Homogenization of ferrofluid flow models in porous media with Langevin magnetization law

Youcef Amirat, Kamel Hamdache

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Abstract

The paper is concerned with the homogenization of the equations describing the flow of a ferrofluid through a heterogeneous porous medium Ω in the presence of an applied magnetic field. We discuss two models where the magnetization M is parallel to the magnetic field H. In the first one M and H satisfy the relation $M = \lambda_0 \, 1_{\Omega_f} H$ in Ω, where λ_0 is a positive constant and 1_{Ω_f} is the characteristic function of Ω_f (the pore space). In the second model, M and H satisfy the Langevin magnetization law $M = M_s \frac{L(b_1 |H|)}{|H|} 1_{\Omega_f} H$, where L is the Langevin function given by $L(x) = \frac{1}{\tanh x} - \frac{1}{x}$, M_s is the saturation magnetization and b_1 is a positive physical constant. The velocity and the pressure satisfy the Stokes equation with a Kelvin magnetic force. We perform the homogenization of the equations of each of the two models. Using the two-scale convergence method, we rigorously derive the homogenized equation for the magnetic potential and determine the asymptotic limit of the magnetization. Then we rigorously derive a Darcy law.

Keywords. Ferrofluid flow in porous media; Stokes equations; Langevin magnetization law; homogenization; two-scale convergence; Darcy law

1 Introduction

Magnetic fluids (also called ferrofluids) are colloidal suspensions of nanoscale magnetic particles in a carrier fluid. Since their physical properties can be easily influenced by an external magnetic field, they have found a wide variety of applications in technology, industry and medicine, see [44]. A potential application of ferrofluids is found in the subsurface environmental engineering, in which externally applied magnetic fields are used to direct and control the flow of ferrofluids under the ground, see [39]. In the past years, ferrofluid flow in porous media has been the subject of various experimental and numerical studies, see [28] and the references therein.

An important tool for modeling flows in heterogeneous porous media is the homogenization theory, that allows to derive equations describing the macroscopic behavior of the flows, from the equations of fluid mechanics valid in the pore space. The most widely used methods for the derivation of macroscopic equations for periodic heterogeneous porous media are the method of multiscale expansions [10, 11, 13, 45], the two-scale convergence method [3, 34, 36], and the periodic unfolding method [17]. Ene and Sanchez-Palencia [23] derived the Darcy law, from the Stokes system, by using a formal multiscale expansion. The rigorous mathematical derivation of the Darcy law was given by Tartar [47], using the method of oscillating test functions. The explicit expression for the pressure extension was given by Lipton and Avellaneda [33]. Several works have been devoted to the derivation of Darcy's law [2, 4, 18, 27, 33]. Homogenization techniques have been developed to treat more

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general problems: porous medium with double porosity [8], nonlinear filtration law [7, 35], multiphase flows [5], non-Newtonian flows [12], interface problems [29], MHD flows [6, 25], etc.

In the present paper we study the homogenization of the equations describing the flow of a ferrofluid through a porous medium in the presence of an applied magnetic field. We consider two simple models where the magnetization is parallel to the magnetic field: a linear model and a nonlinear model.

2 Problem formulation

We consider the flow of a ferrofluid through a porous medium in the presence of an applied magnetic field. The porous medium is denoted by Ω, a smooth bounded and simply connected domain of $\mathbb{R}^3$, and is composed of a solid part and a pore space that is filled by the ferrofluid. The solid part is denoted by Ω_s, the pore space by Ω_f, and the fluid/solid interface is denoted by Γ. We assume for simplicity that Ω_s is strictly included in Ω.

We denote by U the velocity of the fluid acting in Ω_f, H_a the applied magnetic field, M the magnetization in the fluid which vanishes outside Ω_f, H_d the demagnetizing magnetic field, and B the magnetic induction given by

$$B = \begin{cases} \mu_f(H_d + H_a + M) \ \text{ in } \Omega_f, \\ \mu_s(H_d + H_a) \ \text{ in } \ \Omega_s, \\ \mu_0(H_d + H_a) \ \text{ in } \ \mathbb{R}^3 \setminus \Omega, \end{cases}$$

where μ_f, μ_s and μ_0 are the magnetic permeabilities in Ω_f, Ω_s and in vacuum, respectively. We assume that μ_f, μ_s and μ_0 are positive constants. We denote by H the magnetizing magnetic field defined by

$$H = H_d + H_a.$$

The magnetic induction and the (magnetizing) magnetic field are assumed to satisfy the equations of magnetostatics

$$\operatorname{div} B = 0, \quad \operatorname{curl} H = 0 \ \text{ in } \mathbb{R}^3. \tag{1}$$

It results from (1) that on the solid-liquid interface $\Gamma = \partial\Omega_s$ we have that

$$[H \times \nu_\Gamma] = 0, \quad [B \cdot \nu_\Gamma] = 0 \ \text{ on } \Gamma. \tag{2}$$

Here ν_Γ is the unit normal vector to Γ pointing from Ω_s to Ω_f. The brackets $[\cdot]$ stand for the jump across the solid-fluid interface. More precisely, denoting by v_f and v_s the values of $\mathbf{v}$ on either side of the surface Γ, respectively, in the fluid and solid domains, we set $[v] = v_f - v_s$. Equation $(2)_1$ expresses the continuity of the tangential component of the magnetic field H and $(2)_2$ expresses the continuity of the normal component of the magnetic induction. We also have

$$[H \times \nu] = 0, \quad [B \cdot \nu] = 0 \ \text{ on } \partial\Omega \tag{3}$$

where ν is the unit outward normal vector to $\partial\Omega$.

We assume that $H_d = 0 \ \text{ in } \mathbb{R}^3 \setminus \Omega$, then we derive from (3) the boundary condition

$$\mu_f(H + M) \cdot \nu = \mu_0 H_a \cdot \nu \ \text{ on } \partial\Omega.$$

We assume that the magnetic field and the magnetization are parallel [38] (Section 2.2), [44] (Section 2.7) and [46]. We consider two cases, in the first one M and H satisfy the relation

$$M = \lambda_0 \, 1_{\Omega_f} H \ \text{ in } \Omega \tag{4}$$

where λ_0 is a positive constant and 1_{Ω_f} is the characteristic function of Ω_f. This case leads to a linear model. In the second case, M and H satisfy the Langevin magnetization law [1, 43]

$$M = M_s \frac{L(b_1 |H|)}{|H|} 1_{\Omega_f} H, \quad L(x) = \frac{1}{\tanh(x)} - \frac{1}{x}, \ x \in \mathbb{R}, \tag{5}$$

where L is the Langevin function, M_s is the saturation magnetization, $b_1 = \frac{\mu_f m}{k_B T}$, m is the magnetic moment of the particule, T denotes the temperature and k_B is the Boltzmann's constant. We assume that M_s and b_1 are positive constants. This case leads to a nonlinear model. Some other relations are used in practical situations, for example M and H satisfy the relation [39]

$$M = a_1 \frac{\arctan(b_1|H|)}{|H|} 1_{\Omega_f} H,$$

with a_1 and b_1 positive constants. This case leads to a nonlinear model.

2.1 The linear model

Assume that M and H satisfy relation (4). Let μ denote the function defined a.e. in Ω by

$$\mu = \begin{cases} \mu_f(1+\lambda_0) & \text{in } \Omega_f, \\ \mu_s & \text{in } \Omega_s, \end{cases} \tag{6}$$

then we have $B = \mu H$ and H is a solution of

$$\begin{cases} \operatorname{div}(\mu H) = 0, \quad \operatorname{curl} H = 0 & \text{in } \Omega, \\ \mu_f(1+\lambda_0) H \cdot \nu = \mu_0 H_a \cdot \nu & \text{on } \partial\Omega. \end{cases}$$

Since Ω is simply connected, there is $\varphi \in H^1(\Omega)$ such that $H = \nabla\varphi$, see for instance [26] (Theorem 2.9, p. 31). Then φ is a solution of

$$\begin{cases} \operatorname{div}(\mu\nabla\varphi) = 0 & \text{in } \Omega, \\ \mu_f(1+\lambda_0)\nabla\varphi \cdot \nu = \mu_0 H_a \cdot \nu & \text{on } \partial\Omega, \quad \displaystyle\int_\Omega \varphi(x)\,dx = 0. \end{cases} \tag{7}$$

The constraint of zero mean on φ is imposed in order to ensure the uniqueness of solutions of this problem.

The fluid velocity U and the pressure p satisfy the incompressible Stokes equations in the absence of inertial terms

$$\begin{cases} -\eta\Delta U + \nabla p = \mu_f (M \cdot \nabla) H + g, \quad \operatorname{div} U = 0 & \text{in } \Omega_f, \\ U = 0 & \text{on } \partial\Omega_f, \end{cases} \tag{8}$$

where η is the dynamic viscosity, g the external force and the term $\mu_f(M \cdot \nabla)H$ represents the Kelvin body force.

2.1.1 Local description and adimensionalization

We assume that the porous medium has a periodic microstructure. Let l and L denote the characteristic sizes of the heterogeneities and the domain Ω, respectively. We set $\varepsilon = \frac{l}{L}$ and assume $\varepsilon << 1$ small enough.

Let $Y = (0,1)^3$ denote the unit cell. Let Y_s (the solid part) be a closed smooth subset of Y with a strictly positive measure. The fluid part is given by $Y_f = Y \backslash Y_s$. We denote for each $k \in \mathbb{Z}^3$:

$$Y^k = Y + k, \quad Y_s^k = Y_s + k, \quad Y_f^k = Y_f + k,$$

then define the sets

$$X_\varepsilon^k = \left\{x : \frac{x}{\varepsilon L} \in Y^k\right\}, \quad X_{\varepsilon,s}^k = \left\{x : \frac{x}{\varepsilon L} \in Y_s^k\right\}, \quad X_{\varepsilon,f}^k = \left\{x : \frac{x}{\varepsilon L} \in Y_f^k\right\}.$$

The physical (i.e. dimensional) solid and fluid regions are defined as

$$X_s^\varepsilon = \cup_k X_{\varepsilon,s}^k, \quad X_f^\varepsilon = \mathbb{R}^3 \backslash X_s^\varepsilon.$$

Obviously, X_s^ε is a closed subset of $\mathbb{R}^3$ and X_f^ε is an open subset of $\mathbb{R}^3$. Moreover, X_f^ε is a connected domain, while X_s^ε is formed by separate closed subsets of $\mathbb{R}^3$. We assume that Ω is an open simply connected domain of class C^2. We introduce the fluid and solid domains

$$\Omega_f^\varepsilon = \Omega \backslash \bigcup_{k \in I^\varepsilon} X_{k,s}^\varepsilon, \quad \Omega_s^\varepsilon = \Omega \backslash \overline{\Omega_f^\varepsilon},$$

where

$$I^\varepsilon = \{k : X_{\varepsilon,s}^k \subset \Omega\}.$$

The solid-fluid interface Γ_x^ε is defined by

$$\Gamma = \partial Y_s, \quad \Gamma^k = \Gamma + k, \quad \Gamma_{x,\varepsilon}^k = \left\{x : \frac{x}{\varepsilon L} \in \Gamma^k\right\}, \quad \Gamma_x^\varepsilon = \cup_k \Gamma_{x,\varepsilon}^k.$$

For convenience, the interface between the solid and fluid domains $\Omega \cap \Gamma_x^\varepsilon$ is still denoted Γ_x^ε.

In order to express equations (4), (6)–(8) in dimensionless form, we introduce a change of variables:

$$\begin{aligned} &x' = x/L,\ U' = U/u_c,\ p' = p/p_c,\ M' = M/m_c,\ H' = H/h_c, \\ &H_a' = H_a/h_c,\ g' = g/g_c,\ \mu_i' = \mu_i/\mu_{ic}\ (i = s, f, 0), \end{aligned} \tag{9}$$

where the variables indexed by c denote reference values and the variables with the prime superscript denote dimensionless values, respectively. We denote by Ω' and Γ' the image of Ω and Γ, respectively, under the change of variable $x \to x'$. We choose $m_c = h_c$ and $\mu_{sc} = \mu_{fc} = \mu_{0c}$. Using the above change of variables, equation (4) reads

$$M' = \lambda_0\, 1_{\Omega_f'} H' \ \text{ in } \Omega'. \tag{10}$$

The dimensionless equations of magnetostatics are

$$\begin{cases} \operatorname{div}'(\mu' H') = 0, \quad \operatorname{curl}' H' = 0 \ \text{ in } \Omega', \\ \mu_f'(1+\lambda_0) H' \cdot \nu = \mu_0' H_a' \cdot \nu \ \text{ on } \partial\Omega', \end{cases}$$

and equation (7) becomes

$$\begin{cases} \operatorname{div}'(\mu' \nabla' \varphi') = 0 \ \text{ in } \Omega', \\ \mu_f'(1+\lambda_0) \nabla' \varphi' \cdot \nu = \mu_0' H_a' \cdot \nu \ \text{ on } \partial\Omega', \quad \displaystyle\int_{\Omega'} \varphi'(x)\, dx = 0, \end{cases} \tag{11}$$

where φ' is the magnetic potential associated with H', *i.e.* $H' = \nabla' \varphi'$. We have

$$-\frac{\eta u_c}{L^2} \Delta' U' + \frac{p_c}{L} \nabla' p' = \frac{\mu_f m_c h_c}{L} (M' \cdot \nabla') H' + g_c\, g',$$

dividing by $\frac{\rho_f u_c^2}{L}$ we obtain

$$-\frac{1}{Re} \Delta' U' + Eu\, \nabla' p' = \frac{1}{Re_m} (M' \cdot \nabla') H' + Fr\, g' \quad \text{in } \Omega_f',$$

with

$$Re = \frac{\rho_f u_c L}{\eta}, \quad Eu = \frac{p_c}{\rho_f u_c^2}, \quad Re_m = \frac{\rho_f u_c^2}{\mu_f h_c^2}, \quad Fr = \frac{L g_c}{\rho_f u_c^2}.$$

Here, Re is the Reynolds number, Rm is the magnetic Reynolds number, Eu is the Euler number and Fr is the Froude number. In the sequel we take $Re = \frac{1}{\varepsilon^2}$, $Fr = Re_m = Eu = 1$, then obtain the dimensionless Stokes equations

$$\begin{cases} -\varepsilon^2 \Delta' U' + \nabla' p' = (M' \cdot \nabla') H' + g' \quad \text{in } \Omega'_f, \\ \operatorname{div}' U' = 0 \quad \text{in } \Omega'_f, \\ U' = 0 \quad \text{on } \partial\Omega'_f. \end{cases} \tag{12}$$

In what follows we omit the prime index.

2.1.2 Problem $(\mathcal{P}_1^\varepsilon)$

Our objective is to perform an asymptotic analysis of equations (10)–(12), as $\varepsilon \to 0$. Let us note that in the Stokes equation (12), the Kelvin force is a nonlinear function of H, since $(M \cdot \nabla)H = \frac{\lambda_0}{2}\nabla(|H|^2)$, as we will see below. Then our study will focus on the asymptotic behavior of $|H|^2$.

