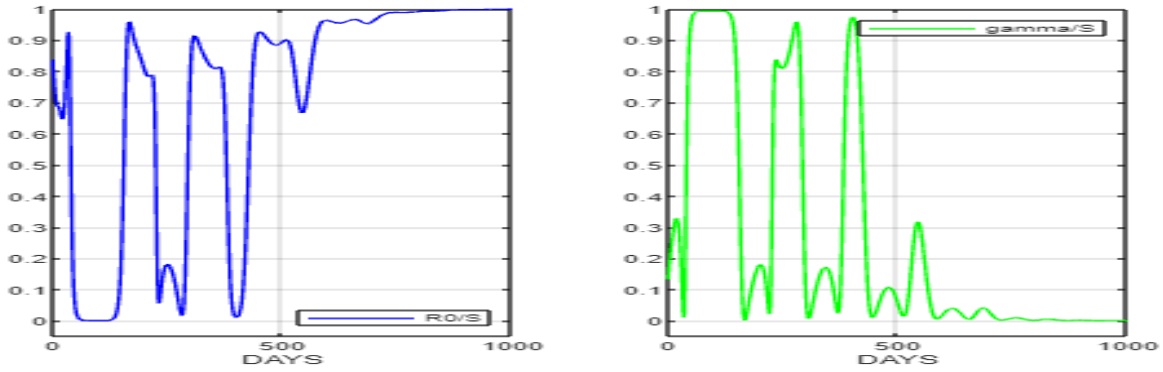


Figure 15: Evolution of compartment SUSCEPTIBLE to the variability of input variables.

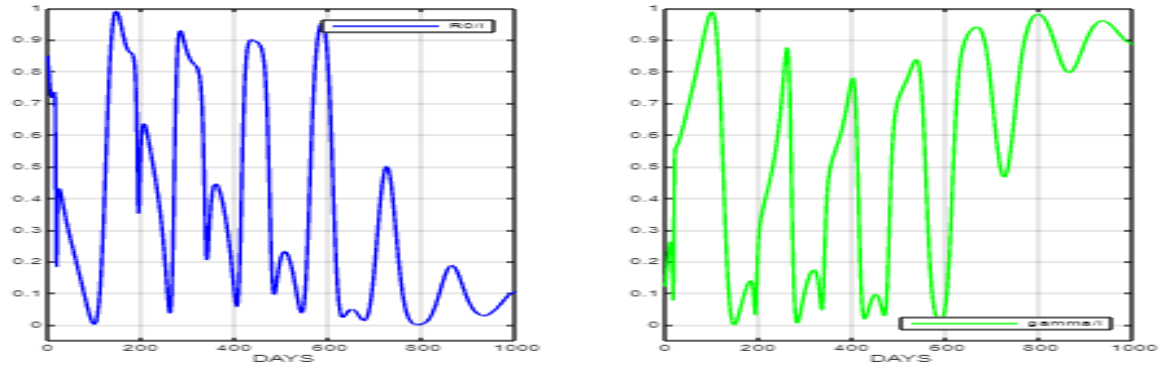


Figure 16: Evolution of compartment INFECTED to the variability of input variables.

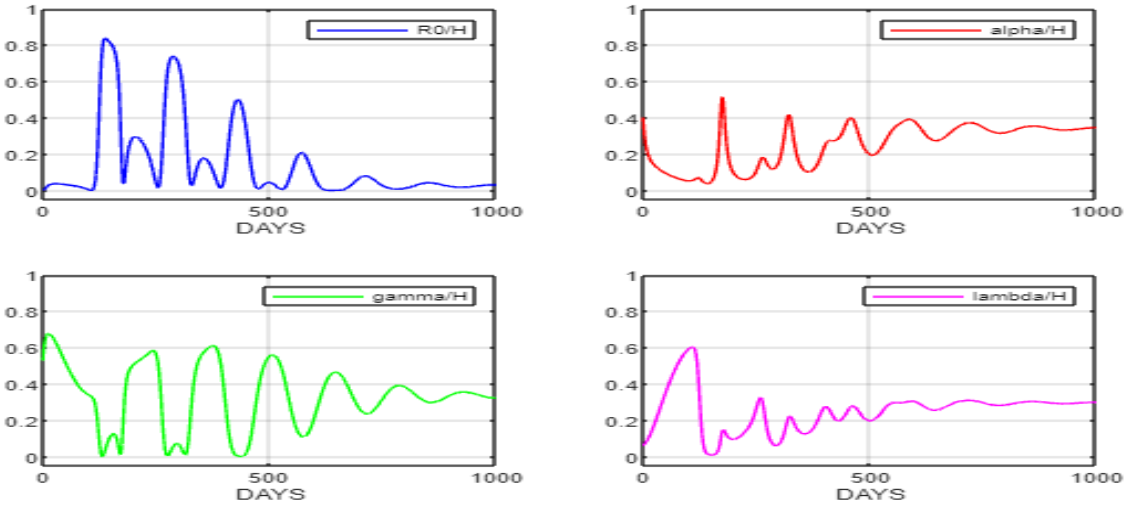


Figure 17: Evolution of compartment HOSPITALIZED to the variability of input variables.

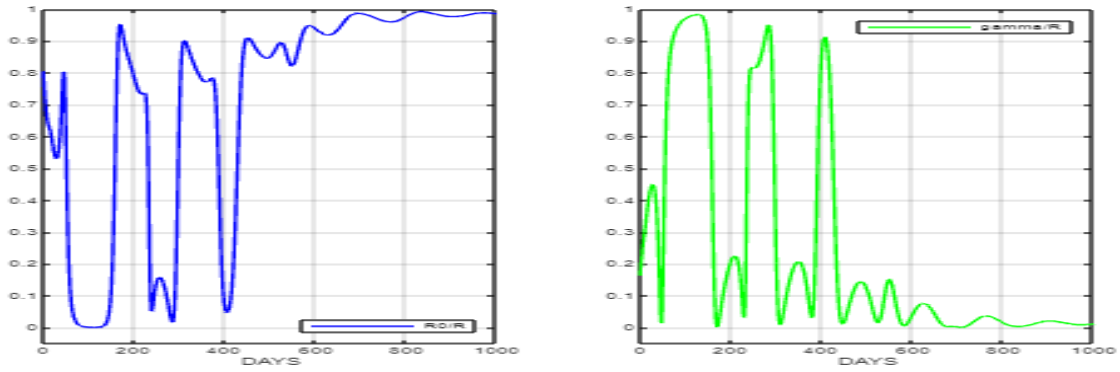


Figure 18: Evolution of compartment RECOVERED to the variability of input variables.

In Figures (15) and (18), we notice that the variable $\mathcal{R}_0$ strongly influences the compartments of susceptible S and recovered R at 99%, it influences the compartments of infected I at 7.1% and hospitalized H at 2.52%, see Table (6).
We also find that the variables α and λ only influence the compartment of hospitalized H people (see Figure (17)). The variable γ influences the compartments of infected I at more than 92% and hospitalized H people at more than 33%, see table (6).

According to the Sobol indices, we can clearly see that the parameters that most influence the compartment of the infected are: γ and $\mathcal{R}_0$. This influence is well represented in Figure (16), which clearly shows that the input variables which influences the compartment of infected persons are γ and $\mathcal{R}_0$.

We see in the Figure (17), that all the input variables influence the compartment of hospitalized people with different intensity.

The Sobol indices of order 2 also inform us that the combined effects on the compartments remains weak. For example the influence of α and γ on I is 0.08%. All the compartment simulations for the 2nd order indices are given in the Figures (19), (20), (21) and (22).

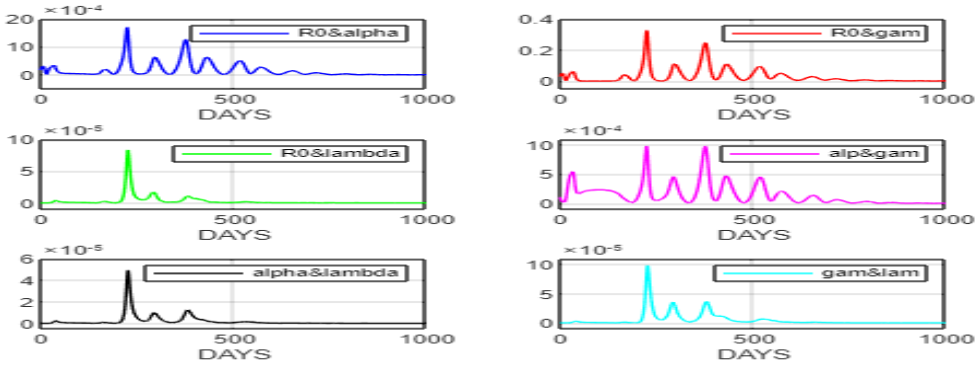


Figure 19: Evolution of the compartment SUSCEPTIBLE in relation to the variability of the mixed-variables.

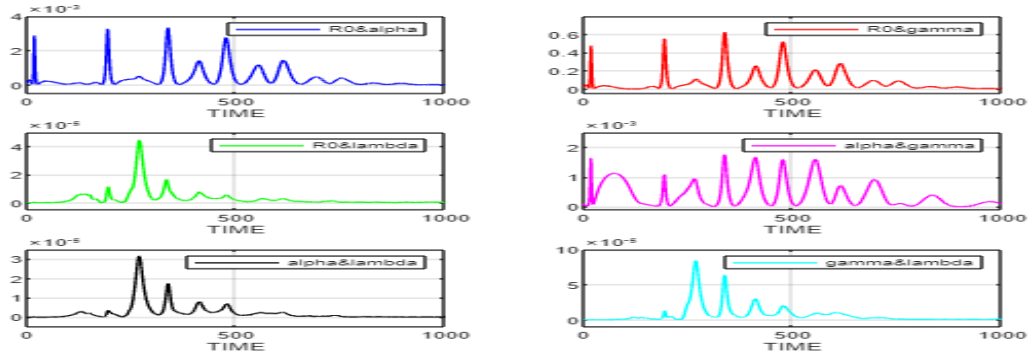


Figure 20: Evolution of the compartment INFECTED in relation to the variability of the mixed-variables .

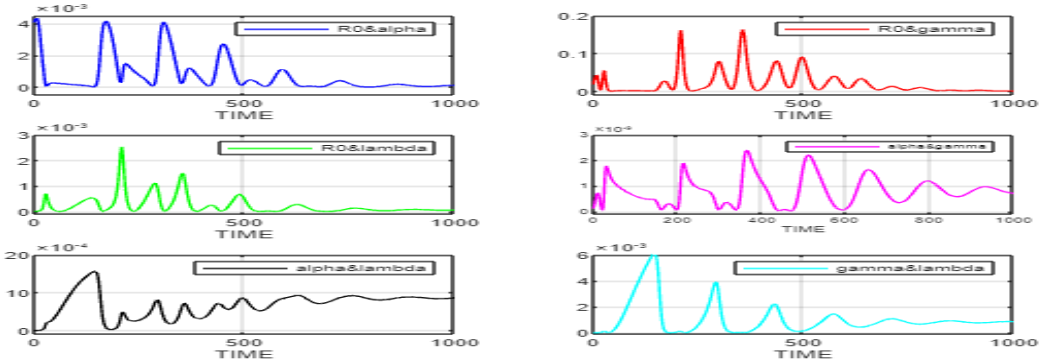


Figure 21: Evolution of the compartment HOSPITALIZED in relation to the variability of the mixed-variables.

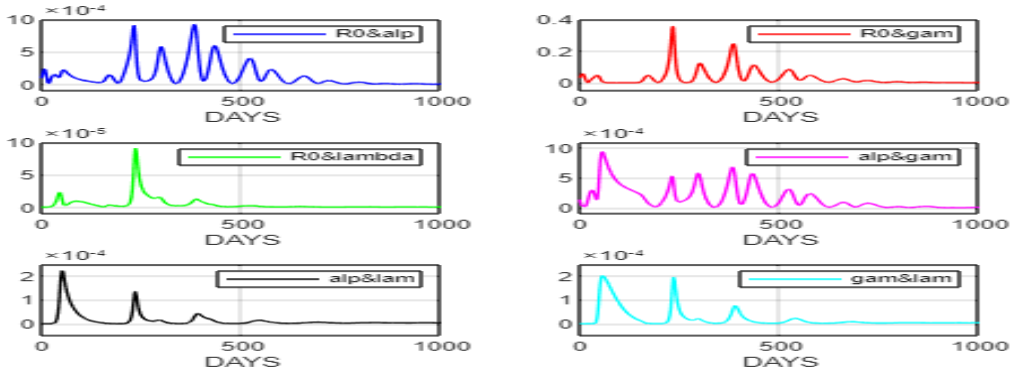


Figure 22: Evolution of the compartment RECOVERED in relation to the variability of the mixed-variables.

