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# GLOBAL QUASI-NEUTRAL LIMIT FOR A TWO-FLUID EULER-POISSON SYSTEM IN SEVERAL SPACE DIMENSIONS

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**Abstract.** This paper concerns the quasi-neutral limit to the Cauchy problem for a two-fluid Euler-Poisson system in several space dimensions. When the initial data are sufficiently close to constant equilibrium states, we prove the global existence of smooth solutions with uniform bounds with respect to the Debye length in Sobolev spaces. This allows to pass to the limit in the system for all time to obtain a compressible Euler system. We also prove global error estimates between the solution of the two-fluid Euler-Poisson system and that of the compressible Euler system. These results are obtained by establishing uniform energy estimates and various dissipation estimates. A key step in the proof is the control of the quasi-neutrality of the velocities. For this purpose, an orthogonal projection operator is used.

**Keywords:** quasi-neutral limit, two-fluid multi-dimensional Euler-Poisson system, energy estimates, global convergence, convergence rate.

**AMS Subject Classification (2000) :** 35B40, 35J05, 35Q35

## 1. INTRODUCTION

In a previous work [33], we considered the global quasi-neutral limit for a two-fluid (bipolar) isentropic Euler-Poisson system in one space dimension. In this paper, we continue to study this limit in several space dimensions. The system arises in the modeling of plasmas (or semiconductors) consisting of electrons of charge  $q_e = -1$  and a single species of ions of charge  $q_i = 1$ . Denote by  $n_e$  and  $u_e$  ( $n_i$  and  $u_i$ , respectively) the scaled density and the velocity of the electrons (ions, respectively), and by  $\phi$  the electric potential. The two-fluid Euler-Poisson system reads (see [6, 29, 20, 10])

$$(1.1) \quad \begin{cases} \partial_t n_\nu + \operatorname{div}(n_\nu u_\nu) = 0, \\ \partial_t(n_\nu u_\nu) + \operatorname{div}(n_\nu u_\nu \otimes u_\nu) + \nabla p_\nu(n_\nu) = -q_\nu n_\nu \nabla \phi - n_\nu u_\nu, & \nu = e, i, \\ -\lambda^2 \Delta \phi = n_i - n_e, & \lim_{|x| \rightarrow +\infty} \phi(t, x) = 0, \end{cases}$$

for  $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$ . Here  $\otimes$  is the tensor product,  $\lambda \in (0, 1]$  is the Debye length. The pressure functions  $p_\nu$  ( $\nu = e, i$ ) are supposed to be smooth and strictly increasing, namely,

$$p'_\nu(\tau) > 0, \quad \forall \tau > 0, \quad \nu = e, i.$$

The system is supplemented by the following initial condition depending on  $\lambda$  :

$$(1.2) \quad t = 0 : \quad (n_\nu, u_\nu) = (n_{\nu,\lambda}^0(x), u_{\nu,\lambda}^0(x)), \quad \nu = e, i, \quad x \in \mathbb{R}^d.$$

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We define  $\phi_\lambda^0$  as the initial data of  $\phi$  by

$$(1.3) \quad -\lambda^2 \Delta \phi_\lambda^0 = n_{i,\lambda}^0 - n_{e,\lambda}^0, \quad \lim_{|x| \rightarrow +\infty} \phi_\lambda^0(x) = 0.$$

We are concerned with smooth solutions to system (1.1) in the non-vacuum field, namely,  $n_\nu > 0$  for  $\nu = e, i$ . Then the momentum equations in (1.1) can be written equivalently as

$$(1.4) \quad \partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n_\nu) = -q_\nu \nabla \phi - u_\nu, \quad \nu = e, i,$$

where the enthalpy functions  $h_\nu$  ( $\nu = e, i$ ) are defined by

$$h_\nu(n) = \int_1^n \frac{p'_\nu(\tau)}{\tau} d\tau, \quad \nu = e, i.$$

System (1.1) is composed of a symmetrizable hyperbolic system for  $(n_\nu, u_\nu)$ ,  $\nu = e, i$ , coupled to a linear Poisson equation. The local well-posedness of classical solutions to the Cauchy problem (1.1)-(1.2) are well-known by applying the results in [22, 28]. Let  $s > \frac{d}{2} + 1$  be an integer. We assume that  $(n_{\nu,\lambda}^0 - 1, u_{\nu,\lambda}^0, \nabla \phi_\lambda^0) \in H^s$  with  $n_{\nu,\lambda}^0 \geq \text{const.} > 0$ , here and in what follows  $H^s$  stands for  $H^s(\mathbb{R}^d)$ . Then there exists  $T_* > 0$  such that the Cauchy problem (1.1)-(1.2) admits a unique solution  $(n_\nu, u_\nu, \phi)$  on the domain  $[0, T_*] \times \mathbb{R}^d$ , and we have

$$n_\nu - 1, u_\nu, \nabla \phi \in \bigcap_{k=0}^s C^k([0, T_*]; H^{s-k}), \quad \inf_{(t,x) \in [0, T_*] \times \mathbb{R}^d} n_\nu(t, x) \geq \text{const.} > 0, \quad \nu = e, i.$$

In general,  $T_*$  depends on the parameter  $\lambda \in (0, 1]$ . For fixed  $\lambda$ , the global existence with long time behavior of smooth solutions near constant equilibrium states was proved in [2, 25], and the stability of solutions was studied in [9].

The one-fluid Euler-Poisson system for electrons (ions, respectively) is described by variables  $(n_e, u_e, \phi)$  ( $(n_i, u_i, \phi)$ , respectively) when the plasma is in a uniform background where  $n_i$  ( $n_e$ , respectively) is given. For fixed  $\lambda$ , the global existence and the stability of solutions were investigated in [1, 17, 15]. When the electron density is replaced by a Boltzmann relation, we obtain a one-fluid model for ions in which the Poisson equation is semilinear [12, 26].

In plasma physics, the Debye length  $\lambda$  is much smaller than the length of physical variables. The quasi-neutrality means  $n_e - n_i \rightarrow 0$  as  $\lambda \rightarrow 0$ . Thus, it is important to study the limit behavior in these systems as  $\lambda \rightarrow 0$ , which is referred to as the quasi-neutral limit. An early analysis of the limit was given in [5] for the Poisson equation coupled to a Boltzmann relation. For the drift-diffusion systems this limit was studied in [21] where it was revealed that the quasi-neutral limit is related to the quasi-Fermi potentials. See also the study of this limit for the Vlasov-Poisson system [3, 16], the pressureless Euler-Poisson system [27] and the Navier-Stokes-Poisson system [11]. It is well-known that the quasi-neutral limit of the one-fluid Euler-Poisson system leads to incompressible Euler equations. The justification of the limit for the one-fluid Euler-Poisson system was carried out in [8, 36, 39] for smooth solutions on a uniform time interval and in [32, 26] for global smooth solutions near constant equilibrium states.

It was already shown in [34, 24, 33] that the quasi-neutral limit of the two-fluid Euler-Poisson system presents a different situation, since its limit system is governed by compressible Euler equations. Indeed, passing formally to the limit in (1.1) as  $\lambda \rightarrow 0$ , we obtain from the Poisson equation that  $n_e = n_i$ , denoted by  $n$ . Therefore, the limit system of the first two equations in (1.1) becomes

$$\begin{cases} \partial_t n + \text{div}(n u_\nu) = 0, \\ \partial_t u_\nu + (u_\nu \cdot \nabla) u_\nu + \nabla h_\nu(n) = -q_\nu \nabla \phi - u_\nu, \quad \nu = e, i. \end{cases}$$

If  $u_e(0, \cdot) = u_i(0, \cdot)$ , we have  $u_e = u_i$ , denoted by  $u$ , at least on a local time interval, see [34]. Adding and subtracting the second equations above for  $\nu = e, i$  yields compressible Euler

equations

$$(1.5) \quad \begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \partial_t u + (u \cdot \nabla)u + \nabla h_+(n) = -u, \end{cases}$$

and

$$\nabla \phi = \nabla h_-(n),$$

where

$$h_{\pm} = \frac{1}{2}(h_e \pm h_i).$$

By the method of asymptotic expansions together with energy estimates, the convergence of system (1.1) to (1.5) as  $\lambda \rightarrow 0$  was justified for smooth solutions on uniform time intervals [18, 24, 19].

The global-in-time convergence of the two-fluid system (1.1) as  $\lambda \rightarrow 0$  is a challenging problem on which very few results are available. In a recent paper [33], we solved this problem in one space dimension. Up to our knowledge, that was the only result on this issue for (1.1). The goal of this paper is to prove this global quasi-neutral limit for (1.1) in several space dimensions.

Let  $(\bar{n}_e, \bar{u}_e, \bar{n}_i, \bar{u}_i, \nabla \bar{\phi}) = (1, 0, 1, 0, 0)$  be a constant equilibrium state. Let  $s > \frac{d}{2} + 1$  be an integer. We assume that  $(n_{\nu, \lambda}^0 - 1, u_{\nu, \lambda}^0, \nabla \phi_{\lambda}^0)$  is uniformly sufficiently small in the norm of  $H^s$  for  $\nu = e, i$ . Moreover, let  $\mu_{\nu}$  be the quasi-Fermi potentials defined by (see [29, 20])

$$(1.6) \quad \mu_{\nu} = h_{\nu}(n_{\nu}) + q_{\nu} \phi, \quad \nu = e, i,$$

and

$$(1.7) \quad \mu = \mu_e - \mu_i = h_e(n_e) - h_i(n_i) - 2\phi.$$

We denote

$$n_- = n_e - n_i, \quad u_- = u_e - u_i.$$

Subtracting (1.4) for  $\nu = e, i$  yields

$$(1.8) \quad \partial_t u_- + ((u_- \cdot \nabla)u_e + (u_i \cdot \nabla)u_-) + u_- + \nabla \mu = 0.$$

We first show our results and assumptions on the initial data. The quasi-neutrality and the formal derivation of system (1.5) from (1.1) suggest that

$$n_{e, \lambda}^0 - n_{i, \lambda}^0 \rightarrow 0, \quad u_{e, \lambda}^0 - u_{i, \lambda}^0 \rightarrow 0, \quad \text{as } \lambda \rightarrow 0.$$

Hence, we assume

$$n_{e, \lambda}^0 - n_{i, \lambda}^0 = O(\lambda^2), \quad u_{e, \lambda}^0 - u_{i, \lambda}^0 = O(\lambda^2).$$

From (1.8) we see formally that  $\nabla \mu$  and  $u_e - u_i$  should have the same order in  $\lambda$ . Therefore, we also assume

$$\nabla \mu(0, \cdot) = O(\lambda^2).$$

Under these conditions we prove that the Cauchy problem (1.1)-(1.2) admits the global smooth solution with uniform bounds in Sobolev spaces with respect to  $\lambda$ , and

$$n_e - n_i = O(\lambda), \quad u_e - u_i = O(\lambda), \quad \nabla \mu = O(\lambda), \quad \forall t > 0.$$

This result is stated in Theorem 2.1. In particular, when  $h_e = h_i$  and the initial data are well-prepared such that  $(n_{e, \lambda}^0, u_{e, \lambda}^0) = (n_{i, \lambda}^0, u_{i, \lambda}^0)$ , then the result in Theorem 2.1 holds, provided that  $(n_{e, \lambda}^0 - 1, u_{e, \lambda}^0)$  is uniformly sufficiently small in the norm of  $H^s$ . The result of the convergence of system (1.1) to (1.5) as  $\lambda \rightarrow 0$  is stated in Theorem 2.2. Furthermore, let  $(n^0, u^0)$  be the initial data of the limit  $(n, u)$  governed by compressible Euler equations (1.5). If

$$n_{\nu, \lambda}^0 - n^0 = O(\lambda), \quad u_{\nu, \lambda}^0 - u^0 = O(\lambda), \quad \nu = e, i,$$

we prove that the solution of (1.1)-(1.2) satisfies global error estimates

$$n_{\nu} - n = O(\lambda), \quad u_{\nu} - u = O(\lambda), \quad \forall t > 0, \quad \nu = e, i.$$

This result is stated in Theorem 2.3. Remark that Theorems 2.1-2.3 are valid for all  $\lambda \in (0, 1]$  without condition  $h_e = h_i$ .

Now let us sketch the proof of these results. First of all, the estimates to establish depend on two quantities : the Debye length  $\lambda$  and the size of the solution in Sobolev norms. The latter is uniformly sufficiently small with respect to  $\lambda$ . From the symmetrizable hyperbolic system (1.1) for  $(n_\nu, u_\nu)$ , we obtain classical uniform energy estimates of solutions and a time dissipation estimate of  $u_\nu$ . From the strict monotonicity of function  $p_\nu$  together with a standard technique, time dissipation estimates of  $\nabla n_\nu$  and  $\lambda^{-1}n_-$  can be further derived. For fixed  $\lambda$ , these estimates are sufficient to prove the global existence of solutions [2]. For the one-fluid Euler-Poisson system, these estimates are also sufficient to prove the uniformly global existence of solutions and pass to the limit in the system [32]. Unfortunately, for the two-fluid Euler-Poisson system, the energy is bounded by a quantity which also depends on  $\lambda^{-1}u_-$ .

Based on the analysis of this problem, the control of the dissipation of  $\lambda^{-1}u_-$  is a key step in the proof. For this purpose, we introduce a linear operator  $\mathcal{P}$  from  $H^s$  to  $H^s$  for all fixed integer  $s \geq 0$ , defined by  $\mathcal{P} = \nabla \Delta^{-1} \text{div}$ . We will see that both  $\mathcal{P}$  and  $I - \mathcal{P}$  are orthogonal projection operators and the latter is the Leray projection, where  $I$  stands for the identity operator. Applying  $\partial_t$  to the Poisson equation and using the density equations in (1.1), we have

$$\lambda^2 \Delta (\partial_t \phi) = \text{div}(n_i u_i - n_e u_e).$$

Therefore, the Poisson equation is equivalent to

$$(1.9) \quad \lambda^2 \partial_t \nabla \phi = \mathcal{P}(n_i u_i - n_e u_e).$$

Notice that

$$n_e u_e - n_i u_i = n_- u_e + n_i u_- = u_- + (n_- u_e + \tilde{n}_i u_-),$$

where  $\tilde{n}_i = n_i - 1$ . Then

$$(1.10) \quad \lambda^2 \partial_t \nabla \phi = -\mathcal{P}u_- - \mathcal{P}(n_- u_e + \tilde{n}_i u_-).$$

Applying  $\partial_t$  to (1.8), we have

$$(1.11) \quad \partial_t^2 u_- + \partial_t u_- + 2\lambda^{-2} \mathcal{P}u_- = F,$$

where

$$F = \partial_t \nabla (h_i(n_i) - h_e(n_e)) - \partial_t ((u_- \cdot \nabla) u_e + (u_i \cdot \nabla) u_-) - 2\lambda^{-2} \mathcal{P}(n_- u_e + \tilde{n}_i u_-).$$

By the density equations in (1.1),  $F$  can be rewritten as

$$F = \nabla (h'_e(n_e) \text{div}(n_e u_e) - h'_i(n_i) \text{div}(n_i u_i)) - \partial_t ((u_- \cdot \nabla) u_e + (u_i \cdot \nabla) u_-) - 2\lambda^{-2} \mathcal{P}(n_- u_e + \tilde{n}_i u_-).$$

In this expression, the first term can be controlled by the dissipation of  $u_\nu$  and the other terms are quadratic and can be treated in a usual way. In particular, the second term depends on the time derivative of solutions. In order to control such kind of terms, we establish estimates in a stronger norm (than that of  $H^s$ ) whose definition contains the mixed derivatives of solutions with respect to  $x$  and  $t$ . Moreover, since  $\mathcal{P}\nabla = \nabla$ , applying  $I - \mathcal{P}$  to (1.8) yields

$$(1.12) \quad \partial_t (I - \mathcal{P})u_- + (I - \mathcal{P})u_- = -(I - \mathcal{P})((u_- \cdot \nabla) u_e + (u_i \cdot \nabla) u_-),$$

where the right-hand side is a quadratic term too. Equations (1.11) and (1.12) provide dissipation estimates of  $\lambda^{-1} \mathcal{P}u_-$  and  $\lambda^{-1} (I - \mathcal{P})u_-$ , which imply dissipation estimates of  $\lambda^{-1} u_-$  by the orthogonal decomposition

$$u_- = \mathcal{P}u_- + (I - \mathcal{P})u_-.$$

The classical energy estimates and the dissipation estimates above imply the result of Theorem 2.1. The proof of Theorem 2.2 follows from the uniform estimates of solutions with respect to  $\lambda$ . These estimates imply strong compactness of the solution sequences and allow to pass to

the limit in (1.1) as  $\lambda \rightarrow 0$ . Finally, a key ingredient in the proof of Theorem 2.3 is to avoid the quadratic term  $(n_e - n)^2$  in the energy estimate. For this purpose, we choose a specific diagonal matrix as a symmetrizer.