We introduce the following formulation of the problem. The magnetization M^ε is given by

$$M^\varepsilon = \lambda_0 \, 1_{\Omega_f^\varepsilon} H^\varepsilon \quad \text{in } \Omega, \tag{13}$$

where $1_{\Omega_f^\varepsilon}$ is the characteristic function of Ω_f^ε. We can write

$$\Omega_f^\varepsilon = \left\{ x \in \Omega : \chi_f \left(\frac{x}{\varepsilon} \right) = 1 \right\},$$

where χ_f is the characteristic function of Y_f; clearly, χ_f is a Y-periodic function. Let μ^ε denote the periodic function defined a.e. in Ω by

$$\mu^\varepsilon(x) = \mu \left(\frac{x}{\varepsilon} \right), \tag{14}$$

with the period ε in the variable x, where

$$\mu(y) = \begin{cases} \mu_f(1 + \lambda_0), & \text{if } y \in Y_f, \\ \mu_s, & \text{if } y \in Y_s. \end{cases} \tag{15}$$

The magnetic field H^ε is such that $H^\varepsilon = \nabla \varphi^\varepsilon$ where φ^ε is a solution of

$$\begin{cases} \operatorname{div}(\mu^\varepsilon \nabla \varphi^\varepsilon) = 0 \quad \text{in } \Omega, \\ \mu^\varepsilon \nabla \varphi^\varepsilon \cdot \nu = \mu_0 H_a \cdot \nu \quad \text{on } \partial\Omega, \quad \displaystyle\int_\Omega \varphi^\varepsilon(x)\, dx = 0. \end{cases} \tag{16}$$

Using (13) and equation $\operatorname{curl} H^\varepsilon = 0$, there holds that

$$\begin{aligned} ((M^\varepsilon \cdot \nabla)H^\varepsilon)_k = \lambda_0((H^\varepsilon \cdot \nabla)H^\varepsilon)_k = \lambda_0 \sum_j H_j^\varepsilon \partial_j H_k^\varepsilon = \lambda_0 \sum_j H_j^\varepsilon \partial_k H_j^\varepsilon \\ = \lambda_0 \sum_j \partial_k \left(\frac{|H_j^\varepsilon|^2}{2} \right) = \frac{\lambda_0}{2} \partial_k (|H^\varepsilon|^2), \end{aligned}$$

then the Kelvin body force can be written as $\mu_f (M^\varepsilon \cdot \nabla) H^\varepsilon = \mu_f \frac{\lambda_0}{2} \nabla \left(|H^\varepsilon|^2 \right)$. Because of its gradient structure the latter term can be included in the new pressure

$$q^\varepsilon = p^\varepsilon - \mu_f \frac{\lambda_0}{2} |H^\varepsilon|^2, \tag{17}$$

then we obtain that the pair $(U^\varepsilon, q^\varepsilon)$ satisfies the Stokes system

$$\begin{cases} -\varepsilon^2 \Delta U^\varepsilon + \nabla q^\varepsilon = g \quad \text{in } \Omega_f^\varepsilon, \\ \operatorname{div} U^\varepsilon = 0 \quad \text{in } \Omega_f^\varepsilon, \\ U^\varepsilon = 0 \ \text{ on } \partial\Omega_f^\varepsilon, \quad \displaystyle\int_{\Omega_f^\varepsilon} q^\varepsilon(x)\, dx = 0. \end{cases} \tag{18}$$

Problem $(\mathcal{P}_1^\varepsilon)$ is formed by equations (13)–(18). We are interested in describing the asymptotic behavior, when ε tends to 0, of the magnetic field H^ε, the magnetization M^ε, the velocity U^ε and the pressure p^ε.

Our goal is the rigorous derivation of a Darcy law including some effect of the applied magnetic field. Clearly, problems (16) and (18) are decoupled; the homogenization of the second order linear elliptic equation (16) is classical, as well as the homogenization of the Stokes system (18), see for instance [4, 27, 45, 47]. But relation (17) gives rise to the study of a nonlinear problem. Let $H^\varepsilon = \nabla\varphi^\varepsilon$ and let Q^ε be the extension of q^ε as defined in Lemma 12 (below). According to (17) we define P^ε in Ω by $P^\varepsilon = Q^\varepsilon + \mu_f \frac{\lambda_0}{2}|H^\varepsilon|^2$.

Our new result is the regularity of the magnetic field H^ε. We prove, by using Meyers' theorem for Neumann problem that, under the assumption $H_a \in \mathbb{H}^1(\mathbb{R}^3)$, there exists a real number $p_M > 2$, such that, for all $2 < r < \inf\{p_M, 6\}$, $H^\varepsilon \in \mathbb{L}^r(\Omega)$, and the sequence (H^ε) is bounded in $\mathbb{L}^r(\Omega)$, see Theorem 1 (below). This implies that the sequence $(|H^\varepsilon|^2)$ is bounded in $L^{r/2}(\Omega)$; note that the exponent $r/2 > 1$ is essential for applying the two-scale convergence to the sequence $(|H^\varepsilon|^2)$. Let $H_0(x, x/\varepsilon)$ be a corrector for $H^\varepsilon(x)$, see Lemma 4 (below). Applying the two-scale convergence, we have $|H^\varepsilon|^2 \overset{2s}{\rightharpoonup} |H_0(x,y)|^2$. It results that

$$\begin{aligned} &P^\varepsilon \overset{2s}{\rightharpoonup} P_0(x,y) = Q(x) + \mu_f \frac{\lambda_0}{2}|H_0(x,y)|^2, \\ &P^\varepsilon \rightharpoonup P(x) = Q(x) + \mu_f \frac{\lambda_0}{2}(I + \mathcal{A})\nabla\varphi(x)\cdot\nabla\varphi(x), \end{aligned}$$

where $\mathcal{A}$ is the constant matrix given by (37), and $\nabla\varphi(x) = \int_Y H_0(x,y)\, dy$. Then we derive a Darcy law, with an additional term representing the effect induced by homogenization, see Theorem 2 and Remark 1 (below).

Let us mention the paper [31] dealing with the linear case. Using the representation $H = \nabla\psi$ for the magnetic field and a formal upscaling technique, the authors derived the macroscopic flow from the description of the physical mechanisms at the pore scale. Our result is in agreement with the model they derived.

Let us also mention some recent works (derived by means of formal power series expansions) where additional terms arising from local variations of fields have been taken into account in the context of Darcy's flow [40], linear elasticity [41], and electromagnetic composites [21].

2.2 The nonlinear model

Assume that M and H satisfy relation (5). Let μ denote the function defined a.e. in Ω by

$$\mu = \begin{cases} \mu_f \left(1 + M_s \dfrac{L(b_1|H|)}{|H|}\right) & \text{in } \Omega_f, \\ \mu_s \quad \text{in } \Omega_s, \end{cases} \tag{19}$$

then we have $B = \mu\, H$ and H is a solution of

$$\begin{cases} \operatorname{div}(\mu\, H) = 0, \quad \operatorname{curl} H = 0 \ \text{ in } \Omega, \\ \mu_f \left(1 + M_s \dfrac{L(b_1|H|)}{|H|}\right) H\cdot\nu = \mu_0 H_a \cdot \nu \ \text{ on } \partial\Omega. \end{cases}$$

Let $\varphi \in H^1(\Omega)$ the magnetic potential associated to H, i.e. $H = \nabla\varphi$; φ is a solution of

$$\begin{cases} \operatorname{div}(\mu\nabla\varphi) = 0 \ \text{ in } \Omega, \\ \mu_f \left(1 + M_s \dfrac{L(b_1|\nabla\varphi|)}{|\nabla\varphi|}\right) \nabla\varphi \cdot \nu = \mu_0 H_a \cdot \nu \ \text{ on } \partial\Omega, \quad \displaystyle\int_\Omega \varphi(x)\, dx = 0. \end{cases} \tag{20}$$

The constraint of zero mean of φ is imposed in order to ensure the uniqueness of solutions of this problem.

The fluid velocity U and the pressure p satisfy the Stokes equations

$$\begin{cases} -\eta\Delta U + \nabla p = \mu_f (M \cdot \nabla) H + g, \quad \operatorname{div} U = 0 \ \text{ in } \Omega_f, \\ U = 0 \ \text{ on } \partial\Omega_f. \end{cases} \tag{21}$$

The Kelvin body force can be written as

$$\mu_f (M \cdot \nabla) H = \mu_f \, M_s \frac{L(b_1|H|)}{|H|} 1_{\Omega_f} \, (H \cdot \nabla) H = \mu_f \, M_s L(b_1|H|) 1_{\Omega_f} \, \nabla(|H|).$$

Using that

$$\int_0^x M_s L(b_1 s)\, ds = \frac{M_s}{b_1} \ln\Big(\frac{\sinh(b_1 x)}{x}\Big), \quad \forall\, x \in \mathbb{R}_+,$$

we can write

$$\mu_f (M \cdot \nabla) H = \mu_f \, 1_{\Omega_f} \, \nabla(\kappa(H)),$$

where $\kappa : \mathbb{R}^3 \to \mathbb{R}$ denote the function defined by

$$\kappa(\xi) = \frac{M_s}{b_1} \ln\Big(\frac{\sinh(b_1|\xi|)}{|\xi|}\Big), \quad \forall\, \xi \in \mathbb{R}^3. \tag{22}$$

As in the linear case the term representing the Kelvin body force can be included in the new pressure $q = p - \mu_f \kappa(H)$. Then we rewrite equation (21) as

$$\begin{cases} -\eta\Delta U + \nabla q = g, \quad \operatorname{div} U = 0 \ \text{ in } \Omega_f, \\ U = 0 \ \text{ on } \partial\Omega_f. \end{cases} \tag{23}$$

We use the change of variables (9) to express equations (5), (19)–(23) in dimensionless form. For notational convenience we take $m_c = h_c = 1$. Omitting the prime index we obtain the following equations. The magnetization M^ε is given by

$$M^\varepsilon = M_s \frac{L(b_1|H^\varepsilon|)}{|H^\varepsilon|} 1_{\Omega_f^\varepsilon} \, H^\varepsilon \ \text{ in } \Omega. \tag{24}$$

Let μ^ε denote the function defined a.e. in Ω by

$$\mu^\varepsilon = \begin{cases} \mu_f \left(1 + M_s \dfrac{L(b_1|H^\varepsilon|)}{|H^\varepsilon|}\right) & \text{in } \Omega_f^\varepsilon, \\ \mu_s \ \text{ in } \Omega_s^\varepsilon, \end{cases} \tag{25}$$

and let $\varphi^\varepsilon \in H^1(\Omega)$ such that $H^\varepsilon = \nabla\varphi^\varepsilon$; φ^ε is a solution of the nonlinear differential equation

$$\begin{cases} \operatorname{div}(\mu^\varepsilon \nabla\varphi^\varepsilon) = 0 \ \text{ in } \Omega, \\ \mu_f \left(1 + M_s \dfrac{L(b_1|\nabla^\varepsilon\varphi|)}{|\nabla\varphi^\varepsilon|}\right) \nabla\varphi^\varepsilon \cdot \nu = \mu_0 H_a \cdot \nu \ \text{ on } \partial\Omega, \quad \displaystyle\int_\Omega \varphi^\varepsilon(x)\, dx = 0. \end{cases} \tag{26}$$

The pair $(U^\varepsilon, q^\varepsilon)$ satisfies the Stokes system

$$\begin{cases} -\varepsilon^2 \Delta U^\varepsilon + \nabla q^\varepsilon = g \quad \text{in } \Omega_f^\varepsilon, \\ \operatorname{div} U^\varepsilon = 0 \quad \text{in } \Omega_f^\varepsilon, \\ U^\varepsilon = 0 \ \text{ on } \partial\Omega_f^\varepsilon, \quad \displaystyle\int_{\Omega_f^\varepsilon} q^\varepsilon(x)\, dx = 0. \end{cases} \tag{27}$$

The function q^ε is linked with the pressure p^ε par the relation

$$q^\varepsilon = p^\varepsilon - \mu_f \kappa(H^\varepsilon). \tag{28}$$

We denote by problem $(\mathcal{P}_{nl}^\varepsilon)$ the system formed by equations (24)–(28). We are interested in describing the asymptotic behavior, when ε tends to 0, of the magnetization M^ε, the magnetic field H^ε, the velocity U^ε and the pressure p^ε.

Here again, our goal is the rigorous derivation of a Darcy law including an effect of the applied magnetic field. Problems (26) and (27) are decoupled, problem (26) is a nonlinear differential equation, and problem (27) is identical to problem (18) considered in the linear case. Let Q^ε be the extension of q^ε as defined in Lemma 12 (below), we define P^ε in Ω by $P^\varepsilon = Q^\varepsilon + \mu_f \kappa(H^\varepsilon)$ where κ is given by (22).

We show that the nonlinear differential operator in (26) is strictly monotone, coercive and hemicontinuous. This allows us to prove that the variational equation associated to (26) has a unique solution $\varphi^\varepsilon \in H^1(\Omega)$ and that φ^ε is uniformly bounded in $H^1(\Omega)$, see Lemma 2 (below). Using the monotonicity of the differential operator and the two-scale convergence method we derive a two-scale homogenized equation, then deduce the homogenized equation of the magnetic potentiel as well as the two-scale limit of the magnetization and its weak limit in $\mathbb{L}^2(\Omega)$, see Theorem 3 (below). Using a corrector for the magnetic field, we deduce the asymptotic behaviour of the sequence $\mu_f \kappa(H^\varepsilon)$. Then we rigorously derive a Darcy law, see Theorem 4 (below). To our knowledge this result is new.

The remainder is organized as follows: In Section 3 we state our main results. Theorems 1 and 2 are concerned with the asymptotic analysis of the linear problem $(\mathcal{P}_l^\varepsilon)$ and Theorems 3 and 4 are concerned with the asymptotic analysis of the nonlinear problem $(\mathcal{P}_{nl}^\varepsilon)$. Section 4 is devoted to the proofs and Section 5 concludes the paper.

3 Main results

We make the following assumptions:

- **a1** Ω is an open simply connected domain of class C^2;
- **a2** Y_s is a closed simply connected domain of class C^2 with a strictly positive measure, and such that $Y_s \subset Y$;
- **a3** μ is the Y-periodic functions in $L^\infty(\mathbb{R}^3)$, given by (6), satisfying $0 < c_0 \leq \mu \leq c_0^{-1}$;
- **a4** $g \in \mathbb{L}^2(\Omega)$;
- **a5** $H_a \in \mathbb{H}^1(\mathbb{R}^3)$.

We introduce the classical function spaces in the theory of the Navier-Stokes equations

$$\begin{aligned} &\mathscr{D}_s(\Omega_f) = \left\{ v \in \mathscr{D}(\Omega_f, \mathbb{R}^3) : \operatorname{div} v = 0 \ \text{ in } \Omega_f \right\}, \\ &V \ = \ \text{closure of } \mathscr{D}_s(\Omega_f) \text{ in } H^1(\Omega_f). \end{aligned}$$

Here $\mathscr{D}(\Omega_f, \mathbb{R}^3)$ is the space of infinitely differentiable functions with compact support in Ω_f and valued in $\mathbb{R}^3$. As is well known,

$$V = \left\{ v \in H_0^1(\Omega_f) : \ \operatorname{div} v = 0 \ \text{ in } \Omega_f \right\}.$$

We also introduce the space

$$H^1_\sharp(\Omega) = \{\psi \in H^1(\Omega) : \int_\Omega \psi(x)\,dx = 0\}.$$

Due to the Poincaré-Wirtinger inequality, there exists a constant c (depending only on Ω) such that

$$\|\varphi\|_{L^2(\Omega)} \le c\|\nabla\varphi\|_{\mathbb{L}^2(\Omega)}, \quad \forall\varphi \in H^1_\sharp(\Omega).$$

3.1 Two-scale convergence

To describe the asymptotic analysis of problems $(\mathcal{P}^\varepsilon_l)$ and $(\mathcal{P}^\varepsilon_{nl})$ we use the two-scale convergence method [3, 34, 36]. We denote by $C^\infty_{per}(Y)$ the space of infinitely differentiable functions in $\mathbb{R}^3$ which are Y-periodic, by $C_{per}(Y)$ the Banach space of continuous and Y-periodic functions, and by $W^{1,q}_{per}(Y)$ $(1 < q < \infty)$ the closure of $C^\infty_{per}(Y)$ in the $W^{1,q}(Y)$-norm. Eventually, $\mathscr{D}(\Omega, C^\infty_{per}(Y))$ denotes the space of infinitely smooth and compactly supported functions in Ω with values in the space $C^\infty_{per}(Y)$.