5.3 Sobol indice with sinusoidal transmission rate after lockdown

In Table (7) we give the Sobol indices of the stationary endemic state with $\mathcal{R}_0^f = 0.95$ that we calculated in an analytical way (see formulae in the Appendix).

Sobol Indices	$X^* = S^*$	$X^* = I^*$	$X^* = H^*$	$X^* = R^*$
$X^*_{\mathcal{R}_0}$	1	0.99663	0.99023	0.9999
X^*_{α}	0	$3.7442 * 10^{-6}$	$6.0829 * 10^{-4}$	$7.6964 * 10^{-8}$
X^*_{γ}	0	$5.8994 * 10^{-4}$	$5.8587 * 10^{-4}$	$4.6120 * 10^{-7}$
X^*_{λ}	0	0	$5.2422 * 10^{-4}$	$5.9159 * 10^{-8}$
$X^*_{\mathcal{R}_0,\alpha}$	0	$1.7486 * 10^{-5}$	0.0028407	$2.9197 * 10^{-6}$
$X^*_{\mathcal{R}_0,\gamma}$	0	0.0027550	0.0027360	$2.8682 * 10^{-6}$
$X^*_{\mathcal{R}_0,\lambda}$	0	0	0.0024481	$2.7627 * 10^{-7}$
$X^*_{\alpha,\gamma}$	0	$4.1472 * 10^{-8}$	$1.4287 * 10^{-6}$	$2.5197 * 10^{-7}$
$X^*_{\alpha,\lambda}$	0	0	$1.5038 * 10^{-6}$	$5.7850 * 10^{-7}$
$X^*_{\gamma,\lambda}$	0	0	$1.4484 * 10^{-6}$	$1.2615 * 10^{-7}$

Table 7: Sobol indices with $\mathcal{R}_0^f = 0.95$ calculated analytically .

In section (5.3), we compute the Sobol indices with a sinusoidal transmission rate (6) and in the section (5.4) with a logistic transmission rate (5). And we compare with the indices which are obtained analytically in table (7).
In this subsection, first we give in the table (8) the steady state Sobol indices using the stochatic collocation method defined in section (4) and the influences of all variable on each compartment with $\mathcal{R}_0$ after the lockdown using a sinusoidal transmission rate.
In figure (23), we have the compartment results for each parameter. Only S_{R0}, I_{R0}, H_{R0} and R_{R0} are represented. The others indices remain close to zero over time.
In table (8), we notice that variable $\mathcal{R}_0$ strongly influences all compartments S, I, H and R at 99%.

Sobol Indices	$X^* = S^*$	$X^* = I^*$	$X^* = H^*$	$X^* = R^*$
$X^*_{\mathcal{R}_0}$	0.9996	0.9944	0.9873	0.9997
X^*_{α}	0.0000	0.0000	0.0009	0.0000
X^*_{γ}	0.0000	0.0014	0.0015	0.0000
X^*_{λ}	0.0000	0.0000	0.0009	0.0000
$X^*_{\mathcal{R}_0,\alpha}$	0.0000	0.0001	0.0025	0.0000
$X^*_{\mathcal{R}_0,\gamma}$	0.0001	0.0036	0.0038	0.0001
$X^*_{\mathcal{R}_0,\lambda}$	0.0000	0.0000	0.0025	0.0000
$X^*_{\alpha,\gamma}$	0.0000	0.0000	0.0000	0.0000
$X^*_{\alpha,\lambda}$	0.0000	0.0000	0.0000	0.0000
$X^*_{\gamma,\lambda}$	0.0000	0.0000	0.0008	0.0000

Table 8: Sobol indices calculated numerically with $\mathcal{R}_0^f = 0.95$ and sinusoidale transmission rate.

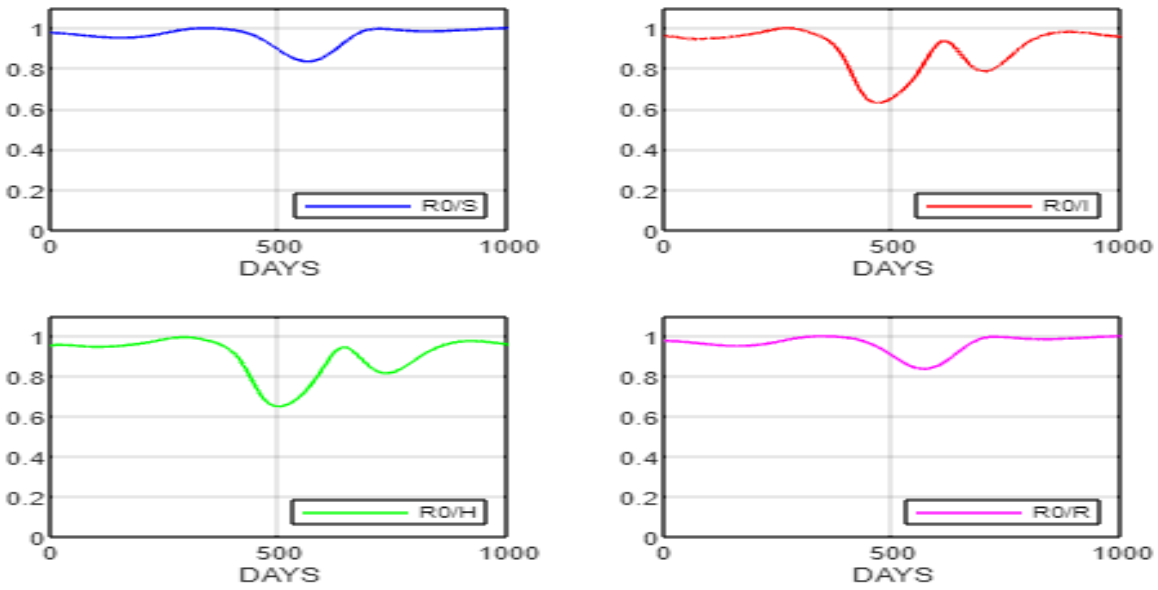


Figure 23: Influence of $\mathcal{R}_0$ on each compartment.

5.4 Sobol indice with logistic transmission rate after lockdown

In this subsection, first we give in the table (9) the steady state Sobol indices using the stochatic collocation method defined in section (4) and the influences of each variable on each compartment with $\mathcal{R}_0$ after the lockdown using a logistic transmission rate.
In figure (24), show the compartment results for each parameter. Indices that remain close to zero over time have not been represented.
In Table (9), we notice the variable $\mathcal{R}_0$ strongly influences all compartments S, I, H and R at 99%.

Sobol Indices	$X^* = S^*$	$X^* = I^*$	$X^* = H^*$	$X^* = R^*$
$X^*_{\mathcal{R}_0}$	0.9999	0.9938	0.9868	0.9999
X^*_{α}	0.0000	0.0000	0.0009	0.0000
X^*_{γ}	0.0000	0.0014	0.0016	0.0000
X^*_{λ}	0.0000	0.0000	0.0009	0.0000
$X^*_{\mathcal{R}_0,\alpha}$	0.0000	0.0000	0.0027	0.0000
$X^*_{\mathcal{R}_0,\gamma}$	0.0000	0.0046	0.0046	0.0000
$X^*_{\mathcal{R}_0,\lambda}$	0.0000	0.0000	0.0025	0.0000
$X^*_{\alpha,\gamma}$	0.0000	0.0002	0.0041	0.0000
$X^*_{\alpha,\lambda}$	0.0000	0.0000	0.0024	0.0000
$X^*_{\gamma,\lambda}$	0.0000	0.0000	0.0008	0.0000

Table 9: Sobol indices calculated numerically with $\mathcal{R}_0^f = 0.95$ and logistic transmission rate.

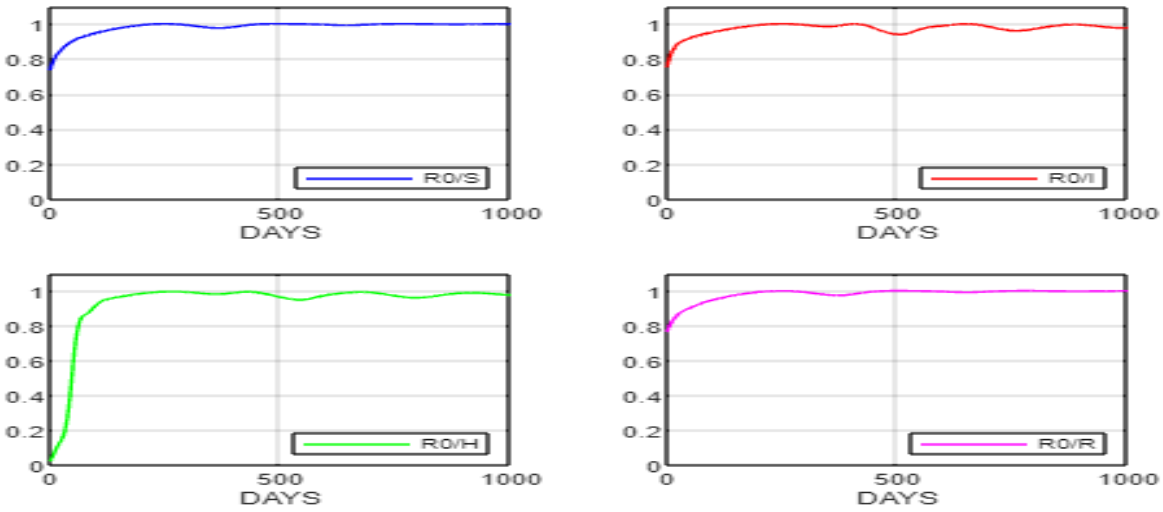


Figure 24: Influence of $\mathcal{R}_0$ on each compartment.

Remark 5.1. *For $\mathcal{R}_0 > 1$, we need more than 500 days to have the convergence of the transient regime towards the point of stationary equilibrium.*
For $\mathcal{R}_0 < 1$, we need at most 150 days to have the convergence of the transient regime towards the point of stationary equilibrium.

6 Conclusion

In this research work, we have formulated a generalized SIHR model to study the dynamics and analyze the epidemic of COVID-19 to describe the evolution of the COVID-19 disease in France in the period from 01 October to 15 November. Knowing that the patient evolves over time, we

consider that the transmission rate (6) and (5) is a function which depends on time and that it is periodic.

We investigate the dynamics of the model and the stability of free-disease and endemic equilibrium states. To fit the proposed model to the reported cumulative data of COVID-19 outbreak in France, we have estimated the model parameters using the real cases of COVID-19 in France. We have observed that with an $\mathcal{R}_0 = 3$, the epidemic will continue to extend over time even after a peak in the number of infected. We also observed that for an $\mathcal{R}_0 = 0.95 < 1$, the disease which decreases and the simulations show us that the system converges towards the free-disease equilibrium state.

We also studied the sensitivity analysis of the parameters $(\mathcal{R}_0, \alpha, \gamma, \lambda)$ on the evolution of the disease when they follow the uniform law. For that, we have used the method of stochastic collocations to calculate the Sobol indices and to see which variables are the most influencial. We have observed that for an $\mathcal{R}_0 = 0.95 < 1$, the basic reproduction rate $\mathcal{R}_0$ is the only parameter which acts on the evolution of the system (compartments). To reduce the evolution of the disease, everything must be done to reduce the value of $\mathcal{R}_0$.

We also suggest that the reduction of the outbreak evolution and the peak prevalence are possible by decreasing the risk of contracting the virus during contact and the number of contacts in a unit of time.

References

[1] World Health Organization. Updated WHO advice for international traffic in relation to the outbreak of the COVID-19. Geneva: WHO 2020 ; Available from: https:// www.who.int/ith/COVID- 19 .