In the case of one space dimension, it is less difficult to obtain the dissipation estimate of  $\lambda^{-1}u_-$ . Indeed, the Poisson equation in (1.1) is equivalent to an evolution equation of the form

$$\lambda^2 \partial_t (\partial_x \phi) = n_i u_i - n_e u_e, \quad t > 0, x \in \mathbb{R}.$$

Then equation (1.11) is reduced to

$$\partial_t^2 u_- + \partial_t u_- + 2\lambda^{-2} u_- = F,$$

where

$$F = \partial_x (h'_e(n_e) \partial_x (n_e u_e) - h'_i(n_i) \partial_x (n_i u_i)) - \partial_t (u_- \partial_x u_e + u_i \partial_x u_-) - 2\lambda^{-2} (n_- u_e + \tilde{n}_i u_-).$$

Thus, the orthogonal decomposition above for  $u_-$  is not necessary. See [33] for more detailed analysis on this issue.

This paper is organized as follows. Theorems 2.1-2.3 are stated in the next section. Sections 3 and 4 are devoted to detailed energy estimates. The proof of Theorems 2.1-2.2 are given in section 5 and that of Theorem 2.3 is given in the last section.

## 2. STATEMENT OF MAIN RESULTS

For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ , we denote

$$\partial_x^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}, \quad \text{with } |\alpha| = \alpha_1 + \dots + \alpha_d.$$

Let  $\|\cdot\|$  and  $\|\cdot\|_\infty$  be the norms of  $L^2(\mathbb{R}^d)$  and  $L^\infty(\mathbb{R}^d)$ , respectively. For an integer  $s \in \mathbb{N}$ , let  $\|\cdot\|_s$  be the usual norm of Sobolev space  $H^s$ . For  $T > 0$ , the mixed space-time space  $B_{s,T}$  and the norm  $\|\!\| \cdot \|\!\|_s$  are defined by

$$B_{s,T} = \bigcap_{k=0}^s C^k([0, T]; H^{s-k}),$$

$$\|\!\|v(t)\|\!\|_s = \left( \sum_{k+|\alpha| \leq s} \|\partial_t^k \partial_x^\alpha v(t, \cdot)\|^2 \right)^{\frac{1}{2}}, \quad v \in B_{s,T}, \quad t \in [0, T].$$

The main results of the paper consist of the following three theorems in  $H^s$  for integer  $s > \frac{d}{2} + 1$ . The first theorem shows the uniformly (with respect to  $\lambda$ ) global existence of smooth solutions to the Cauchy problem (1.1)-(1.2). The second one concerns the global-in-time convergence of the solution of (1.1) to that of the compressible Euler equations (1.5) as  $\lambda \rightarrow 0$ . In the last theorem, we give global convergence rates of the solutions in  $H^{s-2}$ .

**Theorem 2.1.** *Let  $d \geq 2$  and  $s > \frac{d}{2} + 1$  be an integer. Assume that  $(n_{\nu,\lambda}^0 - 1, u_{\nu,\lambda}^0, \nabla \phi_\lambda^0) \in H^s$  for  $\nu = e, i$ . Define*

$$(2.1) \quad \mathcal{N} = \mathcal{N}_0 + \mathcal{N}_1,$$

with

$$(2.2) \quad \begin{cases} \mathcal{N}_0 = \sum_{\nu=e,i} (\|n_{\nu,\lambda}^0 - 1\|_s + \|u_{\nu,\lambda}^0\|_s) + \|\nabla \phi_\lambda^0\|_s, \\ \mathcal{N}_1 = \lambda^{-2} (\|n_{e,\lambda}^0 - n_{i,\lambda}^0\|_{s-1} + \|u_{e,\lambda}^0 - u_{i,\lambda}^0\|_{s-1} + \|\!\|\nabla \mu(0)\|\!\|_{s-2}), \end{cases}$$

where  $\mu$  is defined in (1.6)-(1.7), and  $\|\!\|\nabla \mu(0)\|\!\|_{s-2}$  is the value of  $\|\!\|\nabla \mu(t)\|\!\|_{s-2}$  at  $t = 0$ .

There exist constants  $\varepsilon_0 > 0$  and  $C > 0$  independent of  $\lambda$  such that if  $\mathcal{N} \leq \varepsilon_0$ , then for all  $\lambda \in (0, 1]$ , problem (1.1)-(1.2) admits a unique global smooth solution  $(n_e, u_e, n_i, u_i, \phi)$  satisfying

$$(2.3) \quad \sup_{t \in \mathbb{R}^+} \sum_{\nu=e,i} (\|n_\nu(t) - 1\|_s + \|u_\nu(t)\|_s + \lambda \|\nabla \phi(t)\|_s) \leq C\mathcal{N},$$

$$(2.4) \quad \sum_{\nu=e,i} \int_0^{+\infty} (\|u_\nu(\tau)\|_s^2 + \|\nabla n_\nu(\tau)\|_{s-1}^2) d\tau \leq C\mathcal{N}^2,$$

and

$$(2.5) \quad \sup_{t \in \mathbb{R}^+} \|(I - \mathcal{P})u_-(t)\|_{s-1} \leq C\lambda,$$

$$(2.6) \quad \int_0^{+\infty} (\|n_e(\tau) - n_i(\tau)\|_{s-1}^2 + \|u_e(\tau) - u_i(\tau)\|_{s-1}^2 + \|\nabla \mu(\tau)\|_{s-2}^2) d\tau \leq C\lambda^2.$$

**Theorem 2.2.** Let  $(n_{e,\lambda}, u_{e,\lambda}, n_{i,\lambda}, u_{i,\lambda}, \phi_\lambda)_{\lambda>0}$  be the sequence of corresponding solutions given by Theorem 2.1. For  $\nu = e, i$ , as  $\lambda \rightarrow 0$ , if

$$n_{\nu,\lambda}^0 \rightharpoonup n^0, \quad u_{\nu,\lambda}^0 \rightharpoonup u^0 \quad \text{weakly in } H^s,$$

then there exist functions  $(n, u)$  with

$$(2.7) \quad n - 1 \in L^\infty(\mathbb{R}^+; H^s), \quad u \in L^\infty(\mathbb{R}^+; H^s) \cap L^2(\mathbb{R}^+; H^s),$$

such that

$$(2.8) \quad (n_{\nu,\lambda}, u_{\nu,\lambda}) \rightharpoonup (n, u) \quad \text{weakly-} * \text{ in } L^\infty(\mathbb{R}^+; H^s),$$

and for all  $T > 0$ ,

$$(2.9) \quad (n_{\nu,\lambda}, u_{\nu,\lambda}) \rightarrow (n, u) \quad \text{strongly in } C([0, T]; H_{loc}^{s-1}).$$

Moreover, as  $\lambda \rightarrow 0$ ,

$$(2.10) \quad \nabla \phi_\lambda \rightarrow \nabla h_-(n) \quad \text{strongly in } L_{loc}^2(\mathbb{R}^+; H_{loc}^{s-2}),$$

where  $(n, u)$  is the global solution to compressible Euler equations

$$(2.11) \quad \begin{cases} \partial_t n + \operatorname{div}(nu) = 0, \\ \partial_t u + (u \cdot \nabla)u + \nabla h_+(n) = -u, \end{cases}$$

subject to the initial condition

$$(2.12) \quad t = 0: \quad (n, u) = (n^0, u^0).$$

**Theorem 2.3.** Let  $(n_e, u_e, n_i, u_i, \phi)$  and  $(n, u)$  be the unique solutions to problem (1.1)-(1.2) and problem (2.11)-(2.12), respectively. Assume the conditions in Theorem 2.1 with  $\lambda \in (0, 1]$ . There are constants  $C_2 \geq C_1 > 0$  independent of  $\lambda$  such that if

$$(2.13) \quad \|n_{\nu,\lambda}^0 - n^0\|_{s-2} + \|u_{\nu,\lambda}^0 - u^0\|_{s-2} \leq C_1\lambda, \quad \nu = e, i,$$

we have the following error estimates:

$$(2.14) \quad \sup_{t \in \mathbb{R}^+} (\|n_\nu(t) - n(t)\|_{s-2}^2 + \|u_\nu(t) - u(t)\|_{s-2}^2) \leq C_2\lambda^2, \quad \nu = e, i,$$

and

$$(2.15) \quad \int_0^{+\infty} (\|u_\nu(\tau) - u(\tau)\|_{s-2}^2 + \|\nabla(n_\nu(\tau) - n(\tau))\|_{s-3}^2) d\tau \leq C_2\lambda^2, \quad \nu = e, i.$$

**Remark 2.1.**

(1) Condition  $\mathcal{N} \leq \varepsilon_0$  in Theorem 2.1 implies that

$$\|n_{e,\lambda}^0 - n_{i,\lambda}^0\|_{s-1} + \|u_{e,\lambda}^0 - u_{i,\lambda}^0\|_{s-1} + \|\|\nabla\mu(0)\|\|_{s-2} \leq \varepsilon_0\lambda^2,$$

which is an initial error condition in  $\lambda$ . The last term on the left-hand side can be expressed as the  $H^{s-1}$  norm of the initial data  $(n_{e,\lambda}^0, u_{e,\lambda}^0)$ . For example, when  $d = 2, 3$ , Theorems 2.1-2.3 can be stated with  $s = 3$ . In this case,  $s - 2 = 1$  and

$$\|\|\nabla\mu(0)\|\|_1^2 = \|\nabla\mu(0, \cdot)\|_1^2 + \|\partial_t\nabla\mu(0, \cdot)\|_1^2.$$

From (1.7), (1.9) and (1.1), we have

$$\partial_t\nabla\mu = \nabla(h'_i(n_i)\operatorname{div}(n_i u_i) - h'_e(n_e)\operatorname{div}(n_e u_e)) + 2\lambda^{-2}\mathcal{P}(n_e u_e - n_i u_i).$$

Therefore,

$$\nabla\mu(0, \cdot) = \nabla(h_e(n_{e,\lambda}^0) - h_i(n_{i,\lambda}^0) - 2\phi_\lambda^0), \quad -\lambda^2\Delta\phi_\lambda^0 = n_{i,\lambda}^0 - n_{e,\lambda}^0,$$

and

$$\partial_t\nabla\mu(0, \cdot) = \nabla(h'_i(n_{i,\lambda}^0)\operatorname{div}(n_{i,\lambda}^0 u_{i,\lambda}^0) - h'_e(n_{e,\lambda}^0)\operatorname{div}(n_{e,\lambda}^0 u_{e,\lambda}^0)) + 2\lambda^{-2}\mathcal{P}(n_{e,\lambda}^0 u_{e,\lambda}^0 - n_{i,\lambda}^0 u_{i,\lambda}^0).$$

(2) If  $h_e = h_i$  and the initial data are well prepared, namely,  $(n_{e,\lambda}^0, u_{e,\lambda}^0) = (n_{i,\lambda}^0, u_{i,\lambda}^0)$ , then

$$(2.16) \quad \mathcal{N}_1 = 0, \quad \mathcal{N}_0 = 2(\|n_{e,\lambda}^0 - 1\|_s + \|u_{e,\lambda}^0\|_s), \quad \mathcal{N} = \mathcal{N}_0.$$

Thus, the results in Theorems 2.1-2.3 hold provided that  $(n_{e,\lambda}^0, u_{e,\lambda}^0)$  is uniformly sufficiently close to  $(1, 0)$  in the  $H^s$  norm.

In order to see this result, we let  $U_- = (n_-, u_-)$ . The Poisson equation shows that  $\nabla\phi_\lambda^0 = 0$ , hence  $\nabla\mu(0, \cdot) = 0$ . From (1.1), we have

$$\partial_t U_- = O(U_-, \nabla U_-, \nabla\phi), \quad \partial_t\nabla\phi = -\lambda^{-2}\mathcal{P}(n_- u_- + n_i u_-).$$

Therefore,

$$\partial_t U_-(0, \cdot) = 0, \quad \partial_t\nabla\phi(0, \cdot) = 0, \quad \partial_t\nabla\mu(0, \cdot) = 0.$$

By induction, we obtain

$$\partial_t^k U_-(0, \cdot) = 0, \quad \partial_t^k \nabla\phi(0, \cdot) = 0, \quad \partial_t^k \nabla\mu(0, \cdot) = 0, \quad 0 \leq k \leq s - 2.$$

This shows (2.16).

(3) If the initial data are periodic with domain  $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ , we replace in (1.1) condition  $\lim_{|x| \rightarrow +\infty} \phi(t, x) = 0$  by  $\int_{\mathbb{T}^d} \phi(t, x) dx = 0$  and  $H^s(\mathbb{R}^d)$  by  $H^s(\mathbb{T}^d)$ . Thus, the results in Theorems 2.1-2.3 still hold.

### 3. PRELIMINARIES

#### 3.1. A projection operator.

Let  $s \in \mathbb{N}$  and  $f$  be a function from  $\mathbb{R}^d$  to  $\mathbb{R}^d$ . We consider the following Poisson equation in the whole space  $\mathbb{R}^d$ :

$$(3.1) \quad \Delta v = \operatorname{div} f \text{ in } \mathbb{R}^d.$$

We denote by  $\mathcal{D}(\mathbb{R}^d)$  the space of functions in  $C^\infty(\mathbb{R}^d)$  with compact support.

**Lemma 3.1.** *Let  $d \geq 2$  and  $f \in H^s$ . The Poisson equation (3.1) admits a solution  $v_f$  (weak solution if  $f \in L^2(\mathbb{R}^d)$ ) satisfying  $v_f \in L^2_{loc}(\mathbb{R}^d)$ ,  $\nabla v_f \in H^s$  and*

$$(3.2) \quad \|\nabla v_f\|_s \leq \|f\|_s.$$

The uniqueness of solutions holds in the class of functions  $v$  satisfying  $\lim_{|x| \rightarrow +\infty} v(x) = 0$ .

**Proof.** Since  $\mathcal{D}(\mathbb{R}^d)$  is dense in  $H^s$  and  $f \in H^s$ , there is a sequence of functions  $(f_n)_{n \in \mathbb{N}}$  with  $f_n \in \mathcal{D}(\mathbb{R}^d)$  and  $f_n \rightarrow f$  strongly in  $H^s$ . Define the Newtonian potential  $v_n$  by the following convolution :

$$v_n = \Gamma * \operatorname{div} f_n = \nabla \Gamma * f_n,$$

where  $\Gamma$  is the fundamental solution of the Laplace equation :

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \ln |x|, & \text{if } d = 2, \\ -\frac{C_d}{|x|^{d-2}}, & \text{if } d \geq 3, \end{cases} \quad \nabla \Gamma(x) = \frac{C'_d x}{|x|^d},$$

with  $C_d$  and  $C'_d$  being positive constants. Obviously,  $\nabla \Gamma \in L^1_{loc}(\mathbb{R}^d)$ . A classical property of the convolution gives  $v_n \in C^\infty(\mathbb{R}^d)$  and  $v_n$  is a solution of (3.1) (see [13]).