A sequence (u^ε) of functions in $L^q(\Omega)$, $1 < q < \infty$, is said two-scale convergent (in L^q) to a function $u_0(x,y)$, $u_0 \in L^q(\Omega \times Y)$, as $\varepsilon \to 0$, if

$$\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon(x)\varphi\left(x, \frac{x}{\varepsilon}\right)\,dx = \int_{\Omega\times Y} u_0(x,y)\varphi(x,y)\,dxdy,$$

for any test function $\varphi \in \mathscr{D}(\Omega, C^\infty_{per}(Y))$; we will write $u^\varepsilon \overset{2s}{\rightharpoonup} u_0(x,y)$. Note that for the space of admissible test functions, the space $C(\bar\Omega, C^\infty_{per}(Y))$ can be also used.

It is a crucial property of the two-scale convergence that for any bounded sequence (u^ε) of $L^q(\Omega)$ there is a subsequence, still denoted (u^ε), and a function $u_0(x,y)$, $u_0 \in L^q(\Omega \times Y)$, such that $u^\varepsilon \overset{2s}{\rightharpoonup} u_0(x,y)$, see [3, 34, 36]. Let also cite the following properties [3, 34, 36]:

(i) If (u^ε) is a bounded sequence of $W^{1,q}(\Omega)$, there is a subsequence, still denoted (u^ε), and there are functions $u \in W^{1,q}(\Omega)$, $u^1 \in L^q(\Omega; W^{1,q}_{per}(Y))$, such that

$$u^\varepsilon \rightharpoonup u \ \text{ in } W^{1,q}(\Omega) \text{ weak}, \quad u^\varepsilon \overset{2s}{\rightharpoonup} u(x), \quad \nabla u^\varepsilon \overset{2s}{\rightharpoonup} \nabla u(x) + \nabla_y u^1(x,y).$$

(ii) If (u^ε) is a bounded sequence of $L^q(\Omega)$, such that $(\varepsilon\nabla u^\varepsilon)$ is bounded in $\mathbb{L}^q(\Omega)$, then there is a subsequence, still denoted (u^ε), and a function $u_0(x,y)$, $u_0 \in L^q(\Omega; W^{1,q}_{per}(Y))$, such that

$$u^\varepsilon \overset{2s}{\rightharpoonup} u_0(x,y), \quad \varepsilon\nabla u^\varepsilon \overset{2s}{\rightharpoonup} \nabla_y u_0(x,y).$$

3.2 Main results in the linear case

The weak formulation of problem (16) is

$$\varphi^\varepsilon \in H^1_\sharp(\Omega), \quad \int_\Omega \mu^\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\psi\,dx = \int_{\partial\Omega}\mu_0 H_a\cdot\nu\,\psi\,d\sigma, \quad \forall\psi\in H^1(\Omega), \tag{29}$$

where μ^ε is defined by (14), (15). By the Lax-Milgram theorem and the Poincaré-Wirtinger inequality, there exists a unique solution to (29).

The weak formulation of problem (18) is

$$U^\varepsilon \in V, \quad \varepsilon^2\int_{\Omega^\varepsilon_f}\nabla U^\varepsilon\cdot\nabla v\,dx = \int_{\Omega^\varepsilon_f} g\cdot v\,dx, \quad \forall v\in V. \tag{30}$$

Under assumption **a4**, this problem has a unique solution.

We define a constant (homogenized) matrix μ^{eff} by

$$\mu^{eff}_{ik} = \int_Y \mu(y)\left(\delta_{ik} + \frac{\partial w^k}{\partial y_i}(y)\right)\,dy, \quad i,k = 1,2,3, \tag{31}$$

where $w^k(y)$ denotes a scalar Y-periodic function which solves the cell problem

$$-\mathrm{div}_y\left(\mu(y)\nabla_y w^k(y)\right) = \mathrm{div}_y\left(\mu(y)\, e^k\right) \quad \text{in } Y, \qquad \int_Y w^k(y)\, dy = 0. \tag{32}$$

Here e^k denotes the k-th standard basis vector of $\mathbb{R}^3$. Clearly, problem (32) has a unique weak solution $w^k \in H^1_{per}(Y)$, where $H^1_{per}(Y)$ denotes the space of functions in $H^1_{loc}(\mathbb{R}^3)$ which are Y-periodic. Let φ be a solution of the homogenized equation

$$\varphi \in H^1_\sharp(\Omega), \quad \int_\Omega \mu^{eff}\nabla\varphi\cdot\nabla\psi\, dx = \mu_0\int_{\partial\Omega} H_a\cdot\nu\,\psi\, d\sigma, \quad \forall\psi\in H^1(\Omega), \tag{33}$$

where μ^{eff} is a constant matrix defined by (31). Note that μ^{eff} is symmetric and positive definite. Let φ^1 be the function defined by

$$\varphi^1(x,y) = \sum_{k=1}^{3}\frac{\partial\varphi}{\partial x_k}(x) w^k(y), \quad (x,y)\in\Omega\times Y. \tag{34}$$

Clearly, $\varphi^1 \in L^2(\Omega; H^1_{per}(Y))$ and we have

$$-\,\mathrm{div}_y\left(\mu(y)\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)\right) = 0 \ \text{ in } Y.$$

Let $(e^i)_{1\le i\le 3}$ denote the canonical basis of $\mathbb{R}^3$. We introduce the differential systems in Y_f, for $1\le i\le 3$:

$$\begin{cases} -\Delta_y\omega^i(y) + \nabla_y\pi^i(y) = e^i \ \text{ in } Y_f, \\ \mathrm{div}_y\,\omega^i(y) = 0 \ \text{ in } Y_f, \\ \omega^i{}_{|\partial Y_s} = 0, \quad \int_{Y_f}\pi^i(y)\, dy = 0. \end{cases} \tag{35}$$

Clearly, problem (35) has a unique solution $(\omega^i,\pi^i)\in \mathbb{H}^1_{per}(Y_f)\times L^2(Y_f)$.

We now state our first main result.

Theorem 1. *Under assumptions* **a1**–**a5**, *let φ^ε be a solution of problem* (29), *and $H^\varepsilon = \nabla\varphi^\varepsilon$. Then the sequence (H^ε) two-scale converges to $H_0(x,y) = \nabla\varphi(x)+\nabla_y\varphi^1(x,y)$ where $\varphi(x)$ is a unique solution of the homogenized equation* (33), *and $\varphi^1(x,y)$ is defined by* (34). *We have $\varphi\in H^2(\Omega)$ and $\varphi^1\in H^1(\Omega; W^{1,6}_{per}(Y))$. Moreover:*

(i) There exists a real number $p_M>2$, such that, for all $2<r<\inf\{p_M,6\}$, $H^\varepsilon\in\mathbb{L}^r(\Omega)$, the sequence (H^ε) is bounded in $\mathbb{L}^r(\Omega)$, and $H_0(x,y)\in\mathbb{L}^r(\Omega\times Y)$.

(ii) We have

$$|H^\varepsilon|^2 \overset{2s}{\rightharpoonup} \alpha_0(x,y) := |\nabla\varphi(x)|^2 + 2\nabla\varphi(x)\cdot\nabla_y\varphi^1(x,y) + |\nabla_y\varphi^1(x,y)|^2,$$
$$|H^\varepsilon|^2 \rightharpoonup \alpha(x) := |\nabla\varphi(x)|^2 + \int_Y |\nabla_y\varphi^1(x,y)|^2\, dy \quad \textit{weakly in } L^{r/2}(\Omega).$$

Our second main result is the following.

Theorem 2. *Under assumptions* **a1**–**a5** *and notations of Theorem 1, let M^ε be the function defined by* (13), *$(U^\varepsilon,q^\varepsilon)$ a solution of problem* (18), *p^ε the function linked with q^ε par relation* (17), *Q^ε the extension of q^ε as defined in Lemma 12 (below). We define P^ε in Ω by $P^\varepsilon = Q^\varepsilon + \mu_f\frac{\lambda_0}{2}|H^\varepsilon|^2$. Then:*

(i) the sequence (M^ε) two-scale converges to $M_0(x,y)$ and converges weakly in $L^2(\Omega)$ to $M(x)=\int_Y M_0(x,y)\, dy$, where $M_0\in\mathbb{L}^r(\Omega\times Y)$, for all $2<r<\inf\{p_M,6\}$, is given by

$$M_0(x,y) = \lambda_0\,\chi_f(y)\, H_0(x,y)$$

and $\chi_f(y)$ is a Y-periodic function defined in Y as the characteristic function of Y_f.

(ii) There exist funtions $U_0 \in L^2(\Omega; \mathbb{H}^1_{per}(Y))$, $Q \in L^2(\Omega)$, $P_0 \in L^{r/2}(\Omega \times Y)$ *such that*

$$U^\varepsilon \overset{2s}{\rightharpoonup} U_0(x,y), \quad \varepsilon \nabla U^\varepsilon \overset{2s}{\rightharpoonup} \nabla_y U_0(x,y), \quad Q^\varepsilon \to Q \quad \textit{strongly in } L^2(\Omega),$$

$$P^\varepsilon \overset{2s}{\rightharpoonup} P_0(x,y) = Q(x) + \mu_f \frac{\lambda_0}{2}\, \alpha_0(x,y),$$

$$P^\varepsilon \rightharpoonup P(x) := Q(x) + P_m(x) \quad \textit{weakly in } L^{r/2}(\Omega),$$

where P_m *is the magnetic pressure given by*

$$P_m(x) = \mu_f \frac{\lambda_0}{2}\, \alpha(x) = \mu_f \frac{\lambda_0}{2} (I + \mathcal{A}) \nabla\varphi(x) \cdot \nabla\varphi(x) \tag{36}$$

and $\mathcal{A} = (\mathcal{A}_{ij})_{1 \le i,j \le 3}$ *is a symmetric and positive definite matrix given by*

$$\mathcal{A}_{ij} = \int_Y \nabla_y w^i(y) \cdot \nabla_y w^j(y)\, dy, \quad (w^k \textit{ defined by } (32)). \tag{37}$$

The pair (U_0, Q) *is such that*

$$\begin{cases} U_0 = 0 \quad \textit{in } \Omega \times Y_s, \quad \textit{div}_y\, U_0 = 0 \quad \textit{in } \Omega \times Y, \\ \textit{div} \left(\displaystyle\int_Y U_0(\cdot, y)\, dy \right) = 0 \quad \textit{in } \Omega, \quad \left(\displaystyle\int_Y U_0(\cdot, y)\, dy \right) \cdot \nu = 0 \quad \textit{on } \partial\Omega, \\ U_0(x,y) \quad Y\textit{-periodic for a.e. } x \in \Omega, \quad \displaystyle\int_\Omega Q(x)\, dx = 0, \end{cases} \tag{38}$$

and is a solution of

$$\begin{cases} \displaystyle\int_\Omega \int_{Y_f} \nabla_y U_0(x,y) \cdot \nabla_y \boldsymbol{\zeta}(x,y)\, dxdy - \int_\Omega \int_{Y_f} Q(x)\, \textit{div}_x\, \boldsymbol{\zeta}(x,y)\, dxdy \\ = \displaystyle\int_\Omega \int_{Y_f} g(x) \cdot \zeta(x,y)\, dxdy, \\ \forall \zeta \in \mathscr{D}(\Omega, C^\infty_{per}(Y)), \ \textit{with } \zeta = 0 \textit{ in } \Omega \times Y_s \textit{ and } \textit{div}_y\, \zeta(x,y) = 0 \textit{ in } \Omega \times Y. \end{cases} \tag{39}$$

Moreover,

$$U_0(x,y) = \sum_{j=1}^3 \omega^j(y) \left(g_j(x) - \frac{\partial P}{\partial x_j}(x) + \frac{\partial P_m}{\partial x_j}(x) \right), \quad x \in \Omega,\, y \in Y_f, \tag{40}$$

$$\int_{Y_f} U_0(x,y)\, dy = K\big(g(x) - \nabla P(x) + \nabla P_m(x)\big), \quad x \in \Omega, \qquad \text{(Darcy law)} \tag{41}$$

where $K = (K_{ij})_{1 \le i,j \le 3}$ *is the permeability matrix given by*

$$K_{ij} = \int_{Y_f} \nabla_y \omega^i \nabla_y \omega^j\, dy = \int_{Y_f} \omega^i_j\, dy, \quad (\omega^i \textit{ defined by } (35)). \tag{42}$$

The permeability matrix K *is symmetric and positive definite.*

The proofs of Theorems 1 and 2 are given in Section 4.

Remark 1. *We can rewrite the Darcy law* (41) *in the following form. From Theorem 1 (ii) and* (36) *we have*

$$P_m(x) = \mu_f \frac{\lambda_0}{2} |\nabla\varphi(x)|^2 + \mu_f \frac{\lambda_0}{2} \mathcal{A} \nabla\varphi(x) \cdot \nabla\varphi(x),$$

where $\varphi(x)$ is a unique solution of the homogenized equation (33)*, and $\mathcal{A}$ is a matrix defined by* (37)*. Define*

$$\overline{H}(x) = \nabla\varphi(x), \quad \overline{M}(x) = \lambda_0 \overline{H}(x), \quad g_K(x) = \mu_f(\overline{M}\cdot\nabla)\overline{H}\,(x),$$
$$R(x) = \mu_f \frac{\lambda_0}{2}\mathcal{A}\nabla\varphi(x)\cdot\nabla\varphi(x).$$

We have $\nabla\big(\mu_f \frac{\lambda_0}{2}|\nabla\varphi(x)|^2\big) = \mu_f(\overline{M}\cdot\nabla)\overline{H}\,(x) = g_K(x)$, then (41) *takes the form*

$$\int_{Y_f} U_0(x,y)\,dy = K\big(g(x) + g_K(x) - \nabla P(x) + \nabla R(x)\big),$$

where $K\nabla R(x)$ represents the effect induced by homogenization.

Remark 2. *It results from* (38) *and* (41) *that the pressure P satisfies the differential equation*

$$\begin{cases} -\,div\big(K(\nabla(P - P_m)\big) = -\,div(Kg) \quad in\ \Omega, \\ K\nabla(P - P_m)\cdot\nu = Kg\cdot\nu \quad on\ \partial\Omega. \end{cases}$$

The weak formulation of this problem is

$$P \in H^1_\sharp(\Omega),$$
$$\int_\Omega K\nabla\,(P - P_m)\cdot\nabla\psi\,dx = \int_\Omega Kg\cdot\nabla\psi\,dx, \quad \forall\psi \in H^1(\Omega).$$

Clearly, this problem has a unique solution.

