Stability of Lie groupoid C*-algebras [1]

by Claire Debord and Georges Skandalis

Laboratoire de Mathématiques, UMR 6620 - CNRS
Université Blaise Pascal, Campus des Cézeaux, BP **80026**
F-63171 Aubière cedex, France
claire.debord@math.univ-bpclermont.fr

Université Paris Diderot, Sorbonne Paris Cité
Sorbonne Universités, UPMC Paris 06, CNRS, IMJ-PRG
UFR de Mathématiques, CP **7012** - Bâtiment Sophie Germain
5 rue Thomas Mann, 75205 Paris CEDEX 13, France
skandalis@math.univ-paris-diderot.fr

Abstract

In this paper we generalize a theorem of M. Hilsum and G. Skandalis stating that the C^*-algebra of any foliation of non zero dimension is stable. Precisely, we show that the C*-algebra of a Lie groupoid is stable whenever the groupoid has no orbit of dimension zero. We also prove an analogous theorem for singular foliations for which the holonomy groupoid as defined by I. Androulidakis and G. Skandalis is not Lie in general.

1 Introduction: Statement of the theorem and the steps of the proof

The aim of this paper is to generalize Theorem 1 of [4] stating that the C^*-algebra of any foliation (of non zero dimension !) is stable.

Theorem 1. *Let G be a Lie groupoid with σ-compact $G^{(0)}$. Assume that at every $x \in G^{(0)}$ the anchor $\natural_x : \mathfrak{g}_x \to T_xG^{(0)}$ is nonzero. Then $C^*(G)$ is stable.*

In other words, $C^*(G)$ is stable whenever G has no orbit of dimension 0.

The converse is also true if G is s-connected. Indeed, if G is s-connected and the anchor at x is the zero map, then the orbit of x is reduced to x. Therefore $C^*(G)$ has a character: the trivial representation of the group G^x_x.

Since the reduced C^*-algebra $C^*_r(G)$ of G is a quotient of $C^*(G)$, it follows that it is also stable when G has no orbit of dimension 0.

Here however, the converse may fail for the reduced C^*-algebra: the reduced C^*-algebra of the group $PSL_2(\mathbb{R})$ is stable!

Our proof is not very different from the one of [4] and based on Kasparov's stabilisation theorem ([5]). Note that, unlike in [4], we do not assume the space $G^{(0)}$ to be compact - but this is actually a rather minor point.

The proof is as follows.

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1. Let $x \in G^{(0)}$. There is a section Y of the algebroid $\mathfrak{A}(G)$ whose image under the anchor is a vector field X satisfying $X(x) \neq 0$. Taking a local exponentiation of X we obtain a relatively compact open neighborhood W diffeomorphic to $U \times \mathbb{R}$ where X is proportional to the vector field along the $\mathbb{R}$ lines $\{u\} \times \mathbb{R}$.

   This step will be clarified in section 2.1

   We thus choose a locally finite cover (W_n) by relatively compact open subsets and diffeomorphisms $f_n : U_n \times \mathbb{R} \to W_n$ such that $W'_n = f_n(U'_n \times \mathbb{R})$ cover $G^{(0)}$ with U'_n relatively compact in U_n. Let $p_n : W_n \to U_n$ be the composition of f_n^{-1} with the projection $U_n \times \mathbb{R} \to U_n$.

2. One may then construct a locally finite family of open subsets V_j of $G^{(0)}$ such that:

   - Every V_j sits in a $W_{n(j)}$ and its intersection with each line $f_{n(j)}(u \times \mathbb{R})$ is an (open) interval. More precisely, $f_{n(j)}(V_j)$ is of the form $\{(x,t) \in U_{n(j)} \times \mathbb{R};\ \varphi_j^-(x) < t < \varphi_j^+(x)\}$ where $\varphi_j^-, \varphi_j^+ : U_{n(j)} \to \mathbb{R}$ are smooth and $\varphi_j^+ - \varphi_j^-$ is nonnegative with compact support.
   - The $\overline{V_j}$ are pairwise disjoint and locally finite: every compact subset of M meets only finitely many $\overline{V_j}$'s.
   - For every n, the $p_n(V_j \cap W_n)$ cover U'_n: we have $U'_n \subset \bigcup_{j;\ n(j)=n} p_n(V_j \cap W_n)$.