Since  $f_n$  is compactly supported, as  $|x|$  is sufficiently large, we have

$$\frac{|x|}{|x-y|} \leq 2, \quad \forall y \in \operatorname{Supp} f_n$$

which implies that

$$|\nabla \Gamma(x-y)| \leq \frac{C'_d 2^{d-1}}{|x|^{d-1}}, \quad \forall y \in \operatorname{Supp} f_n.$$

Therefore, as  $|x|$  is sufficiently large,

$$|v_n(x)| \leq \int_{\mathbb{R}^d} |\nabla \Gamma(x-y) f_n(y)| dy \leq \frac{C'_d 2^{d-1}}{|x|^{d-1}} \|f_n\|_{L^1(\mathbb{R}^d)}.$$

Let  $\alpha \in \mathbb{N}^d$ . Similarly as above, we have

$$\partial_x^\alpha v_n = \nabla \Gamma * \partial_x^\alpha f_n, \quad \Delta(\partial_x^\alpha v_n) = \operatorname{div}(\partial_x^\alpha f_n),$$

and

$$|\partial_x^\alpha v_n(x)| \leq \frac{C'_d 2^{d-1}}{|x|^{d-1}} \|\partial_x^\alpha f_n\|_{L^1(\mathbb{R}^d)}.$$

Let  $B_R \subset \mathbb{R}^d$  be the ball centered at zero of radius  $R > 0$  and  $S_R$  be its sphere. The Green formula yields

$$\int_{B_R} |\partial_x^\alpha \nabla v_n|^2 dx = \int_{S_R} \partial_x^\alpha v_n (\nabla \partial_x^\alpha v_n \cdot \nu) ds - \int_{B_R} \partial_x^\alpha v_n \operatorname{div} \partial_x^\alpha f_n dx.$$

Since the surface area of  $S_R$  is of order  $O(R^{d-1})$ , by the estimates above, there is a constant  $C''' > 0$  such that, as  $R$  is sufficiently large,

$$\left| \int_{S_R} \partial_x^\alpha v_n (\nabla \partial_x^\alpha v_n \cdot \nu) ds \right| \leq \frac{C'''}{R^{d-1}},$$

which tends to zero as  $R \rightarrow +\infty$ . Passing to the limit in the Green formula above as  $R \rightarrow +\infty$ , we obtain

$$\|\partial_x^\alpha \nabla v_n\|^2 = - \int_{\mathbb{R}^d} \partial_x^\alpha v_n \operatorname{div} \partial_x^\alpha f_n dx = \int_{\mathbb{R}^d} \partial_x^\alpha \nabla v_n \cdot \partial_x^\alpha f_n dx.$$

Therefore,

$$\|\partial_x^\alpha \nabla v_n\| \leq \|\partial_x^\alpha f_n\|,$$

which implies that

$$(3.3) \quad \|\nabla v_n\|_s \leq \|f_n\|_s.$$

By the linearity of the Poisson equation together with (3.3), we have

$$\|\nabla(v_n - v_{n'})\|_s \leq \|f_n - f_{n'}\|_s, \quad \forall n, n' \in \mathbb{N}.$$

This shows that  $(\nabla v_n)_{n \in \mathbb{N}}$  is a Cauchy sequence of  $H^s$ . Consequently, there exists  $V \in H^s$  such that

$$\nabla v_n \longrightarrow V \text{ strongly in } H^s, \text{ as } n \rightarrow +\infty.$$

Let  $\phi \in \mathcal{D}(\mathbb{R}^d)$  satisfying  $\operatorname{div} \phi = 0$ . We have

$$\langle V, \phi \rangle = \lim_{n \rightarrow +\infty} \langle \nabla v_n, \phi \rangle = - \lim_{n \rightarrow +\infty} \langle v_n, \operatorname{div} \phi \rangle = 0.$$

By a theorem of G. de Rham ([38, 14]), there exists  $v_f \in L^2_{loc}(\mathbb{R}^d)$  such that  $\nabla v_f = V \in H^s$ . Finally, passing to the limit in (3.3) and  $\operatorname{div}(\nabla v_n) = \operatorname{div} f_n$  (in the sense of distributions if  $f \in L^2(\mathbb{R}^d)$ ), we see that  $v_f$  satisfies (3.2) and the Poisson equation (3.1).

The uniqueness of solutions to (3.1) is obvious with condition  $\lim_{|x| \rightarrow +\infty} v(x) = 0$ , by the Liouville's theorem for the harmonic function.  $\square$

For all fixed integer  $s \geq 0$ , Lemma 3.1 allows us to define a linear operator  $\mathcal{P}$  from  $H^s$  to  $H^s$  by

$$\mathcal{P}f = \nabla v_f, \quad f \in H^s.$$

For all  $j = 1, \dots, d$ , we denote by  $\mathcal{P}_j$  the  $j$ -th component of  $\mathcal{P}$ , namely,

$$\mathcal{P}_j f = \partial_{x_j} v_f.$$

By the Fourier transform, these operators can be formally expressed as

$$\mathcal{P}f(\xi) = \mathcal{F}^{-1}(|\xi|^{-2} \xi (\xi \cdot \mathcal{F}f(\xi))), \quad \mathcal{P}_j f(\xi) = \mathcal{F}^{-1}(|\xi|^{-2} \xi_j (\xi \cdot \mathcal{F}f(\xi))),$$

for  $\xi = (\xi_1, \dots, \xi_d)$ .

In the next lemma we show that  $\mathcal{P}$  is an orthogonal projection operator in  $H^s$  with useful properties, which will be used to prove the dissipation estimate of  $\lambda^{-1}u_-$  in the next section. For the sake of completeness, we give the proof of the lemma. We mention that the technique of projection operators is frequently used to study the Navier-Stokes equations, see for example [38, 14, 7].

**Lemma 3.2.** *Let  $s \in \mathbb{N}$  and  $f, g : \mathbb{R}^d \rightarrow \mathbb{R}^d$  be functions satisfying  $f, g \in H^s$ . It holds*

- (i)  $\operatorname{div}(\mathbf{I} - \mathcal{P})f = 0$ ,
- (ii)  $\langle (\mathbf{I} - \mathcal{P})f, \mathcal{P}g \rangle_{H^s} = 0$ , where  $\langle \cdot, \cdot \rangle_{H^s}$  denotes the inner product of  $H^s$ ,
- (iii)  $\|f\|_s^2 = \|\mathcal{P}f\|_s^2 + \|(\mathbf{I} - \mathcal{P})f\|_s^2$ ,
- (iv)  $(g \cdot \nabla)\mathcal{P}f = \nabla(g \cdot \mathcal{P}f) - (\nabla g)^\top \mathcal{P}f$  for  $s \geq 1$ , where  $\nabla g = \left( \frac{\partial g_i}{\partial x_j} \right)_{1 \leq i, j \leq d}$ ,
- (v)  $\mathcal{P}(\nabla f) = \nabla f$  and  $(\mathbf{I} - \mathcal{P})\nabla f = 0$  for a real valued function  $f$  and  $s \geq 1$ ,
- (vi)  $\mathcal{P}f \in B_{s,T}$  and  $\partial_t^k \partial_x^\alpha \mathcal{P}f = \mathcal{P} \partial_t^k \partial_x^\alpha f$  for  $f \in B_{s,T}$  with  $\forall k + |\alpha| \leq s$ .
- (vii) The range of  $\mathcal{P}$ , denoted by  $M$ , is a closed subspace of  $H^s$  and  $\mathcal{P}$  is an orthogonal projection operator onto  $M$ .

**Proof.** It is easy to check properties (i), (v) and (vi) from the definition of  $\mathcal{P}$ , (iii) follows from (ii).

It remains to prove (ii), (iv) and (vii). For  $f, g \in H^s$ , there exist two sequences of functions  $(f_n)_n$  and  $(g_n)_n$  with  $f_n, g_n \in \mathcal{D}(\mathbb{R}^d)$  such that, as  $n \rightarrow +\infty$ ,

$$f_n \longrightarrow f, \quad g_n \longrightarrow g \text{ strongly in } H^s.$$

According to the proof of Lemma 3.1 and the definition of  $\mathcal{P}$ , we have

$$\mathcal{P}f_n = \nabla v_n \longrightarrow \nabla v_f = \mathcal{P}f \text{ strongly in } H^s$$

and

$$\mathcal{P}g_n \longrightarrow \mathcal{P}g \text{ strongly in } H^s.$$

Let  $v_{g_n}$  be defined by  $\nabla v_{g_n} = \mathcal{P}g_n$ . Since  $v_{g_n} \in C^\infty(\mathbb{R}^d)$ , by (i) we obtain

$$\langle (\mathbf{I} - \mathcal{P})f_n, \mathcal{P}g_n \rangle_{H^s} = - \langle \operatorname{div}(\mathbf{I} - \mathcal{P})f_n, v_{g_n} \rangle_{\mathcal{D}(\mathbb{R}^d), \mathcal{D}'(\mathbb{R}^d)} = 0.$$

Passing to the limit as  $n \rightarrow +\infty$  in this relation yields (ii).

To prove (iv), we first notice that

$$\partial_{x_j} \mathcal{P}f = \nabla \mathcal{P}_j f.$$

Then, with  $g = (g_1, \dots, g_d)$ ,

$$\begin{aligned} (g \cdot \nabla) \mathcal{P}f &= \sum_{j=1}^d g_j \partial_{x_j} \mathcal{P}f = \sum_{j=1}^d g_j \nabla \mathcal{P}_j f \\ &= \sum_{j=1}^d \left[ \nabla (g_j \mathcal{P}_j f) - \mathcal{P}_j f (\nabla g_j) \right] \\ &= \nabla (g \cdot \mathcal{P}f) - (\nabla g)^\top \mathcal{P}f, \end{aligned}$$

which proves (iv).

Now we prove (vii). By Lemma 3.1,  $M$  is a subspace of  $H^s$ . Let  $(\nabla v_n)_n$  be a sequence of  $M$  with  $\nabla v_n = \mathcal{P}f_n$  and  $f_n \in H^s$ . Assume

$$\nabla v_n \longrightarrow f \text{ strongly in } H^s,$$

with  $f \in H^s$ . Similarly to the proof in Lemma 3.1, there exists  $v \in \mathcal{D}'(\mathbb{R}^d)$  satisfying  $f = \nabla v$ . Therefore,  $f = \mathcal{P}f \in M$ , which implies that  $M$  is closed. Applying the projection theorem to the Hilbert space  $H^s$  (see [4]), condition (ii) implies that  $\mathcal{P}$  is an orthogonal projection operator onto  $M$ .  $\square$

### 3.2. Estimates related to the electric field.

In what follows,  $s$  is an integer satisfying  $s > \frac{d}{2} + 1$ . We first give Moser-type calculus inequalities in  $H^s$ .

**Lemma 3.3.** (Moser-type calculus inequalities in  $H^s$ , see [28, 40])

Let  $\alpha \in \mathbb{N}^d$ . If  $u \in H^{s-1}$  and  $v \in H^{|\alpha|}$  with  $|\alpha| \leq s-1$ , then

$$(3.4) \quad \|\partial_x^\alpha (uv)\| \leq C \|u\|_{s-1} \|v\|_{|\alpha|}.$$

If  $u \in H^s$  and  $v \in H^{|\alpha|-1}$  with  $1 \leq |\alpha| \leq s$ , then

$$(3.5) \quad \|\partial_x^\alpha (uv) - u \partial_x^\alpha v\| \leq C \|\nabla u\|_{s-1} \|v\|_{|\alpha|-1}.$$

If  $f$  is a smooth function and  $u \in H^{|\alpha|}$  with  $1 \leq |\alpha| \leq s-1$ , then

$$(3.6) \quad \|\partial_x^\alpha f(u)\| \leq C_f \|u\|_{|\alpha|},$$

where  $C_f$  is a constant which may depend continuously on  $\|u\|_{s-1}$  and  $f$ .

Inequality (3.6) can be found in [23, 28] where more calculus inequalities of Moser-type are available. Inequality (3.5) was given in [40] in the case of a bounded domain with an explanation of the proof by using a lemma of Kato, and (3.4) is just a variant of (3.5). In a similar way, we get Moser-type calculus inequalities in  $B_{s,T}$  as follows. In case  $d \leq 3$ , similar inequalities in  $B_{s,T}$  were proved in [31].

**Lemma 3.4.** (Moser-type calculus inequalities in  $B_{s,T}$ )

Let  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^d$ . If  $u, v \in B_{s-1,T}$  and  $k + |\alpha| \leq s-1$ , then

$$(3.7) \quad \|\partial_t^k \partial_x^\alpha (uv)\| \leq C \|u\|_{s-1} \|v\|_{k+|\alpha|}.$$

If  $u \in B_{s,T}$ ,  $v \in B_{s-1,T}$  and  $1 \leq k + |\alpha| \leq s$ , then

$$(3.8) \quad \|\partial_t^k \partial_x^\alpha (uv) - u \partial_t^k \partial_x^\alpha v\| \leq C \|\partial_t u\|_{s-1} \|v\|_{k+|\alpha|-1}.$$

If  $f$  is a smooth function and  $u \in B_{s,T}$ ,  $k \geq 1$  and  $k + |\alpha| \leq s$ , then

$$(3.9) \quad \|\partial_t^k \partial_x^\alpha f(u)\| \leq C_f \|\partial_t u\|_{k+|\alpha|-1},$$

where  $\partial u$  stands for the first order partial derivative with respect to  $t$  or  $x$ ,  $C_f$  is a constant which may depend continuously on  $\|u\|_s$  and  $f$ .

**Proof.** We first prove (3.7). By the Leibniz formula, we have

$$\partial_t^k \partial_x^\alpha (uv) = \sum_{l+l'=k, \beta+\beta'=\alpha} c_{kl} c'_{\alpha\beta} \partial_t^l \partial_x^\beta u \partial_t^{l'} \partial_x^{\beta'} v.$$

Recall the lemma of Kato ([22], Lemma 2.1) as follows. Let  $s_0 = [d/2] + 1$ ,  $s_1 \geq 0$  and  $s_2 \geq 0$  be integers such that  $s_1 + s_2 - s_0 \geq 0$ . Define  $s_3 = \min\{s_1, s_2, s_1 + s_2 - s_0\}$ . Then the imbedding from  $H^{s_1} H^{s_2}$  to  $H^{s_3}$  is continuous, where  $H^{s_1} H^{s_2}$  is the set of all functions  $ab$  with  $a \in H^{s_1}$  and  $b \in H^{s_2}$ .

Applying this lemma with

$$s_1 = s - 1 - (l + |\beta|), \quad s_2 = l + |\beta|, \quad s_1 + s_2 - s_0 \geq 0,$$

since  $l + l' = k$  and  $\beta + \beta' = \alpha$ , we have

$$\begin{aligned} \|\partial_t^l \partial_x^\beta u \partial_t^{l'} \partial_x^{\beta'} v\| &\leq \|\partial_t^l \partial_x^\beta u \partial_t^{l'} \partial_x^{\beta'} v\|_{s_3} \\ &\leq C \|\partial_t^l \partial_x^\beta u\|_{s_1} \|\partial_t^{l'} \partial_x^{\beta'} v\|_{s_2} \\ &\leq C \|u\|_{s-1} \|v\|_{k+|\alpha|}, \end{aligned}$$

which implies (3.7).

To prove (3.8), we still use the Leibniz formula :

$$\partial_t^k \partial_x^\alpha (uv) - u \partial_t^k \partial_x^\alpha v = \sum_{\substack{l+l'=k, \beta+\beta'=\alpha \\ l+|\beta| \geq 1}} c_{kl} c'_{\alpha\beta} \partial_t^l \partial_x^\beta u \partial_t^{l'} \partial_x^{\beta'} v.$$

Then

$$\|\partial_t^k \partial_x^\alpha (uv) - u \partial_t^k \partial_x^\alpha v\| \leq \sum_{\substack{l+l'=k, \beta+\beta'=\alpha \\ l+|\beta| \geq 1}} c_{kl} c'_{\alpha\beta} \|\partial_t^l \partial_x^\beta u \partial_t^{l'} \partial_x^{\beta'} v\|.$$

Since  $l + |\beta| \geq 1$ , we may write

$$\partial_t^l \partial_x^\beta u = \partial_t^{l_1} \partial_x^{\beta_1} (\partial u), \quad \text{with } l_1 + |\beta_1| = l + |\beta| - 1.$$

Thus, we can proceed as in the proof of (3.7) to obtain

$$\|\partial_t^l \partial_x^\beta u \partial_t^{l'} \partial_x^{\beta'} v\| \leq \|\partial u\|_{s-1} \|v\|_{k+|\alpha|-1},$$

which implies (3.8).

Finally, (3.9) can be proved in a similar way as in the case  $d \leq 3$ , see [31].  $\square$

Now we introduce perturbed variables

$$\tilde{n}_\nu = n_\nu - 1, \quad U_\nu = \begin{pmatrix} \tilde{n}_\nu \\ u_\nu \end{pmatrix}, \quad \nu = e, i, \quad U = (U_e, U_i, \lambda \nabla \phi).$$

Recall that

$$n_- = n_e - n_i, \quad u_- = u_e - u_i.$$

Let  $T > 0$  and  $(U_e, U_i, \phi)$  be a smooth solution of problem (1.1)-(1.2) defined on time interval  $[0, T]$ . We denote

$$(3.10) \quad \|U(t)\|_s = \sum_{\nu=e,i} \|U_\nu(t)\|_s + \lambda \|\nabla \phi(t)\|_s,$$

and

$$U_T = \sup_{t \in [0, T]} \|U(t)\|_s.$$

When  $U_T$  is sufficiently small, from the continuous embedding  $H^{s-1} \hookrightarrow L^\infty(\mathbb{R}^d)$ , we see that  $(n_\nu, u_\nu)$  is sufficiently close to  $(1, 0)$ . Hence, we may assume

$$(3.11) \quad \frac{1}{2} \leq n_\nu \leq \frac{3}{2}, \quad |u_\nu| \leq \frac{1}{2}.$$

The following Lemma plays an important role in energy estimates in sections 4-5.