[2] World Health Organization, Clinical Management of Severe Acute Respiratory Infection When Infection is Suspected, 2020. Available from: https://www.who.int/publications-detail/ .

[3] World Health Organization, Middle East respiratory syndrome coronavirus (MERS-CoV), 2019, https://www.who.int/news-room/fact-sheets/detail/.

[4] A. Saltelli, K. Chan and E.M. Scott,*Sensitivity Analysis* , 2000.

[5] A.W. Vaart,*Asymptotic Statistics,* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 1998 : https://doi.org/10.1017/CBO9780511802256

[6] B. Efron and C. Stein, *The jackknife estimate of variance.* The Annals of Statistics 9, 3 (1981), 586-596.

[7] B. Iooss, *Revue sur l'analyse de sensibilité globale de modèles numériques*, hal-00503179, 7 Dec 2010.

[8] C. Castillo-Chavez, B. Song, *Dynamical model of tuberculosis and their applications*, Mathematical Biosciences and Engineering, 1(2), 361–404, (2004).

[9] C. Castillo-Chavez and Z. Feng, *Mathematical models for the disease dynamics of tuberculosis. Advances in Mathematical Population Dynamics-Molecules*, Cells and Man. 1996; 1–28.

[10] C. Chauvière, H. Djellout, *An efficient spectral method for the numerical solution to stochastic differential equations*, hal-01666095, 18 Dec 2017.

[11] D. Gottlieb and S.A. Orszag, *Numerical analysis of spectral methods: theory and applications*, Society for Industrial and Applied Mathematics, Philadelphia, 1977, CBMS-NSF Regional Conference Series in Applied Mathematics, No. 26.

[12] G. Chowell , P.W. Fenimore, M.A. Castillo-Garsow and al., *SARS outbreak in Ontario, Hong Kong and Singapore: the role of diagnosis and isolation as a control mechanism*, Los Alamos Unclassified Report LA-UR-03-2653, 2003.

[13] G. Dahlquist and A.K. Bjorck, *Numerical methods in scientific computing*, Vol. I. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2008.

[14] I.M. Sobol, *Sensitivity estimates for nonlinear mathematical models,* Mathematical Modelling and Computational Experiment 1 (1993), 407-414.

[15] J. Jacques, *Pratique de l'analyse de sensibilité : comment évaluer l'impact des entrées aléatoires sur la sortie d'un modèle mathématique*, 25 mars 2011.

[16] J. Li , Y. Hu, and Zhen., *Rumor Spreading of an SIHR Model in Heterogeneous Networks Based on Probability Generating Function*, Hindawi Complexity Volume 2019, Article ID 4268393, https://doi.org/10.1155/2019/4268393.

[17] J. Murray, *Mathematical Biology*, third edition, Springer-Verlag, Heidelberg.

[18] J.P. LaSalle , *An invariance principe in the theory of stability* , Center for Dynamical Systems, Brown University, APRIL, 1968 .

[19] J.T. Wu, K. Leung, G.M. Leung, *Nowcasting and forecasting the potential domestic and international spread of the 2019-nCoV outbreak originating inWuhan*, China, 29 February 2020.

[20] J.T. Wu, K. Leung, M. Bushman, N. Kishore, R. Niehus, P.M. de Salazar and al., *Estimating clinical severity of COVID-19 from the transmission dynamics in Wuhan*, China. Nature Medicine. (2020). DOI: 10.1038/s41591-020-0822-7.[20]

[21] J. Wallinga and M. Lipsitch,*How generation intervals shape the relationship between growth rates and reproductive numbers*, Proc. R. Soc. B (2007) 274.

[22] M. Martcheva, *An Introduction to Mathematical Epidemiology*, Springer, New York (2015)

[23] O. Diekmann, J.A.P. Heesterbeek and M.G. Roberts, *The Construction of Next Generation Matrices for Compartmental Epidemic Models*, Journal of The Royal So- ciety Interface. 2009; 7(47): 873–885.

[24] P.J. Davis and P. Rabinowitz, *Methods of numerical integration*, second edition, Computer Science and Applied Mathematics. Academic Press, Inc., Orlando, FL, 1984.

[25] P. van den Driessche and J. Watmough, *Reproduction numbers and subthreshold endemic equilibria for compartmental models of disease transmission*, Math. Biosci., 180 (2002), 29-48.

[26] S. Boatto, C. Bonnet, B. Cazelles and F. Mazenc, *SIR model with time dependent infectivity parameter : approximating the epidemic attractor and the importance of the initial phase*, 8 Jan 2018, Id: hal-01677886.

[27] S. Jiao and M. Huang,*An SIHR epidemic model of the COVID-19 with general population-size dependent contact rate*, Published: 28 August 2020, AIMS Mathematics, 5(6): 6714–6725, DOI:10.3934/math.2020431.

[28] S. K. Biswas, J. K. Ghosh, S. Sarkar, and U.Ghosh, *COVID-19 Pandemic in India: A Mathematical Model Study*, May 30, 2020,doi:10.20944/preprints202005.0508.v1.

[29] S. Rahman,*Extended Polynomial Dimensional Decomposition for Arbitrary Probability Distributions.*, J. Eng. Mech. 135, 12 (2009), 1439-1451.

[30] S.T.R. Pinho, C.P. Ferreira, L. Esteva, F.R. Barreto, V.C. Morato e Silva and M.G.L. Teixeira, *Modelling the dynamics of dengue real epidemics*, Phil. Trans. R. Soc. A, 28 December 2010, 368, 5679–5693.

[31] T. Tiana, W. Luoa, Y. Jianga, M. Chena , C. Wenb, W. L. Pana, and X. Wang, *The Effects of Stringent Interventions for Coronavirus Pandemic.*, https://doi.org/10.1101/2020.05.16.20103879.

[32] T. Tian, J. Zhang, S. Lin, Y. Jiang, J. Tan, Z. Li, X. Wang,*Data-driven analysis on the simulations of the spread of COVID-19 under different interventions of China.* : https://doi.org/10.1101/2020.05.15.20103051

[33] T. Turanyi., *Sensitivity analysis of complex kinetic system, tools and applications*, Journal of Mathematical Chemistry, 5 :203–248, 1990.

[34] W.O. Kermack and A.G. McKendrick,*A Contribution to the Mathematical Theoryof Epidemics*, Proceedings of the Royal Society. 1927; 115(772): 700–722

[35] Z. Ma and J. Li, *Dynamical Modeling and Anaylsis of Epidemics*, World Scientific, (2009) .

A Appendix: Sobol indices calculations

In this section, we give the explicit expressions of Sobol indices computed for the stationary state of the model (1). To simplify the equations, we consider the following notations and primitives:

Notations

- $T_1(X) = (\mu_2 + X)\ln(\mu_2 + X)$,
- $T_2(X) = (\mu_2 + X)^2\ln(\mu_2 + X)$,
- $T_3(X) = (\mu_2 + X)\big(\ln(\mu_2 + X)\big)^2$,
- $T_4(X) = \ln(\mu_2 + X)$
- $\int X^2\big(\ln(X)\big)^2 dX = \dfrac{X^3}{3}\big[\big(\ln(X)\big)^2 - \dfrac{2}{3}\ln(X) + \dfrac{2}{9}\big]$,
- $\int X\big(\ln(X)\big)^2 dX = \dfrac{X^2}{2}\big[\big(\ln(X)\big)^2 - \ln(X) + \dfrac{1}{2}\big]$,
- $\int \big(\ln(X)\big)^2 dX = X\big[\big(\ln(X)\big)^2 - 2\ln(X) + 2\big]$,
- $\int X\ln(X)dX = \frac{X^2}{2}\big(\ln(X) - \frac{1}{2}\big)$
- $d_\alpha = b_\alpha - a_\alpha\,,\ d_\gamma = b_\gamma - a_\gamma\,,\ d_\lambda = b_\lambda - a_\lambda$ and $d_{R_0} = b_{R_0} - a_{R_0}$.

(29)

A.1 Sobol indices for S^*

Definition A.1. *The Sobol indices of order 1 and 2 for S^* are defined:*

$$S^*_{X_i} = \frac{\mathbb{V}\big(\mathbb{E}(S^*|X_i)\big)}{\mathbb{V}(S^*)} \qquad i = 1, ..., 4 \tag{30}$$

$$S^*_{X_i,X_j} = \frac{\mathbb{V}\big(\mathbb{E}(S^*|X_i, X_j)\big)}{\mathbb{V}(S^*)} - S^*_{X_i} - S^*_{X_j}. \tag{31}$$

To calculate the Sobol indices for S^*, we will need:

$$\begin{cases} \mathbb{E}\left(\frac{1}{\mathcal{R}_0}\right) = \dfrac{\ln(b_{R_0}) - \ln(a_{R_0})}{d_{R_0}} \\ \mathbb{E}\left(\frac{1}{\mathcal{R}_0^2}\right) = \dfrac{1}{b_{R_0} \times a_{R_0}} \\ \mathbb{E}(S^*) = \dfrac{\tau}{\mu_1}\left(\dfrac{\ln(b_{R_0}) - \ln(a_{R_0})}{d_{R_0}}\right) \\ \mathbb{V}(S^*) = \left(\frac{\tau}{\mu_1}\right)^2\left[\frac{1}{a_{R_0}\times b_{R_0}} - \left(\frac{\ln(b_{R_0})-\ln(a_{R_0})}{d_{R_0}}\right)^2\right]. \end{cases} \tag{32}$$

We obtain:

$$S^*_{\mathcal{R}_0} = \frac{\mathbb{V}(\mathbb{E}(S^*|\mathcal{R}_0))}{\mathbb{V}(S^*)} = \frac{\mathbb{V}(S^*)}{\mathbb{V}(S^*)} = 1,$$

the other Sobol indices for S^* are zero.

A.2 Sobol indices for I^*

Definition A.2. *The Sobol indices of order 1 and 2 for I^* are defined:*

$$I^*_{X_i} = \frac{\mathbb{V}\big(\mathbb{E}(I^*|X_i)\big)}{\mathbb{V}(I^*)} \qquad i = 1, ..., 4 \tag{33}$$

$$I^*_{X_i,X_j} = \frac{\mathbb{V}\big(\mathbb{E}(I^*|X_i, X_j)\big)}{\mathbb{V}(I^*)} - I^*_{X_i} - I^*_{X_j}. \tag{34}$$

Sobol indices of order 1

First of all we calculate,

$$\mathbb{E}\left(\frac{1}{\mu_2+\gamma+\alpha}\right) = \frac{1}{d_\alpha \times d_\gamma}\Big[T_1(b_\alpha+b_\gamma) - T_1(a_\alpha+b_\gamma) - T_1(b_\alpha+a_\gamma) + T_1(a_\alpha+a_\gamma)\Big], \tag{35}$$

The mean of I_1 becomes:

$$\mathbb{E}(I^*) = \tau\mathbb{E}\left(\frac{1}{\mu_2+\gamma+\alpha}\right)\left(1-\mathbb{E}(\frac{1}{\mathcal{R}_0})\right) = \tau\mathbb{E}\left(\frac{1}{\mu_2+\gamma+\alpha}\right)\left(1-\frac{\ln(b_{R_0})-\ln(a_{R_0})}{d_{R_0}}\right). \tag{36}$$

To calculate the variance of I_1, we need:

$$\mathbb{E}\left(\frac{1}{(\mu_2+\gamma+\alpha)^2}\right) = \frac{1}{d_\alpha \times d_\gamma}\Big[-T_4(b_\alpha+b_\gamma)+T_4(a_\alpha+b_\gamma)+T_4(b_\alpha+a_\gamma)-T_4(a_\alpha+a_\gamma)\Big], \tag{37}$$

$$\mathbb{E}\left(\left(1-\frac{1}{\mathcal{R}_0}\right)^2\right) = 1 + 2\frac{\ln(a_{R_0})-\ln(b_{R_0})}{d_{R_0}} + \frac{1}{a_{R_0}\times b_{R_0}}. \tag{38}$$

With the equations (37), (38) and (36), the variance of I^* is then equal

$$\mathbb{V}(I^*) = \tau^2\mathbb{E}\left(\frac{1}{(\mu_2+\gamma+\alpha)^2}\right)\mathbb{E}\left(\left(1-\frac{1}{\mathcal{R}_0}\right)^2\right) - \big(\mathbb{E}(I^*)\big)^2. \tag{39}$$