   The details of the constructions of the V_j's are given in section 2.2.

3. Let then q_j be the characteristic function of V_j. We prove that q_j is a multiplier of $C^*(G)$. By local finiteness, the characteristic function $q = \sum q_j$ of $V = \bigcup V_j$ is also a multiplier of $C^*(G)$.

   See section section 3.3.

4. We show that $qC^*(G)$ is a full Hilbert submodule of $C^*(G)$ (see corollary 9 - section 3.2).

5. Considering a natural diffeomorphism $V_j \simeq p_n(V_j) \times]0,1[$, it follows that the Hilbert $C^*(G)$-modules $q_jC^*(G)$ and $qC^*(G)$ are stable.

6. Using Kasparov's stabilisation Theorem ([5]), it follows that $C^*(G)$ is stable.

   This follows from corollary 6 - see section 3.4.

In section 4, we prove an analogous theorem for singular foliations in the sense of [1]. We prove:

Theorem 2. *The C^*-algebra of a singular foliation (as defined in [1]) which has no leaves reduced to a single point is stable.*

The main steps of the proof are the same as for theorem 1. Using vector fields along the foliation, we construct the same small open subsets V_j. Note that, in proving that the characteristic functions of these V_j are multipliers of $C^*(M, \mathcal{F})$, we chose to take a somewhat different path in order to shed a new light to it. This led us to construct groupoid homomorphisms between singular foliation groupoids (section 4.2). Of course, we could have used the same kind of proof as for the Lie groupoid case.

2 Geometric constructions

2.1 Nonzero vector fields in the algebroid

Let M be a smooth (open) manifold, x a point of M, and let $X \in \mathcal{X}(M)$ be a smooth vector field with compact support on M such that $X(x) \neq 0$. Denote by $\Psi_X = (\Psi_X^t)_{t\in\mathbb{R}}$ the flow of X. One can find a codimension one submanifold U of M and a neighborhood I of 0 in $\mathbb{R}$ such that the restriction of Ψ_X to $U \times I$ is a diffeomorphism onto an open (tubular) neighborhood W of U in M. In other words, U is a codimension one submanifold of M which contains x and which is transverse to the integral curves of X.

If G is a Lie groupoid and Y a section with compact support of its Lie algebroid such that the vector field $X := \natural(Y)$ doesn't vanish on a point $x \in G^{(0)}$, let Z_Y be the associated right invariant vector field on G and Ψ_{Z_Y} its flow. We have $r \circ \Psi_{Z_Y}^t = \Psi^t \circ r$.

Applying the construction above, one finds a codimension one submanifold U of $G^{(0)}$ and a neighborhood I of 0 in $\mathbb{R}$ such that the composition map

$$U \times I \xrightarrow{\Psi_{Z_Y}} G \xrightarrow{r} G^{(0)}$$

is precisely the restriction of the flow Ψ_X of X to $U \times I$ and thus a diffeomorphism onto an open neighborhood W of x in $G^{(0)}$. Note that $s \circ \Psi_{Z_Y}$ is the projection $U \times I \to U$.

Now the following maps are diffeomorphisms :

$$\begin{array}{ccc} G^U \times I & \to & G^W \\ (\gamma, t) & \mapsto & \Psi_{Z_Y}(r(\gamma), t)\gamma \end{array} \quad \text{and} \quad \begin{array}{ccc} G_U^U \times I \times I & \to & G_W^W \\ (\gamma, t, \lambda) & \mapsto & \Psi_{Z_Y}(r(\gamma), t)\gamma\Psi_{Z_Y}(s(\gamma), \lambda)^{-1} \end{array} .$$

2.2 Construction of the family V_j

In this section, we explain the construction of the V_j's.

The construction above yields a locally finite cover (W_n) by relatively compact open subsets and diffeomorphisms $f_n : U_n \times \mathbb{R} \to W_n$ such that $W_n' = f_n(U_n' \times \mathbb{R})$ cover $G^{(0)}$ with U_n' relatively compact in U_n. We will often identify W_n and $U_n \times \mathbb{R}$ under f_n.