**Lemma 3.5.** *Let  $U_T$  be small enough. Let  $k \in \mathbb{N}$  and  $\alpha, \beta \in \mathbb{N}^d$ . If  $1 \leq k + |\alpha| \leq s$ , it holds*

$$(3.12) \quad \lambda^2 \|\partial_t^k \partial_x^\alpha \nabla \phi\| \leq C(\|n_-\|_{s-1} + \|u_-\|_{s-1}),$$

$$(3.13) \quad |\langle n_-\partial_t^k \partial_x^\alpha u_e - \partial_t^k \partial_x^\alpha (n_- u_e), \partial_t^k \partial_x^\alpha (\nabla \phi) \rangle| \leq C\lambda^{-2} \|u_e\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2),$$

and

$$(3.14) \quad |\langle \tilde{n}_i \partial_t^k \partial_x^\alpha u_- - \partial_t^k \partial_x^\alpha (\tilde{n}_i u_-), \partial_t^k \partial_x^\alpha (\nabla \phi) \rangle| \leq C\lambda^{-2} \|\tilde{n}_i\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2),$$

where  $\langle \cdot, \cdot \rangle$  denotes the inner product of  $L^2(\mathbb{R}^d)$ .

**Proof.** We first prove (3.12). When  $k = 0$  and  $1 \leq |\alpha| \leq s$ , applying the Calderon-Zygmund inequality (see [13]) together with the Poisson equation  $\lambda^2 \Delta \phi = n_-$ , we have

$$\lambda^2 \|\partial_x^\alpha \nabla \phi\| \leq C\lambda^2 \|\Delta \phi\|_{|\alpha|-1} = C\|n_-\|_{|\alpha|-1},$$

which implies that

$$\lambda^2 \|\partial_x^\alpha \nabla \phi\| \leq C\|n_-\|_{s-1}, \quad 1 \leq |\alpha| \leq s.$$

When  $k \geq 1$  and  $k + |\alpha| \leq s$ , by (1.10) and Lemma 3.2 (vi) we have

$$\lambda^2 \partial_t^k \partial_x^\alpha (\nabla \phi) = \mathcal{P} f^{k,\alpha},$$

with

$$f^{k,\alpha} = -\partial_t^{k-1} \partial_x^\alpha (n_- u_e + n_i u_-).$$

Since  $\mathcal{P}$  is a projection operator, by the triangle inequality and (3.7) in Lemma 3.4, we get

$$\lambda^2 \|\partial_t^k \partial_x^\alpha \nabla \phi\| \leq \|f^{k,\alpha}\| \leq C(\|u_-\|_{s-1} + \|n_-\|_{s-1}).$$

This prove (3.12).

To prove (3.13), we write

$$n_-\partial_t^k \partial_x^\alpha u_e - \partial_t^k \partial_x^\alpha (n_- u_e) = (u_e \partial_t^k \partial_x^\alpha n_- - \partial_t^k \partial_x^\alpha (n_- u_e) + n_-\partial_t^k \partial_x^\alpha u_e) - u_e \partial_t^k \partial_x^\alpha n_-.$$

By the continuous embedding  $H^{s-1} \hookrightarrow L^\infty(\mathbb{R}^d)$  and (3.8) in Lemma 3.4, we have

$$\begin{aligned} \|u_e \partial_t^k \partial_x^\alpha n_- - \partial_t^k \partial_x^\alpha (n_- u_e) + n_-\partial_t^k \partial_x^\alpha u_e\| &\leq C\|n_-\|_{s-1} \|u_e\|_s + C\|n_-\|_\infty \|u_e\|_s \\ &\leq C\|n_-\|_{s-1} \|u_e\|_s. \end{aligned}$$

It follows from (3.12) that

$$(3.15) \quad \begin{aligned} |\langle n_-\partial_t^k \partial_x^\alpha u_e + u_e \partial_t^k \partial_x^\alpha n_- - \partial_t^k \partial_x^\alpha (n_- u_e), \partial_t^k \partial_x^\alpha (\nabla \phi) \rangle| &\leq C\|n_-\|_{s-1} \|u_e\|_s \|\partial_t^k \partial_x^\alpha (\nabla \phi)\| \\ &\leq C\lambda^{-2} \|u_e\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2). \end{aligned}$$

Next, using the Poisson equation  $\lambda^2 \Delta \phi = n_-$  and a straightforward calculation, we have

$$\begin{aligned} \langle -u_e \partial_t^k \partial_x^\alpha n_-, \partial_t^k \partial_x^\alpha (\nabla \phi) \rangle &= -\lambda^2 \langle u_e \Delta \partial_t^k \partial_x^\alpha \phi, \partial_t^k \partial_x^\alpha (\nabla \phi) \rangle \\ &= -\lambda^2 \langle \operatorname{div} u_e, |(\partial_t^k \partial_x^\alpha \nabla \phi)|^2 \rangle + \lambda^2 \sum_{j=1}^d \left\langle \frac{\partial(\partial_t^k \partial_x^\alpha \phi)}{\partial x_j}, \nabla(\partial_t^k \partial_x^\alpha \phi) \cdot \frac{\partial u}{\partial x_j} \right\rangle. \end{aligned}$$

By (3.12), we get

$$\begin{aligned} |\langle u_e \partial_t^k \partial_x^\alpha n_-, \partial_t^k \partial_x^\alpha (\nabla \phi) \rangle| &\leq C\lambda^2 \|\nabla u_e\|_\infty \|\partial_t^k \partial_x^\alpha (\nabla \phi)\|^2 \\ &\leq C\lambda^{-2} \|u_e\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2). \end{aligned}$$

This together with (3.15) implies (3.13).

Finally, applying (3.8) in Lemma 3.4 yields

$$\begin{aligned} |\langle \tilde{n}_i \partial_t^k \partial_x^\alpha u_- - \partial_t^k \partial_x^\alpha (\tilde{n}_i u_-), \partial_t^k \partial_x^\alpha (\nabla \phi) \rangle| &\leq C \|\tilde{n}_i \partial_t^k \partial_x^\alpha u_- - \partial_t^k \partial_x^\alpha (\tilde{n}_i u_-)\| \|\partial_t^k \partial_x^\alpha (\nabla \phi)\| \\ &\leq C \lambda^{-2} \|\tilde{n}_i\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2), \end{aligned}$$

which proves (3.14).  $\square$

#### 4. ENERGY ESTIMATES

In this section, we assume that  $U_T$  is sufficiently small.

##### 4.1. Classical energy estimates.

The first lemma concerns an  $L^2$  energy equality for system (1.1).

**Lemma 4.1.** *For all  $t \in [0, T]$ , it holds that*

$$(4.1) \quad \frac{d}{dt} \int_{\mathbb{R}^d} \left( \sum_{\nu=e,i} (h'_\nu(n_\nu^*) |\tilde{n}_\nu|^2 + n_\nu |u_\nu|^2) + \lambda^2 |\nabla \phi|^2 \right) dx + 2 \sum_{\nu=e,i} \int_{\mathbb{R}^d} n_\nu |u_\nu|^2 dx = 0,$$

where  $n_\nu^*$  is between 1 and  $n_\nu$ .

**Proof.** The energy conservation of the Euler equations is

$$\partial_t \left( n_\nu |u_\nu|^2 + 2H_\nu(n_\nu) \right) + \operatorname{div} \left( n_\nu |u_\nu|^2 u_\nu + 2n_\nu h_\nu(n_\nu) u_\nu \right) + 2n_\nu |u_\nu|^2 = -2q_\nu n_\nu u_\nu \cdot \nabla \phi,$$

where

$$H'_\nu(n_\nu) = h_\nu(n_\nu), \quad \nu = e, i.$$

By the Taylor formula, we have

$$H_\nu(n_\nu) = H_\nu(1) + h_\nu(1)(n_\nu - 1) + \frac{1}{2} h'_\nu(n_\nu^*) (n_\nu - 1)^2,$$

hence,

$$\begin{aligned} \partial_t (h'_\nu(n_\nu^*) |\tilde{n}_\nu|^2 + n_\nu |u_\nu|^2) + \operatorname{div} (n_\nu |u_\nu|^2 u_\nu + 2n_\nu (h_\nu(n_\nu) - h_\nu(1)) u_\nu) + 2n_\nu |u_\nu|^2 \\ = -2q_\nu n_\nu u_\nu \cdot \nabla \phi. \end{aligned}$$

Recall (1.9), i.e.

$$\lambda^2 \partial_t \nabla \phi = \mathcal{P}(n_i u_i - n_e u_e).$$

Then

$$\lambda^2 \partial_t |\nabla \phi|^2 = 2\mathcal{P}(n_i u_i - n_e u_e) \cdot \nabla \phi.$$

It follows that

$$\begin{aligned} \partial_t \left( \sum_{\nu=e,i} (h'_\nu(n_\nu^*) |\tilde{n}_\nu|^2 + n_\nu |u_\nu|^2) + \lambda^2 |\nabla \phi|^2 \right) \\ + \sum_{\nu=e,i} \operatorname{div} (n_\nu |u_\nu|^2 u_\nu + 2n_\nu (h_\nu(n_\nu) - h_\nu(1)) u_\nu) + 2 \sum_{\nu=e,i} n_\nu |u_\nu|^2 \\ = -2(\mathbf{I} - \mathcal{P})(n_i u_i - n_e u_e) \cdot \nabla \phi. \end{aligned}$$

Integrating this equation on  $\mathbb{R}^d$  and applying Lemma 3.2 (ii) and (v) yield (4.1).  $\square$

Next, we study higher order energy estimates of the solution in space  $B_{s,T}$ . For this purpose, we write the equations for  $(n_\nu, u_\nu)$  in (1.1) as

$$(4.2) \quad \partial_t U_\nu + \sum_{j=1}^d A_j^\nu(U_\nu) \partial_{x_j} U_\nu = K_\nu(u_\nu, \nabla \phi), \quad \nu = e, i,$$

where

$$A_j^\nu(U_\nu) = \begin{pmatrix} u_{\nu j} & n_\nu e_j^\top \\ h'_\nu(n_\nu) e_j & u_{\nu j} \mathbf{I}_d \end{pmatrix}, \quad j = 1, \dots, d,$$

and

$$K_\nu = - \begin{pmatrix} 0 \\ q_\nu \nabla \phi + u_\nu \end{pmatrix}.$$

Here  $\mathbf{I}_d$  stands for the unit matrix of order  $d$ ,  $u_{\nu j}$  is the  $j$ -th component of  $u_\nu$ ,  $(e_1, \dots, e_d)$  is the canonical basis of  $\mathbb{R}^d$  and  $e_j^\top$  is the transpose of  $e_j$ .

Introduce the symmetric and positive definite matrix  $A_0^\nu(n_\nu)$  as follows

$$A_0^\nu(n_\nu) = \begin{pmatrix} h'_\nu(n_\nu) & 0 \\ 0 & n_\nu \mathbf{I}_d \end{pmatrix}, \quad \nu = e, i.$$

Then the matrix  $\tilde{A}_j^\nu(U_\nu)$  defined by

$$\tilde{A}_j^\nu(U_\nu) \stackrel{\text{def}}{=} A_0^\nu(n_\nu) A_j^\nu(U_\nu) = \begin{pmatrix} h'_\nu(n_\nu) u_{\nu j} & p'_\nu(n_\nu) e_j^\top \\ p'_\nu(n_\nu) e_j & n_\nu u_{\nu j} \mathbf{I}_d \end{pmatrix}$$

is symmetric for all  $1 \leq j \leq d$ . Consequently, system (4.2) for  $U_\nu$  is symmetrizable hyperbolic.

**Lemma 4.2.** *For all  $t \in [0, T]$ , it holds that*

$$\begin{aligned} & \frac{d}{dt} \sum_{1 \leq k+|\alpha| \leq s} \left( \sum_{\nu=e,i} \langle A_0^\nu \partial_t^k \partial_x^\alpha U_\nu, \partial_t^k \partial_x^\alpha U_\nu \rangle + \lambda^2 \|\partial_t^k \partial_x^\alpha \nabla \phi\|^2 \right) + \sum_{\substack{\nu=e,i \\ 1 \leq k+|\alpha| \leq s}} \|\partial_t^k \partial_x^\alpha u_\nu\|^2 \\ (4.3) \quad & \leq C \sum_{\nu=e,i} \|U\|_s (\|u_\nu\|_s^2 + \|\nabla n_\nu\|_{s-1}^2) + C \lambda^{-2} \|U\|_s (\|n_\nu\|_{s-1}^2 + \|u_\nu\|_{s-1}^2). \end{aligned}$$

**Proof.** Let  $1 \leq k+|\alpha| \leq s$ . Applying the differential operator  $\partial_t^k \partial_x^\alpha$  to (4.2), we get

$$\partial_t \partial_t^k \partial_x^\alpha U_\nu + \sum_{j=1}^d A_j^\nu(U_\nu) \partial_{x_j} \partial_t^k \partial_x^\alpha U_\nu = \partial_t^k \partial_x^\alpha K_\nu + I_\nu^{k,\alpha},$$

supplemented by the Poisson equation

$$\lambda^2 \Delta \partial_t^k \partial_x^\alpha \phi = \partial_t^k \partial_x^\alpha n_\nu,$$

where

$$I_\nu^{k,\alpha} = \sum_{j=1}^d (A_j^\nu(U_\nu) \partial_t^k \partial_x^\alpha (\partial_{x_j} U_\nu) - \partial_t^k \partial_x^\alpha (A_j^\nu(U_\nu) \partial_{x_j} U_\nu)).$$

Taking the inner product of the above equation for  $\partial_t^k \partial_x^\alpha U_\nu$  with  $A_0^\nu(n_\nu) \partial_t^k \partial_x^\alpha U_\nu$  in  $L^2(\mathbb{R}^d)$  yields a classical energy equality

$$\begin{aligned} & \frac{d}{dt} \langle A_0^\nu(n_\nu) \partial_t^k \partial_x^\alpha U_\nu, \partial_t^k \partial_x^\alpha U_\nu \rangle = \langle \operatorname{div} \vec{A}^\nu(U_\nu) \partial_t^k \partial_x^\alpha U_\nu, \partial_t^k \partial_x^\alpha U_\nu \rangle + 2 \langle A_0^\nu(n_\nu) \partial_t^k \partial_x^\alpha K_\nu, \partial_t^k \partial_x^\alpha U_\nu \rangle \\ (4.4) \quad & \quad \quad \quad + 2 \langle A_0^\nu(n_\nu) I_\nu^{k,\alpha}, \partial_t^k \partial_x^\alpha U_\nu \rangle \\ & = R_{1\nu}^{k,\alpha} + R_{2\nu}^{k,\alpha} + R_{3\nu}^{k,\alpha}, \end{aligned}$$

where

$$\operatorname{div} \vec{A}^\nu(U_\nu) = \partial_t A_0^\nu(n_\nu) + \sum_{j=1}^d \partial_{x_j} \tilde{A}_j^\nu(U_\nu).$$

In what follows, we control  $R_{1\nu}^{k,\alpha}$ ,  $R_{2\nu}^{k,\alpha}$  and  $R_{3\nu}^{k,\alpha}$  in (4.4).