Remark 3. *Theorem 2 can be generalized to the case where the load g also depends on the variable $y = x/\varepsilon$. Let $g \in L^2(\Omega; C_{per}(Y)^3)$. Define $g^\varepsilon(x) = g(x, \frac{x}{\varepsilon})$. We have $\|g^\varepsilon\|_{\mathbb{L}^2(\Omega)} \le \|g\|_{L^2(\Omega;C_{per}(Y)^3)}$. The Stokes system*

$$\begin{cases} -\varepsilon^2\Delta U^\varepsilon + \nabla q^\varepsilon = g^\varepsilon \quad in\ \Omega_f^\varepsilon, \\ div\ U^\varepsilon = 0 \quad in\ \Omega_f^\varepsilon, \\ U^\varepsilon = 0 \quad on\ \partial\Omega_f^\varepsilon, \quad \displaystyle\int_{\Omega_f^\varepsilon} q^\varepsilon(x)\,dx = 0. \end{cases}$$

has a unique solution $(U^\varepsilon, q^\varepsilon)$. We extend U^ε by zero in $\Omega\setminus\Omega_f^\varepsilon$ and still denote by U^ε the extended function. Let Q^ε be the extension of q^ε as defined in Lemma 12 (below). We define P^ε in Ω by $P^\varepsilon = Q^\varepsilon + \mu_f\frac{\lambda_0}{2}|H^\varepsilon|^2$. We have

$$\varepsilon\|\nabla U^\varepsilon\|_{\mathbb{L}^2(\Omega)} \le c, \quad \|U^\varepsilon\|_{\mathbb{L}^2(\Omega)} \le c, \quad \|Q^\varepsilon\|_{\mathbb{L}^2(\Omega)} \le c,$$

where c is a constant independent of ε. Using the two-scale convergence, we show that the sequence $(U^\varepsilon, Q^\varepsilon)$ two-scale converges to the unique solution $(U_0, Q) \in L^2(\Omega; \mathbb{H}^1_{per}(Y)) \times L^2(\Omega)$ of the two-scale homogenized problem

$$\begin{cases} -\,\Delta_y U_0(x,y) + \nabla Q(x) + \nabla_y Q^1(x,y) = g(x,y) \quad in\ \Omega\times Y_f, \\ U_0 = 0 \quad in\ \Omega\times Y_s, \quad div_y\, U_0 = 0 \quad in\ \Omega\times Y, \\ div\left(\displaystyle\int_Y U_0(\cdot,y)\,dy\right) = 0 \quad in\ \Omega, \quad \left(\int_Y U_0(\cdot,y)\,dy\right)\cdot\nu = 0 \quad on\ \partial\Omega, \\ U_0(x,y),\ Q^1(x,y)\ \ Y\text{-periodic for a.e. } x\in\Omega, \\ \displaystyle\int_\Omega Q(x)\,dx = 0, \quad \int_Y Q^1(x,y)\,dy = 0 \quad for\ a.e.\ x\in\Omega. \end{cases}$$

Let $(\omega^0, \pi^0) \in L^2(\Omega; \mathbb{H}^1_{per}(Y)) \times L^2(\Omega \times Y_f)$ *the unique solution of the cell problem*

$$\begin{cases} -\Delta_y \omega^0(x,y) + \nabla_y \pi^0(x,y) = g(x,y) \quad \text{in } \Omega \times Y_f, \\ div_y\, \omega^0(x,y) = 0 \quad \text{in } \Omega \times Y_f, \\ \omega^0{}_{|\partial Y_s} = 0, \quad \displaystyle\int_{Y_f} \pi^0(x,y)\, dy = 0 \quad \text{for a.e. } x \in \Omega. \end{cases}$$

We show that

$$U_0(x,y) = \omega^0(x,y) + \sum_{j=1}^{3} \omega^j(y) \left(-\frac{\partial Q}{\partial x_j}(x) \right), \quad x \in \Omega,\ y \in Y_f.$$

Using that

$$P^\varepsilon \rightharpoonup P(x) := Q(x) + P_m(x) \quad \text{weakly in } L^{r/2}(\Omega),$$

with $P_m = Q + \mu_f \frac{\lambda_0}{2} |\nabla \varphi|^2 + \mu_f \frac{\lambda_0}{2} \mathcal{A} \nabla \varphi \cdot \nabla \varphi$, *we derive the Darcy law*

$$\int_{Y_f} U_0(x,y)\, dy = \int_{Y_f} \omega^0(x,y)\, dy + K\big(-\nabla P(x) + \nabla P_m(x) \big), \quad x \in \Omega.$$

3.3 Main results in the nonlinear case

Let $a : Y \times \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$a(y,\xi) = \begin{cases} \widetilde{\mu}(y)\xi + \mu_f\, \chi_f(y) M_s \frac{L(b_1|\xi|)}{|\xi|}\, \xi \quad \text{for a.e. } y \in Y, \text{ for all } \xi \in \mathbb{R}^3 \setminus \{0\}, \\ 0 \ \text{ if } \ \xi = 0, \end{cases} \tag{43}$$

where χ_f is the characteristic function of Y_f and $\tilde{\mu}$ the Y-periodic function defined in Y by

$$\tilde{\mu} = \begin{cases} \mu_f \quad \text{a.e. in } Y_f, \\ \mu_s \quad \text{a.e. in } Y_s. \end{cases} \tag{44}$$

We have the following result.

Lemma 1. *The function* $a(y,\xi)$ *defined by* (43) *satisfies the following properties:*

(i) $a(y,0) = 0$ *for a.e.* $y \in Y$*;*

(ii) $a(y,\xi)$ *is Lipschitz continuous with respect to* ξ*, i.e. there exists a constant* c *such that*

$$|a(y,\xi_1) - a(y,\xi_2)| \le c|\xi_1 - \xi_2|, \ \text{for a.e. } y \in Y, \text{ for all } \xi_1, \xi_2 \in \mathbb{R}^3;$$

(iii) $a(y,\xi)$ *is monotone and coercive, more precisely,*

$$(a(y,\xi_1) - a(y,\xi_2)) \cdot (\xi_1 - \xi_2) \ge \inf\{\mu_f, \mu_s\} |\xi_1 - \xi_2|^2, \text{ for a.e. } y \in Y, \text{ for all } \xi_1, \xi_2 \in \mathbb{R}^3.$$

The weak formulation of problem (26) reads

$$\varphi^\varepsilon \in H^1_\sharp(\Omega), \quad \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla \varphi^\varepsilon(x) \right) \cdot \nabla \psi\, dx = \int_{\partial\Omega} \mu_0 H_a \cdot \nu\, \psi\, d\sigma, \quad \forall \psi \in H^1(\Omega). \tag{45}$$

We have the following result.

Lemma 2. *Problem* (45) *has a unique solution* φ^ε*. Moreover, the sequence* (φ^ε) *is bounded in* $H^1_\sharp(\Omega)$*.*

The proofs of Lemmas 1 and 2 are provided in Section 4. Problem (27) is identical to problem (18) whose weak formulation is

$$U^\varepsilon \in V, \quad \varepsilon^2 \int_{\Omega_f^\varepsilon} \nabla U^\varepsilon \cdot \nabla v \, dx = \int_{\Omega_f^\varepsilon} g \cdot v \, dx, \quad \forall v \in V.$$

Let us mention that homogenization of nonlinear problems of the form

$$-\operatorname{div} a\left(\frac{x}{\varepsilon}, \nabla u^\varepsilon\right) = f \quad \text{in } \Omega, \quad u^\varepsilon \in W_0^{1,r}(\Omega), \tag{46}$$

where $f \in W^{-1,r}(\Omega)$, $1 < r < \infty$, and $a(y,\xi)$ is a Y-periodic function in y and satisfies suitable assumptions of uniform Lipschitz continuity and uniform strict monotonicity in ξ, have been studied in [3, 9, 14, 15, 16, 20, 34]. In these papers, oscillating test functions are used to derive the homogenized equation. A corrector result has been proven in [19]. Let us also mention a paper on homogenization of boundary value problems for monotone operators in perforated domains with rapidly oscillating boundary conditions of Fourier type [42].

We now state our third main result. Using the two-scale convergence method we derive a two-scale homogenized equation. In addition we derive the homogenized equation of the magnetic potentiel as well as the two-scale limit of the magnetization and its weak limit in $\mathbb{L}^2(\Omega)$.

Theorem 3. *Under assumptions* **a1**–**a5**, *let a be defined by* (43), *φ^ε a solution of problem* (45), *$H^\varepsilon = \nabla\varphi^\varepsilon$, and let M^ε be the function defined by* (24). *Then:*

(i) The sequence (φ^ε) of solutions of problem (45) *converges weakly in $H^1_\sharp(\Omega)$ to a function φ and the sequence (H^ε) two-scale converges to $H_0(x,y) = \nabla\varphi(x) + \nabla_y\varphi_1(x,y)$, where (φ, φ_1) is the unique solution in $H^1_\sharp(\Omega) \times L^2(\Omega; H^1_{per}(Y))$ of the two-scale equation*

$$\int_\Omega \int_Y a\big(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)\big) \cdot \Big(\nabla\phi(x) + \nabla_y\phi_1(x,y)\Big) \, dxdy = \int_{\partial\Omega} \mu_0 H_a \cdot \nu\phi \, d\sigma, \tag{47}$$

for any $\phi \in H^1(\Omega)$, $\phi_1 \in L^2(\Omega; H^1_{per}(Y))$. Moreover,

$$a^\varepsilon(x) := a\left(\frac{x}{\varepsilon}, \nabla\varphi^\varepsilon(x)\right) \overset{2s}{\rightharpoonup} a\big(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)\big),$$

$$a^\varepsilon(x) \rightharpoonup a^*(x) := \int_Y a\big(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)\big) \, dy \quad \textit{in } L^2(\Omega) \textit{ weak.}$$

(ii) The function φ is a unique solution in $H^1_\sharp(\Omega)$ of the homogenized equation

$$\int_\Omega b(\nabla\varphi(x)) \cdot \nabla\phi(x) \, dx = \int_{\partial\Omega} \mu_0 H_a \cdot \nu\phi \, d\sigma, \quad \forall \phi \in H^1(\Omega). \tag{48}$$

The operator $b : \mathbb{R}^3 \to \mathbb{R}^3$ is defined for every $\xi \in \mathbb{R}^3$ by

$$b(\xi) = \int_Y a\big(y, \xi + \nabla_y w(y,\xi)\big) dy \tag{49}$$

where $w(\cdot,\xi)$ is a unique (up to an additive constant) solution in $H^1_{per}(Y)$ of the cell problem

$$\int_Y a(y, \xi + \nabla_y w(y,\xi)) \cdot \nabla v(y) \, dy = 0 \quad \textit{for all } v \in H^1_{per}(Y). \tag{50}$$

(iii) The sequence (M^ε) two-scale converges to $M_0(x,y)$ and converges weakly in $\mathbb{L}^2(\Omega)$ to $M(x) = \int_Y M_0(x,y) \, dy$ where $M_0 \in \mathbb{L}^2(\Omega \times Y)$ is given by

$$M_0(x,y) = \mu_f \, \chi_f(y) \, w(H_0(x,y))$$

and $\chi_f(y)$ is a Y-periodic function, defined in Y as the characteristic function of Y_f, and $w : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by $w(\xi) = M_s \frac{L(b_1|\xi|)}{|\xi|} \, \xi$ if $\xi \neq 0$ and $w(0) = 0$.

We now study the asymptotic behaviour of the sequence $\mu_f \kappa(H^\varepsilon)$ where κ is defined by (22). To this aim we use a corrector for the magnetic field. Following [19], we introduce a sequence (m_ε) of approximations of the identity map in $\mathbb{L}^2(\Omega)$ such that $m_\varepsilon \Phi$ is a step function for every $\Phi \in \mathbb{L}^2(\Omega)$. More precisely, for every $\varepsilon > 0$ and every $\Phi \in \mathbb{L}^2(\Omega)$, we define a function $m_\varepsilon \Phi : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$m_\varepsilon \Phi(x) = \sum_{i \in I_\varepsilon} 1_{Y_\varepsilon^i}(x) \frac{1}{|Y_\varepsilon^i|} \int_{Y_\varepsilon^i} \Phi(y)\, dy \tag{51}$$

where $Y_\varepsilon^i = \varepsilon(i + Y)$ (for $i \in \mathbb{Z}^3$), $I_\varepsilon = \{i \in \mathbb{Z}^3 : Y_\varepsilon^i \subset \Omega\}$ and 1_A is the characteristic function of a set $A \subset \mathbb{R}^3$. Let us note that, as $\varepsilon \to 0$, we have [48] (Chapter 8)

$$m_\varepsilon \Phi \to \Phi \quad \text{a.e. in } \Omega \text{ and strongly in } \mathbb{L}^2(\Omega). \tag{52}$$

We define a function $h : Y \times \mathbb{R}^3 \to \mathbb{R}^3$ by

$$h(y, \xi) = \xi + \nabla_y w(y, \xi) \tag{53}$$

where for every $\xi \in \mathbb{R}^3$, $w(\cdot, \xi)$ is a unique (up to an additive constant) solution in $H^1_{per}(Y)$ of the cell problem (50). Let us note that $w(\cdot, \xi)$ can be extended by periodicity to a function of $\mathbb{H}^1_{loc}(\mathbb{R}^3)$ and that problem (50) is equivalent to $\mathrm{div}_x(a(x, \xi + \nabla_x w(x, \xi))) = 0$ in $\mathcal{D}'(\mathbb{R}^3)$. Finally, we set

$$h_\varepsilon(x, \xi) = \xi + \nabla_y w\big(\frac{x}{\varepsilon}, \xi\big), \quad (x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{54}$$

Our fourth main result is the following.

Theorem 4. *Under assumptions* **a1**–**a5** *and notations of Theorem 3, let* $(U^\varepsilon, q^\varepsilon)$ *be a solution of problem* (27), p^ε *the function linked with* q^ε *par relation* (28), Q^ε *the extension of* q^ε *as defined in Lemma 12 (below). We define* P^ε *in* Ω *by* $P^\varepsilon = Q^\varepsilon + \mu_f \kappa(H^\varepsilon)$ *where* κ *is defined by* (22). *Then:*

(i) We have

$$H^\varepsilon(x) - h_\varepsilon\left(x, (m_\varepsilon \nabla \varphi)(x)\right) \to 0 \text{ a.e. in } \Omega \text{ and strongly in } \mathbb{L}^2(\Omega),$$

where $h_\varepsilon(x, \xi)$ *is defined by* (54).

(ii) We have from Theorem 2 (ii)

$$U^\varepsilon \overset{2s}{\rightharpoonup} U_0(x, y), \quad \varepsilon \nabla U^\varepsilon \overset{2s}{\rightharpoonup} \nabla_y U_0(x, y), \quad Q^\varepsilon \to Q \quad \text{strongly in } L^2(\Omega),$$

where the pair (U_0, Q) *satisfies* (38) *and is a solution of problem* (39).

(iii) Define

$$\kappa^0(\xi) = \int_Y \kappa\big(h(y, \xi)\big) dy, \quad \forall\, \xi \in \mathbb{R}^3, \tag{55}$$

where $h(\cdot, \xi)$ *is defined by* (53). *We have*

$$P^\varepsilon \rightharpoonup P(x) := Q(x) + \mu_f \kappa_1(x) \quad \text{weakly in } L^2(\Omega), \tag{56}$$

with $\kappa_1(x) = \kappa^0(\nabla \varphi(x))$, $x \in \Omega$. *Moreover,*

$$U_0(x, y) = \sum_{j=1}^3 \omega^j(y) \left(g_j(x) - \frac{\partial P}{\partial x_j}(x) + \mu_f \frac{\partial \kappa_1}{\partial x_j}(x) \right), \quad x \in \Omega,\ y \in Y_f, \tag{57}$$

$$\int_{Y_f} U_0(x, y)\, dy = K\left(g(x) - \nabla P(x) + \mu_f \nabla \kappa_1(x)\right), \quad x \in \Omega. \qquad \text{(Darcy law)} \tag{58}$$

The proofs of Theorems 3 and 4 are given in Section 4.

4 Proofs

4.1 Proof of Theorem 1

4.1.1 Limit of the magnetic field

We have the following result whose proof is immediate.

Lemma 3. *Let (φ^ε) be the sequence of solutions of equation (29). Then (φ^ε) converges weakly in $H^1_\sharp(\Omega)$ to the solution φ of the homogenized equation (33). Moreover, the sequence $(\nabla\varphi^\varepsilon)$ two-scale converges to $\nabla\varphi(x)+\nabla_y\varphi^1(x,y)$ where $\varphi^1(x,y)$ is given by (34) and we have*

$$\int_\Omega\int_Y \mu(y)\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)\cdot\left(\nabla\psi(x)+\nabla_y\psi^1(x,y)\right)dx = \mu_0\int_{\partial\Omega} H_a\cdot\nu\psi\,d\sigma, \quad (59)$$

$\forall(\psi,\psi^1)\in H^1_\sharp(\Omega)\times\in L^2(\Omega;H^1_{per}(Y))$.