Let $p_n : W_n \to U_n$ be the composition of f_n^{-1} with the projection $U_n \times \mathbb{R} \to U_n$. As (W_n) is locally finite and $G^{(0)}$ is σ-compact, the set of indices is countable; we identify it with $\mathbb{N}$.

- First, choose a (Riemannian) metric d on $G^{(0)}$.
- For every $n \in \mathbb{N}$, let $\varepsilon_n > 0$ be small enough so that, for every $m \in \mathbb{N}$ such that $W_n \cap W_m \neq \emptyset$, and every $x \in U_m'$, $d(f_m(x,-1), f_m(x,1)) > \varepsilon_n$. Thus if $W_n \cap W_m \neq \emptyset$, for every $x \in U_m'$: $\operatorname{diam}(f_m(\{x\} \times \mathbb{R})) > \varepsilon_n$.
- Let $n \in \mathbb{N}$. Assume that we have already constructed a finite set J_n with a map, $j \mapsto n(j)$ from J_n to $\mathbb{N}$ with $n(j) < n$ and a family of open sets $(V_j)_{j\in J_n}$ with pairwise disjoint closures satisfying,

$$\forall \ell < n, \quad U_\ell' \subset \bigcup_{j;\ n(j)=\ell} p_\ell(V_j), \quad \operatorname{diam}(V_j) \leq \varepsilon_{n(j)}.$$

By the diameter assumption, for every $x \in U'_n$, the set $f_n(\{x\} \times \mathbb{R})$ is not contained in a $\overline{V_j}$ with $j \in J_n$; therefore $f_n(\{x\} \times \mathbb{R})$ is not contained in $\bigcup_{j \in J_n} \overline{V_j}$ (by connectedness of $\mathbb{R}$).

By compactness, we may then construct a finite cover $(A_\ell)_{\ell \in J'_n}$ of $\overline{U'_n}$ by open subsets of U_n and open intervals I_ℓ in $\mathbb{R}$ such that $\overline{f_n(A_\ell \times I_\ell)} \cap \bigcup_{j \in J_n} \overline{V_j} = \emptyset$ and $\operatorname{diam}(f_n(A_\ell \times I_\ell)) \leq \varepsilon_n$.

Replacing the I_ℓ's by smaller intervals, we now further assume that the $\overline{I_\ell}$ with $\ell \in J'_n$ are pairwise disjoint. One then constructs for each $\ell \in J'_n$ two smooth functions $\varphi_\ell^\pm : U_n \to I_\ell$ such that $\varphi_\ell^+ \geq \varphi_\ell^-$ and such that $A_\ell = \{x \in U_n;\ \varphi_\ell^-(x) < \varphi_\ell^+(x)\}$.

Let then J_{n+1} be the disjoint union of J_n end J'_n. For $\ell \in J'_n$, put $V_\ell = f_n(\{(x,t) \in U_n \times \mathbb{R};\ \ \varphi_\ell^-(x) < t < \varphi_\ell^+(x)\})$ and define $n(\ell) = n$.

- We will also use the diffeomorphism $h_j : A_j \times]0,1[\to V_j$ given by

$$h_j(x,t) = f_{n(j)}\Big(x, (1-t)\varphi_j^-(x) + t\varphi_\ell^+(x)\Big). \qquad (\diamondsuit)$$

- We thus construct the family (V_j) inductively. Every compact set meets only finitely many W_n's and therefore finitely many $\overline{V_j}$'s since $\overline{V_j} \subset W_{n(j)}$ and $\{j;\ n(j) = n\}$ is finite.

3 Hilbert modules

3.1 Stable Hilbert modules; full Hilbert modules

We start by briefly recalling some now classical general facts on Hilbert modules. The basic reference for them is [5]. See also e.g. [8, 6].

Let A be a separable C^*-algebra.

Definition 3. A separable Hilbert A-module E is said to be

Stable if it is isomorphic to $\ell^2(\mathbb{N}) \otimes E$.

Full if the vector span of the set of products $\langle x|y\rangle$ with $x, y \in E$ is dense in A.