**Estimates of  $R_{1\nu}^{k,\alpha}$  and  $R_{3\nu}^{k,\alpha}$ .** Using classical Sobolev embedding theorem, we have

$$\|\partial_t n_\nu\|_\infty = \|\operatorname{div}(n_\nu u_\nu)\|_\infty \leq C \|u_\nu\|_s.$$

Then

$$\begin{aligned} \left| \langle \operatorname{div} \vec{A}^\nu(U_\nu) \partial_t^k \partial_x^\alpha U_\nu, \partial_t^k \partial_x^\alpha U_\nu \rangle \right| &\leq C (\|\partial_t n_\nu\|_\infty + \|\nabla U_\nu\|_\infty) (\|\partial_t^k \partial_x^\alpha \tilde{n}_\nu\|^2 + \|u_\nu\|_s^2) \\ &\leq C \|U\|_s (\|\partial_t^k \partial_x^\alpha \tilde{n}_\nu\|^2 + \|u_\nu\|_s^2). \end{aligned}$$

When  $\alpha = 0$ , by  $\partial_t \tilde{n}_\nu = -\operatorname{div}(n_\nu u_\nu)$ , we get

$$\|\partial_t^k \tilde{n}_\nu\| = \|\partial_t^{k-1} \operatorname{div}(n_\nu u_\nu)\| \leq C \|u_\nu\|_s.$$

When  $|\alpha| \geq 1$ , we easily see that

$$\|\partial_t^k \partial_x^\alpha \tilde{n}_\nu\| \leq \|\nabla n_\nu\|_{s-1}.$$

It follows that

$$(4.5) \quad |R_{1\nu}^{k,\alpha}| = \left| \langle \operatorname{div} \vec{A}^\nu(U_\nu) \partial_t^k \partial_x^\alpha U_\nu, \partial_t^k \partial_x^\alpha U_\nu \rangle \right| \leq C \|U\|_s (\|u_\nu\|_s^2 + \|\nabla n_\nu\|_{s-1}^2).$$

In view of the expression of  $I_\nu^{k,\alpha}$ , by (3.5)-(3.6) in Lemma 3.3 and (3.8)-(3.9) in Lemma 3.4, we get

$$(4.6) \quad |R_{3\nu}^{k,\alpha}| = 2 \left| \langle A_0^\nu(n_\nu) I_\nu^{k,\alpha}, \partial_t^k \partial_x^\alpha U_\nu \rangle \right| \leq C \|U\|_s (\|u_\nu\|_s^2 + \|\nabla n_\nu\|_{s-1}^2).$$

**Estimate of  $R_{2\nu}^{k,\alpha}$ .** From the explicit expressions of  $A_0^\nu(n_\nu)$ ,  $K_\nu$  and  $U_\nu$ , we have

$$\begin{aligned} R_{2\nu}^{k,\alpha} &= 2 \langle A_0^\nu(n_\nu) \partial_t^k \partial_x^\alpha K_\nu, \partial_t^k \partial_x^\alpha U_\nu \rangle \\ &= -2 \langle n_\nu \partial_t^k \partial_x^\alpha u_\nu, \partial_t^k \partial_x^\alpha u_\nu \rangle - 2 \langle q_\nu n_\nu \partial_t^k \partial_x^\alpha u_\nu, \partial_t^k \partial_x^\alpha \nabla \phi \rangle. \end{aligned}$$

From (3.11), we have

$$-2 \langle n_\nu \partial_t^k \partial_x^\alpha u_\nu, \partial_t^k \partial_x^\alpha u_\nu \rangle \leq -\|\partial_t^k \partial_x^\alpha u_\nu\|^2.$$

Since  $n_- = n_e - n_i$  and  $u_- = u_e - u_i$ , we obtain

$$\begin{aligned} & - \sum_{\nu=e,i} \langle q_\nu n_\nu \partial_t^k \partial_x^\alpha u_\nu, \partial_t^k \partial_x^\alpha \nabla \phi \rangle \\ &= \langle n_e \partial_t^k \partial_x^\alpha u_e - n_i \partial_t^k \partial_x^\alpha u_i, \partial_t^k \partial_x^\alpha \nabla \phi \rangle \\ &= \langle \tilde{n}_e \partial_t^k \partial_x^\alpha u_e - \partial_t^k \partial_x^\alpha (\tilde{n}_e u_e) - \tilde{n}_i \partial_t^k \partial_x^\alpha u_i + \partial_t^k \partial_x^\alpha (\tilde{n}_i u_i), \partial_t^k \partial_x^\alpha \nabla \phi \rangle \\ & \quad + \langle \partial_t^k \partial_x^\alpha (n_e u_e - n_i u_i), \partial_t^k \partial_x^\alpha \nabla \phi \rangle \\ &= \langle n_- \partial_t^k \partial_x^\alpha u_e - \partial_t^k \partial_x^\alpha (n_- u_e), \partial_t^k \partial_x^\alpha \nabla \phi \rangle + \langle \tilde{n}_i \partial_t^k \partial_x^\alpha u_- - \partial_t^k \partial_x^\alpha (\tilde{n}_i u_-), \partial_t^k \partial_x^\alpha \nabla \phi \rangle \\ (4.7) \quad & + \langle \partial_t^k \partial_x^\alpha (n_e u_e - n_i u_i), \partial_t^k \partial_x^\alpha \nabla \phi \rangle. \end{aligned}$$

For the first two terms on the right-hand side of (4.7), by (3.13)-(3.14) in Lemma 3.5, we have

$$\begin{aligned} \left| \langle n_- \partial_t^k \partial_x^\alpha u_e - \partial_t^k \partial_x^\alpha (n_- u_e), \partial_t^k \partial_x^\alpha \nabla \phi \rangle \right| &\leq C \lambda^{-2} \|u_e\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2), \\ \left| \langle \tilde{n}_i \partial_t^k \partial_x^\alpha u_- - \partial_t^k \partial_x^\alpha (\tilde{n}_i u_-), \partial_t^k \partial_x^\alpha \nabla \phi \rangle \right| &\leq C \lambda^{-2} \|\tilde{n}_i\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2). \end{aligned}$$

For the last term in (4.7), we apply  $\partial_t^k \partial_x^\alpha$  to (1.9) and Lemma 3.2 (vi) to get

$$\lambda^2 \partial_t (\partial_t^k \partial_x^\alpha \nabla \phi) = \mathcal{P} \partial_t^k \partial_x^\alpha (n_i u_i - n_e u_e).$$

Therefore,

$$\frac{\lambda^2}{2} \frac{d}{dt} \|\partial_t^k \partial_x^\alpha \nabla \phi\|^2 = \langle \mathcal{P} \partial_t^k \partial_x^\alpha (n_i u_i - n_e u_e), \partial_t^k \partial_x^\alpha \nabla \phi \rangle.$$

Thus, Lemma 3.2 (ii) yields

$$\frac{\lambda^2}{2} \frac{d}{dt} \|\partial_t^k \partial_x^\alpha \nabla \phi\|^2 = \langle \partial_t^k \partial_x^\alpha (n_i u_i - n_e u_e), \partial_t^k \partial_x^\alpha \nabla \phi \rangle.$$

These estimates imply that

$$(4.8) \quad \sum_{\nu=e,i} (R_{2\nu}^{k,\alpha} + \|\partial_t^k \partial_x^\alpha u_\nu\|^2) + \lambda^2 \frac{d}{dt} \|\partial_t^k \partial_x^\alpha \nabla \phi\|^2 \leq C \lambda^{-2} \|U\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2).$$

Summing (4.4) for  $\nu = e, i$  and all indices  $k \in \mathbb{N}$  and  $\alpha \in \mathbb{N}^d$  with  $1 \leq k + |\alpha| \leq s$ , by (4.5), (4.6) and (4.8), we obtain (4.3).  $\square$

#### 4.2. Dissipation estimates of $\nabla n_\nu$ and $\lambda^{-1} n_-$ .

Formulas (4.1) and (4.3) give a time-dissipation estimate of  $u_\nu$ . However, some other terms, such as  $\|\nabla n_\nu\|_{s-1}^2$ ,  $\lambda^{-2} \|n_-\|_{s-1}^2$  and  $\lambda^{-2} \|u_-\|_{s-1}^2$ , still arise on the right-hand side of (4.3). Now we focus on the estimation of these terms. The following Lemma gives a time-dissipation of  $\tilde{n}_\nu$  and  $\lambda^{-1} n_-$ .

**Lemma 4.3.** *There exists a positive constant  $a_0$  independent of  $\lambda$  and  $t$  such that*

$$(4.9) \quad \begin{aligned} & \frac{d}{dt} \sum_{\substack{\nu=e,i \\ k+|\beta| \leq s-1}} \langle \partial_t^k \partial_x^\beta u_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle + a_0 \left( \sum_{\nu=e,i} \|\nabla n_\nu\|_{s-1}^2 + \lambda^{-2} \|n_-\|_{s-1}^2 \right) \\ & \leq C \sum_{\nu=e,i} \|u_\nu\|_s^2 + C \sum_{\nu=e,i} \|U\|_s (\|u_\nu\|_s^2 + \|\nabla n_\nu\|_{s-1}^2). \end{aligned}$$

**Proof.** For  $k \in \mathbb{N}$  and  $\beta \in \mathbb{N}^d$  with  $0 \leq k + |\beta| \leq s - 1$ , applying  $\partial_t^k \partial_x^\beta$  to (1.4) and taking the inner product with  $\partial_t^k \partial_x^\beta \nabla n_\nu$  in  $L^2(\mathbb{R}^d)$ , we obtain

$$(4.10) \quad \begin{aligned} & \langle h'_\nu(n_\nu) \partial_t^k \partial_x^\beta \nabla n_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle + \langle q_\nu \partial_t^k \partial_x^\beta \nabla \phi, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle \\ & = -\langle \partial_t \partial_t^k \partial_x^\beta u_\nu + \partial_t^k \partial_x^\beta u_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle + \langle S^{k,\beta}, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle, \end{aligned}$$

where

$$S^{k,\beta} = -\partial_t^k \partial_x^\beta ((u_\nu \cdot \nabla) u_\nu) + h'_\nu(n_\nu) \partial_t^k \partial_x^\beta \nabla n_\nu - \partial_t^k \partial_x^\beta (h'_\nu(n_\nu) \nabla n_\nu).$$

Since  $h'_\nu(\tau) > 0$  for  $\tau > 0$ , we can choose a positive constant  $a_0 \leq 1$  independent of  $t$  and  $\lambda$  such that

$$h'_\nu(n_\nu) \geq 2a_0, \quad \nu = e, i.$$

It follows that

$$\langle h'_\nu(n_\nu) \partial_t^k \partial_x^\beta \nabla n_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle \geq 2a_0 \|\partial_t^k \partial_x^\beta \nabla n_\nu\|^2.$$

By the triangle inequality and (3.7)-(3.9) in Lemma 3.4, we have

$$|\langle S^{k,\beta}, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle| \leq C \|U\|_s (\|u_\nu\|_s^2 + \|\nabla n_\nu\|_{s-1}^2).$$

For the first term on the right-hand side of (4.10), by an integration by parts, we get

$$\begin{aligned} & -\langle \partial_t \partial_t^k \partial_x^\beta u_\nu + \partial_t^k \partial_x^\beta u_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle \\ & = -\frac{d}{dt} \langle \partial_t^k \partial_x^\beta u_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle + \langle \partial_t^k \partial_x^\beta \operatorname{div} u_\nu, \partial_t^k \partial_x^\beta \operatorname{div}(n_\nu u_\nu) \rangle - \langle \partial_t^k \partial_x^\beta u_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle, \end{aligned}$$

with

$$\begin{aligned} & |\langle \partial_t^k \partial_x^\beta \operatorname{div} u_\nu, \partial_t^k \partial_x^\beta \operatorname{div}(n_\nu u_\nu) \rangle| + |\langle \partial_t^k \partial_x^\beta u_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle| \\ & \leq C \|u_\nu\|_s^2 + C \|u_\nu\|_s \|\partial_t^k \partial_x^\beta \nabla n_\nu\| \\ & \leq C \|u_\nu\|_s^2 + a_0 \|\partial_t^k \partial_x^\beta \nabla n_\nu\|^2. \end{aligned}$$

For the term  $\langle q_\nu \partial_t^k \partial_x^\beta \nabla \phi, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle$ , by the Poisson equation  $\lambda^2 \Delta \phi = n_-$ , we have

$$\begin{aligned} \sum_{\nu=e,i} \langle q_\nu \partial_t^k \partial_x^\beta \nabla n_\nu, \partial_t^k \partial_x^\beta \nabla \phi \rangle & = -\langle \partial_t^k \partial_x^\beta \nabla n_-, \partial_t^k \partial_x^\beta \nabla \phi \rangle \\ & = \langle \partial_t^k \partial_x^\beta n_-, \partial_t^k \partial_x^\beta \Delta \phi \rangle \\ & = \lambda^{-2} \|\partial_t^k \partial_x^\beta n_-\|^2. \end{aligned}$$

Adding (4.10) for  $\nu = e, i$  and all  $k \in \mathbb{N}, \beta \in \mathbb{N}^d$  with  $k + |\beta| \leq s - 1$  together with the estimates above gives (4.9).  $\square$

### 4.3. Dissipation estimates of $\lambda^{-1}u_-$ .

We make the orthogonal decomposition of  $u_-$  as

$$u_- = \mathcal{P}u_- + (\mathbf{I} - \mathcal{P})u_-.$$

The dissipation estimates of  $\lambda^{-1}\mathcal{P}u_-$  and  $\lambda^{-1}(\mathbf{I} - \mathcal{P})u_-$  are studied in Lemma 4.4 and Lemma 4.5 below, respectively.

**Lemma 4.4.** *For any  $t \in [0, T]$ , it holds*

$$(4.11) \quad \begin{aligned} & \frac{d}{dt} \sum_{k+|\beta| \leq s-1} \left( \langle \partial_t^{k+1} \partial_x^\beta u_- + \partial_t^k \partial_x^\beta G + \frac{1}{2} \partial_t^k \partial_x^\beta u_-, \partial_t^k \partial_x^\beta u_- \rangle \right) + 2\lambda^{-2} \|\mathcal{P}u_-\|_{s-1}^2 \\ & \leq C \sum_{\nu=e,i} (\|u_\nu\|_s^2 + \|\nabla n_\nu\|_{s-1}^2) + C\lambda^{-2} \|U\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2), \end{aligned}$$

where

$$(4.12) \quad G = \nabla(h_e(n_e) - h_i(n_i)) + (u_- \cdot \nabla)u_e + (u_i \cdot \nabla)u_-.$$

**Proof.** With the definition of  $G$  above, equation (1.11) can be written as

$$\partial_t^2 u_- + \partial_t(u_- + G) + 2\lambda^{-2}\mathcal{P}u_- = -2\lambda^{-2}\mathcal{P}(n_-u_e + \tilde{n}_i u_-).$$

Let  $k \in \mathbb{N}$  and  $\beta \in \mathbb{N}^d$  with  $k + |\beta| \leq s - 1$ . Applying  $\partial_t^k \partial_x^\beta$  to this equation, by Lemma 3.2 (vi), we have

$$\partial_t(\partial_t^{k+1} \partial_x^\beta u_-) + \partial_t^{k+1} \partial_x^\beta (u_- + G) + 2\lambda^{-2} \partial_t^k \partial_x^\beta \mathcal{P}u_- = q_1^{k,\beta},$$

where

$$q_1^{k,\beta} = -2\lambda^{-2} \mathcal{P} \partial_t^k \partial_x^\beta (n_-u_e + \tilde{n}_i u_-).$$

Taking the inner product of this equation with  $\partial_t^k \partial_x^\beta u_-$  in  $L^2(\mathbb{R}^d)$  yields

$$(4.13) \quad \begin{aligned} & \frac{d}{dt} \left( \langle \partial_t^{k+1} \partial_x^\beta u_- + \partial_t^k \partial_x^\beta G + \frac{1}{2} \partial_t^k \partial_x^\beta u_-, \partial_t^k \partial_x^\beta u_- \rangle \right) + 2\lambda^{-2} \|\partial_t^k \partial_x^\beta \mathcal{P}u_-\|^2 \\ & = \langle \partial_t^{k+1} \partial_x^\beta u_- + \partial_t^k \partial_x^\beta G, \partial_t^{k+1} \partial_x^\beta u_- \rangle + \langle q_1^{k,\beta}, \partial_t^k \partial_x^\beta u_- \rangle. \end{aligned}$$

In view of  $u_- = u_e - u_i$  and  $k + |\beta| \leq s - 1$ , we have

$$|\langle \partial_t^{k+1} \partial_x^\beta u_-, \partial_t^{k+1} \partial_x^\beta u_- \rangle| \leq C \sum_{\nu=e,i} \|u_\nu\|_s^2.$$

By (4.12), the highest derivative order of  $\nabla n_\nu$  and  $u_\nu$  in  $\partial_t^k \partial_x^\beta G$  is  $s - 1$  and  $s$ , respectively. Applying (3.4)-(3.6) in Lemma 3.3 and (3.7)-(3.9) in Lemma 3.4, it yields

$$\begin{aligned} |\langle \partial_t^k \partial_x^\beta G, \partial_t^{k+1} \partial_x^\beta u_- \rangle| & \leq \|\partial_t^k \partial_x^\beta G\| \|\partial_t^{k+1} \partial_x^\beta u_-\| \\ & \leq C \sum_{\nu=e,i} (\|u_\nu\|_s^2 + \|\nabla n_\nu\|_{s-1}^2). \end{aligned}$$

Since  $\mathcal{P}$  is a projection operator in  $L^2(\mathbb{R}^d)$  and  $k + |\beta| \leq s - 1$ , again by (3.7) in Lemma 3.4, we have

$$\begin{aligned} \|q_1^{k,\beta}\| & \leq 2\lambda^{-2} \|\partial_t^k \partial_x^\beta (n_-u_e + \tilde{n}_i u_-)\| \\ & \leq C\lambda^{-2} \|U\|_s (\|n_-\|_{s-1} + \|u_-\|_{s-1}). \end{aligned}$$

Therefore,

$$\begin{aligned} |\langle q_1^{k,\beta}, \partial_t^k \partial_x^\beta u_- \rangle| & \leq \|q_1^{k,\beta}\| \|\partial_t^k \partial_x^\beta u_-\| \\ & \leq C\lambda^{-2} \|U\|_s (\|n_-\|_{s-1}^2 + \|u_-\|_{s-1}^2). \end{aligned}$$

Combining the above inequalities, and summing (4.13) for all indices  $k, \beta$  with  $k + |\beta| \leq s - 1$  yields the estimate (4.11).  $\square$

**Lemma 4.5.** *For all  $t \in [0, T]$ , it holds*

$$(4.14) \quad \lambda^{-2} \frac{d}{dt} \|(I - \mathcal{P})u_-\|_{s-1}^2 + 2\lambda^{-2} \|(I - \mathcal{P})u_-\|_{s-1}^2 \leq C\lambda^{-2} \|U\|_s \|u_-\|_{s-1}^2.$$

**Proof.** For all index  $k \in \mathbb{N}$  and  $\beta \in \mathbb{N}^d$  with  $0 \leq k + |\beta| \leq s - 1$ , applying  $\partial_t^k \partial_x^\beta$  to (1.12), we have

$$\partial_t(\partial_t^k \partial_x^\beta (I - \mathcal{P})u_-) + \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- = (I - \mathcal{P})(q_2^{k,\beta} + q_3^{k,\beta}),$$

where

$$q_2^{k,\beta} = -\partial_t^k \partial_x^\beta ((u_- \cdot \nabla)u_-), \quad q_3^{k,\beta} = -\partial_t^k \partial_x^\beta ((u_i \cdot \nabla)u_-).$$

Taking the inner product of this equation with  $2\partial_t^k \partial_x^\beta (I - \mathcal{P})u_-$  in  $L^2(\mathbb{R}^d)$  yields

$$(4.15) \quad \frac{d}{dt} \|\partial_t^k \partial_x^\beta (I - \mathcal{P})u_-\|^2 + 2\|\partial_t^k \partial_x^\beta (I - \mathcal{P})u_-\|^2 = 2\langle (I - \mathcal{P})(q_2^{k,\beta} + q_3^{k,\beta}), \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle.$$

Noticing that the highest derivative order of  $u_-$  in  $q_2^{k,\beta}$  is  $s - 1$ , by (3.7) in Lemma 3.4, we have

$$(4.16) \quad |\langle (I - \mathcal{P})q_2^{k,\beta}, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle| \leq \|q_2^{k,\beta}\| \|\partial_t^k \partial_x^\beta (I - \mathcal{P})u_-\| \leq C \|U\|_s \|u_-\|_{s-1}^2.$$

By Lemma 3.2 (ii), we also have

$$\langle (I - \mathcal{P})q_3^{k,\beta}, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle = \langle q_3^{k,\beta}, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle.$$

This last term on the right-hand side is estimated as follows.