We have the following result.

Lemma 4. *Let (φ^ε) be the sequence of solutions of (29), $\varphi(x)$ be defined by (33) and $\varphi^1(x,y)$ by (34). Then*

$$\nabla\varphi^\varepsilon(x)-\nabla\varphi(x)-\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\to 0 \quad \text{in } \mathbb{L}^2(\Omega). \quad (60)$$

Remark 4. *The function $\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)$ is a corrector for $H^\varepsilon(x)$.*

Proof of Lemma 4. We have

$$\begin{aligned}
&\mu_-\int_\Omega\left|\nabla\varphi^\varepsilon(x)-\nabla\varphi(x)-\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right|^2dx\\
&\le\int_\Omega\mu^\varepsilon(x)\left(\nabla\varphi^\varepsilon(x)-\nabla\varphi(x)-\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)\cdot\left(\nabla\varphi^\varepsilon(x)-\nabla\varphi(x)-\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx\\
&=\int_\Omega\mu^\varepsilon(x)\nabla\varphi^\varepsilon(x)\cdot\nabla\varphi^\varepsilon(x)-2\int_\Omega\mu^\varepsilon(x)\nabla\varphi^\varepsilon(x)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx\\
&\quad+\int_\Omega\mu^\varepsilon(x)\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx,
\end{aligned}$$

where $\mu_-=\inf\{\mu_f(1+\lambda_0),\mu_s\}$. We have from (29)

$$\int_\Omega\mu^\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\varphi^\varepsilon\,dx=\int_{\partial\Omega}\mu_0H_a\cdot\nu\,\varphi^\varepsilon\,d\sigma,$$

then

$$\lim_{\varepsilon\to0}\int_\Omega\mu^\varepsilon\nabla\varphi^\varepsilon\cdot\nabla\varphi^\varepsilon\,dx=\int_{\partial\Omega}\mu_0H_a\cdot\nu\,\varphi\,d\sigma.$$

Using (34) we have

$$\begin{aligned}
&\int_\Omega\mu^\varepsilon(x)\nabla\varphi^\varepsilon(x)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx\\
&=\int_\Omega\mu^\varepsilon(x)\nabla\varphi^\varepsilon(x)\cdot\left(\nabla\varphi(x)+\sum_{k=1}^3\frac{\partial\varphi}{\partial x_k}(x)\nabla_yw^k\left(\frac{x}{\varepsilon}\right)\right)dx,
\end{aligned}$$

then, passing to the two-scale limit we get

$$\begin{aligned}
&\lim_{\varepsilon\to0}\int_\Omega\mu^\varepsilon(x)\nabla\varphi^\varepsilon(x)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx\\
&=\int_\Omega\int_Y\mu(y)\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)\cdot\left(\nabla\varphi(x)+\sum_{k=1}^3\frac{\partial\varphi}{\partial x_k}(x)\nabla_yw^k(y)\right)dxdy\\
&=\int_\Omega\int_Y\mu(y)\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)dxdy.
\end{aligned}$$

We have

$$\int_\Omega \mu^\varepsilon(x)\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx$$
$$=\int_\Omega \mu^\varepsilon(x)\left(\nabla\varphi(x)+\sum_{k=1}^3\frac{\partial\varphi}{\partial x_k}(x)\nabla_y w^k\left(\frac{x}{\varepsilon}\right)\right)\cdot\left(\nabla\varphi(x)+\sum_{k=1}^3\frac{\partial\varphi}{\partial x_k}(x)\nabla_y w^k\left(\frac{x}{\varepsilon}\right)\right)dx,$$

then, passing to the two-scale limit we get

$$\lim_{\varepsilon\to0}\int_\Omega \mu^\varepsilon(x)\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right)dx$$
$$=\int_\Omega\int_Y \mu(y)\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)dxdy.$$

We then get

$$\lim_{\varepsilon\to0}\mu_-\int_\Omega\left|\nabla\varphi^\varepsilon(x)-\nabla\varphi(x)-\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)\right|^2dx$$
$$\le-\int_\Omega\int_Y \mu(y)\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)\cdot\left(\nabla\varphi(x)+\nabla_y\varphi^1(x,y)\right)dxdy+\int_{\partial\Omega}\mu_0H_a\cdot\nu\,\varphi\,d\sigma$$
$$=0,$$

where we used (59) in the last equality. Then (60) follows and the proof of Lemma 4 is achieved. □

4.1.2 Regularity and two-scale convergence of $(|\mathbf{H}^\varepsilon|^2)$

We have the following result.

Lemma 5. *The solution w^k of problem* (32) *belongs to $W^{1,6}_{per}(Y)$.*

Proof. We extend by periodicity the function w^k to $\mathbb{R}^3$ and still denote by w^k the extended function. We take a cut-off function $\theta\in\mathscr{D}(\mathbb{R}^3)$ such that $\theta=1$ in a neighborhood of Y. Let $\tilde{Y}$ denote an open smooth bounded domain containing the support of θ. We set $u=\theta w^k$. We have

$$\begin{aligned}-\operatorname{div}(\mu\nabla u)&=-\operatorname{div}(\mu w^k\nabla\theta)-\operatorname{div}(\mu\theta\nabla w^k)\\&=-\operatorname{div}(\mu w^k\nabla\theta)-\theta\operatorname{div}(\mu\nabla w^k)-\mu\nabla\theta\cdot\nabla w^k\\&=-\operatorname{div}(\mu w^k\nabla\theta)+\theta\operatorname{div}(\mu\boldsymbol{e}^k)-\mu\nabla\theta\cdot\nabla w^k.\end{aligned}\tag{61}$$

Using the continuous Sobolev embedding $H^1(\tilde{Y})\hookrightarrow L^6(\tilde{Y})$, we have that $\mu w^k\nabla\theta\in L^6(\tilde{Y})$. Moreover, $\mu\boldsymbol{e}^k\in L^6(\tilde{Y})$ and $\mu\nabla\theta\cdot\nabla w^k\in L^2(\tilde{Y})$. Using the Sobolev embedding $W^{1,6/5}_0(\tilde{Y})\hookrightarrow L^2(\tilde{Y})$, it holds that $L^2(\tilde{Y})\hookrightarrow W^{-1,6}(\tilde{Y})$. Thus the right-hand side of (61) belongs to $W^{-1,6}(\tilde{Y})$. We also have $u=0$ on the boundary $\partial\tilde{Y}$ of $\tilde{Y}$. Applying the result in [22] we obtain that $u\in W^{1,6}_0(\tilde{Y})$. Therefore $w^k\in W^{1,6}(Y)$. The proof of the lemma is complete. □

We have the following result.

Lemma 6. *Under assumption* **a5**, *let φ be the solution of the homogenized equation* (33), *and φ^1 defined by* (34). *Then $\varphi\in H^2(\Omega)$ and $\varphi^1\in H^1(\Omega;W^{1,6}_{per}(Y))$.*

Proof. The function φ is a weak solution of the equation

$$-\operatorname{div}(\mu^{eff}\nabla\varphi)=0\ \text{ in }\Omega,\quad \mu^{eff}\nabla\varphi\cdot\nu=\mu_0H_a\cdot\nu\ \text{ on }\partial\Omega,\quad\int_\Omega\varphi(x)\,dx=0.$$

Under assumption **a5**, $H_a \cdot \nu \in H^{1/2}(\partial\Omega)$, then by a classical result on the regularity of the solution of an elliptic equation with Neumann condition, we have $\varphi \in H^2(\Omega)$.

We have from (34)

$$\varphi^1(x,y) = \sum_{k=1}^{3} \frac{\partial \varphi}{\partial x_k}(x) w^k(y), \quad \nabla_y \varphi^1(x,y) = \sum_{k=1}^{3} \frac{\partial \varphi}{\partial x_k}(x) \nabla_y w^k(y),$$

where $w^k(y) \in H^1_{per}(Y)$ $(k = 1,2,3)$ and solves the cell problem (32). Thanks to the H^2-regularity of φ and Lemma 5, we have $\varphi^1 \in H^1(\Omega; W^{1,6}_{per}(Y))$. This completes the proof of Lemma 6. □

We have the following result.

Lemma 7. *There is a number $2 < p_M$ such that for all $2 < r < p_M$, $H^\varepsilon \in \mathbb{L}^r(\Omega)$, and the sequence (H^ε) is bounded in $\mathbb{L}^r(\Omega)$.*

Proof. Let us rewrite problem (29) as a homogeneous Neumann problem. Consider the equation

$$-\Delta \rho = \operatorname{div} H_a \quad \text{in } \Omega, \quad \nabla \rho \cdot \nu = H_a \cdot \nu \quad \text{on } \partial\Omega.$$

Under assumption **a5**, and the continuous Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, this equation has a weak solution $\rho \in W^{1,6}(\Omega)$ satisfying $\|\nabla \rho\|_{\mathbb{L}^6(\Omega)} \le c(\Omega) \|H_a\|_{\mathbb{L}^6(\Omega)}$, see [37] (Lemma 4.27). We impose $\int_\Omega \rho(x)\,dx = 0$, then define $\widetilde{\rho} = \frac{\mu_0}{\mu_f(1+\lambda_0)}\rho$ and $\widetilde{\varphi}^\varepsilon = \varphi^\varepsilon - \widetilde{\rho}$. We have

$$\|\nabla \widetilde{\rho}\|_{\mathbb{L}^6(\Omega)} \le c(\Omega) \frac{\mu_0}{\mu_f(1+\lambda_0)} \|H_a\|_{\mathbb{L}^6(\Omega)}. \tag{62}$$

We also have

$$\begin{aligned} -\operatorname{div}(\mu^\varepsilon \nabla \widetilde{\varphi}^\varepsilon) &= \operatorname{div}(\mu^\varepsilon \nabla \widetilde{\rho}) \ \text{ in } \Omega, \\ \mu_f(1+\lambda_0)\nabla \widetilde{\varphi}^\varepsilon \cdot \nu &= (\mu_f(1+\lambda_0)\nabla \varphi^\varepsilon - \mu_f(1+\lambda_0)\nabla \widetilde{\rho}) \cdot \nu \\ &= (\mu_f(1+\lambda_0)\nabla \varphi^\varepsilon - \mu_0 \nabla \rho) \cdot \nu \\ &= (\mu_f(1+\lambda_0)\nabla \varphi^\varepsilon - \mu_0 H_a) \cdot \nu = 0 \ \text{ on } \partial\Omega. \end{aligned}$$

Then $\widetilde{\varphi}^\varepsilon$ belongs to $H^1_\sharp(\Omega)$ and satisfies a homogeneous Neumann problem

$$\begin{cases} -\operatorname{div}(\mu^\varepsilon \nabla \widetilde{\varphi}^\varepsilon) = \operatorname{div}(\mu^\varepsilon \nabla \widetilde{\rho}) & \text{in } \Omega, \\ \mu^\varepsilon \nabla \widetilde{\varphi}^\varepsilon \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

The weak formulation of this problem is

$$\widetilde{\varphi}^\varepsilon \in H^1_\sharp(\Omega), \quad \int_\Omega \mu^\varepsilon \nabla \widetilde{\varphi}^\varepsilon \cdot \nabla \psi \, dx = -\int_\Omega \mu^\varepsilon \nabla \widetilde{\rho} \cdot \nabla \psi \, dx, \quad \forall \psi \in H^1(\Omega). \tag{63}$$

Let $f \in (H^1(\Omega))'$ be defined by

$$\langle f, \psi \rangle_{(H^1(\Omega))', H^1(\Omega))} = -\int_\Omega \mu^\varepsilon \nabla \widetilde{\rho} \cdot \nabla \psi \, dx, \quad \forall \psi \in H^1(\Omega). \tag{64}$$

We have that $\nabla \widetilde{\rho} \in \mathbb{L}^6(\Omega)$, then $f \in (W^{1,6/5}(\Omega))'$ and satisfies $\langle f, 1 \rangle_{(H^1(\Omega))', H^1(\Omega))} = 0$. Note that $(W^{1,6/5}(\Omega))' \subset (H^1(\Omega))')$. The operator T which associates with every function $H_a \in \mathbb{L}^6(\Omega)$ the solution $\widetilde{\varphi}^\varepsilon$ of (63) can be decomposed as follows: $T = T_3 \circ T_2 \circ T_1$ where T_1 is the linear operator defined by $T_1(H_a) = \nabla \widetilde{\rho} \in \mathbb{L}^6(\Omega)$ for all $H_a \in \mathbb{L}^6(\Omega)$, T_2 is the linear operator defined by $T_2(\nabla \widetilde{\rho}) = f \in (W^{1,6/5}(\Omega))'$, with f given by (64) and T_3 is the linear operator defined by $T_3(f) = u^\varepsilon$ for all $f \in (W^{1,6/5}(\Omega))'$ with $\langle f, 1 \rangle_{(H^1(\Omega))', H^1(\Omega))} = 0$, where u^ε is the unique solution to

$$u^\varepsilon \in H^1_\sharp(\Omega), \quad \int_\Omega \mu^\varepsilon \nabla u^\varepsilon \cdot \nabla \psi \, dx = \langle f, \psi \rangle_{(H^1(\Omega))', H^1(\Omega))}, \quad \forall \psi \in H^1(\Omega).$$

Thanks to (62) the operator T_1 is continuous from $\mathbb{L}^6(\Omega)$ to $\mathbb{L}^6(\Omega)$ with a norm bounded by $c(\Omega)\frac{\mu_f(1+\lambda_0)}{\mu_0}$. We easily verify that the operator T_2 is continuous from $\mathbb{L}^6(\Omega)$ to $(W^{1,6/5}(\Omega))'$ with a norm bounded by a constant depending only on c_0 and Ω. For the operator T_3 we can apply Meyers' theorem for Neumann problem, proved in [24]: There is a real number $p_M > 2$ such that for all r, $2 < r < p_M$, the operator T_3 is linear continuous from $(W^{1,6/5}(\Omega))'$ to $W^{1,r}(\Omega)$. Moreover, the norm of T_3 only depends on c_0 and Ω, and p_M on c_0 and Ω, not on f. This implies that $\widetilde{\varphi}^\varepsilon \in W^{1,r}(\Omega)$ and that $(\widetilde{\varphi}^\varepsilon)$ is bounded in $W^{1,r}(\Omega)$. We conclude that $H^\varepsilon \in \mathbb{L}^r(\Omega)$ and the sequence (H^ε) is bounded in $\mathbb{L}^r(\Omega)$. The proof of Lemma 7 is complete. □

We have the following result.