We have:

Proposition 4. *1. Let J be a countable set and $(E_j)_{j \in J}$ a family of (separable) stable Hilbert A modules, then $\bigoplus_{j \in J} E_j$ is stable.*

2. A separable Hilbert A-module E is stable if and only if the C^*-algebra $\mathcal{K}(E)$ is stable (see e.g. [3, Facts 4.7]). □

Let us also recall Kasparov's stabilization Theorem. Put $\mathcal{H}_A = \ell^2(\mathbb{N}) \otimes A$.

Theorem 5 (Kasparov's stabilization Theorem)**.** *For every separable Hilbert A-module E, the Hilbert A-modules $\mathcal{H}_A$ and $E \oplus \mathcal{H}_A$ are isomorphic.* □

The following statement is an immediate consequence of Kasparov's stabilization theorem. We outline a proof for completeness.

Corollary 6. *1. A separable full and stable Hilbert A-module is isomorphic to $\mathcal{H}_A$ (cf. [6, Prop. 7.4, p. 73]).*

2. If E is a separable Hilbert A-module and $p \in \mathcal{L}(E)$ is a projection such that pE is full and stable, then E is isomorphic to $\mathcal{H}_A$.

Proof. 1. Put $B = \mathcal{K}(E)$ and $E^* = \mathcal{K}(E, A)$ considered as a Hilbert B module. Since E is full, we find $\mathcal{K}(E^*) = E^* \otimes_B E = A$. Since E is stable, the Hilbert B-module B is isomorphic to $\mathcal{H}_B$, therefore, using Kasparov's stabilization theorem, we find an isomorphism $u : B \to B \oplus (\ell^2(\mathbb{N}) \otimes E^*)$. We thus obtain an isomorphism of Hilbert A-modules $u \otimes_B 1_E$ between $E = B \otimes_B E$ and $E \oplus (\ell^2(\mathbb{N}) \otimes E^* \otimes_B E) \simeq \mathcal{H}_A$.

2. Write $E = pE \oplus (1-p)E \simeq \mathcal{H}_A \oplus (1-p)E \simeq \mathcal{H}_A$. □

3.2 Hilbert module associated with a transverse map

We now recall the rather well known construction of generalized smooth transverse maps $M \to G$ from a manifold M to a smooth groupoid G. This construction is useful in steps 4 and 5 of the proof of our theorem as explained in section 1. Note that we actually only use it for a (locally closed) submanifold $M \subset G^{(0)}$ transverse to G.

Definition 7. Let M be a smooth manifold and G a smooth groupoid with algebroid $\mathfrak{A}$ and anchor $\natural$. A smooth map $f : M \to G^{(0)}$ is said to be *transverse* to G if for every $x \in M$, $df_x(T_xM) + \natural_{f(x)}\mathfrak{A}_{f(x)} = T_{f(x)}G^{(0)}$.

The *graph* of a smooth map $f : M \to G^{(0)}$ is the set $\Gamma^f = M \times_{G^{(0)}} G = \{(x, \gamma) \in M \times G;\ f(x) = r(\gamma)\}$.

Equivalently, f is transverse if and only if the *source map* s_f is a submersion, where $s_f : \Gamma^f \to G^{(0)}$ is the map $(x, \gamma) \mapsto s(\gamma)$.

We will in fact only use the case where $f : M \to G^{(0)}$ is the inclusion of a (locally closed) submanifold M of $G^{(0)}$. Then, $\Gamma^f = G^M = \{\gamma \in G;\ r(\gamma) \in M\}$.

For the sake of completeness, we recall also the notion of a *generalized transverse map.*

Let G be a smooth groupoid, M a smooth manifold. A generalized morphism $f : M \to G$ is given either by:

A cocycle : An open cover $(U_i)_{i\in I}$ of M, smooth maps $f_i : U_i \to G^{(0)}$ and smooth maps $f_{ij} : U_i \cap U_j \to G$ satisfying $r \circ f_{ij} = f_i$, $s \circ f_{ij} = f_j$ and, for all $x \in U_i \cap U_j \cap U_k$, $f_{ik}(x) = f_{ij}(x)f_{jk}(x)$.