Using the decomposition

$$u_- = \mathcal{P}u_- + (I - \mathcal{P})u_-,$$

$q_3^{k,\beta}$  can be written as

$$q_3^{k,\beta} = -\partial_t^k \partial_x^\beta ((u_i \cdot \nabla)\mathcal{P}u_-) - \partial_t^k \partial_x^\beta ((u_i \cdot \nabla)(I - \mathcal{P})u_-).$$

Applying Lemma 3.2 (iv), we have

$$q_3^{k,\beta} = -\partial_t^k \partial_x^\beta \nabla(u_i \cdot \mathcal{P}u_-) + \partial_t^k \partial_x^\beta ((\nabla u_i)^\top \mathcal{P}u_-) - \partial_t^k \partial_x^\beta ((u_i \cdot \nabla)(I - \mathcal{P})u_-).$$

By an integration by parts together with Lemma 3.2 (i) yields

$$\begin{aligned} \langle q_3^{k,\beta}, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle &= \langle \partial_t^k \partial_x^\beta ((\nabla u_i)^\top \mathcal{P}u_-), \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle \\ &\quad - \langle \partial_t^k \partial_x^\beta ((u_i \cdot \nabla)(I - \mathcal{P})u_-), \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle \\ &= J_1^{k,\beta} + J_2^{k,\beta}. \end{aligned}$$

Using again the fact that  $\mathcal{P}$  is a projection operator in  $L^2(\mathbb{R}^d)$  together with (3.7) in Lemma 3.4 and  $k + |\beta| \leq s - 1$ , we get

$$\begin{aligned} |J_1^{k,\beta}| &\leq \|\partial_t^k \partial_x^\beta ((\nabla u_i)^\top \mathcal{P}u_-)\| \|\partial_t^k \partial_x^\beta (I - \mathcal{P})u_-\| \\ &\leq C \|U\|_s \|\mathcal{P}u_-\|_{s-1} \|(I - \mathcal{P})u_-\|_{s-1} \\ &\leq C \|U\|_s \|u_-\|_{s-1}^2. \end{aligned}$$

Notice that

$$\langle (u_i \cdot \nabla) \partial_t^k \partial_x^\beta (I - \mathcal{P})u_-, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle = -\frac{1}{2} \langle (\operatorname{div} u_i) \partial_t^k \partial_x^\beta (I - \mathcal{P})u_-, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle.$$

It follows that

$$\begin{aligned} J_2^{k,\beta} &= -\langle \partial_t^k \partial_x^\beta (u_i \cdot \nabla)(I - \mathcal{P})u_- - (u_i \cdot \nabla) \partial_t^k \partial_x^\beta (I - \mathcal{P})u_-, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle \\ &\quad + \frac{1}{2} \langle (\operatorname{div} u_i) \partial_t^k \partial_x^\beta (I - \mathcal{P})u_-, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle. \end{aligned}$$

By (3.8) in Lemma 3.4 and the Cauchy-Schwarz inequality, it gives

$$\begin{aligned} & \left| \langle \partial_t^k \partial_x^\beta ((u_i \cdot \nabla)(I - \mathcal{P})u_-) - (u_i \cdot \nabla) \partial_t^k \partial_x^\beta (I - \mathcal{P})u_-, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle \right| \\ & \leq C \| \|U\| \|_s \| \| (I - \mathcal{P})u_- \| \|_{s-1}^2 \\ & \leq C \| \|U\| \|_s \| \| u_- \| \|_{s-1}^2, \end{aligned}$$

and

$$\begin{aligned} \left| \langle (\operatorname{div} u_i) \partial_t^k \partial_x^\beta (I - \mathcal{P})u_-, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle \right| & \leq \| \operatorname{div}(u_i) \|_\infty \| \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \|^2 \\ & \leq C \| \|U\| \|_s \| \| u_- \| \|_{s-1}^2. \end{aligned}$$

The last two estimates imply

$$|J_2^{k,\beta}| \leq C \| \|U\| \|_s \| \| u_- \| \|_{s-1}^2.$$

Therefore,

$$(4.17) \quad \left| \langle q_3^{k,\beta}, \partial_t^k \partial_x^\beta (I - \mathcal{P})u_- \rangle \right| = |J_1^{k,\beta} + J_2^{k,\beta}| \leq C \| \|U\| \|_s \| \| u_- \| \|_{s-1}^2.$$

Finally, multiplying (4.15) by  $\lambda^{-2}$  and summing it for all indices  $k \in \mathbb{N}, \beta \in \mathbb{N}^d$  with  $k + |\beta| \leq s - 1$ , using (4.16)-(4.17), we obtain (4.14).  $\square$

From Lemmas 4.1-4.5, we obtain the following estimate.

**Lemma 4.6.** *Let  $U_T$  be sufficiently small. For all  $t \in [0, T]$  and  $\lambda \in (0, 1]$ , it holds*

$$(4.18) \quad \| \|U(t)\| \|_s^2 + \lambda^{-2} \| \| (I - \mathcal{P})u_-(t) \| \|_{s-1}^2 + \int_0^t D_\lambda(\tau) d\tau \leq C (\| \|U(0)\| \|_s^2 + \lambda^{-2} \| \| u_-(0) \| \|_{s-1}^2),$$

where

$$(4.19) \quad D_\lambda(t) = \sum_{\nu=e,i} (\| \|u_\nu(t)\| \|_s^2 + \| \nabla n_\nu(t) \| \|_{s-1}^2) + \lambda^{-2} (\| \|n_-(t)\| \|_{s-1}^2 + \| \|u_-(t)\| \|_{s-1}^2).$$

**Proof.** Let  $\pi_1$  and  $\pi_2$  be two sufficiently small positive constants to be chosen. We introduce a Lyapunov function

$$\begin{aligned} L_\lambda(t) & = \int_{\mathbb{R}^d} \left( \sum_{\nu=e,i} (h'_\nu(n_\nu^*) |\tilde{n}_\nu|^2 + n_\nu |u_\nu|^2) + \lambda^2 |\nabla \phi|^2 \right) dx \\ & + \sum_{1 \leq k+|\alpha| \leq s} \left( \sum_{\nu=e,i} \langle A_0^\nu \partial_t^k \partial_x^\alpha U_\nu, \partial_t^k \partial_x^\alpha U_\nu \rangle + \lambda^2 \| \partial_t^k \partial_x^\alpha \nabla \phi \|^2 \right) + \pi_1 \sum_{\substack{\nu=e,i \\ k+|\beta| \leq s-1}} \langle \partial_t^k \partial_x^\beta u_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle \\ & + \pi_2 \sum_{k+|\beta| \leq s-1} \langle \partial_t^{k+1} \partial_x^\beta u_- + \partial_t^k \partial_x^\beta G + \frac{1}{2} \partial_t^k \partial_x^\beta u_-, \partial_t^k \partial_x^\beta u_- \rangle + \lambda^{-2} \| \| (I - \mathcal{P})u_- \| \|_{s-1}^2. \end{aligned}$$

Adding (4.1), (4.3), (4.9) multiplied by  $\pi_1$  and (4.11) multiplied by  $\pi_2$  and (4.14) yields the following energy estimate

$$(4.20) \quad L'_\lambda(t) + \tilde{D}_\lambda(t) \leq R_\lambda(t),$$

where

$$\begin{aligned} \tilde{D}_\lambda(t) & = \sum_{\nu=e,i} \left( 2 \int_{\mathbb{R}^d} n_\nu |u_\nu|^2 dx + \sum_{1 \leq k+|\alpha| \leq s} \| \partial_t^k \partial_x^\alpha u_\nu \|^2 + a_0 \pi_1 \| \nabla n_\nu \| \|_{s-1}^2 \right) \\ & + \lambda^{-2} (a_0 \pi_1 \| \|n_-\| \|_{s-1}^2 + 2\pi_2 \| \| \mathcal{P}u_- \| \|_{s-1}^2 + 2 \| \| (I - \mathcal{P})u_- \| \|_{s-1}^2), \\ R_\lambda(t) & = C \sum_{\nu=e,i} ((\pi_1 + \pi_2) \| \|u_\nu\| \|_s^2 + \pi_2 \| \nabla n_\nu \| \|_{s-1}^2) + C \| \|U\| \|_s \sum_{\nu=e,i} (\| \|u_\nu\| \|_s^2 + \| \nabla n_\nu \| \|_{s-1}^2) \\ & + C \lambda^{-2} \| \|U\| \|_s (\| \|n_-\| \|_{s-1}^2 + \| \|u_-\| \|_{s-1}^2). \end{aligned}$$

Since  $A_0^\nu(n_\nu)$  is positive definite, from (3.10), the first two terms in the expression of  $L_\lambda(t)$  are uniformly equivalent to  $\|U(t)\|_s^2$ . From (4.12) and  $u_- = u_e - u_i$ , we have

$$\sum_{\substack{\nu=e,i \\ k+|\beta|\leq s-1}} |\langle \partial_t^k \partial_x^\beta u_\nu, \partial_t^k \partial_x^\beta \nabla n_\nu \rangle| + \sum_{k+|\beta|\leq s-1} |\langle \partial_t^{k+1} \partial_x^\beta u_- + \partial_t^k \partial_x^\beta G + \frac{1}{2} \partial_t^k \partial_x^\beta u_-, \partial_t^k \partial_x^\beta u_- \rangle| \leq C \|U(t)\|_s^2.$$

Therefore, when  $\pi_1$  and  $\pi_2$  are sufficiently small,  $L_\lambda(t)$  is uniformly equivalent to

$$\|U(t)\|_s^2 + \lambda^{-2} \|(I - \mathcal{P})u_-(t)\|_{s-1}^2.$$

Next, by Lemma 3.2 (iii), we have

$$\|u_-\|_{s-1}^2 = \|\mathcal{P}u_-\|_{s-1}^2 + \|(I - \mathcal{P})u_-\|_{s-1}^2.$$

From the definition of  $\tilde{D}_\lambda(t)$  and  $n_\nu \geq 1/2$ , it is easy to see that

$$\tilde{D}_\lambda(t) \geq \sum_{\nu=e,i} (\|u_\nu(t)\|_s^2 + a_0 \pi_1 \|\nabla n_\nu(t)\|_{s-1}^2) + \lambda^{-2} (a_0 \pi_1 \|n_-(t)\|_{s-1}^2 + 2\pi_2 \|u_-(t)\|_{s-1}^2).$$

This together with (4.19) shows that  $\tilde{D}_\lambda(t)$  is uniformly equivalent to  $D_\lambda(t)$ . Now we choose  $\pi_1$  and  $\pi_2$  such that

$$C(\pi_1 + \pi_2) \leq \frac{1}{4}, \quad C\pi_2 \leq \frac{1}{4} a_0 \pi_1 \leq \frac{1}{4}.$$

If

$$CU_T \leq \frac{1}{4} \min\{a_0 \pi_1, \pi_2\},$$

we also have  $CU_T \leq 1/4$ , hence,

$$R_\lambda(t) \leq \frac{1}{2} \tilde{D}_\lambda(t).$$

It follows from (4.20) that

$$L'_\lambda(t) + \frac{1}{2} \tilde{D}_\lambda(t) \leq 0.$$

Integrating it in time yields (4.18). □

## 5. PROOF OF THEOREMS 2.1-2.2

### 5.1. Estimates of the initial energy.

Recall that, for all integer  $s \geq 1$ ,

$$\|v\|_s^2 = \|v\|_s^2 + \sum_{k=1}^s \|\partial_t^k v\|_{s-k}^2.$$

Then

$$\|v\|_s \leq \|v\|_s + \sum_{k=1}^s \|\partial_t^k v\|_{s-k},$$

which will be used in the proof below.

**Lemma 5.1.** *It holds*

$$(5.1) \quad \|U(0)\|_s \leq C\mathcal{N}_0 + C\lambda^{-2} (\|n_-(0)\|_{s-1} + \|u_-(0)\|_{s-1}),$$

where  $\mathcal{N}_0$  is defined by (2.2).

**Proof.** We first prove

$$(5.2) \quad \|\|\|\nabla\phi(0)\|\|\|_s \leq \mathcal{N}_0 + C\lambda^{-2}(\|\|n_-(0)\|\|_{s-1} + \|\|u_-(0)\|\|_{s-1}).$$

Obviously,

$$\|\|\|\nabla\phi(0, \cdot)\|\|\|_s = \|\|\|\nabla\phi_\lambda^0\|\|\|_s \leq \mathcal{N}_0,$$

and by (3.12) in Lemma 3.5, we have

$$\sum_{k=1}^s \|\|\|\partial_t^k \nabla\phi(0, \cdot)\|\|\|_{s-k} \leq C\lambda^{-2}(\|\|n_-(0)\|\|_{s-1} + \|\|u_-(0)\|\|_{s-1}).$$

This proves (5.2).

Next, we prove

$$(5.3) \quad \|\|U_\nu(0)\|\|_s \leq C\mathcal{N}_0 + C\lambda^{-2}(\|\|n_-(0)\|\|_{s-1} + \|\|u_-(0)\|\|_{s-1}), \quad \nu = e, i.$$

Obviously,

$$\|\|U_\nu(0, \cdot)\|\|_s = \|(n_{\nu,\lambda}^0 - 1, u_{\nu,\lambda}^0)\|_s \leq C\mathcal{N}_0.$$

From (4.2), for  $\nu = e, i$ , we have

$$\partial_t U_\nu = - \sum_{j=1}^d A_j^\nu(U_\nu) \partial_{x_j} U_\nu + \begin{pmatrix} 0 \\ -q_\nu \nabla\phi - u_\nu \end{pmatrix},$$

which can be written in the form

$$\partial_t U_\nu = O(U_\nu) + O(\nabla U_\nu) + O(\nabla\phi).$$

Therefore,

$$\|\|\|\partial_t^k U_\nu\|\|\|_{s-k} \leq C(\|U_\nu\|_s + \|\|\|\nabla\phi\|\|\|_s), \quad \forall 1 \leq k \leq s,$$

which implies that

$$\|\|\|\partial_t^k U_\nu(0, \cdot)\|\|\|_{s-k} \leq C(\|(n_{\nu,\lambda}^0 - 1, u_{\nu,\lambda}^0)\|_s + \|\|\|\nabla\phi(0)\|\|\|_s), \quad \forall 1 \leq k \leq s.$$

This together with (5.2) yields (5.3).

Finally, (5.1) follows from (5.2) and (5.3) together with the definition of  $U$  and  $\lambda \leq 1$ .  $\square$

**Lemma 5.2.** *It holds*

$$(5.4) \quad \lambda^{-2}(\|\|n_-(0)\|\|_{s-1} + \|\|u_-(0)\|\|_{s-1}) \leq C\mathcal{N}_1,$$

where  $\mathcal{N}_1$  is defined by (2.2).