Lemma 8. *The sequence $(|H^\varepsilon|^2)$ is bounded in $L^{r/2}(\Omega)$ and we have*

$$|H^\varepsilon|^2 \overset{2s}{\rightharpoonup} |\nabla\varphi(x)|^2 + 2\nabla\varphi(x)\cdot\nabla_y\varphi^1(x,y) + |\nabla_y\varphi^1(x,y)|^2$$

where $\varphi(x)$ and $\varphi^1(x,y)$ are defined by (33) *and* (34)*, respectively. We also have*

$$|H^\varepsilon|^2 \rightharpoonup |\nabla\varphi(x)|^2 + \int_Y |\nabla_y\varphi^1(x,y)|^2\,dy \quad \textit{weakly in } L^{r/2}(\Omega).$$

Proof. By Lemma 7, there exists $r > 2$ such that $(|H^\varepsilon|^2)$ is bounded in $L^{r/2}(\Omega)$. Then there is a function $\alpha(x,y) \in L^{r/2}(\Omega\times Y)$ and a subsequence, still indexed by ε, such that

$$|H^\varepsilon|^2 \overset{2s}{\rightharpoonup} \alpha(x,y).$$

Writing $H^\varepsilon = a^\varepsilon + b^\varepsilon$, with $a^\varepsilon(x) = H^\varepsilon(x) - (\nabla\varphi(x) + \nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)$, $b^\varepsilon(x) = \nabla\varphi(x) + \nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)$, we have

$$|H^\varepsilon(x)|^2 = |a^\varepsilon(x)|^2 + 2a^\varepsilon(x)\cdot b^\varepsilon(x) + |b^\varepsilon(x)|^2.$$

Thanks to Lemma 6, we have that $\nabla\varphi(x)$ and $\nabla_y\varphi^1(x,y)$ belong to $\mathbb{L}^6(\Omega)$ and $\mathbb{L}^6(\Omega\times Y)$, respectively. Moreover, $\nabla_y\varphi^1(x,\frac{x}{\varepsilon})$ is bounded in $\mathbb{L}^6(\Omega\times Y)$. Let $\psi \in \mathscr{D}(\Omega, C^\infty_{per}(Y))$. We have

$$\int_\Omega \left||a^\varepsilon(x)|^2\psi\left(x,\frac{x}{\varepsilon}\right)\right| dx \le \max_{(x,y)\in\overline{\Omega\times Y}}|\psi(x,y)| \int_\Omega |a^\varepsilon(x)|^2 dx,$$

then

$$\lim_{\varepsilon\to 0}\int_\Omega \left||a^\varepsilon(x)|^2\psi\left(x,\frac{x}{\varepsilon}\right)\right| dx = 0,$$

thanks to Lemma 4. Consequently, $|a^\varepsilon|^2 \overset{2s}{\rightharpoonup} 0$. Using the Cauchy-Schwarz inequality we have

$$\int_\Omega \left|a^\varepsilon(x)b^\varepsilon(x)\psi\left(x,\frac{x}{\varepsilon}\right)\right| dx \le \max_{(x,y)\in\overline{\Omega\times Y}}|\psi(x,y)|\|a^\varepsilon\|_{\mathbb{L}^2(\Omega)}\|b^\varepsilon\|_{\mathbb{L}^2(\Omega)},$$

then

$$\lim_{\varepsilon\to 0}\int_\Omega \left|a^\varepsilon(x)b^\varepsilon(x)\psi\left(x,\frac{x}{\varepsilon}\right)\right| dx = 0,$$

consequently, $a^\varepsilon b^\varepsilon \overset{2s}{\rightharpoonup} 0$. We have

$$|b^\varepsilon(x)|^2 = |\nabla\varphi(x)|^2 + 2\nabla\varphi(x)\cdot\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right) + |\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right)|^2.$$

We have from (34) that

$$\nabla_y\varphi^1\left(x,\frac{x}{\varepsilon}\right) = \sum_{k=1}^3 \frac{\partial\varphi}{\partial x_k}(x)\nabla_y w^k\left(\frac{x}{\varepsilon}\right).$$

Passing to the two-scale limit there holds that

$$|b^\varepsilon|^2 \overset{2s}{\rightharpoonup} \alpha_0(x,y) = |\nabla\varphi(x)|^2 + 2\nabla\varphi(x)\cdot\nabla_y\varphi^1(x,y) + |\nabla_y\varphi^1(x,y)|^2.$$

Consequently, $|H^\varepsilon|^2 \overset{2s}{\rightharpoonup} \alpha_0(x,y)$ and $|H^\varepsilon|^2 \rightharpoonup \alpha(x) = \int_Y \alpha_0(x,y)\,dy$ weakly in $L^{r/2}(\Omega)$. Since each extracted sequence of (H^ε) has the same limit, we deduce that the whole sequence converges. The proof of Lemma 8 is complete. □

4.2 Proof of Theorem 2

4.2.1 Homogenization of the Stokes equations

The homogenization of the Stokes equations has been studied by Tartar, using the method of oscillating test functions, and by Allaire, using the two-scale convergence method. Here we present the results and refer to [4, 47] for the proofs. We first mention a version of the Poincaré inequality [47] which is used for proving uniform estimates of the solution of problem (30).

Lemma 9. *There exists a positive constant c, depending only on Y_f, such that, for evrey $v \in V$, we have*

$$\|v\|_{\mathbb{L}^2(\Omega_f} \leq c\varepsilon\|\nabla v\|_{\mathbb{L}^2(\Omega_f)}.$$

Then we easily show the following estimates of the velocity.

Lemma 10. *Under assumptions* **a1**–**a5**, *the velocity U^ε satisfies the uniform estimates*

$$\varepsilon\|\nabla U^\varepsilon\|_{\mathbb{L}^2(\Omega_f)} \leq c\|g\|_{\mathbb{L}^2(\Omega)}, \quad \|U^\varepsilon\|_{\mathbb{L}^2(\Omega)} \leq c\|g\|_{\mathbb{L}^2(\Omega)}, \tag{65}$$

where c is a constant independent of ε.

Remark 5. *The function U^ε can be extended by zero in $\Omega\setminus\Omega_f^\varepsilon$ because of its zero trace on $\partial\Omega_f^\varepsilon$. It is well known that extension by zero preserves $\mathbb{L}^q$ and $\mathbb{W}_0^{1,q}$ norms for $1 < q < \infty$.*

A uniformly bounded extension of the pressure to the whole domain Ω was introduced by Tartar [47]; he has defined a restriction operator satisfying the following properties.

Lemma 11. *There is a restriction operator*

$$R_\varepsilon : \mathbb{H}_0^1(\Omega) \longrightarrow \mathbb{H}_0^1(\Omega_f^\varepsilon) \quad (1 < q < \infty)$$

such that

$$W \in \mathbb{H}_0^1(\Omega_f^\varepsilon) \Longrightarrow R_\varepsilon W = W,$$

(elements of $\mathbb{H}_0^1(\Omega_f^\varepsilon)$ are extended by 0 to Ω)

$$\begin{aligned} &div\, W = 0 \Longrightarrow div\, R_\varepsilon W = 0,\\ &\|R_\varepsilon W\|_{\mathbb{L}^2(\Omega_f^\varepsilon)} \leq c\|W\|_{\mathbb{L}^2(\Omega)} + c\varepsilon\|\nabla W\|_{\mathbb{L}^2(\Omega)},\\ &\|\nabla R_\varepsilon W\|_{\mathbb{L}^2(\Omega_f^\varepsilon)} \leq \frac{c}{\varepsilon}\|W\|_{\mathbb{L}^2(\Omega)} + c\|\nabla W\|_{\mathbb{L}^2(\Omega)}, \end{aligned}$$

where c is a constant independent of ε.

Then the following result was derived [47].

Lemma 12. *There is an extension Q^ε of q^ε which satisfies, for any $W \in \mathscr{D}(\Omega,\mathbb{R}^3)$,*

$$\varepsilon^2\int_\Omega \nabla U^\varepsilon\cdot\nabla R_\varepsilon W\,dx - \int_\Omega Q^\varepsilon\, div W\,dx = \int_\Omega g R_\varepsilon W\,dx.$$

Moreover, there is a constant c independent of ε such that

$$\|Q^\varepsilon\|_{L^2(\Omega)/\mathbb{R}} + \|\nabla Q^\varepsilon\|_{\mathbb{H}^{-1}(\Omega)} \leq c\|g\|_{\mathbb{L}^2(\Omega)}.$$

Remark 6. *Following [33] one can define the extension of* q^ε *by*

$$Q^\varepsilon(x) = \begin{cases} q^\varepsilon \quad \text{in } \Omega_f^\varepsilon, \\ \dfrac{1}{|Y_{\varepsilon,f}^k|}\displaystyle\int_{Y_{\varepsilon,f}^k} q^\varepsilon(x)\,dx \quad \text{in each } Y_{\varepsilon,s}^k, \end{cases}$$

where $Y_{\varepsilon,f}^k = \varepsilon(Y_f + k)$ *and* $Y_{\varepsilon,s}^k = \varepsilon(Y_s + k)$.

Then we can pass to the limit in the Stokes equations. We have from [47] that there is a subsequence, not relabelled for convenience, and a function Q such that $Q^\varepsilon \to Q$ in $L^2(\Omega)$ strong. We extend the function U^ε by zero in $\Omega \setminus \Omega_f^\varepsilon$ and use the same notation for the extended function. We deduce from (65) that U^ε and $\varepsilon\nabla U^\varepsilon$ are bounded in $\mathbb{L}^2(\Omega)$. Then there is a subsequence, not relabelled for convenience, and a function $U^0(x,y) \in \mathbb{L}^2(\Omega \times Y)$, such that

$$U^\varepsilon \overset{2s}{\rightharpoonup} U^0(x,y), \quad \varepsilon\nabla U^\varepsilon \overset{2s}{\rightharpoonup} \nabla_y U^0(x,y).$$

Using appropriate test functions one can derive the equations (38) and (39) satisfied by U^0 and Q. Finally, we have

$$U^0(x,y) = \sum_{j=1}^3 \omega^j(y)\left(g_j(x) - \frac{\partial Q}{\partial x_j}(x)\right), \quad x \in \Omega,\ y \in Y_f, \tag{66}$$

$$\int_{Y_f} U^0(x,y)\,dy = K\,(g(x) - \nabla Q(x)), \quad x \in \Omega, \qquad \text{(Darcy law)} \tag{67}$$

where $K = (K_{ij})_{1\le i,j\le 3}$ is the permeability matrix defined by (42).

4.2.2 End of the proof of Theorem 2

We have from (13) that (M^ε) is bounded in $\mathbb{L}^r(\Omega)$, then

$$M^\varepsilon \overset{2s}{\rightharpoonup} M_0(x,y) = \lambda_0\,\chi_f(y)\,H_0(x,y),$$
$$M^\varepsilon \rightharpoonup M(x) = \lambda_0 \int_Y \chi_f(y)\,H_0(x,y)\,dy \ \text{ weakly in } \mathbb{L}^r(\Omega).$$

We have clearly

$$Q(x) = P(x) - \mu_f\frac{\lambda_0}{2}\,\alpha(x), \quad \alpha(x) = |\nabla\varphi(x)|^2 + \int_Y |\nabla_y\varphi^1(x,y)|^2\,dy.$$

Using (34) we have that

$$\begin{aligned}\alpha(x) &= |\nabla\varphi(x)|^2 + \int_Y \Big|\sum_{k=1}^3 \frac{\partial\varphi}{\partial x_k}(x)\nabla_y w^k(y)\Big|^2\,dy \\ &= \sum_{i=1}^3 \Big|\frac{\partial\varphi}{\partial x_i}(x)\Big|^2 + \sum_{j,k=1}^3 \frac{\partial\varphi}{\partial x_j}(x)\frac{\partial\varphi}{\partial x_k}(x)\int_Y \nabla_y w^j(y)\cdot\nabla_y w^k(y)\,dy \\ &= \sum_{i=1}^3 (1+\mathcal{A}_{ii})\Big|\frac{\partial\varphi}{\partial x_i}(x)\Big|^2 + \sum_{j\ne k}\mathcal{A}_{jk}\,\frac{\partial\varphi}{\partial x_j}(x)\frac{\partial\varphi}{\partial x_k}(x),\end{aligned}$$

then $\alpha(x) = (I+\mathcal{A})\nabla\varphi(x)\cdot\nabla\varphi(x)$ where $\mathcal{A}$ is the matrix given by (37), hence relation (36).

To finish the proof of Theorem 2, it remains to establish relations (40) and (41). These are direct consequences of (66) and (67), respectively, together with the relationship $P(x) = Q(x) + P_m(x)$. The proof of Theorem 2 is complete.

4.3 Proofs of Lemmas 1 and 2

4.3.1 Proof of Lemma 1

(i) Clearly, $a(y,0)=0$ for a.e. $y\in Y$.

(ii) Let us introduce the functions $u,v:\mathbb{R}\to\mathbb{R}$ by

$$u(x)=\begin{cases} M_sL(b_1x) & \text{if } x\neq 0,\\ 0 & \text{if } x=0\end{cases}\qquad v(x)=\begin{cases}\dfrac{u(x)}{x} & \text{if } x\neq 0,\\ \frac{1}{3}M_sb_1 & \text{if } x=0.\end{cases}$$

The following properties of the Langevin function are useful:

$$0\le L(x)\le 1,\ \ \forall x\ge 0,\quad L'(x)=-\frac{1}{\sinh^2(x)}+\frac{1}{x^2},$$
$$L'(x)\ge 0,\ \ \forall x\ge 0,\quad \lim_{x\to+\infty}L(x)=1,\quad \lim_{x\to+\infty}L'(x)=0,$$
$$\text{for } x\to 0:\quad L(x)=\frac{1}{3}x-\frac{1}{45}x^3+\mathcal{O}(x^4),\quad -\frac{1}{\sinh^2(x)}+\frac{1}{x^2}=\frac{1}{3}x-\frac{1}{15}x^2+\mathcal{O}(x^3).$$

We easily verify that $u\in C^1(\mathbb{R})$ and there exists a constant c such that $|u'(x)|\le c$, for all $x\in\mathbb{R}$. Let us verify that $v\in C^1(\mathbb{R})$. We have for $x\to 0$

$$\frac{v(x)-v(0)}{x}=\frac{M_s\frac{L(b_1x)}{x}-\frac{1}{3}M_sb_1}{x}=\frac{M_s\frac{\frac{b_1x}{3}-\frac{(b_1x)^3}{45}}{x}-\frac{1}{3}M_sb_1+\mathcal{O}(x^3)}{x}=-\frac{M_sb_1^3x}{45}+\mathcal{O}(x^2),$$

then v is derivable at $x=0$ and $v'(0)=0$. For $x\neq 0$ we have

$$v'(x)=M_s\frac{b_1x\left(-\frac{1}{\sinh^2(b_1x)}+\frac{1}{(b_1x)^2}\right)-L(b_1x)}{x^2}=\frac{M_s}{x}\left\{\frac{-b_1}{\sinh^2(b_1x)}+\frac{b_1}{(b_1x)^2}-\frac{L(b_1x)}{x}\right\}$$
$$=\mathcal{O}(x^2)\to 0,\quad \text{for } x\to 0.$$

Finally we have

$$v'(x)=\begin{cases}M_s\frac{b_1xL'(b_1x)-L(b_1x)}{x^2} & \text{if } \ x\neq 0,\\ 0 & \text{if } x=0,\end{cases}\tag{68}$$

and $v\in C^1(\mathbb{R})$. We deduce from (68) that there exists a constant c such that $|v'(x)|\le c$, for all $x\in\mathbb{R}$. Then u and v are Lipschitz continuous on $\mathbb{R}$.

Now define the function $w:\mathbb{R}^3\to\mathbb{R}^3$ by

$$w(\xi)=(w_i(\xi))_{1\le i\le 3},\quad w_i(\xi)=v(|\xi|)\xi_i,\ \ \forall\xi\in\mathbb{R}^3.$$

Let us show that w is Lipschitz continuous. We have

$$\frac{\partial w_i}{\partial\xi_j}(\xi)=v'(|\xi|)\frac{\xi_i\xi_j}{|\xi|}+v(|\xi|)\delta_{ij},\ \ \forall\xi\neq 0$$

where δ_{ij} is the Kronecker symbol. Let $\hat{\xi}^j$ be the vector of $\mathbb{R}^3$ whose components are all 0 except the jth which equals ξ_j. For $\xi=0$, the partial derivative of w_i with respect to ξ_j is

$$\frac{\partial w_i}{\partial\xi_j}(0)=\lim_{\xi_j\to 0}\frac{w_i(\hat{\xi}^j)-w_i(0)}{\xi_j}=\lim_{\xi_j\to 0}\frac{w_i(\hat{\xi}^j)}{\xi_j}=\lim_{\xi_j\to 0}\frac{v(|\xi_j|)\hat{\xi}^j_i}{\xi_j}=v(0)\delta_{ij}=\frac{1}{3}M_sb_1\delta_{ij}.$$

Moreover for $\xi\neq 0$, $\frac{\xi_i\xi_j}{|\xi|}\le\frac{1}{2}|\xi|$, then

$$\left|\frac{\partial w_i}{\partial\xi_j}(\xi)\right|\le\frac{1}{2}|\xi||v'(|\xi|)|+|v(|\xi|)|.$$

Since v is bounded as well as $|\xi|v'(|\xi|)$, it results that there exists a constant c such that

$$\left|\frac{\partial w_i}{\partial x_j}(\xi)\right| \leq c, \ \ \forall \xi \in \mathbb{R}^3 \ (i,j=1,2,3).$$

We conclude that w is Lipschitz continuous. Since $a(y,\xi) = \tilde{\mu}(y)\xi + \mu_f \chi_f(y) w(\xi)$, for all $(y,\xi) \in Y \times \mathbb{R}^3$, where $\tilde{\mu}$ is defined by (44), we deduce that $a(y,\xi)$ is Lipschitz continuous with respect to ξ. Point *(ii)* is proven.