The graph of f : A set Γ^f which is a G-principal bundle over M. We therefore are given maps $r_f : \Gamma^f \to M$ which is a smooth surjective submersion and $s_f : \Gamma^f \to G^{(0)}$ with a right action of G which is a smooth map $(x, \gamma) \mapsto x\gamma$ from $\Gamma^f \times_{G^{(0)}} G = \{(x, \gamma) \in \Gamma^f \times G;\ s_f(x) = r(\gamma)\}$ to Γ^f. This action is assumed to be proper and free with quotient M:

Proper means that the map $(x, \gamma) \mapsto (x, x\gamma)$ is proper from $\Gamma^f \times_{G^{(0)}} G$ to $\Gamma^f \times \Gamma^f$.

Free means that the map $(x,\gamma) \mapsto (x, x\gamma)$ is injective.

The quotient is M means that for a pair $(x,y) \in \Gamma^f \times \Gamma^f$ there exists $\gamma \in G$ with $x\gamma = y$ if and only $r_f(x) = r_f(y)$. Note that by freeness, this γ is unique. We will denote it by $x^{-1}y$.

These three conditions altogether mean that the map $(x,\gamma) \mapsto (x, x\gamma)$ is a diffeomorphism from $\Gamma^f \times_{G^{(0)}} G$ onto $\Gamma^f \times_M \Gamma^f = \{(x,y) \in \Gamma^f \times \Gamma^f;\ r_f(x) = r_f(y)\}$ - whose inverse is $(x,y) \mapsto (x, x^{-1}y)$.

Given a cocycle (f_i), $(f_{i,j})$, we obtain the graph Γ^f by gluing the graphs Γ^{f_i} thanks to the $f_{i,j}$'s. Conversely, we obtain a cocycle out of Γ^f by means of local sections of the submersion $r_f : \Gamma^f \to M$.

The generalized morphism f is *transverse* to G (or a submersion from M to the 'bad manifold' $G^{(0)}/G$) if $s_f : \Gamma^f \to G^{(0)}$ is a submersion. This is equivalent to saying in the cocycle vision that the maps f_i are transverse to G.

If f is transverse to G, the map $(x,\gamma) \mapsto \gamma$ is a submersion from $\Gamma^f \times_{G^{(0)}} G$ into G, whence the map $(x,y) \mapsto x^{-1}y$ is a submersion from $\Gamma^f \times_M \Gamma^f$ to G

It then defines a Hilbert-$C^*(G)$-module $C^*(\Gamma^f)$: this is the completion of $C_c(\Gamma^f)$ with respect to the $C^*(G)$-valued inner product given (using 1/2-densities) by

$$\langle g|h\rangle(\gamma) = \int_{x\in\Gamma^f;\ s_f(x)=r(\gamma)} \overline{g}(x)h(x\gamma).$$

Let us state the following easy fact which is important for our constructions:

Proposition 8. *Let $f : M \to G$ be a generalized* transverse *morphism with graph Γ^f. The module $C^*(\Gamma^f)$ is full if and only if $s_f : \Gamma^f \to G^{(0)}$ is onto.*

Proof. The image of the submersion $s_f : \Gamma^f \to G^{(0)}$ is an open subset U of $G^{(0)}$. Using the action of G on Γ^f, we deduce that U is saturated in $G^{(0)}$ - i.e. if $r(\gamma) \in U$, there exists $x \in \Gamma^f$ with $r(\gamma) = s_f(x)$; then $s(\gamma) = s_f(x\gamma) \in U$.

The image of the map $(x,y) \mapsto x^{-1}y$ from (defined on $\Gamma^f \times_M \Gamma^f$) is G^U_U.

It follows that the inner products $\langle g|h\rangle$ with $g,h \in C^\infty_c(\Gamma^f)$, span a dense subset of $C^\infty_c(G^U_U)$.

If $U = G^{(0)}$, these scalar products span a dense subspace in $C^*(G)$; if $U \neq G^{(0)}$, they all sit in the kernel of a regular representation associated to any point $x \notin U$. □

Note that if M is a submanifold of $G^{(0)}$, this condition means that M meets all the G orbits.

Let us state this proposition in the precise way we will need to use it:

Corollary 9. *Let G be smooth and V an open subset of $G^{(0)}$. Consider $C_0(V)$ as sitting in $C_0(G^{(0)})$ and therefore in the multiplier algebra of $C^*(G)$. Assume that every orbit of G has a nonempty intersection with V. Then $C_0(V)C^*(G)$ is a full submodule of $C^*(G)$.* □

3.3 The characteristic function of V_j is a projection

We prove here that the characteristic function q_j of V_j is a multiplier of $C^*(G)$, *i.e.* that $C_0(V_j)C^*(G)$ is *orthocomplemented* in $C^*(G)$.