For Sobol indices of order 1 of I^*, we will need:

$$\begin{cases} \mathbb{E}_{\mathcal{R}_0}(I^*) = \tau\,(1-1/\mathcal{R}_0)\,\mathbb{E}\left(\frac{1}{\mu_2+\gamma+\alpha}\right), \\ \mathbb{E}_{\gamma}(I^*) = \tau\mathbb{E}_{\gamma}(\frac{1}{\mu_2+\gamma+\alpha})\mathbb{E}\,(1-1/\mathcal{R}_0) \\ \mathbb{E}_{\alpha}(I^*) = \tau\mathbb{E}_{\alpha}\left(\frac{1}{\mu_2+\gamma+\alpha}\right)\mathbb{E}(1-1/\mathcal{R}_0). \end{cases} \tag{40}$$

with

$$\begin{aligned} \mathbb{E}_{\gamma}\left(\frac{1}{\mu_2+\gamma+\alpha}\right) &= \frac{T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)}{d_\alpha}. \\ \mathbb{E}_{\alpha}\left(\frac{1}{\mu_2+\gamma+\alpha}\right) &= \frac{T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)}{d_\gamma}. \end{aligned} \tag{41}$$

We obtain

$$I^*_{\mathcal{R}_0} = \frac{\mathbb{V}(\mathbb{E}\,(I^*|\mathcal{R}_0))}{\mathbb{V}(I^*)} = \frac{\tau\,\mathbb{E}\left(\frac{1}{\mu_2+\gamma+\alpha}\right)\mathbb{V}\big(1-1/\mathcal{R}_0\big)}{\mathbb{V}(I^*)}. \tag{42}$$

Sobol indice for α:

$$I^*_{\alpha} = \frac{\mathbb{V}\big(\mathbb{E}(I^*|\alpha)\big)}{\mathbb{V}(I^*)} = \frac{\big(\tau\,\mathbb{E}(1-1/\mathcal{R}_0)\big)^2\mathbb{V}\big(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\big)}{d_\gamma^2\,\mathbb{V}(I^*)} \tag{43}$$

Now we calculate,

$$\mathbb{V}\bigg(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\bigg) = \mathbb{E}\bigg[\bigg(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\bigg)^2\bigg]-\bigg[\mathbb{E}\bigg(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\bigg)\bigg]^2. \tag{44}$$

The second term of (44) :

$$\begin{aligned} \mathbb{E}\bigg(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\bigg) = \frac{1}{d_\alpha}\bigg[&T_1(b_\alpha+b_\gamma)-T_1(a_\alpha+b_\gamma) \\ &-T_1(b_\alpha+a_\gamma)+T_1(a_\alpha+a_\gamma)+a_\gamma-b_\gamma\bigg]. \end{aligned} \tag{45}$$

The first term of (44):

$$\begin{aligned} \mathbb{E}\bigg[\bigg(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma\bigg)^2\bigg] = \mathbb{E}\bigg[\bigg(T_4(\alpha+b_\gamma)\bigg)^2\bigg] + \mathbb{E}\bigg[\bigg(T_4(\alpha+a_\gamma)\bigg)^2\bigg] \\ -2\,\mathbb{E}\bigg[T_4(\alpha+b_\gamma)\times T_4(\alpha+a_\gamma)\bigg] \end{aligned} \tag{46}$$

Thanks to the primitive of $(\ln(x))^2$, the first two terms of (46) are easy to calculate:

$$\mathbb{E}\Bigg[\bigg(\ln(\mu_2+\alpha+b_\gamma)\bigg)^2\Bigg] = \frac{1}{d_\alpha}\Bigg[T_3(b_\alpha+b_\gamma) - T_3(a_\alpha+b_\gamma) - 2T_1(b_\alpha+b_\gamma) + 2\,T_1(a_\alpha+b_\gamma) + 2d_\alpha\Bigg],$$

$$\mathbb{E}\Bigg[\bigg(\ln(\mu_2+\alpha+b_\gamma)\bigg)^2\Bigg] = \frac{1}{d_\alpha}\Bigg[T_3(b_\alpha+a_\gamma) - T_3(a_\alpha+a_\gamma) - 2T_1(b_\alpha+a_\gamma) + 2T_1(a_\alpha+a_\gamma) + 2d_\alpha\Bigg],$$

We consider $F_1(\alpha)$ a primitive of the function $\ln(\mu_2+\alpha+b_\gamma)\ln(\mu_2+\alpha+a_\gamma)$. We obtain

$$\mathbb{E}\Bigg[\ln(\mu_2+\alpha+b_\gamma)\times\ln(\mu_2+\alpha+a_\gamma)\Bigg] = F_1(b_\alpha) - F_1(a_\alpha). \tag{47}$$

With (45), (46) and (47), we have (44) which gives us the Sobol indice I^*_α.

We redo exactly the same work for the Sobol indice I^*_γ: you just have to permute α and γ.

$$I^*_\gamma = \frac{\mathbb{V}\big(\mathbb{E}(I^*|\gamma)\big)}{\mathbb{V}(I^*)} = \frac{\bigg(\tau\ \mathbb{E}(1-1/R_0)\bigg)^2\mathbb{V}\bigg(T_4(\gamma+b_\alpha) - T_4(\gamma+a_\alpha)\bigg)}{d_\alpha^2\ \mathbb{V}(I^*)}.$$

Sobol indices of order 2

We start with the indice $I_{1,\alpha,\gamma}$:

$$I^*_{\alpha,\gamma} = \frac{\mathbb{V}\big(\mathbb{E}(I^*|\alpha,\gamma)\big)}{\mathbb{V}(I^*)} \tag{48}$$

We known that :

$$\mathbb{E}(I^*|\alpha,\gamma) = \frac{\tau}{\mu_2+\gamma+\alpha}\mathbb{E}(1-1/\mathcal{R}_0) = \frac{\tau}{\mu_2+\gamma+\alpha}\left(1-\frac{\ln(b_R)-\ln(a_R)}{d_{R_0}}\right)$$

The Sobol indice $I^*_{\alpha,\gamma}$:

$$I^*_{\alpha,\gamma} = \left(\tau(1-\frac{\ln(b_{R_0})-\ln(a_{R_0})}{d_{R_0}})\right)^2\frac{\mathbb{V}(1/(\mu_2+\alpha+\gamma))}{\mathbb{V}(I^*)},$$

The term $\mathbb{V}(1/(\mu_2+\alpha+\gamma))$ which is equal to:

$$\mathbb{V}(1/(\mu_2+\alpha+\gamma)) = \mathbb{E}\bigg(\frac{1}{(\mu_2+\gamma+\alpha)^2}\bigg) - \bigg(\mathbb{E}(\frac{1}{\mu_2+\gamma+\alpha})\bigg)^2,$$

is calculated using the equations (35) and (37).
For the Sobol indice I^*_{α,R_0}, we calculate first:

$$\mathbb{E}(I^*|\alpha,\mathcal{R}_0) = \tau(1-1/\mathcal{R}_0)\frac{T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)}{d_\gamma}.$$

We finally obtain:

$$I^*_{\alpha,\mathcal{R}_0} = \frac{\tau^2}{d_\gamma^2}\frac{\mathbb{V}\big((1-1/\mathcal{R}_0)(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma))\big)}{\mathbb{V}(I_1)},$$

We calculate the term :

$$\mathbb{V}\bigg((1-\frac{1}{\mathcal{R}_0})(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma))\bigg) = \mathbb{E}\bigg[(1-1/\mathcal{R}_0)^2\bigg]\mathbb{E}\bigg[\bigg(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\bigg)^2\bigg] - \bigg[\mathbb{E}(1-1/\mathcal{R}_0)\bigg]^2\bigg[\mathbb{E}\big(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\big)\bigg]^2$$

With $\mathbb{E}\big[\big(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\big)^2\big]$ which is given by (46), $\mathbb{E}\big[(1-1/\mathcal{R}_0)^2\big]$ which is given by (32) et $\mathbb{E}\big(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\big)$ which is given by (45).

We redo exactly the same work for the Sobol index $I^*_{\gamma,\mathcal{R}_0}$:

$$\mathbb{E}(I^*|\gamma,\mathcal{R}_0) = \tau(1-1/\mathcal{R}_0)\frac{T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)}{d_\alpha}.$$

Finally we obtain :

$$I^*_{\gamma,\mathcal{R}_0} = \frac{\tau^2}{d_\alpha^2}\frac{\mathbb{V}\big((1-1/\mathcal{R}_0)(T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha))\big)}{\mathbb{V}(I_1)},$$

A.3 Sobol indices for H^*

Definition A.3. *The Sobol indices of order 1 and 2 for H^* are defined:*

$$H^*_{X_i} = \frac{\mathbb{V}\big(\mathbb{E}(H^*|X_i)\big)}{\mathbb{V}(H^*)} \qquad i=1,...,4 \tag{49}$$

$$H^*_{X_i,X_J} = \frac{\mathbb{V}\big(\mathbb{E}(H^*|X_i,X_j)\big)}{\mathbb{V}(H^*)} - H^*_{X_i} - H^*_{X_j}. \tag{50}$$

First we calculate the mean and the variance of H^*,

$$\begin{aligned}\mathbb{E}\bigg(\frac{1}{\mu_2+\lambda}\bigg) &= \frac{T_4(b_\lambda)-T_4(a_\lambda)}{d_\lambda},\\ \mathbb{E}\bigg(\frac{1}{(\mu_2+\lambda)^2}\bigg) &= \frac{1}{(\mu_2+b_\lambda)(\mu_2+a_\lambda)},\end{aligned} \tag{51}$$

The mean H^* is obtained:

$$\mathbb{E}(H^*) = \tau\,\mathbb{E}\bigg(\frac{1}{\mu_2+\lambda}\bigg)\mathbb{E}\bigg(1-1/\mathcal{R}_0\bigg)\mathbb{E}\bigg(\frac{\alpha}{\mu_2+\gamma+\alpha}\bigg),$$

with

$$\mathbb{E}\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}\Big) = \frac{1}{2\times d_\alpha \times d_\gamma}\Big[T_2(b_\gamma+b_\alpha) - T_2(b_\gamma+a_\alpha) - T_2(a_\gamma+b_\alpha) + T_2(a_\gamma+a_\alpha)\Big] \\ - \frac{\mu_2+b_\gamma}{d_\alpha\times d_\gamma}\Big[T_1(b_\gamma+b_\alpha) - T_1(b_\gamma+a_\alpha) - d_\alpha\Big] \\ + \frac{\mu_2+a_\gamma}{d_\alpha\times d_\gamma}\Big[T_1(a_\gamma+b_\alpha) - T_1(a_\gamma+a_\alpha) - d_\alpha\Big] - \frac{1}{2}. \tag{52}$$

For the variance of H^*:

$$\mathbb{V}(H^*) = \tau^2\mathbb{E}\Big(\frac{1}{(\mu_2+\lambda)^2}\Big)\mathbb{E}\Big[(1-1/\mathcal{R}_0)^2\Big]\mathbb{E}\Big(\frac{\alpha^2}{(\mu_2+\gamma+\alpha)^2}\Big) - \Big(\mathbb{E}(H^*)\Big)^2$$

with

$$\mathbb{E}\Big(\frac{\alpha^2}{(\mu_2+\gamma+\alpha)^2}\Big) = \mathbb{E}\Big((1-\frac{\gamma+\mu_2}{\mu_2+\gamma+\alpha})^2\Big) \\ = \frac{1}{d_\alpha\times d_\gamma}\int\int\Big(1-2\frac{\mu_2+\gamma}{\mu_2+\gamma+\alpha}+\frac{(\mu_2+\gamma)^2}{(\mu_2+\gamma+\alpha)^2}\Big)d\alpha d\gamma. \tag{53}$$

The second term of (53) is equal to:

$$\int\int\frac{\mu_2+\gamma}{\mu_2+\gamma+\alpha}d\alpha d\gamma = \frac{1}{2}\Big[T_2(b_\gamma+b_\alpha) - T_2(a_\gamma+b_\alpha) - T_2(b_\gamma+a_\alpha) + T_2(a_\gamma+a_\alpha)\Big] \\ - b_\alpha\Big[T_1(b_\gamma+b_\alpha) - T_1(a_\gamma+b_\alpha)\Big] + a_\alpha\Big[T_1(b_\gamma+a_\alpha) \\ - T_1(a_\gamma+a_\alpha)\Big] + \frac{1}{2}d_\alpha\times d_\gamma.$$