**Proof.** Let

$$U_- = \begin{pmatrix} n_- \\ u_- \end{pmatrix}, \quad B_j^\nu(U_\nu) = \begin{pmatrix} u_{\nu j} & n_\nu e_j^\top \\ 0 & u_{\nu j} \mathbf{I}_d \end{pmatrix}, \quad j = 1, \dots, d.$$

We rewrite system (4.2) as

$$\partial_t U_\nu = - \sum_{j=1}^d B_j^\nu(U_\nu) \partial_{x_j} U_\nu - \begin{pmatrix} 0 \\ \nabla(q_\nu \phi + h_\nu(n_\nu)) + u_\nu \end{pmatrix}, \quad \nu = e, i.$$

Then

$$\partial_t U_- = - \sum_{j=1}^d B_j^e(U_e) \partial_{x_j} U_- - \sum_{j=1}^d (B_j^e(U_e) - B_j^i(U_i)) \partial_{x_j} U_i - \begin{pmatrix} 0 \\ \nabla\mu + u_- \end{pmatrix},$$

where  $\mu$  is defined in (1.6)-(1.7) and recalled here

$$\mu = h_e(n_e) - h_i(n_i) - 2\phi.$$

Let  $u_{-j}$  be the  $j$ -th component of  $u_-$ . Since

$$B_j^e(U_e) - B_j^i(U_i) = \begin{pmatrix} u_{-j} & n_- e_j^\top \\ 0 & u_{-j} I_d \end{pmatrix},$$

the equation for  $U_-$  is of the form

$$\partial_t U_- = O(U_-) + O(\nabla U_-) + O(\nabla \mu).$$

Therefore,

$$\|\partial_t^k U_-\|_{s-1-k} \leq C(\|U_-\|_{s-1} + \|\nabla \mu\|_{s-2}), \quad \forall 1 \leq k \leq s-1,$$

which implies that

$$\|\partial_t^k U_-(0, \cdot)\|_{s-1-k} \leq C(\|n_-(0, \cdot)\|_{s-1} + \|u_-(0, \cdot)\|_{s-1}) + C\|\nabla \mu(0)\|_{s-2}, \quad \forall 1 \leq k \leq s-1.$$

This proves (5.4) by the definition of  $\mathcal{N}_1$ .  $\square$

From Lemmas 5.1-5.2, since  $\lambda \leq 1$ , we easily get the following Proposition.

**Proposition 5.1.** *It holds*

$$(5.5) \quad \|\|U(0)\|\|_s + \lambda^{-1} \|u_-(0)\|_{s-1} \leq C\mathcal{N},$$

where  $\mathcal{N}$  is defined by (2.1).

### 5.2. Proof of Theorem 2.1.

Since  $\|\cdot\|_s \leq \|\|\cdot\|\|_s$  for all integer  $s$ , estimates (4.18) and (5.5) show that the smooth solution  $U$  is uniformly bounded in  $L^\infty([0, T]; H^s)$  with respect to  $\lambda$  and  $T$ . By the bootstrap principle (see [30, 37]), it yields uniformly global solution. In particular, these estimates give (2.3)-(2.5) and (2.6) in which the estimate for  $\nabla \mu$  follows from equation (1.8), i.e.

$$\nabla \mu = -\partial_t u_- - (u_e \cdot \nabla) u_- - (u_- \cdot \nabla) u_i - u_-.$$

This proves Theorem 2.1.  $\square$

### 5.3. Proof of Theorem 2.2.

Let  $\nu = e, i$ . From (2.3)-(2.4) and (2.6) together with the definition of  $\|\|\cdot\|\|_s$ , we have the following bounds and convergences of the solution sequences :

- (i)  $(n_{\nu, \lambda} - 1)_{\lambda > 0}$  and  $(u_{\nu, \lambda})_{\lambda > 0}$  are bounded in  $L^\infty(\mathbb{R}^+; H^s)$  and  $L^\infty(\mathbb{R}^+; H^s) \cap L^2(\mathbb{R}^+; H^s)$ , respectively,
- (ii)  $(\partial_t n_{\nu, \lambda})_{\lambda > 0}$  and  $(\partial_t u_{\nu, \lambda})_{\lambda > 0}$  are bounded in  $L^\infty(\mathbb{R}^+; H^{s-1})$  and  $L^\infty(\mathbb{R}^+; H^{s-1}) \cap L^2(\mathbb{R}^+; H^{s-1})$ , respectively,

$$(iii) \quad n_{e, \lambda} - n_{i, \lambda} \rightarrow 0, \quad u_{e, \lambda} - u_{i, \lambda} \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}^+; H^{s-1}),$$

$$(iv) \quad \partial_t (u_{e, \lambda} - u_{i, \lambda}) \rightarrow 0 \quad \text{strongly in } L^2(\mathbb{R}^+; H^{s-2}).$$

By (i)-(ii) and a classical compactness theorem (see [35]), for all  $T > 0$ ,  $(n_{\nu, \lambda} - 1)_{\lambda > 0}$  and  $(u_{\nu, \lambda})_{\lambda > 0}$  are relatively compact in  $C([0, T]; H_{loc}^{s-1})$ . This together with (iii) implies that there exist functions  $n$  and  $u$  with  $n - 1 \in L^\infty(\mathbb{R}^+; H^s)$  and  $u \in L^\infty(\mathbb{R}^+; H^s) \cap L^2(\mathbb{R}^+; H^s)$  such that (up to subsequences) (2.8) and (2.9) hold.

By the first equation in (1.1) and by adding equations (1.4) for  $\nu = e, i$ , we have

$$\begin{cases} \partial_t n_{e, \lambda} + \operatorname{div}(n_{e, \lambda} u_{e, \lambda}) = 0, \\ \partial_t (u_{e, \lambda} + u_{i, \lambda}) + u_{e, \lambda} \cdot \nabla u_{e, \lambda} + u_{i, \lambda} \cdot \nabla u_{i, \lambda} + \nabla (h_e(n_{e, \lambda}) + h_i(n_{i, \lambda})) = -u_{e, \lambda} - u_{i, \lambda}. \end{cases}$$

The strong convergence of  $(n_{\nu, \lambda})_{\lambda > 0}$  and  $(u_{\nu, \lambda})_{\lambda > 0}$  in (2.9) allows to pass to the limit in this system in the sense of distributions to obtain (2.11). Since this convergence is uniform in time, we also have (2.12). Next, subtracting equations (1.4) for  $\nu = e, i$ , we have

$$\begin{aligned} 2\nabla \phi_\lambda &= \nabla (h_e(n_{e, \lambda}) - h_i(n_{i, \lambda})) + \partial_t (u_{e, \lambda} - u_{i, \lambda}) + (u_{e, \lambda} - u_{i, \lambda}) \cdot \nabla u_{e, \lambda} \\ &\quad + u_{i, \lambda} \cdot \nabla (u_{e, \lambda} - u_{i, \lambda}) + u_{e, \lambda} - u_{i, \lambda}. \end{aligned}$$

Again by (2.9) and the convergences (iii)-(iv) above, we can pass to the limit in this equality and this together with the definition of  $h_-$  yields (2.10).

Finally, the uniqueness of global smooth solutions to the Cauchy problem (2.11)-(2.12) implies the convergence of the whole sequences  $(n_{\nu,\lambda}, u_{\nu,\lambda})_{\lambda>0}$  and  $(\nabla\phi_\lambda)_{\lambda>0}$ .  $\square$

## 6. PROOF OF THEOREM 2.3

Since the proof is the same for  $\nu = e$  and  $\nu = i$ , we only prove (2.14)-(2.15) for  $\nu = e$ . We introduce new variables

$$\mathcal{N}_e = n_e - n, \quad \mathcal{U}_e = u_e - u, \quad \mathcal{W}_e = \begin{pmatrix} \mathcal{N}_e \\ \mathcal{U}_e \end{pmatrix}.$$

Let  $T > 0$  and  $t \in [0, T]$ . We denote

$$\mathcal{E}(t) = (\|\mathcal{N}_e(t)\|_{s-2}^2 + \|\mathcal{U}_e(t)\|_{s-2}^2)^{\frac{1}{2}}, \quad \mathcal{D}(t) = (\|\mathcal{U}_e(t)\|_{s-2}^2 + \|\nabla\mathcal{N}_e(t)\|_{s-3}^2)^{\frac{1}{2}}.$$

It is clear that the following Lemma implies the result of Theorem 2.3.

**Lemma 6.1.** *If  $\varepsilon_0$  introduced in Theorem 2.1 is sufficiently small and (2.13) holds, then*

$$(6.1) \quad \sup_{0 \leq t \leq T} \mathcal{E}^2(t) + \int_0^T \mathcal{D}^2(t) dt \leq C\lambda^2, \quad \forall T > 0.$$

In order to prove Lemma 6.1, we rewrite (1.1) for  $(n_e, u_e)$  as (the subscript  $\lambda$  is dropped)

$$\begin{cases} \partial_t n_e + u_e \cdot \nabla n_e + n_e \operatorname{div} u_e = 0, \\ \partial_t u_e + (u_e \cdot \nabla) u_e + \nabla h_+(n_e) = -u_e + r_e, \end{cases}$$

where  $h_+ = \frac{1}{2}(h_e + h_i)$  and

$$r_e = \frac{1}{2}(\partial_t u_- + (u_e \cdot \nabla) u_- + (u_- \cdot \nabla) u_i + \nabla(h_i(n_e) - h_i(n_i)) + u_-).$$

From (2.11) we also have

$$\begin{cases} \partial_t n + u \cdot \nabla n + n \operatorname{div} u = 0, \\ \partial_t u + u \cdot \nabla u + \nabla h_+(n) = -u. \end{cases}$$

Hence,

$$(6.2) \quad \begin{cases} \partial_t \mathcal{N}_e + u_e \cdot \nabla \mathcal{N}_e + n \operatorname{div} \mathcal{U}_e = -\mathcal{U}_e \cdot \nabla n - \mathcal{N}_e \operatorname{div} u_e, \\ \partial_t \mathcal{U}_e + (u \cdot \nabla) \mathcal{U}_e + \nabla(h_+(n_e) - h_+(n)) = -\mathcal{U}_e - (\mathcal{U}_e \cdot \nabla) u_e + r_e, \end{cases}$$

where

$$\nabla(h_+(n_e) - h_+(n)) = a_e \nabla \mathcal{N}_e + \mathcal{N}_e \nabla a_e,$$

with

$$a_e = \int_0^1 h'_+(sn_e + (1-s)n) ds.$$

We first prove that

$$(6.3) \quad \int_0^{+\infty} \|r_e(t)\|_{s-2}^2 dt \leq C\lambda^2.$$

Indeed, by (2.6) in Theorem 2.1 and the continuous embedding  $H^{s-1} \hookrightarrow L^\infty(\mathbb{R}^d)$ , we have

$$(6.4) \quad \int_0^{+\infty} \|\partial_t u_- + (u_e \cdot \nabla) u_- + (u_- \cdot \nabla) u_i + u_-\|_{s-2}^2 dt \leq C \int_0^{+\infty} \|u_-\|_{s-1}^2 dt \leq C\lambda^2.$$

On the other hand,

$$h_i(n_e) - h_i(n_i) = bn_-, \quad \nabla(h_i(n_e) - h_i(n_i)) = n_- \nabla b + b \nabla n_-,$$

where

$$b = \int_0^1 h'_i(sn_e + (1-s)n_i)ds.$$

Since  $p_i$  is sufficiently smooth,  $b$  is a smooth function of  $(n_e, n_i)$ . Therefore, by (3.6) in Lemma 3.3 and (2.6), we have

$$(6.5) \quad \int_0^{+\infty} \|\nabla(h_i(n_e) - h_i(n_i))\|_{s-2}^2 dt \leq C \int_0^{+\infty} \|n_-\|_{s-1}^2 dt \leq C\lambda^2.$$

Thus, (6.3) follows from (6.4)-(6.5).

System (6.2) can be written under the form

$$(6.6) \quad \partial_t \mathcal{W}_e + \sum_{j=1}^d \mathcal{A}_j \partial_{x_j} \mathcal{W}_e = \mathcal{K},$$

where

$$\mathcal{A}_j = \begin{pmatrix} u_{ej} & ne_j^\top \\ a_e e_j & u_j \mathbf{I}_d \end{pmatrix}, \quad j = 1, \dots, d,$$

and

$$\mathcal{K} = - \begin{pmatrix} \mathcal{U}_e \cdot \nabla n + \mathcal{N}_e \operatorname{div} u_e \\ \mathcal{U}_e + (\mathcal{U}_e \cdot \nabla) u_e + \mathcal{N}_e \nabla a_e - r_e \end{pmatrix}.$$

Recall that  $h_\nu$  ( $\nu = e, i$ ) are strictly increasing functions on  $(0, +\infty)$ , so is  $h_+$ . Since the solutions of (1.1)-(1.2) and of (2.11)-(2.12) are sufficiently close to the equilibrium state, there exists a positive constant  $a_1$  such that

$$(6.7) \quad a_0 \leq a_e(t, x) \leq a_1, \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d,$$

where  $a_0$  is the positive constant satisfying  $h'_\nu(n_\nu) \geq 2a_0$  for  $\nu = e, i$ . By estimates (2.3)-(2.4) in Theorem 2.1 and the convergence results in Theorem 2.2, we have

$$(6.8) \quad \sup_{t \in \mathbb{R}^+} (\|n_e(t) - 1\|_s + \|u_e(t)\|_s + \|n(t) - 1\|_s + \|u(t)\|_s + \|\nabla a_e(t)\|_{s-1}) \leq C\varepsilon_0,$$

and

$$(6.9) \quad \int_0^{+\infty} d_e^2(\tau) d\tau \leq C\varepsilon_0^2,$$

with

$$d_e(t) = \|u_e(t)\|_s + \|\nabla n(t)\|_{s-1} + \|\nabla a_e(t)\|_{s-1}.$$

These estimates will be used in the following discussion.

Introduce matrices

$$\mathcal{A}_0 = \begin{pmatrix} 1 & 0 \\ 0 & na_e^{-1} \mathbf{I}_d \end{pmatrix}, \quad \tilde{\mathcal{A}}_j \stackrel{\text{def}}{=} \mathcal{A}_0 \mathcal{A}_j = \begin{pmatrix} u_{ej} & ne_j^\top \\ ne_j & nu_j a_e^{-1} \mathbf{I}_d \end{pmatrix}.$$

Obviously,  $\mathcal{A}_0$  is symmetric positive definite and  $\tilde{\mathcal{A}}_j$  is symmetric for all  $1 \leq j \leq d$ . Consequently, system (6.6) is symmetrizable hyperbolic. Remark that for (6.2) there is no time-dissipation effect on  $\mathcal{N}_e$  in  $L^2(\mathbb{R}^d)$ , hence the quadratic term  $\mathcal{N}_e^2$  cannot be controlled in energy estimates. The choice of  $\mathcal{A}_0$  above avoids the appearance of such a term.