(iii) Let $F : Y \times \mathbb{R}^3 \to \mathbb{R}$ be the function defined by

$$F(y,\xi) = \frac{\widetilde{\mu}(y)}{2}|\xi|^2 + \mu_f \chi_f(y) \frac{M_s}{b_1} \ln\left(\frac{\sinh(b_1|\xi|)}{|\xi|}\right) \ \text{ if } \ \xi \neq 0.$$

We have

$$a(y,\xi) = \nabla_\xi F(y,\xi) \ \text{ if } \ \xi \neq 0.$$

The Hessian matrix $\mathcal{H}(y,\xi) = (\mathcal{H}_{ij}(y,\xi))$ of the function $\xi \mapsto F(y,\xi)$ is given by $\mathcal{H}_{ij}(y,\xi) = \frac{\partial a_i(y,\xi)}{\partial \xi_j}$. For $\xi \neq 0$ we have

$$\begin{aligned}\frac{\partial a_i(y,\xi)}{\partial \xi_j} &= \widetilde{\mu}(y)\delta_{ij} + \mu_f \chi_f(y) M_s \Big(\frac{L(b_1|\xi|)}{|\xi|}\delta_{ij} + \frac{b_1|\xi|L'(b_1|\xi|) - L(b_1|\xi|)}{|\xi|^2}\Big)\frac{\xi_i\xi_j}{|\xi|} \\ &= \widetilde{\mu}(y)\delta_{ij} + \mu_f \chi_f(y) M_s \frac{L(b_1|\xi|)}{|\xi|}\Big(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\Big) + \mu_f \chi_f(y) M_s b_1 L'(b_1|\xi|)\frac{\xi_i\xi_j}{|\xi|^2}.\end{aligned}$$

Let $\eta \in \mathbb{R}^3$. Since $|\xi|^2|\eta|^2 - (\xi\cdot\eta)^2 \geq 0$ and $L'(b_1|\xi|^2) \geq 0$ we deduce that

$$\Big(\mathcal{H}(y,\xi)\eta;\eta\Big) \geq \widetilde{\mu}(y)|\eta|^2 \geq \inf\{\mu_f,\mu_s\}|\eta|^2.$$

It results that the function $\xi \mapsto a(y,\xi)$ is strictly monotone and satisfies for all $\xi_1,\xi_2 \in \mathbb{R}^3$,

$$\big(a(y,\xi_1) - a(y,\xi_2)\big)\cdot\big(\xi_1 - \xi_2\big) \geq \inf\{\mu_f,\mu_s\}|\xi_1 - \xi_2|^2,$$

then $\xi \mapsto a(y,\xi)$ is monotone and coercive. Point *(iii)* is proven. The proof of Lemma 1 is achieved.

4.3.2 Proof of Lemma 2

We define the operator $A : H^1_\sharp(\Omega) \to (H^1_\sharp(\Omega))'$ by

$$\langle A(\varphi),\psi\rangle = \int_\Omega a\left(\frac{x}{\varepsilon},\nabla\varphi(x)\right)\cdot\nabla\psi(x)\,dx, \ \ \forall(\varphi,\psi)\in H^1_\sharp(\Omega)\times H^1_\sharp(\Omega), \tag{69}$$

where $a(\cdot,\cdot)$ is defined by (43). Using the Hölder inequality, we easily verify that the operator A is well-defined. Define

$$l(\psi) = \mu_0 \int_{\partial\Omega} H_a\cdot\nu\,\psi\,ds, \quad \forall\,\psi\in H^1_\sharp(\Omega).$$

Clearly $l \in (H^1_\sharp(\Omega))'$.

(i) Let us show that the operator A is strictly monotone [30](Chapter III), [32](Chapter 2, Section 2). Let $\varphi_1,\varphi_2 \in H^1_\sharp(\Omega)$. Using Lemma 1 (iii) we have

$$\langle A(\varphi_1) - A(\varphi_2), \varphi_1 - \varphi_2\rangle \geq \inf\{\mu_f,\mu_s\}\int_\Omega |\nabla\varphi_1 - \nabla\varphi_2|^2\,dx.$$

This inequality, together with the Poincaré-Wirtinger inequality, shows that the operator A is strictly monotone.

(ii) Let us now check that the operator A satisfies the coerciveness property:

$$\frac{\langle A(\varphi),\varphi\rangle}{\|\varphi\|_W}\to+\infty \quad \text{as } \|\varphi\|_{H^1_\sharp(\Omega)}\to+\infty. \tag{70}$$

We have

$$\langle A(\varphi),\varphi\rangle \geq \inf\{\mu_f,\mu_s\}\int_\Omega |\nabla\varphi|^2\,dx; \quad \forall\,\varphi\in H^1_\sharp(\Omega),$$

then using the Poincaré-Wirtinger inequality there holds that $\langle A(\varphi),\varphi\rangle \geq c\|\varphi\|^2_{H^1_\sharp(\Omega)}$ where c is a constant independent of ε, hence (70).

(iii) It is clear that A is hemicontinuous. We conclude, see e.g. [32] (Chapter 2, Section 2), [30] (Chapter III), that there is a unique $\varphi\in H^1_\sharp(\Omega)$ such that $A(\varphi)=l$.

(iv) Taking $\psi=\varphi^\varepsilon$ in (45) there holds that

$$\inf\{\mu_f,\mu_s\}\int_\Omega |\nabla\varphi^\varepsilon|^2\,dx \leq \langle A(\varphi^\varepsilon),\varphi^\varepsilon\rangle = \mu_0\int_{\partial\Omega} H_a\cdot\nu\,\varphi^\varepsilon\,d\sigma.$$

Using the Poincaré-Wirtinger inequality and the trace theorem, we deduce that the sequence (φ^ε) is bounded in $H^1_\sharp(\Omega)$. The proof of Lemma 2 is complete.

4.4 Proof of Theorem 3

4.4.1 Proof of (i)

Define

$$a^\varepsilon(x)=a\left(\frac{x}{\varepsilon},\nabla\varphi^\varepsilon(x)\right)$$

where $a(y,\xi)$ is defined by (43). We have from (26) that

$$\begin{cases} \operatorname{div} a^\varepsilon = 0 \ \text{ in } \Omega,\\ a^\varepsilon\cdot\nu=\mu_0 H_a\cdot\nu \ \text{ on } \partial\Omega. \end{cases} \tag{71}$$

We have from Lemma 2 that (φ^ε) is bounded in $H^1_\sharp(\Omega)$ and we easily verify that (a^ε) is bounded in $\mathbb{L}^2(\Omega)$. Then there are subsequences, still indexed by ε, and functions $\varphi\in H^1_\sharp(\Omega)$, $\varphi_1(x,y)\in L^2(\Omega,H^1_{per}(Y))$, and $a_0(x,y)\in\mathbb{L}^2(\Omega\times Y)$, such that (φ^ε) converges to φ weakly in $H^1(\Omega)$, $(\nabla\varphi^\varepsilon)$ two-scale converges to $\nabla\varphi(x)+\nabla_y\varphi_1(x,y)$, and a^ε two-scale converges to $a_0(x,y)$. As is classical we have that

$$\begin{cases} \operatorname{div}_y a_0 = 0 \quad \text{in } \Omega\times Y,\\ \operatorname{div}\left(\displaystyle\int_Y a_0(\cdot,y)\,dy\right)=0 \quad \text{in } \Omega, \quad \left(\displaystyle\int_Y a_0(\cdot,y)\,dy\right)\cdot\nu=\mu_0 H_a\cdot\nu \ \text{ on } \partial\Omega,\\ a_0(x,y) \ \text{ Y-periodic for a.e. } x\in\Omega. \end{cases}$$

In order to express $a_0(x,y)$ in terms of $\varphi(x)$ and $\varphi_1(x,y)$ we consider the test function

$$\Psi^\varepsilon(x)=\nabla\psi^\varepsilon(x)+t\Phi(x,\left(\frac{x}{\varepsilon}\right)), \qquad \psi^\varepsilon(x)=\psi(x)+\varepsilon\psi_1\left(x,\frac{x}{\varepsilon}\right),$$

with $\psi\in\mathscr{D}(\overline{\Omega})$, $\psi_1\in\mathscr{D}(\overline{\Omega},C^\infty_{per}(Y))$, $\Phi\in\mathscr{D}(\Omega,C^\infty_{per}(Y))$, and t a positive real number. Using the monotonicity of the operator A (defined by (69)), we have

$$\int_\Omega\left(a^\varepsilon(x)-a\left(\frac{x}{\varepsilon},\Psi^\varepsilon(x)\right)\right)\cdot(\nabla\varphi^\varepsilon(x)-\Psi^\varepsilon(x))\,dx\geq 0. \tag{72}$$

Using (71) and the Green formula we have

$$\int_\Omega a^\varepsilon(x)\cdot(\nabla\varphi^\varepsilon(x)-\Psi^\varepsilon(x))\,dx=\mu_0\int_{\partial\Omega}H_a\cdot\nu(\varphi^\varepsilon-\psi^\varepsilon)\,d\sigma-\int_\Omega a^\varepsilon(x)\cdot t\Phi(x,\left(\frac{x}{\varepsilon}\right)dx$$

then (72) becomes

$$\mu_0 \int_{\partial\Omega} H_a \cdot \nu(\varphi^\varepsilon - \psi^\varepsilon)\, d\sigma - \int_\Omega a^\varepsilon(x) \cdot t\Phi(x, \left(\frac{x}{\varepsilon}\right) dx$$
$$- \int_\Omega a\left(\frac{x}{\varepsilon}, \Psi^\varepsilon(x)\right) \cdot \left(\nabla\varphi^\varepsilon(x) - \Psi^\varepsilon(x)\right) dx \geq 0. \tag{73}$$

Let us write the last term of (73) as

$$\int_\Omega a\left(\frac{x}{\varepsilon}, \Psi^\varepsilon(x)\right) \cdot \left(\nabla\varphi^\varepsilon(x) - \Psi^\varepsilon(x)\right) dx = I_1^\varepsilon + I_2^\varepsilon,$$

with

$$I_1^\varepsilon = \int_\Omega \left[a\left(\frac{x}{\varepsilon}, \Psi^\varepsilon(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla\psi(x) + \nabla_y\psi_1\left(x, \frac{x}{\varepsilon}\right) + t\Phi(x, \left(\frac{x}{\varepsilon}\right)\right)\right] \cdot \left[\nabla\varphi^\varepsilon(x) - \Psi^\varepsilon(x)\right] dx,$$
$$I_2^\varepsilon = \int_\Omega a\left(\frac{x}{\varepsilon}, \nabla\psi(x) + \nabla_y\psi_1\left(x, \frac{x}{\varepsilon}\right) + t\Phi(x, \left(\frac{x}{\varepsilon}\right)\right) \cdot \left(\nabla\varphi^\varepsilon(x) - \Psi^\varepsilon(x)\right) dx.$$

We can pass to the limit in I_2^ε, as $\varepsilon \to 0$, since $a(y, \nabla\psi(x) + \nabla_y\psi_1(x,y) + t\Phi(x,y))$ belongs to $\mathbb{L}^2_{per}(Y; C(\overline{\Omega}))$. Indeed

$$I_2^\varepsilon \to \int_\Omega \int_Y a(y, \nabla\psi(x) + \nabla_y\psi_1(x,y) + t\Phi(x,y)) \cdot$$
$$\cdot \left(\nabla(\varphi(x) - \psi(x)) + \nabla_y(\varphi_1(x,y) - \psi_1(x,y)) - t\Phi(x,y)\right) dxdy.$$

We have by Lemma 1

$$\left|a\left(\frac{x}{\varepsilon}, \Psi^\varepsilon(x)\right) - a\left(\frac{x}{\varepsilon}, \nabla\psi(x) + \nabla_y\psi_1\left(x, \frac{x}{\varepsilon}\right) + t\Phi(x, \left(\frac{x}{\varepsilon}\right)\right)\right|$$
$$\leq c\left|\Psi^\varepsilon(x) - \left(\nabla\psi(x) + \nabla_y\psi_1\left(x, \frac{x}{\varepsilon}\right) + t\Phi\left(x, \frac{x}{\varepsilon}\right)\right)\right|$$
$$= c\varepsilon\nabla_x\psi_1\left(x, \frac{x}{\varepsilon}\right).$$

Here and in the sequel we denote by c a positive constant independent of $\boldsymbol{\varepsilon}$. Using the Cauchy-Schwarz inequality it then holds that $|I_1^\varepsilon| \leq c\varepsilon$, since $(\nabla\varphi^\varepsilon)$ and (Ψ^ε) are bounded in $\mathbb{L}^2(\Omega)$. Therefore, passing to the two-scale limit in (73) as $\varepsilon \to 0$, yields

$$\int_{\partial\Omega} \mu_0 H_a \cdot \nu(\varphi - \psi)\, d\sigma - \int_\Omega \int_Y a_0(x,y) \cdot t\Phi(x,y) dxdy$$
$$- \int_\Omega \int_Y a(y, \nabla\psi(x) + \nabla_y\psi_1(x,y) + t\Phi(x,y)) \cdot$$
$$\cdot \left(\nabla(\varphi(x) - \psi(x)) + \nabla_y(\varphi_1(x,y) - \psi_1(x,y)) - t\Phi(x,y)\right) dxdy \geq 0.$$

Now we take in the previous inequality $\psi = \varphi - t\phi$, $\psi_1 = \varphi_1 - t\phi_1$, $\phi = \phi(x) \in \mathscr{D}(\overline{\Omega})$ and $\phi_1 = \phi_1(x,y) \in \mathscr{D}(\overline{\Omega}, C^\infty_{per}(Y))$. After dividing by t, there holds that

$$\int_{\partial\Omega} \mu_0 H_a \cdot \nu\phi\, d\sigma - \int_\Omega \int_Y a_0(x,y) \cdot \Phi(x,y) dxdy$$
$$- \int_\Omega \int_Y a(y, \nabla(\varphi - t\phi)(x) + \nabla_y(\varphi_1 - t\phi_1)(x,y) + t\Phi(x,y)) \cdot$$
$$\cdot \left(\nabla\phi(x) + \nabla_y\phi_1(x,y) - \Phi(x,y)\right) dxdy \geq 0.$$

Letting $t \to 0$, it results that

$$\int_{\partial\Omega} \mu_0 H_a \cdot \nu\phi\, d\sigma - \int_\Omega \int_Y a_0(x,y) \cdot \Phi(x,y) dxdy$$
$$- \int_\Omega \int_Y a(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)) \cdot \left(\nabla\phi(x) + \nabla_y\phi_1(x,y) - \Phi(x,y)\right) dxdy = 0,$$

for any $\phi \in \mathscr{D}(\overline{\Omega})$, $\phi_1 \in \mathscr{D}(\overline{\Omega}, C^\infty_{per}(Y))$ and $\Phi \in \mathscr{D}(\Omega, C^\infty_{per}(Y))$. We deduce that

$$a_0(x,y) = a\Big(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)\Big)$$

and

$$\int_\Omega \int_Y a(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)) \cdot \Big(\nabla\phi(x) + \nabla_y\phi_1(x,y)\Big)\, dxdy = \int_{\partial\Omega} \mu_0 H_a \cdot \nu\phi \, d\sigma, \quad (74)$$

for any $\phi \in \mathscr{D}(\overline{\Omega})$, $\phi_1 \in \mathscr{D}(\overline{\Omega}, C^\infty_{per}(Y))$. Using a density argument we deduce that (74) holds for any $\phi \in H^1(\Omega)$, $\phi_1 \in L^2(\Omega, H^1_{per}(Y))$.