The third term: just use the following decomposition

$$(\mu_2+\gamma)^2 = (\mu_2+\gamma+a_\alpha-a_\alpha)^2 = (\mu_2+\gamma+a_\alpha)^2 - 2(\mu_2+\gamma+a_\alpha)a_\alpha + a_\alpha^2 \tag{54}$$

and we obtain :

$$\int_{a_\gamma}^{b_\gamma}\int_{a_\alpha}^{b_\alpha}\frac{(\mu_2+\gamma)^2}{(\mu_2+\gamma+\alpha)^2}\big)d\alpha d\gamma = \int_{a_\gamma}^{b_\gamma}(\mu_2+\gamma)^2\big[-\frac{1}{\mu_2+\gamma+b_\alpha}+\frac{1}{\mu_2+\gamma+a_\alpha}\big]d\gamma \\ = -\int_{a_\gamma}^{b_\gamma}\frac{(\mu_2+\gamma)^2}{\mu_2+\gamma+b_\alpha}d\gamma + \int_{a_\gamma}^{b_\gamma}\frac{(\mu_2+\gamma)^2}{\mu_2+\gamma+b_\alpha}d\gamma \\ = -b_\alpha^2\big(T_4(b_\gamma+b_\alpha) - T_4(a_\gamma+b_\alpha)\big) \\ + a_\alpha^2\big(T_4(b_\gamma+a_\alpha) - T_4(a_\gamma+a_\alpha)\big) - d_\alpha\times d_\gamma.$$

Sobol indices of order 1

For Sobol indices $H^*_{R_0}$ and H^*_λ, we calculate firstly:

$$\mathbb{E}(H^*|\mathcal{R}_0) = \tau(1-1/\mathcal{R}_0)\mathbb{E}\Big(\frac{1}{\mu_2+\lambda}\Big)\mathbb{E}\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}\Big),$$
$$\mathbb{E}(H^*|\lambda) = \frac{\tau}{\mu_2+\lambda}\mathbb{E}\Big(1-1/\mathcal{R}_0\Big)\mathbb{E}\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}\Big).$$

and we have:

$$H^*_{\mathcal{R}_0} = \frac{\mathbb{V}\Big(\mathbb{E}(H^*|\mathcal{R}_0)\Big)}{\mathbb{V}(H^*)} = \Big(\tau\mathbb{E}\Big(\frac{1}{\mu_2+\lambda}\Big)\mathbb{E}\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}\Big)\Big)^2\frac{\mathbb{V}(1/\mathcal{R}_0)}{\mathbb{V}(H^*)},$$
$$H^*_\lambda = \frac{\mathbb{V}\Big(\mathbb{E}(H^*|\lambda)\Big)}{\mathbb{V}(H^*)} = \Big(\tau\mathbb{E}(1-1/\mathcal{R}_0)\mathbb{E}\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}\Big)\Big)^2\frac{\mathbb{V}(1/(\mu_2+\lambda))}{\mathbb{V}(H^*)}$$

We have already calculated all the terms in the equations (32), (51) and (52).

For Sobol indices H^*_γ et H^*_α, we calculate:

$$\mathbb{E}_\gamma(H^*) = \tau\mathbb{E}(1-1/\mathcal{R}_0)\mathbb{E}\Big(\frac{1}{\mu_2+\lambda}\Big)\mathbb{E}_\gamma\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}\Big),$$
$$\mathbb{E}_(H^*|\alpha) = \tau\alpha\mathbb{E}(1-1/\mathcal{R}_0)\mathbb{E}\Big(\frac{1}{\mu_2+\lambda}\Big)\mathbb{E}_\alpha\Big(\frac{1}{\mu_2+\gamma+\alpha}\Big).$$

We will need the following means:

$$\mathbb{E}_\gamma\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}\Big) = \mathbb{E}\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}|\gamma\Big) = 1-\frac{\mu_2+\gamma}{d_\alpha}\Big(T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)\Big),$$
$$\mathbb{E}_\alpha\Big(\frac{1}{\mu_2+\gamma+\alpha}\Big) = \mathbb{E}\Big(\frac{1}{\mu_2+\gamma+\alpha}|\alpha\Big) = \frac{1}{d_\gamma}\Big(T_4(\alpha+b_\gamma)-T_4(\alpha+a_\gamma)\Big). \tag{55}$$

We also calculate the following two variances:

$$\mathbb{V}\Big[\mathbb{E}_\gamma\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\Big] = \mathbb{E}\Big[\big(\mathbb{E}_\gamma\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\big)^2\Big] - \Big[\mathbb{E}\mathbb{E}_\gamma\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\Big]^2.$$

$$\begin{aligned}\mathbb{V}\Big[\alpha\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\Big)\Big] &= \mathbb{E}\Big[\big(\alpha\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\big)^2\Big] - \Big[\mathbb{E}\big(\alpha\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\big)\Big]^2 \\ &= \mathbb{E}\Big[\alpha^2\big(\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\big)^2\Big] - \Big[\mathbb{E}\big(\alpha\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\big)\Big]^2.\end{aligned} \tag{56}$$

with $\mathbb{E}\mathbb{E}_\Sigma = \mathbb{E}[\mathbb{E}(X|\Sigma)] = \mathbb{E}(X)$.

To calculate the first variance, just calculate the first term:

$$\begin{aligned}\mathbb{E}\Big[\big(\mathbb{E}_\gamma(\frac{\alpha}{\mu_2+\gamma+\alpha})\big)^2\Big] &= \mathbb{E}\Big[\big[1-\frac{\mu_2+\gamma}{d_\alpha}\big(T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)\big)\big]^2\Big]\\ &= \mathbb{E}\Big[1-2\frac{\mu_2+\gamma}{d_\alpha}\big(T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)\big)\\ &\quad+\frac{(\mu_2+\gamma)^2}{d_\alpha^2}\big(T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)\big)^2\Big] \qquad (57)\end{aligned}$$

For the second term of (57), we obtain:

$$\begin{aligned}\mathbb{E}\Big[(\mu_2+\gamma)T_4(\gamma+b_\alpha)\Big] &= \mathbb{E}\Big[T_1(\gamma+b_\alpha)\big]-b_\alpha\mathbb{E}\big(T_4(\gamma+b_\alpha)\big)\\ &= \frac{1}{2*d_g}\big[T_2(b_\gamma+b_\alpha)-T_2(a_\gamma+b_\alpha)-\frac{1}{2}b_\gamma(b_\gamma+2\mu_2+2b_\alpha)\\ &\quad+\frac{1}{2}a_\gamma(a_\gamma+2\mu_2+2b_\alpha)\big]-b_\alpha\big[T_1(b_\gamma+b_\alpha)-T_1(a_\gamma+b_\alpha)-d_\gamma\Big].\end{aligned}$$

$$\begin{aligned}\mathbb{E}\Big[(\mu_2+\gamma)T_4(\gamma+a_\alpha)\Big] &= \mathbb{E}\Big[T_1(\gamma+a_\alpha)\big]-a_\alpha\mathbb{E}\big(T_4(\gamma+a_\alpha)\big)\\ &= \frac{1}{2\times d_\gamma}\Big[T_2(b_\gamma+a_\alpha)-T_2(a_\gamma+a_\alpha)-\frac{1}{2}b_\gamma(b_\gamma+2\mu_2+2a_\alpha)\\ &\quad+\frac{1}{2}a_\gamma(a_\gamma+2\mu_2+2a_\alpha)\big]-a_\alpha\big[T_1(b_\gamma+a_\alpha)-T_1(a_\gamma+a_\alpha)-d_\gamma\Big].\end{aligned}$$

For the third term of (57), we develop the term squared then we first calculate the term

$$\begin{aligned}\mathbb{E}\Big[(\mu_2+\gamma)^2\big(T_4(\gamma+b_\alpha)\big)^2\Big] &= \mathbb{E}\Big[(\mu_2+\gamma+b_\alpha-b_\alpha)^2\big(T_4(\gamma+b_\alpha)\big)^2\Big]\\ &= \mathbb{E}\big[\big((\mu_2+\gamma+b_\alpha)^2-2b_\alpha(\mu_2+\gamma+b_\alpha)+b_\alpha^2\big)\big(\ln(T_4(\gamma+b_\alpha)\big)^2\big]\\ &= \mathbb{E}\Big[(\mu_2+\gamma+b_\alpha)^2\big(T_4(\gamma+b_\alpha)\big)^2\Big]\\ &\quad-2b_\alpha\mathbb{E}\Big[T_3(\gamma+b_\alpha)\big]+b_\alpha^2\mathbb{E}\big[\big(T_4(\gamma+b_\alpha)\big)^2\Big].\end{aligned}$$

To continue the calculations, we use the primitives of functions $X^2(\ln(X))^2$, $X(\ln(X))^2$ and $(\ln(X))^2$ which are given in (29)
We do the same for the term $\mathbb{E}\big[(\mu_2+\gamma)^2\big(T_4(\gamma+a_\alpha)\big)^2\big]$.
To complete the calculation of (57), we consider the function

$$F_2(x) = \int x^2\big(\ln(x+A)\ln(x+B)\big)dx \qquad (58)$$

for all A and B two reals. The last term of (57) then becomes :

$$\mathbb{E}\Big[(\mu_2+\gamma)^2\big(\ln(\mu_2+\gamma+b_\alpha)\ln(\mu_2+\gamma+a_\alpha)\big)\Big] = F_2(b_\gamma) - F_2(a_\gamma),$$

with $A = b_\alpha$ and $B = a_\alpha$.

For the second variance, we start by calculating the first term:

$$\mathbb{E}\Big[\big(\alpha\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\big)^2\Big] = \frac{1}{d_\gamma^2 * d_\alpha}\int \alpha^2\big(T_4(\alpha+b_\gamma) - T_4(\alpha+a_\gamma)\big)^2 d\alpha \tag{59}$$

With:

$$\begin{aligned}
\int \alpha^2\big(T_4(\alpha+b_\gamma)\big)^2 d\alpha &= \int (\mu_2+\alpha+b_\gamma)^2\big(T_4(\alpha+b_\gamma)\big)^2 d\alpha \\
&\quad + (\mu_2+b_\gamma)^2\int \big(T_4(\alpha+b_\gamma)\big)^2 d\alpha + 2(\mu_2+b_\gamma)\int T_3(\alpha+b_\gamma)d\alpha \\
&= \frac{1}{3}\Big[X^3\big((\ln(x))^2 - \frac{1}{3}\ln(X) + \frac{1}{9}\big)\Big]_{a_\alpha}^{b_\alpha} \\
&\quad + (\mu_2+b_\gamma)\Big[X^2\big((\ln(X))^2 - \ln(X) + \frac{1}{2}\big)\Big]_{a_\alpha}^{b_\alpha} \\
&\quad + (\mu_2+b_\gamma)^2\Big[X\big((\ln(X))^2 - 2\ln(X) + 2\big)\Big]_{a_\alpha}^{b_\alpha}
\end{aligned}$$

with $X = \mu_2 + \alpha + b_\gamma$
For

$$\int \alpha^2\big(T_4(\alpha+a_\gamma)\big)^2 d\alpha,$$

we do exactly the same way a_γ instead of b_γ.

For the last term

$$\int \alpha^2\big(\ln(\mu_2+\alpha+b_\gamma)\ln(\mu_2+\alpha+a_\gamma)\big)d\alpha,$$

we use the function (58).
The second term of the second variance in (56)

$$\Big[\mathbb{E}\big[\alpha\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\big]\Big]^2 = \frac{1}{d_\gamma^2 \times d_\alpha^2}\Big[\int \alpha\big(T_4(\alpha+b_\gamma) - T_4(\alpha+a_\gamma)\big)d\alpha\Big]^2 \tag{60}$$

Knowing that :

$$\int \alpha\big(\ln(\mu_2+\alpha+b_\gamma)-\ln(\mu_2+\alpha+a_\gamma)\big)d\alpha = \Big[\frac{X^2}{2}\big(\ln(X)-\frac{1}{2}\big)-(\mu_2+b_\gamma)\big(X\ln(X)-X\big)\Big]_{a_\alpha}^{b_\alpha} \tag{61}$$
$$-\Big[\frac{Y^2}{2}\big(\ln(Y)-\frac{1}{2}\big)-(\mu_2+a_\gamma)\big(Y\ln(Y)-Y\big)\Big]_{a_\alpha}^{b_\alpha},$$

with $X=\mu_2+\alpha+b_\gamma$ and $Y=\mu_2+\alpha+a_\gamma$.