**Lemma 6.2.** *(Energy estimate of  $\mathcal{W}_e$ ) Under the assumption of Lemma 6.1, there exists a positive constant  $b_0$  independent of  $\lambda$  and  $t$  such that*

$$(6.10) \quad \sum_{|\gamma| \leq s-2} \frac{d}{dt} \langle \mathcal{A}_0 \partial_x^\gamma \mathcal{W}_e, \partial_x^\gamma \mathcal{W}_e \rangle + b_0 \|\mathcal{U}_e\|_{s-2}^2 \leq C\varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t) + C \|r_e\|_{s-2}^2.$$

**Proof.** Let  $\gamma \in \mathbb{N}^d$  with  $|\gamma| \leq s - 2$ . Applying  $\partial_x^\gamma$  to (6.6), we get

$$\partial_t \partial_x^\gamma \mathcal{W}_e + \sum_{j=1}^d \mathcal{A}_j \partial_{x_j} \partial_x^\gamma \mathcal{W}_e = \partial_x^\gamma \mathcal{K} + \mathcal{I}^\gamma,$$

with

$$\mathcal{I}^\gamma = \sum_{j=1}^d (\mathcal{A}_j \partial_x^\gamma (\partial_{x_j} \mathcal{W}_e) - \partial_x^\gamma (\mathcal{A}_j \partial_{x_j} \mathcal{W}_e)).$$

Taking the inner product of this equation with  $\mathcal{A}_0 \partial^\gamma \mathcal{W}_e$  in  $L^2(\mathbb{R}^d)$  gives

$$(6.11) \quad \frac{d}{dt} \langle \mathcal{A}_0 \partial_x^\gamma \mathcal{W}_e, \partial_x^\gamma \mathcal{W}_e \rangle = \langle (\operatorname{div} \vec{\mathcal{A}}) \partial_x^\gamma \mathcal{W}_e, \partial_x^\gamma \mathcal{W}_e \rangle + 2 \langle \mathcal{A}_0 \mathcal{I}^\gamma, \partial_x^\gamma \mathcal{W}_e \rangle + 2 \langle \mathcal{A}_0 \partial_x^\gamma \mathcal{K}, \partial_x^\gamma \mathcal{W}_e \rangle,$$

where

$$\operatorname{div} \vec{\mathcal{A}} = \partial_t \mathcal{A}_0 + \sum_{j=1}^d \partial_{x_j} \tilde{\mathcal{A}}_j.$$

A straightforward calculation gives

$$\langle (\operatorname{div} \vec{\mathcal{A}}) \partial_x^\gamma \mathcal{W}_e, \partial_x^\gamma \mathcal{W}_e \rangle = \langle \operatorname{div} u_e, (\partial_x^\gamma \mathcal{N}_e)^2 \rangle + 2 \langle \partial_x^\gamma \mathcal{U}_e \cdot \nabla n, \partial_x^\gamma \mathcal{N}_e \rangle + \langle s_e \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \mathcal{U}_e \rangle,$$

where

$$s_e = \partial_t (n a_e^{-1}) + \operatorname{div} (n u a_e^{-1}).$$

By an integration by parts, we have

$$\begin{aligned} |\langle \operatorname{div} u_e, (\partial_x^\gamma \mathcal{N}_e)^2 \rangle| &= 2 |\langle u_e \partial_x^\gamma \mathcal{N}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle| \\ &\leq C \|u_e\|_s \mathcal{E}(t) \mathcal{D}(t). \end{aligned}$$

Obviously,

$$|\langle \partial_x^\gamma \mathcal{U}_e \cdot \nabla n, \partial_x^\gamma \mathcal{N}_e \rangle| \leq C \|\nabla n\|_{s-1} \mathcal{E}(t) \mathcal{D}(t).$$

By the density equations

$$\partial_t n + \operatorname{div}(nu) = 0, \quad \partial_t n_e + \operatorname{div}(n_e u_e) = 0,$$

we have

$$s_e = -n a_e^{-2} (\partial_t a_e + u \cdot \nabla a_e).$$

Remark that  $a_e$  is a smooth function of  $(n_e, n)$ . Then  $\partial_t a_e$  can be further expressed as a linear combination of  $\operatorname{div}(nu)$  and  $\operatorname{div}(n_e u_e)$ . Therefore, (6.8) gives

$$\|s_e\|_\infty \leq C \|s_e\|_{s-1} \leq C \varepsilon_0,$$

which implies that

$$|\langle s_e \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \mathcal{U}_e \rangle| \leq C \varepsilon_0 \mathcal{D}^2(t).$$

We conclude from the above estimates that

$$(6.12) \quad |\langle (\operatorname{div} \vec{\mathcal{A}}) \partial_x^\gamma \mathcal{W}_e, \partial_x^\gamma \mathcal{W}_e \rangle| \leq C \varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t).$$

Next, we remark that  $\mathcal{I}^\gamma = 0$  if  $\gamma = 0$ . For  $1 \leq |\gamma| \leq s - 2$ , by (3.5)-(3.6) in Lemma 3.3 and the expression of  $\mathcal{I}^\gamma$ , we have

$$(6.13) \quad |\langle \mathcal{A}_0 \mathcal{I}^\gamma, \partial_x^\gamma \mathcal{W}_e \rangle| \leq C \sum_{j=1}^d \|\nabla_x \mathcal{A}_j\|_{s-1} \|\partial_{x_j} \mathcal{W}_e\|_{s-3} \|\partial_x^\gamma \mathcal{W}_e\| \leq C \varepsilon_0 \mathcal{D}^2(t).$$

For the last term in (6.11), a straightforward calculation shows that

$$\begin{aligned} \langle \mathcal{A}_0 \partial_x^\gamma \mathcal{K}, \partial_x^\gamma \mathcal{W}_e \rangle &= -\langle n a_e^{-1} \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \mathcal{U}_e \rangle - \langle \partial_x^\gamma (\mathcal{N}_e \operatorname{div} u_e), \partial_x^\gamma \mathcal{N}_e \rangle \\ &\quad - \langle \partial_x^\gamma (\mathcal{U}_e \cdot \nabla n), \partial_x^\gamma \mathcal{N}_e \rangle - \langle n a_e^{-1} \partial_x^\gamma ((\mathcal{U}_e \cdot \nabla) u_e), \partial_x^\gamma \mathcal{U}_e \rangle \\ &\quad - \langle n a_e^{-1} \partial_x^\gamma (\mathcal{N}_e \nabla a_e), \partial_x^\gamma \mathcal{U}_e \rangle + \langle n a_e^{-1} \partial_x^\gamma r_e, \partial_x^\gamma \mathcal{U}_e \rangle. \end{aligned}$$

In view of  $n \geq 1/2$  and (6.7), there exists a positive constant  $b_0$  independent of  $\lambda$  such that

$$\langle na_e^{-1} \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \mathcal{U}_e \rangle \geq b_0 \|\partial_x^\gamma \mathcal{U}_e\|^2.$$

By (3.4) in Lemma 3.3, we have

$$\begin{aligned} |\langle \partial_x^\gamma (\mathcal{N}_e \operatorname{div} u_e), \partial_x^\gamma \mathcal{N}_e \rangle| &\leq C \|\operatorname{div} u_e\|_{s-1} \|\mathcal{N}_e\|_{|\gamma|} \|\partial_x^\gamma \mathcal{N}_e\| \\ &\leq C d_e(t) \mathcal{E}(t) \mathcal{D}(t). \end{aligned}$$

Similarly,

$$\begin{aligned} |\langle \partial_x^\gamma (\mathcal{U}_e \cdot \nabla n), \partial_x^\gamma \mathcal{N}_e \rangle| + |\langle na_e^{-1} \partial_x^\gamma (\mathcal{N}_e \nabla a_e), \partial_x^\gamma \mathcal{U}_e \rangle| &\leq C d_e(t) \mathcal{E}(t) \mathcal{D}(t), \\ |\langle na_e^{-1} \partial_x^\gamma ((\mathcal{U}_e \cdot \nabla) u_e), \partial_x^\gamma \mathcal{U}_e \rangle| &\leq C \varepsilon_0 \mathcal{D}^2(t). \end{aligned}$$

Finally, by the Young inequality, we easily get

$$|\langle na_e^{-1} \partial_x^\gamma r_e, \partial_x^\gamma \mathcal{U}_e \rangle| \leq \frac{b_0}{2} \|\partial_x^\gamma \mathcal{U}_e\|^2 + C \|r_e\|_{s-2}^2.$$

These estimates imply

$$(6.14) \quad 2 \langle \mathcal{A}_0 \partial_x^\gamma \mathcal{K}, \partial_x^\gamma \mathcal{W}_e \rangle + b_0 \|\partial_x^\gamma \mathcal{U}_e\|^2 \leq C \varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t) + C \|r_e\|_{s-2}^2.$$

Combining (6.11)-(6.14), we obtain

$$\frac{d}{dt} \langle \mathcal{A}_0 \partial_x^\gamma \mathcal{W}_e, \partial_x^\gamma \mathcal{W}_e \rangle + b_0 \|\partial_x^\gamma \mathcal{U}_e\|^2 \leq C \varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t) + C \|r_e\|_{s-2}^2.$$

Summing these inequalities for all  $\gamma \in \mathbb{N}^d$  with  $|\gamma| \leq s-2$ , we obtain (6.10). This proves Lemma 6.2.  $\square$

**Lemma 6.3.** (*Dissipation estimate of  $\nabla \mathcal{N}_e$* ) Under the assumption of Lemma 6.1, it holds

$$(6.15) \quad \begin{aligned} &\sum_{|\gamma| \leq s-3} \frac{d}{dt} \langle \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle + \frac{a_0}{2} \|\nabla \mathcal{N}_e\|_{s-3}^2 \\ &\leq C \varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t) + C (\|\mathcal{U}_e\|_{s-2}^2 + \|r_e\|_{s-3}^2). \end{aligned}$$

**Proof.** Let  $\gamma \in \mathbb{N}^d$  with  $|\gamma| \leq s-3$ . Applying  $\partial_x^\gamma$  to the second equation in (6.2), we get

$$\begin{aligned} a_e \partial_x^\gamma \nabla \mathcal{N}_e &= \partial_x^\gamma (r_e - \mathcal{U}_e) - \partial_x^\gamma ((u \cdot \nabla) \mathcal{U}_e + (\mathcal{U}_e \cdot \nabla) u_e + \mathcal{N}_e \nabla a_e) \\ &\quad + (a_e \partial_x^\gamma \nabla \mathcal{N}_e - \partial_x^\gamma (a_e \nabla \mathcal{N}_e)) - \partial_t \partial_x^\gamma \mathcal{U}_e. \end{aligned}$$

Taking the inner product of this equality with  $\partial_x^\gamma \nabla \mathcal{N}_e$  in  $L^2(\mathbb{R}^d)$  leads to

$$(6.16) \quad \begin{aligned} \langle a_e \partial_x^\gamma \nabla \mathcal{N}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle &= \langle \partial_x^\gamma (r_e - \mathcal{U}_e), \partial_x^\gamma \nabla \mathcal{N}_e \rangle \\ &\quad - \langle \partial_x^\gamma ((u \cdot \nabla) \mathcal{U}_e + (\mathcal{U}_e \cdot \nabla) u_e + \mathcal{N}_e \nabla a_e), \partial_x^\gamma \nabla \mathcal{N}_e \rangle \\ &\quad + \langle a_e \partial_x^\gamma \nabla \mathcal{N}_e - \partial_x^\gamma (a_e \nabla \mathcal{N}_e), \partial_x^\gamma \nabla \mathcal{N}_e \rangle - \langle \partial_t \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle. \end{aligned}$$

Obviously,

$$\begin{aligned} \langle a_e \partial_x^\gamma \nabla \mathcal{N}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle &\geq a_0 \|\partial_x^\gamma \nabla \mathcal{N}_e\|^2, \\ |\langle \partial_x^\gamma (r_e - \mathcal{U}_e), \partial_x^\gamma \nabla \mathcal{N}_e \rangle| &\leq \frac{a_0}{2} \|\partial_x^\gamma \nabla \mathcal{N}_e\|^2 + C (\|\mathcal{U}_e\|_{s-3}^2 + \|r_e\|_{s-3}^2). \end{aligned}$$

Similarly to the proof in Lemma 6.2, we also have

$$\begin{aligned} |\langle \partial_x^\gamma ((u \cdot \nabla) \mathcal{U}_e + (\mathcal{U}_e \cdot \nabla) u_e + \mathcal{N}_e \nabla a_e), \partial_x^\gamma \nabla \mathcal{N}_e \rangle| &\leq C \varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t), \\ |\langle a_e \partial_x^\gamma \nabla \mathcal{N}_e - \partial_x^\gamma (a_e \nabla \mathcal{N}_e), \partial_x^\gamma \nabla \mathcal{N}_e \rangle| &\leq C \varepsilon_0 \mathcal{D}^2(t). \end{aligned}$$

Using the equation of  $\mathcal{N}_e$  in (6.2), the last term in (6.16) can be written as

$$\begin{aligned} & -\langle \partial_t \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle \\ = & -\frac{d}{dt} \langle \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle - \langle \operatorname{div} \partial_x^\gamma \mathcal{U}_e, \partial_t \partial_x^\gamma \mathcal{N}_e \rangle \\ = & -\frac{d}{dt} \langle \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle + \langle \operatorname{div} \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma (u_e \cdot \nabla \mathcal{N}_e + \mathcal{N}_e \operatorname{div} u_e + \mathcal{U}_e \cdot \nabla n + n \operatorname{div} \mathcal{U}_e) \rangle. \end{aligned}$$

By (3.4) in Lemma 3.3, we have

$$\begin{aligned} |\langle \operatorname{div} \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma (u_e \cdot \nabla \mathcal{N}_e + \mathcal{N}_e \operatorname{div} u_e + \mathcal{U}_e \cdot \nabla n) \rangle| & \leq C \varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t), \\ |\langle \operatorname{div} \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma (n \operatorname{div} \mathcal{U}_e) \rangle| & \leq C \|\mathcal{U}_e\|_{s-2}^2. \end{aligned}$$

Combining the above estimates together with (6.16) yields

$$\frac{d}{dt} \langle \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle + \frac{a_0}{2} \|\partial_x^\gamma \nabla \mathcal{N}_e\|^2 \leq C \varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t) + C (\|\mathcal{U}_e\|_{s-2}^2 + \|r_e\|_{s-3}^2).$$

Summing these inequalities for all  $\gamma \in \mathbb{N}^d$  with  $|\gamma| \leq s-3$ , we obtain (6.15).  $\square$

**Proof of Lemma 6.1.** Adding (6.10) and (6.15) multiplied by a positive constant  $\kappa_0$ , it gives the following estimate

$$\begin{aligned} (6.17) \quad \mathcal{L}'_e(t) + (b_0 - C\kappa_0) \|\mathcal{U}_e\|_{s-2}^2 + \frac{\kappa_0 a_0}{2} \|\nabla \mathcal{N}_e\|_{s-3}^2 \\ \leq C \varepsilon_0 \mathcal{D}^2(t) + C d_e(t) \mathcal{E}(t) \mathcal{D}(t) + C \|r_e\|_{s-2}^2, \end{aligned}$$

where

$$\mathcal{L}_e(t) = \sum_{|\gamma| \leq s-2} \langle \mathcal{A}_0 \partial_x^\gamma \mathcal{W}_e, \partial_x^\gamma \mathcal{W}_e \rangle + \kappa_0 \sum_{|\gamma| \leq s-3} \langle \partial_x^\gamma \mathcal{U}_e, \partial_x^\gamma \nabla \mathcal{N}_e \rangle.$$

In view of the expression of  $\mathcal{A}_0$  and  $\mathcal{E}(t)$ , we have

$$(2a_2 - \kappa_0) \mathcal{E}^2(t) \leq \mathcal{L}_e(t) \leq C \mathcal{E}^2(t),$$

where

$$a_2 = \min \{1/2, 1/4a_1\}.$$

Now we choose  $\kappa_0 > 0$  such that

$$2C\kappa_0 \leq b_0 \quad \text{and} \quad \kappa_0 \leq a_2.$$

Then

$$a_2 \mathcal{E}^2(t) \leq \mathcal{L}_e(t) \leq C \mathcal{E}^2(t),$$

and by the expression of  $\mathcal{D}(t)$ , we have

$$\frac{b_0}{2} \|\mathcal{U}_e\|_{s-2}^2 + \frac{\kappa_0 a_0}{2} \|\nabla \mathcal{N}_e\|_{s-3}^2 \geq 2c_0 \mathcal{D}^2(t),$$

where

$$c_0 = \frac{1}{4} \min \{b_0, \kappa_0 a_0\}.$$

It follows from (6.17) that

$$\mathcal{L}'_e(t) + (2c_0 - C\varepsilon_0) \mathcal{D}^2(t) \leq C d_e(t) \mathcal{E}(t) \mathcal{D}(t) + C \|r_e\|_{s-2}^2.$$

Integrating this inequality over  $[0, t]$  and using (6.3) yields

$$a_2 \mathcal{E}^2(t) + (2c_0 - C\varepsilon_0) \int_0^t \mathcal{D}^2(\tau) d\tau \leq C \mathcal{E}_T \int_0^t d_e(\tau) \mathcal{D}(\tau) d\tau + C \lambda^2,$$

where

$$\mathcal{E}_T = \sup_{0 \leq t \leq T} \mathcal{E}(t).$$

By the Cauchy-Schwarz inequality and the Young inequality together with (6.9), we have

$$\begin{aligned} \mathcal{E}_T \int_0^t d_e(\tau) \mathcal{D}(\tau) d\tau &\leq \mathcal{E}_T \left( \int_0^t d_e^2(\tau) d\tau \right)^{\frac{1}{2}} \left( \int_0^t \mathcal{D}^2(\tau) d\tau \right)^{\frac{1}{2}} \\ &\leq C\varepsilon_0 \mathcal{E}_T \left( \int_0^t \mathcal{D}^2(\tau) d\tau \right)^{\frac{1}{2}} \\ &\leq C\varepsilon_0 \mathcal{E}_T^2 + C\varepsilon_0 \int_0^t \mathcal{D}^2(\tau) d\tau. \end{aligned}$$

Therefore,

$$a_2 \mathcal{E}^2(t) + 2(c_0 - C\varepsilon_0) \int_0^t \mathcal{D}^2(\tau) d\tau \leq C\varepsilon_0 \mathcal{E}_T^2 + C\lambda^2.$$

Thus, when  $\varepsilon_0$  is sufficiently small such that

$$C\varepsilon_0 \leq \frac{1}{2} \min\{c_0, a_2\},$$

we obtain

$$a_2 \mathcal{E}^2(t) + c_0 \int_0^t \mathcal{D}^2(\tau) d\tau \leq \frac{a_2}{2} \mathcal{E}_T^2 + C\lambda^2, \quad \forall t \in [0, T],$$

which implies (6.1).  $\square$

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