Using the strict monotonicity of the operator A (defined by (69)), we easily show that the variational equation (74) has a unique solution (φ, φ_1) in $H^1_\sharp(\Omega) \times L^2(\Omega, H^1_{per}(Y))$. This implies that the whole sequence (φ^ε) converges. We also have that the whole sequence (a^ε) two-scale converges to $a_0(x,y)$. Then (a^ε) converges in $\mathbb{L}^2(\Omega)$ weak to $a^*(x) = \int_Y a_0(x,y)\, dy$. Clearly, the mean null condition $\int_\Omega \varphi(x)\, dx = 0$ is satisfied. The proof of **(i)** is complete.

Remark 7. *We have shown in the above proof that the sequence* $\Big(a\left(\frac{x}{\varepsilon}, \nabla\varphi^\varepsilon(x)\right)\Big)$ *two-scale converges to* $a\big(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)\big)$.

4.4.2 Proof of (ii)

We have from **(i)** above that

$$a^\varepsilon(x) := a\left(\frac{x}{\varepsilon}, \nabla\varphi^\varepsilon(x)\right) \rightharpoonup a^*(x) := \int_Y a\big(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)\big)\, dy \quad \text{in } L^2(\Omega) \text{ weak}$$

and

$$\int_\Omega a^*(x) \cdot \nabla\phi(x)\, dx = \int_{\partial\Omega} \mu_0 H_a \cdot \nu\phi \, d\sigma, \quad \forall \phi \in H^1(\Omega). \quad (75)$$

Moreover, taking $\phi = 0$ and $\phi_1 \in H_{per}(Y)$ in (47) we obtain that $\varphi(x)$ and $\varphi_1(x,y)$ satisfy the cell problem

$$\int_Y a(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)) \cdot \nabla_y\phi_1(y)\, dy = 0, \quad \text{for all } \phi_1 \in H^1_{per}(Y). \quad (76)$$

Then, according to the definition of the operator b (see (49) and (50)) it holds that

$$b(\nabla\varphi(x)) = \int_Y a\big(y, \nabla\varphi(x) + \nabla_y\varphi_1(x,y)\big)\, dy = a^*(x),$$

and (75) reads

$$\int_\Omega b(\nabla\varphi(x)) \cdot \nabla\phi(x)\, dx = \int_{\partial\Omega} \mu_0 H_a \cdot \nu\phi \, d\sigma, \quad \text{for all } \phi \in H^1(\Omega),$$

which is equation (48). We prove as in [20] that the operator b satisfies the following properties:

(i) $b(0) = 0$;

(ii) b is Lipschitz continuous, i.e. there exists a constant c such that

$$|b(\xi_1) - b(\xi_2)| \le c|\xi_1 - \xi_2|, \quad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^3;$$

(iii) b is monotone and coercive, more precisely,

$$\big(b(\xi_1) - b(\xi_2)\big) \cdot (\xi_1 - \xi_2) \ge \inf\{\mu_f, \mu_s\}|\xi_1 - \xi_2|^2, \quad \text{for every } \xi_1, \xi_2 \in \mathbb{R}^3.$$

It results that equation (48) has a unique solution in $H^1_\sharp(\Omega)$. Point **(ii)** is proven.

4.4.3 Proof of (iii)

Thanks to Remark 7 we have that $a\left(\frac{x}{\varepsilon}, \nabla\varphi^\varepsilon(x)\right) \overset{2s}{\rightharpoonup} a\big(y, H_0(x,y)\big)$. Using that $a(y,\xi) = \tilde{\mu}(y)\xi + \mu_f\,\chi_f(y) M_s \frac{L(b_1|\xi|)}{|\xi|}\,\xi$, we deduce that

$$\mu_f\,\chi_f(\frac{x}{\varepsilon})\,w\big(\nabla\varphi^\varepsilon(x)\big) \overset{2s}{\rightharpoonup} \mu_f\,\chi_f(y)\,w\big(H_0(x,y)\big), \quad w(\xi) = M_s\frac{L(b_1|\xi|)}{|\xi|}\,\xi,$$

and the result follows readily. The proof of Theorem 3 is complete.

4.5 Proof of Theorem 4

4.5.1 Proof of (i)

In [19] the authors studied the homogenization of a nonlinear problem of the form (46) (with a Dirichlet boundary condition). They constructed a corrector [19] (Theorem 2.1). The same arguments allow to prove that their result is also valid for our problem.

Theorem 5. *Let φ^ε be a solution of problem* (45)*, $H^\varepsilon = \nabla\varphi^\varepsilon$, h_ε be defined by* (54)*, and let φ be a solution of problem* (48)*. Then*

$$H^\varepsilon - h_\varepsilon(\cdot, (m_\varepsilon\nabla\varphi)) \to 0 \quad \textit{a.e. and strongly in } \mathbb{L}^2(\Omega).$$

4.5.2 Proof of (ii)

Problem (27) is the same as problem (18) then we have from Theorem 2 (ii)

$$U^\varepsilon \overset{2s}{\rightharpoonup} U_0(x,y), \quad \varepsilon\nabla U^\varepsilon \overset{2s}{\rightharpoonup} \nabla_y U_0(x,y), \quad Q^\varepsilon \to Q \quad \text{strongly in } L^2(\Omega),$$

where the pair (U_0, Q) satisfies (38) and is a solution of problem (39).

4.5.3 Proof of (iii)

Let κ be the function defined by (22). We have $\dfrac{\partial\kappa}{\partial\xi_j}(\xi) = M_s L(b_1|\xi|)\dfrac{\xi_j}{|\xi|}$ then $\left|\dfrac{\partial\kappa}{\partial\xi_j}(\xi)\right| \le M_s$, $j = 1,2,3$, which implies that κ is Lipchitz continuous. Moreover, $|\kappa(\xi)| \le M_s|\xi|$, then $(\kappa(H^\varepsilon))$ is bounded in $L^2(\Omega)$ since (H^ε) is bounded in $\mathbb{L}^2(\Omega)$.

We have from Theorem 5 that

$$H^\varepsilon(x) - h_\varepsilon\big(x, (m_\varepsilon\nabla\varphi)(x)\big) \to 0 \ \text{ a.e. in } \Omega \text{ and strongly in } \mathbb{L}^2(\Omega),$$

then

$$\kappa\big(H^\varepsilon(x)\big) - \kappa\big(h_\varepsilon\big(x, (m_\varepsilon\nabla\varphi)(x)\big)\big) \to 0 \ \text{ a.e. in } \Omega \text{ and strongly in } \mathbb{L}^2(\Omega),$$

since κ is Lipchitz continuous. We are then led to study the limit of $\kappa\big(h_\varepsilon\big(\cdot, (m_\varepsilon\nabla\varphi)\big)\big)$, as $\varepsilon \to 0$.

Let $\xi_1, \xi_2 \in \mathbb{R}^3$ and let $h(y,\xi)$ be the function defined by (53). Using Lemma 1 and the relation

$$\begin{aligned}&\int_Y \big(a(y, h(y,\xi_1)) - a(y, h(y,\xi_2))\big)\cdot(h(y,\xi_1) - h(y,\xi_2))\,dy\\ &= \int_Y \big(a(y, h(y,\xi_1)) - a(y, h(y,\xi_2))\big)\cdot(\xi_1 - \xi_2)\,dy,\end{aligned}$$

we deduce that

$$\int_Y |h(y,\xi_1) - h(y,\xi_2)|^2 dy \le c\int_Y |h(y,\xi_1) - h(y,\xi_2)||\xi_1 - \xi_2|dy,$$

hence, using the Cauchy-Schwarz inequality,

$$\|h(\cdot,\xi_1)-h_(\cdot,\xi_1)\|_{\mathbb{L}^2(Y)}\le c|\xi_1-\xi_2|. \tag{77}$$

Here and in the sequel we denote by c a positive constant independent of $\boldsymbol{\varepsilon}$. Let us now consider the function κ^0 given by (55). Since κ is Lipschitz continuous, we deduce by using (77) that

$$\begin{aligned}|\kappa^0(\xi_1)-\kappa^0(\xi_2)|&\le\int_Y|\kappa\big(h(y,\xi_1)\big)-\kappa\big(h(y,\xi_2)\big)|dy\le c\int_Y|h(y,\xi_1)-h(y,\xi_2)|dy\\&\le c\|h(\cdot,\xi_1)-h(\cdot,\xi_2)\|_{\mathbb{L}^2(Y)}\le c|\xi_1-\xi_2|,\ \ \forall\xi_1,\xi_2\in\mathbb{R}^3.\end{aligned} \tag{78}$$

Let us now study the limit of $\kappa\big(h_\varepsilon(\cdot,(m_\varepsilon\nabla\varphi))\big)$, as $\varepsilon\to 0$. Since for every $\xi\in\mathbb{R}^3$, $y\mapsto\kappa\big(h(y,\xi)\big)\in L^2(Y)$ and $\kappa\big(h_\varepsilon(\cdot,\xi)\big)$ is ε-periodic it holds that

$$\kappa\big(h_\varepsilon(\cdot,\xi)\big)\rightharpoonup\kappa^0(\xi)\ \text{ weakly in } L^2(\Omega),\ \text{for every } \xi\in\mathbb{R}^3. \tag{79}$$

Let us prove that

$$\kappa\big(h_\varepsilon(\cdot,m_\varepsilon\Psi)\big)\rightharpoonup\kappa^0(\Psi)\ \text{ weakly in } L^2(\Omega), \tag{80}$$

for every $\Psi\in\mathbb{L}^2(\Omega)$. To this aim we use the following result [19] (Lemma 3.5).

Lemma 13. *Let $\Phi\in\mathbb{L}^2(\Omega)$ and let Ψ be a step function of the form*

$$\Psi(x)=\sum_{j=1}^n\eta_j 1_{\Omega_j}(x)$$

with $\eta_j\in\mathbb{R}^3\setminus\{0\}$, $\Omega_j\subset\subset\Omega$, $|\partial\Omega_j|=0,\Omega_j\cap\Omega_k=\emptyset$ for $j\neq k$. Then

$$\limsup_{\varepsilon\to 0}\|h_\varepsilon(\cdot,m_\varepsilon\Phi)-h_\varepsilon(\cdot,\Psi)\|_{\mathbb{L}^2(\Omega)}\le c\|\Phi-\Psi\|_{\mathbb{L}^2(\Omega)} \tag{81}$$

where the constant c is independent of ε and n.

To prove (80) we first note that, by (79),

$$\kappa\big(h_\varepsilon(\cdot,\Psi)\big)\rightharpoonup\kappa^0(\Psi)\ \text{ weakly in } L^2(\Omega),$$

for every step function $\Psi\in\mathbb{L}^2(\Omega)$. Let then $\Psi\in\mathbb{L}^2(\Omega)$. For every $\delta>0$, there exists a step function $\eta(x)=\sum_{j=1}^n\eta_j 1_{\Omega_j}(x)$ with $\eta_j\in\mathbb{R}^3\setminus\{0\}$, $\Omega_j\subset\subset\Omega$, $|\partial\Omega_j|=0,\Omega_j\cap\Omega_k=\emptyset$ for $j\neq k$, such that

$$\|\Psi-\eta\|_{\mathbb{L}^2(\Omega)}\le\delta. \tag{82}$$

Let us write

$$\begin{aligned}\kappa\big(h_\varepsilon(\cdot,m_\varepsilon\Psi)\big)-\kappa^0(\Psi)=&\big[\kappa\big(h_\varepsilon(\cdot,m_\varepsilon\Psi)\big)-\kappa\big(h_\varepsilon(\cdot,\eta)\big)\big]+\big[\kappa\big(h_\varepsilon(\cdot,\eta)\big)-\kappa^0(\eta)\big]\\&+\big[\kappa^0(\eta)-\kappa^0(\Psi)\big].\end{aligned} \tag{83}$$

By (79) we have that

$$\kappa\big(h_\varepsilon(\cdot,\eta)\big)\rightharpoonup\kappa^0(\eta)\ \text{ weakly in } L^2(\Omega). \tag{84}$$

Since κ is Lipchitz continuous there is a constant c independent of ε such that

$$\big|\kappa\big(h_\varepsilon(x,(m_\varepsilon\Psi)(x))\big)-\kappa\big(h_\varepsilon(x,\eta(x))\big)\big|\le c\,|h_\varepsilon(x,(m_\varepsilon\Psi)(x))-h_\varepsilon(x,\eta(x))|,\quad\text{a.e. in }\Omega.$$

Using Lemma 13 and the previous inequality we deduce that

$$\limsup_{\varepsilon\to 0}\|\kappa(h_\varepsilon(\cdot,m_\varepsilon\Psi)-\kappa\big(h_\varepsilon(\cdot,\eta)\big)\|_{L^2(\Omega)}\le c\|\Psi-\eta\|_{\mathbb{L}^2(\Omega)}\le c\delta \tag{85}$$

where in the last inequality we used (82).

Let us now consider $\|\kappa^0(\eta) - \kappa^0(\Psi)\|_{L^2(\Omega)}$. We deduce from (78) that

$$\int_\Omega |\kappa^0(\eta(x)) - \kappa^0(\Psi(x))|^2 dx \leq c \int_\Omega |\eta(x) - \Psi(x)|^2 dx \leq c\delta^2 \tag{86}$$

where we used (82) in the last inequality. From (83)–(86) we deduce (80). Applying (80) with $\Psi = \nabla\varphi$ and using Theorem 4 (ii) we obtain (56).

To finish the proof of Theorem 4, it remains to establish relations (40) and (41). These are direct consequences of (57) and (58), respectively, together with the relationship

$$Q(x) = P(x) - \mu_f \kappa_1(x), \quad \kappa_1(x) = \kappa^0(\nabla\varphi(x)).$$

Point **(iii)** is proven and the proof of Theorem 4 is complete.

5 Concluding remarks

We considered the equations describing the flow of a ferrofluid through a heterogeneous porous medium Ω in the presence of an applied magnetic field. We discussed two models where the magnetization M is parallel to the magnetic field H: a linear model and a nonlinear model where the magnetization and the magnetic field satisfy the Langevin law. The velocity and the pressure satisfy the Stokes equation with a Kelvin magnetic force. Choosing the characteristic parameters of the flow as $Re = \frac{1}{\varepsilon^2}$, $Fr = Re_m = Eu = 1$, we investigated in each of the two models the homogenization of the differential system with the use of the two-scale convergence method. We rigorously derived the homogenized equation for the magnetic potential and determined the asymptotic limit of the magnetization. Then we rigorously derived a Darcy law.

Let us mention that the analysis done for the Langevin model can be applied by using similar arguments to the model where the magnetization and the magnetic field satisfy the relation

$$M = a_1 \frac{\arctan(b_1|H|)}{|H|} 1_{\Omega_f} H,$$

where a_1 and b_1 are positive constants and 1_{Ω_f} is the characteristic function of the pore space Ω_f. The extension of this study to take a more general law, e.g. $M = f(H)$, is limited to the cases where we can homogenize the nonlinear problem satisfied by the potential φ associated with the magnetic field H and that $(f(H)\cdot\nabla)H$ has a gradient structure.

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