Thus we conclude the Sobol indice H^*_γ and we redo exactly the same procedures to assess the Sobol indice H^*_α.

$$H^*_\gamma = \frac{\mathbb{V}\Big(\mathbb{E}(H^*|\gamma)\Big)}{\mathbb{V}(H^*)} = \Big(\tau\mathbb{E}\big(\frac{1}{\mu_2+\lambda}\big)\mathbb{E}\Big(1-1/\mathcal{R}_0\Big)^2\frac{\mathbb{V}\Big[\mathbb{E}_\gamma\Big(\frac{\alpha}{\mu_2+\gamma+\lambda}\Big)\Big]}{\mathbb{V}(H^*)}$$
$$H^*_\alpha = \frac{\mathbb{V}\Big(\mathbb{E}(H^*|\alpha)\Big)}{\mathbb{V}(H^*)} = \Big[\tau\mathbb{E}\Big(\frac{1}{\mu_2+\lambda}\big)\mathbb{E}(1-1/\mathcal{R}_0)\Big]^2\frac{\mathbb{V}\Big[\alpha\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\lambda}\Big)\Big]}{\mathbb{V}(H^*)}$$

Sobol indices of order 2
For Sobol indices $H^*_{\mathcal{R}_0,\alpha}$ and $H^*_{\mathcal{R}_0,\gamma}$:

$$H^*_{\mathcal{R}_0,\alpha} = \frac{\mathbb{V}\big(\mathbb{E}(H^*|\mathcal{R}_0,\alpha)\big)}{\mathbb{V}(H^*)} = \Big[\tau\mathbb{E}\big(\frac{1}{\mu_2+\lambda}\big)\Big]^2\frac{\mathbb{V}\Big[\alpha(1-1/\mathcal{R}_0)\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\Big]}{\mathbb{V}(H^*)}$$

$$H^*_{\mathcal{R}_0,\gamma} = \frac{\mathbb{V}\big(\mathbb{E}(H^*|\mathcal{R}_0,\gamma)\big)}{\mathbb{V}(H^*)} = \Big[\tau\mathbb{E}\big(\frac{1}{\mu_2+\lambda}\big)\Big]^2\frac{\mathbb{V}\Big[(1-1/\mathcal{R}_0)\mathbb{E}_\gamma\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\Big]}{\mathbb{V}(H^*)}$$

we have already calculated in (32),(51),(55),(37) and (56) all the terms that are present.

For Sobol indices $H^*_{\mathcal{R}_0,\lambda}$ and $H^*_{\alpha,\gamma}$:

$$H^*_{\mathcal{R}_0,\lambda} = \frac{\mathbb{V}\big(\mathbb{E}(H^*|\mathcal{R}_0,\lambda)\big)}{\mathbb{V}(H^*)} = \Big[\tau\mathbb{E}\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\Big]^2\frac{\mathbb{V}\big[\frac{1}{\mu_2+\lambda}(1-1/\mathcal{R}_0)\big]}{\mathbb{V}(H^*)}$$

$$H^*_{\alpha,\gamma} = \frac{\mathbb{V}\big(\mathbb{E}(H^*|\alpha,\gamma)\big)}{\mathbb{V}(H^*)} = \Big[\tau\mathbb{E}(1-1/\mathcal{R}_0)\mathbb{E}\big(\frac{1}{\mu_2+\lambda}\big)\Big]^2\frac{\mathbb{V}\Big[\frac{\alpha}{\mu_2+\gamma+\alpha}\Big]}{\mathbb{V}(H^*)}$$

we have already calculated all the terms in (52), (53), (32) and (51).
For Sobol indices $H^*_{\alpha,\lambda}$ et $H^*_{\gamma,\lambda}$:

$$H^*_{\alpha,\lambda} = \frac{\mathbb{V}\big(\mathbb{E}(H^*|\alpha,\lambda)\big)}{\mathbb{V}(H^*)} = \Big[\tau\mathbb{E}(1-1/\mathcal{R}_0)\Big]^2 \frac{\mathbb{V}\Big[\frac{\alpha}{\mu_2+\lambda}\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\Big]}{\mathbb{V}(H^*)}$$

$$H^*_{\gamma,\lambda} = \frac{\mathbb{V}\big(\mathbb{E}(H^*|\gamma,\lambda)\big)}{\mathbb{V}(H^*)} = \Big[\tau\mathbb{E}(1-1/\mathcal{R}_0)\Big]^2 \frac{\mathbb{V}\Big[\frac{1}{\mu_2+\lambda}\mathbb{E}_\gamma\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\Big]}{\mathbb{V}(H^*)}$$

we have everything we need in (32), (51), (41) and (56).

A.4 Sobol indices for R^*

Definition A.4. *Sobol indices of order 1 and 2 for R^* are defined:*

$$R^*_{X_i} = \frac{\mathbb{V}\big(\mathbb{E}(R^*|X_i)\big)}{\mathbb{V}(R^*)} \qquad i = 1, ..., 4 \tag{62}$$

$$R^*_{X_i,X_J} = \frac{\mathbb{V}\big(\mathbb{E}(R^*|X_i,X_j)\big)}{\mathbb{V}(R^*)} - R^*_{X_i} - R^*_{X_j}. \tag{63}$$

Remark A.1. *The expression of R^* is written in the following form:*

$$R^* = \frac{\tau}{\mu}\bigg(\frac{\gamma}{\mu_2+\gamma+\alpha} + \frac{\lambda}{(\mu_2+\lambda)}\frac{\alpha}{\mu_2+\gamma+\alpha}\bigg)\bigg(1-\frac{1}{\mathcal{R}_0}\bigg). \tag{64}$$

First we calculate the mean and the variance of R^*,

$$\begin{aligned}
&\mathbb{E}\bigg(\frac{\lambda}{\mu_2+\lambda}\bigg) = \mathbb{E}\bigg(1-\frac{\mu_2}{\mu_2+\lambda}\bigg) = 1-\mu_2\frac{T_4(b_\lambda)-T_4(a_\lambda)}{d_\lambda}, \\
&\mathbb{E}\big(\frac{\lambda^2}{(\mu_2+\lambda)^2}\big) = \frac{1}{d_\lambda}\bigg[d_\lambda - 2\mu_2\big[T_4(b_\lambda)-T_4(a_\lambda)\big] + \mu_2^2\frac{d_\lambda}{(\mu_2+b_\lambda)(\mu_2+a_\lambda)}\bigg].
\end{aligned} \tag{65}$$

For $\mathbb{E}\bigg(1-\frac{1}{\mathcal{R}_0}\bigg)$ see (32). For $\mathbb{E}\bigg(\frac{\alpha}{\mu_2+\gamma+\alpha}\bigg)$ see (52). And for $\mathbb{E}\bigg(\frac{\gamma}{\mu_2+\gamma+\alpha}\bigg)$ just permute α and γ in (52).

With these results, we have

$$\mathbb{E}(R^*) = \frac{\tau}{\mu}\bigg[\mathbb{E}\bigg(\frac{\gamma}{\mu_2+\gamma+\alpha}\bigg) + \mathbb{E}\bigg(\frac{\lambda}{(\mu_2+\lambda)}\bigg)\mathbb{E}\bigg(\frac{\alpha}{\mu_2+\gamma+\alpha}\bigg)\bigg]\mathbb{E}\bigg(1-\frac{1}{\mathcal{R}_0}\bigg)$$

For the variance of R^*, we calculate $\mathbb{E}\big((R^*)^2\big)$:

$$\begin{aligned}\mathbb{E}\big((R^*)^2\big) &= \frac{\tau^2}{\mu^2}\mathbb{E}\bigg((1-\frac{1}{\mathcal{R}_0})^2\bigg)\mathbb{E}\bigg[\big(\frac{\gamma}{(\mu_2+\gamma+\alpha)}\big)^2+\big(\frac{\lambda}{(\mu_2+\lambda)}\frac{\alpha}{(\mu_2+\gamma+\alpha)}\big)^2\\ &+2\frac{\lambda}{(\mu_2+\lambda)}\frac{\alpha}{(\mu_2+\gamma+\alpha)}\frac{\gamma}{(\mu_2+\gamma+\alpha)}\bigg]\\ &= \frac{\tau^2\mathbb{E}\big((1-\frac{1}{\mathcal{R}_0})^2\big)}{\mu^2}\big[\mathbb{E}\big(\frac{\gamma^2}{(\mu_2+\gamma+\alpha)^2}\big)+\mathbb{E}\big(\frac{\lambda^2}{(\mu_2+\lambda)^2}\big)\mathbb{E}\bigg(\frac{\alpha^2}{(\mu_2+\gamma+\alpha)^2}\bigg)\\ &+2\mathbb{E}\bigg(\frac{\lambda}{(\mu_2+\lambda)}\bigg)\mathbb{E}\bigg(\frac{\alpha}{(\mu_2+\gamma+\alpha)}\frac{\gamma}{(\mu_2+\gamma+\alpha)}\bigg)\bigg]\end{aligned}$$

with $\mathbb{E}\big[\big(\frac{\gamma}{\mu_2+\gamma+\alpha}\big)^2\big]$ et $\mathbb{E}\big[\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)^2\big]$ which are given in (53) and $\mathbb{E}\big((1-\frac{1}{\mathcal{R}_0})^2\big)$ given in (38).

Now let's calculate the last term:

$$\begin{aligned}\mathbb{E}\bigg(\frac{\alpha\gamma}{(\mu_2+\gamma+\alpha)^2}\bigg) &= \frac{1}{d_\alpha\times d_\gamma}\int_{a_\alpha}^{b_\alpha}\alpha\int_{a_\gamma}^{b_\gamma}\frac{\gamma}{(\mu_2+\gamma+\alpha)^2}d\gamma d\alpha\\ &= \frac{1}{d_\alpha\times d_\gamma}\int_{a_\alpha}^{b_\alpha}\alpha\big[\big(T_4(b_\gamma+\alpha)-T_4(a_\gamma+\alpha)\big)\\ &-(\mu_2+\alpha)\big(\frac{1}{\mu_2+b_\gamma+\alpha}-\frac{1}{\mu_2+a_\gamma+\alpha}\big)\big]d\alpha\end{aligned}\tag{66}$$

We calculate the first term:

$$\begin{aligned}\int_{a_\alpha}^{b_\alpha}\alpha\big(T_4(b_\gamma+\alpha)-T_4(a_\gamma+\alpha)\big)d\alpha &= \int_{a_\alpha}^{b_\alpha}\big[T_1(\alpha+b_\gamma)-(b_\gamma+\mu_2)\ln(\mu_2+b_\gamma+\alpha)\big]d\alpha\\ &-\int_{a_\alpha}^{b_\alpha}\big[T_1(\alpha+a_\gamma)-(a_\gamma+\mu_2)\ln(\mu_2+a_\gamma+\alpha)\big]d\alpha.\end{aligned}$$

To continue it is necessary to use the primitives of $\ln(x)$ et $x\ln(x)$. For the second term:

$$\begin{aligned}\int_{a_\alpha}^{b_\alpha}\frac{\alpha(\mu_2+\alpha)}{\mu_2+b_\gamma+\alpha}d\alpha &= \int_{a_\alpha}^{b_\alpha}\big[\alpha-b_\gamma\big(1-\frac{\mu_2+b_\gamma}{\mu_2+b_\gamma+\alpha}\big)\big]d\alpha\\ &= \frac{1}{b_\alpha-a_\alpha}\big[\alpha^2/2-b_\gamma\big(\alpha-(\mu_2+b_\gamma)\ln(\mu_2+b_\gamma+\alpha)\big)\big]_{a_\alpha}^{b_\alpha}.\end{aligned}$$

And the same for the last term we replace b_α by a_α.

Sobol indices of order 1

For Sobol indices $R^*_{\mathcal{R}_0}$ and R^*_λ, we have the following result for the mean:

$$\mathbb{E}(R^*|\mathcal{R}_0) = \frac{\tau(1-1/\mathcal{R}_0)}{\mu}\left[\mathbb{E}\left(\frac{\gamma}{\mu_2+\gamma+\alpha}\right)+\mathbb{E}\left(\frac{\lambda}{\mu_2+\lambda}\right)\mathbb{E}\left(\frac{\alpha}{\mu_2+\gamma+\alpha}\right)\right]$$

$$\mathbb{E}(R^*|\lambda) = \frac{\tau}{\mu}\mathbb{E}(1-\frac{1}{\mathcal{R}_0})\Big[\mathbb{E}\big(\frac{\gamma}{\mu_2+\gamma+\alpha}\big)+\frac{\lambda}{\mu_2+\lambda}\mathbb{E}\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\Big]$$

All these terms are well calculated. So the Sobol indices are:

$$R^*_{\mathcal{R}_0} = \frac{\mathbb{V}\big(\mathbb{E}(R^*|R_0)\big)}{\mathbb{V}(R^*)} = \frac{\tau^2}{\mu^2}\Big[\big[\mathbb{E}\big(\frac{\gamma}{\mu_2+\gamma+\alpha}\big)+\mathbb{E}\big(\frac{\lambda}{(\mu_2+\lambda)}\big)\mathbb{E}\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\big]\Big]^2\frac{\mathbb{V}\big(\frac{1}{\mathcal{R}_0}\big)}{\mathbb{V}(R^*)}$$

$$R^*_{\lambda} = \frac{\mathbb{V}\big(\mathbb{E}(R^*|\lambda)\big)}{\mathbb{V}(R^*)} = \frac{\tau^2}{\mu^2}\Big(\mathbb{E}(1-\frac{1}{\mathcal{R}_0})\Big)^2\frac{\mathbb{V}\Big[\big[\mathbb{E}\big(\frac{\gamma}{\mu_2+\gamma+\alpha}\big)+\big(\frac{\lambda}{\mu_2+\lambda}\big)\mathbb{E}\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\big]\Big]^2}{\mathbb{V}(R^*)}$$

where $\mathbb{V}\big(\frac{1}{\mathcal{R}_0}\big)$ and $\mathbb{V}\big(\frac{\lambda}{\mu_2+\lambda}\big)$ are given in (32) et (65).

For Sobol indices R^*_γ and R^*_α, we start by calculating the mean:

$$\mathbb{E}(R^*|\gamma) = \frac{\tau}{\mu_1}\Big[\gamma\mathbb{E}_\gamma\Big(\frac{1}{\mu_2+\gamma+\alpha}\Big)+\mathbb{E}\big(\frac{\lambda}{(\mu_2+\lambda)}\big)\mathbb{E}_\gamma\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)\Big]\mathbb{E}(1-\frac{1}{\mathcal{R}_0})$$

$$\mathbb{E}(R^*|\alpha) = \frac{\tau}{\mu_1}\Big[\mathbb{E}_\alpha\big(\frac{\gamma}{\mu_2+\gamma+\alpha}\big)+\mathbb{E}\big(\frac{\lambda}{(\mu_2+\lambda)}\big)\alpha\mathbb{E}_\alpha\big(\frac{1}{\mu_2+\gamma+\alpha}\big)\Big]\mathbb{E}(1-\frac{1}{\mathcal{R}_0})$$

To facilitate the calculation and the notations, we consider

$$M_1 = \mathbb{E}\Big(\frac{\lambda}{\mu_2+\lambda}\Big)$$

$$Q_1 = \gamma\,\mathbb{E}_\gamma\Big(\frac{1}{\mu_2+\gamma+\alpha}\Big)+\mathbb{E}\Big(\frac{\lambda}{\mu_2+\lambda}\Big)\mathbb{E}_\gamma\Big(\frac{\alpha}{\mu_2+\gamma+\alpha}\Big)$$

$$Q_2 = \mathbb{E}_\alpha\Big(\frac{\gamma}{\mu_2+\gamma+\alpha}\Big)+\mathbb{E}\Big(\frac{\lambda}{\mu_2+\lambda}\Big)\alpha\mathbb{E}_\alpha\Big(\frac{1}{\mu_2+\gamma+\alpha}\Big)$$

For expressions $\mathbb{E}_\alpha$ and $\mathbb{E}_\gamma$ see (55).

To calculate $\mathbb{V}\big(Q_1\big)$, just calculate $\mathbb{E}\big[\big(Q_1\big)^2\big]$ ($\mathbb{E}\big[Q_1\big]$ can be easily computed)

$$\mathbb{E}\big[\big(Q_1\big)^2\big] = \frac{1}{d_\alpha^2}\mathbb{E}\Big[\big((\gamma-M_1(\mu_2+\gamma))(T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)\big)+M_1\times d_\alpha\big)^2\Big]. \tag{67}$$

After simplification we find: ‘

$$\begin{aligned}\mathbb{E}\big[\big(Q_1\big)^2\big] = \frac{1}{d_\alpha^2}\mathbb{E}\Big[\big(\big(T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)\big)^2\big(\gamma(1-M_1)-M_1\mu_2\big)^2\Big],\\ +\frac{2M_1}{d_a}\mathbb{E}\Big[\big((T_4(\gamma+b_\alpha)T_4(\gamma+a_\alpha)\big)\big(\gamma(1-M_1)-M_1\mu_2\big)\Big]+M_1^2.\end{aligned}$$

To calculate the second term, it suffices to reduce under the form $X\ln(X)$.

$$\begin{aligned}
&\mathbb{E}\Big[\big((T_4(\gamma+b_\alpha)\big)\big(\gamma(1-M_1)-M_1\mu_2\big)\Big]=\\
&\frac{(1-M_1)}{d_g}\Big[\frac{1}{2}\Big[T_2(b_\gamma+b_\alpha)-T_2(a_\gamma+b_\alpha)\Big]-(\mu_2+b_\alpha)\Big[T_1(b_\gamma+b_\alpha)-T_1(a_\gamma+b_\alpha)\Big]\\
&+\frac{1}{4}\big(-b_\gamma(b_\gamma+2\mu_2+2b_\alpha)+a_\gamma(a_\gamma+2\mu_2+2b_\alpha)\big)+d_\gamma\Big]\\
&\frac{-M_1\mu_2}{d_g}\Big[T_1(b_\gamma+b_\alpha)-T_1(a_\gamma+b_\alpha)-d_g\Big].
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\big[\big((T_4(\gamma+a_\alpha)\big)\big(\gamma(1-M_1)-M_1\mu_2\big)\big]=\\
&\frac{(1-M_1)}{d_g}\Big[\frac{1}{2}\big[T_2(b_\gamma+a_\alpha)-T_2(a_\gamma+a_\alpha)\big]-(\mu_2+a_\alpha)\big[T_1(b_\gamma+a_\alpha)-T_1(a_\gamma+a_\alpha)\Big]\\
&+\frac{1}{4}\big(-b_\gamma(b_\gamma+2\mu_2+2a_\alpha)+a_\gamma(a_\gamma+2\mu_2+2a_\alpha)\big)+d_\gamma\Big]\\
&\frac{-M_1\mu_2}{d_g}\Big[T_1(b_\gamma+a_\alpha)-T_1(a_\gamma+a_\alpha)-d_\gamma\Big].
\end{aligned}$$

We make the difference and we get:

$$\begin{aligned}
&Q_3=\mathbb{E}\Big[\big((T_4(\gamma+b_\alpha)-T_4(\gamma+a_\alpha)\big)\big(\gamma(M_1-1)-M_1\mu_2\big)\Big]=\\
&\frac{(1-M_1)}{d_g}\Big[\frac{1}{2}\big[T_2(b_\gamma+b_\alpha)-T_2(a_\gamma+b_\alpha)-T_2(b_\gamma+a_\alpha)+T_2(a_\gamma+a_\alpha)\Big]\\
&-(\mu_2+b_\alpha)\big[T_1(b_\gamma+b_\alpha)-T_1(a_\gamma+b_\alpha)\Big]+(\mu_2+a_\alpha)\big[T_1(b_\gamma+a_\alpha)-T_1(a_\gamma+a_\alpha)\Big]\\
&-\frac{1}{2}d_\alpha\times d_\gamma\Big]-\frac{M_1\mu_2}{d_\gamma}\Big[T_1(b_\gamma+b_\alpha)-T_1(a_\gamma+b_\alpha)+T_1(b_\gamma+a_\alpha)-T_1(a_\gamma+a_\alpha)\big)\Big].
\end{aligned}$$

For the first term, we consider $F_3(X)$ a primitive of the function $\big(\ln(X+A)-\ln(X+B)\big)^2\big(X.C+D\big)^2$. Finally we have,

$$\mathbb{E}\big[\big(Q_1\big)^2\big]=\frac{1}{d_\alpha^2}\big(F_3(b_\gamma)-F_3(a_\alpha)\big)+\frac{2M_1}{d_\alpha}Q_3+M_1^2$$

with $A=\mu_2+b_\alpha$, $B=\mu_2+a_\alpha$, $C=d_\alpha-M_1$ and $D=-M_1\mu_2$.
Finally, we obtain :

$$\mathbb{V}\big[\mathbb{E}(R^*|\gamma)\big]=\frac{\tau^2}{\mu^2}\Big[\mathbb{E}(1-1/\mathcal{R}_0)\Big]^2\Big[\mathbb{E}\big(Q_1^2\big)-\big[\mathbb{E}\big(Q_1\big)\big]^2\Big]$$

The Sobol indice R^*_γ:

$$R^*_\gamma = \frac{\mathbb{V}\big[\mathbb{E}(R^*|\gamma)\big]}{\mathbb{V}(R^*)}$$

The Sobol indice R^*_α: we proceed in the same way with:

$$\mathbb{V}\bigg[\mathbb{E}(R^*|\alpha)\bigg] = \frac{\tau^2}{\mu^2}\bigg[\mathbb{E}(1-1/\mathcal{R}_0)\bigg]^2\bigg[\mathbb{E}\Big(Q_2^2\Big)\bigg] - \bigg[\mathbb{E}\big(Q_2\big)\big]^2\bigg],$$

$$R^*_\alpha = \frac{\mathbb{V}\big[\mathbb{E}(R^*|\alpha)\big]}{\mathbb{V}(R^*)}$$

For the calculation of $\mathbb{E}\big(Q_2^2\big)$ we proceed as for $\mathbb{E}\big(Q_1^2\big)$ knowing that :

$$\mathbb{E}\bigg[\big(Q_2\big)^2\bigg] = \frac{1}{d_\gamma^2}\mathbb{E}\bigg[\Big((\alpha.M_1 - (\mu_2+\alpha))(T_4(\alpha+b_\gamma) - T_4(\alpha+a_\gamma)\Big) + d_\gamma\Big)^2\bigg]. \tag{68}$$

Sobol indices of order 2

We consider some notations of known expressions:

- $K_0 = \mathbb{E}\bigg(\frac{1}{\mu_2+\gamma+\alpha}\bigg)$
- $K_1 = \mathbb{E}\bigg(\frac{\gamma}{\mu_2+\gamma+\alpha}\bigg)$
- $K_2 = \mathbb{E}\big(\frac{\alpha}{\mu_2+\gamma+\alpha}\big)$
- $K_{1,1} = \mathbb{E}\bigg(\frac{\gamma^2}{(\mu_2+\gamma+\alpha)^2}\bigg)$
- $K_{2,2} = \mathbb{E}\bigg(\frac{\alpha^2}{(\mu_2+\gamma+\alpha)^2}\bigg)$
- $K_3 = \mathbb{E}\bigg[(1-1/R_0)^2\bigg]$
- $K_4 = \mathbb{E}\bigg[1-1/R_0\bigg]$
- $K_5 = \mathbb{E}\bigg[\frac{\lambda^2}{(\mu_2+\lambda)^2}\bigg]$
- $K_6 = \mathbb{E}\bigg[\frac{\lambda}{\mu_2+\lambda}\bigg]$
- $K_7 = \mathbb{E}\bigg(\frac{\alpha\gamma}{(\mu_2+\gamma+\alpha)^2}\bigg)$

- $K_8 = \mathbb{E}_\alpha\left(\frac{\gamma}{\mu_2+\gamma+\alpha}\right)$
- $K_{8,8} = \mathbb{E}_\alpha\left(\frac{1}{\mu_2+\gamma+\alpha}\right)$
- $K_9 = \mathbb{E}_\gamma\left(\frac{\alpha}{\mu_2+\gamma+\alpha}\right)$
- $K_{9,9} = \mathbb{E}_\gamma\left(\frac{1}{\mu_2+\gamma+\alpha}\right)$
- $K_{10} = \mathbb{E}\left[\left(\mathbb{E}_\gamma\left(\frac{\alpha}{\mu_2+\gamma+\alpha}\right)\right)^2\right]$
- $K_{11} = \mathbb{E}\left[\left(\mathbb{E}_\alpha\left(\frac{\gamma}{\mu_2+\gamma+\alpha}\right)\right)^2\right]$
- $K_{12} = \mathbb{E}\left[\left(\mathbb{E}_\gamma\left(\frac{1}{\mu_2+\gamma+\alpha}\right)\right)^2\right]$
- $K_{13} = \mathbb{E}\left[\left(\mathbb{E}_\alpha\left(\frac{1}{\mu_2+\gamma+\alpha}\right)\right)^2\right]$

K_0 is given by(35), K_1 and K_2 are given by (52), K_3 is given by (38), K_4 is given by (32), K_5 and K_6 are given by (65), K_7 is given by (66), for K_8 and K_9 are given by (55), for $K_{8,8}$ and $K_{9,9}$ are given byr (41), $K_{1,1}$ and $K_{2,2}$ given by (53), for K_{10} and K_{11} are given by (57), for K_{12} and K_{13} are given by (55).

We start with the indice $R^*_{\mathcal{R}_0,\lambda}$. We have:

$$\mathbb{E}\big(R^*|\mathcal{R}_0,\lambda\big) = \frac{\tau}{\mu}\left(1-1/\mathcal{R}_0\right)\left(K_1 + \frac{\lambda}{\mu_2+\lambda}K_2\right)$$

The variance of the previous term:

$$\mathbb{V}\left[\mathbb{E}\big(R^*|\mathcal{R}_0,\lambda\big)\right] = \frac{\tau^2}{\mu^2}\left[K_3\mathbb{E}\big[\big(K_1+\frac{\lambda}{\mu_2+\lambda}K_2\big)^2\big] - \big[K_4\big(K_1+K_6.K_2\big)\big]^2\right]$$

Then :

$$R^*_{\mathcal{R}_0,\lambda} = \frac{\mathbb{V}\big[\mathbb{E}\big(R^*|\mathcal{R}_0,\lambda\big)\big]}{\mathbb{V}(R^*)}$$

For the Sobol indice $R^*_{\alpha,\gamma}$. We have :

$$\mathbb{E}\big(R^*|\alpha,\gamma\big) = \frac{\tau}{\mu}K_4\left(\frac{\gamma}{\mu_2+\gamma+\alpha} + \frac{\alpha}{\mu_2+\gamma+\alpha}K_6\right)$$

Its variance is equal:

$$\mathbb{V}\big[\mathbb{E}\big(R^*|\alpha,\gamma\big)\big] = \frac{\tau^2 K_4^2}{\mu^2}\bigg[\bigg(K_{1,1} + K_6^2 K_{2,2} + 2K_6 K_7\bigg) - (K_1 + K_6 K_2)^2\bigg]$$

$$R^*_{\alpha,\gamma} = \frac{\mathbb{V}\big[\mathbb{E}\big(R^*|\alpha,\gamma\big)\big]}{\mathbb{V}(R^*)}$$

For the Sobol indice $R^*_{\mathcal{R}_0,\alpha}$.

$$\mathbb{E}\bigg(R^*|\mathcal{R}_0,\alpha\bigg) = \frac{\tau}{\mu}\bigg(1 - 1/\mathcal{R}_0\bigg)\bigg(K_8 + \alpha K_6 K_{8,8}\bigg)$$

Its variance is equal:

$$\mathbb{V}\big[\mathbb{E}\big(R^*|\mathcal{R}_0,\alpha\big)\big] = \frac{\tau^2}{\mu^2}\bigg[K_3\mathbb{E}\bigg(K_8^2 + \alpha^2 K_6^2 K_{8,8}^2 + 2\alpha K_6 K_8 K_{8,8}\bigg) - K_4^2(K_1 + K_6\mathbb{E}(\alpha K_{8,8}))^2\bigg],$$

because $\mathbb{E}(K_8) = K_1$ and $\mathbb{E}(\alpha K_{8,8})$ is given by (61).

As $\mathbb{E}(K_8^2) = K_{11}$ then the variance becomes:

$$\mathbb{V}\big[\mathbb{E}\big(R^*|\mathcal{R}_0,\alpha\big)\big] = \frac{\tau^2}{\mu^2}\bigg[K_3\bigg(K_{11} + K_6^2\mathbb{E}[\alpha^2 K_{8,8}^2] + 2K_6\mathbb{E}[\alpha K_8 K_{8,8}]\bigg) - K_4^2(K_1 + K_6\mathbb{E}(\alpha K_{8,8}))^2\bigg],$$

The term $\mathbb{E}[\alpha^2 K_{8,8}^2]$ is given in (59) and we now calculate $\mathbb{E}[\alpha K_8 K_{8,8}]$.

$$\begin{aligned}
\mathbb{E}[\alpha K_8 K_{8,8}] &= \mathbb{E}\bigg[\alpha\big(1 - \frac{\mu_2 + \alpha}{d_\gamma}\big[T_4(\alpha + b_\gamma) - T_4(\alpha + a_\gamma)\big]\big)\frac{1}{d_\gamma}\big[T_4(\alpha + b_\gamma) - T_4(\alpha + a_\gamma)\big]\bigg] \\
&= \frac{1}{d_\gamma}\mathbb{E}\bigg[\alpha\big[T_4(\alpha + b_\gamma) - T_4(\alpha + a_\gamma)\big]\bigg] \\
&- \frac{1}{d_\gamma^2}\mathbb{E}\bigg[\alpha(\mu_2 + \alpha)\big[T_4(\alpha + b_\gamma) - T_4(\alpha + a_\gamma)\big]^2\bigg]
\end{aligned}$$

The term $\mathbb{E}\bigg[\alpha\big[T_4(\alpha + b_\gamma) - T_4(\alpha + a_\gamma)\big]\bigg]$ is given in (61). For the second term, we do exactly the same way we treated the term in (59).
We note $P_1 = \mathbb{E}[\alpha^2 K_{8,8}^2]$, $P_2 = \mathbb{E}[\alpha K_8 K_{8,8}]$ et $P_3 = \mathbb{E}(\alpha K_{8,8})$ The Sobol indice $R^*_{\mathcal{R}_0,\alpha}$:

$$R^*_{\mathcal{R}_0,\alpha} = \frac{\mathbb{V}\big[\mathbb{E}\big(R^*|\mathcal{R}_0,\alpha\big)\big]}{\mathbb{V}(R^*)} = \frac{\tau^2}{\mu^2}\frac{\bigg[K_3\bigg(K_{11}^2 + K_6^2 P_1 + 2K_6 P_2\bigg) - K_4^2(K_1 + K_6 P_3)^2\bigg]}{\mathbb{V}(R^*)}$$

For the Sobol indice $R^*_{\mathcal{R}_0,\gamma}$,

$$\mathbb{E}\bigg(R^*|\mathcal{R}_0,\gamma\bigg) = \frac{\tau}{\mu}\bigg(1 - 1/\mathcal{R}_0\bigg)\bigg(\gamma K_{9,9} + K_6 K_9\bigg)$$

Its variance is equal:

$$\mathbb{V}\big[\mathbb{E}\big(R^*|\mathcal{R}_0,\gamma\big)\big] = \frac{\tau^2}{\mu^2}\bigg[K_3\mathbb{E}\bigg(\gamma^2K_{9,9}^2 + K_6^2K_9^2 + 2\gamma K_6K_{9,9}K_9\bigg) - K_4^2(\mathbb{E}(\gamma K_{9,9}) + K_6K_2)^2\bigg],$$

We note $U_1 = \mathbb{E}(\gamma K_{9,9})$ is given in (61). The term $\mathbb{E}(K_9^2) = K_{10}$. We also note $U_2 = \mathbb{E}[\gamma^2K_{9,9}^2]$ which is given in (59) and we calculate $U_3 = \mathbb{E}[\gamma K_9K_{9,9}]$ the same way as (61). The Sobol indice $R^*_{\mathcal{R}_0,\gamma}$:

$$R^*_{\mathcal{R}_0,\gamma} = \frac{\mathbb{V}\big[\mathbb{E}\big(G_1|\mathcal{R}_0,\gamma\big)\big]}{\mathbb{V}(G_1)} = \frac{\tau^2}{\mu^2}\frac{\bigg[K_3\bigg(U_2 + K_6^2K_{10} + 2K_6U_3\bigg) - K_4^2(U_1 + K_6K_2)^2\bigg]}{\mathbb{V}(R^*)}$$

For the Sobol indice $R^*_{\alpha,\lambda}$,

$$\mathbb{E}\big(R^*|\alpha,\lambda\big) = \frac{\tau}{\mu}K_4\bigg(K_8 + \frac{\lambda}{\mu_2+\lambda}\alpha K_{8,8}\bigg)$$

Its variance is equal:

$$\begin{aligned}\mathbb{V}\bigg[\mathbb{E}\bigg(R^*|\alpha,\lambda\bigg)\bigg] &= \frac{\tau^2}{\mu^2}K_4^2\bigg[\mathbb{E}\bigg(K_8^2 + \frac{\lambda^2}{(\mu_2+\lambda)^2}\alpha^2K_{8,8}^2 + 2\alpha K_8K_{8,8}\frac{\lambda}{\mu_2+\lambda}\bigg) - (K_1 + K_6P_3)^2\bigg]\\ &= \frac{\tau^2}{\mu^2}K_4^2\bigg[\bigg(K_{11} + K_5P_1 + 2P_2K_6\bigg) - (K_1 + K_6P_3)^2\bigg].\end{aligned}$$

The Sobol indice $R^*_{\alpha,\lambda}$:

$$R^*_{\alpha,\lambda} = \frac{\mathbb{V}\big[\mathbb{E}\big(R^*|\alpha,\lambda\big)\big]}{\mathbb{V}(R^*)} = \frac{\tau^2}{\mu^2}K_4^2\frac{\bigg[\bigg(K_{11} + K_5P_1 + 2P_2K_6\bigg) - (K_1 + K_6P_3)^2\bigg]}{\mathbb{V}(R^*)}$$

For the Sobol indice $R^*_{\gamma,\lambda}$,

$$\mathbb{E}\bigg(R^*|\gamma,\lambda\bigg) = \frac{\tau}{\mu}K_4\bigg(\gamma K_{9,9} + \frac{\lambda}{\mu_2+\lambda}\alpha K_9\bigg)$$

Its variance is equal:

$$\begin{aligned}\mathbb{V}\big[\mathbb{E}\big(R^*|\gamma,\lambda\big)\big] &= \frac{\tau^2}{\mu^2}K_4^2\bigg[\mathbb{E}\bigg(\gamma^2K_{9,9}^2 + \frac{\lambda^2}{(\mu_2+\lambda)^2}K_9^2 + 2\gamma K_9K_{9,9}\frac{\lambda}{\mu_2+\lambda}\bigg) - (U_1 + K_6k_2)^2\bigg]\\ &= \frac{\tau^2}{\mu^2}K_4^2\bigg[\bigg(U_2 + K_5K_{10} + 2U_3K_6\bigg) - (K_1 + K_6P_3)^2\bigg].\end{aligned}$$

The Sobol indice $R^*_{\gamma,\lambda}$:

$$R^*_{\gamma,\lambda} = \frac{\mathbb{V}\bigg[\mathbb{E}\bigg(R^*|\gamma,\lambda\bigg)\bigg]}{\mathbb{V}(R^*)} = \frac{\tau^2}{\mu^2}K_4^2\frac{\bigg[\bigg(U_2 + K_5K_{10} + 2U_3K_6\bigg) - (K_1 + K_6P_3)^2\bigg]}{\mathbb{V}(R^*)}$$