Indeed, put $n = n(j)$. We may write $q_j = \vartheta_j q_j \vartheta_j$ where ϑ_j is a smooth real valued function on $G^{(0)}$ with support in W_n which is equal to 1 on $\overline{V}_j$. It is then enough to prove that $q_j \in \mathcal{L}(C^*(G^{W_n}))$ viewing ϑ_j as an element $T_{\vartheta_j} \in \mathcal{L}(C^*(G); C^*(G^{W_n}))$ and write $q_j = T^*_{\vartheta_j} q_j T_{\vartheta_j}$.

Using the identification between W_n and $U_n \times \mathbb{R}$ coming from the diffeomorphism f_n, we may write $C^*(G^{W_n}) = (C_0(U_n) \otimes L^2(\mathbb{R})) \otimes_{C_0(U_n)} C^*(G^{U_n})$.

The characteristic function of V_j is given by a ($*$-)strongly continuous map from U_n to $\mathcal{L}(L^2(\mathbb{R}))$, and therefore is an element $Q_j \in \mathcal{L}(C_0(U_n) \otimes L^2(\mathbb{R}))$. Therefore $q_j = Q_j \otimes_{C_0(U_n)} 1$ is in $\mathcal{L}(C^*(G^{W_n}))$.

Note also that Q_j is the $*$-strong limit of a sequence (θ_k) of smooth functions on $U_n \times \mathbb{R}$: put $\theta_k(x,t) = \phi(k(t - \varphi^-(x)))\phi(k(\varphi^+(x) - t))$ where $\phi : \mathbb{R} \to [0,1]$ is a continuous function vanishing for $t \leq 0$ and equal to 1 for $t \geq 1$. It follows that the range of q_j is (the closure of) $C_0(V_j)C^*(G)$. It is the Hilbert $C^*(G)$-module $C^*(G^{V_j})$ corresponding to the inclusion $V_j \to G^{(0)}$.

Now the projections q_j are pairwise orthogonal and, by local finiteness, the sum $\sum q_j$ is strictly convergent. Indeed, the sum $\sum q_j \xi$ has only finitely many nonzero terms for $\xi \in C_c(G)$.

3.4 Stability

Stability of $q_j C^*(G)$. Using the diffeomorphism $h_j : A_j \times]0,1[\to V_j$ (see section 2.2, formula ($\diamondsuit$)), we deduce a diffeomorphism $G^{V_j} \simeq G^{A_j} \times]0,1[$, whence an isomorphism of Hilbert $C^*(G)$-modules $C^*(G^{V_j}) \simeq C^*(G^{A_j}) \otimes L^2(]0,1[)$. Therefore, the Hilbert C^*-module $q_j C^*(G)$ is stable.

Stability of $qC^*(G)$. Since $q = \sum q_j$ ($*$-strong convergence), it follows that $qC^*(G) = \bigoplus_j q_j C^*(G)$. Therefore $qC^*(G)$ is stable too.

Conclusion. It then follows from corollary 6 that the Hilbert $C^*(G)$-module $C^*(G)$ is isomorphic to $\mathcal{H}_{C^*(G)}$. The C^*-algebra $C^*(G) = \mathcal{K}(C^*(G)) = \mathcal{K}(\mathcal{H}_{C^*(G)})$ is stable.

This ends the proof of theorem 1.

4 Stability of the C*-algebra of a singular foliation

4.1 The holonomy groupoid of a singular foliation

The C*-algebra of a singular foliation was defined in [1]. Let us briefly recall a few facts and constructions from [1].

Recall that a foliation on a manifold M is defined in [1] to be a (locally) finitely generated submodule $\mathcal{F}$, stable by Lie brackets, of the $C^\infty(M)$-module $\mathcal{X}_c(M)$ of smooth vector fields on M with compact support.

A *bi-submersion* of $\mathcal{F}$ is the data of (N, r_N, s_N) where N is a smooth manifold, $r_N, s_N : N \to M$ are smooth submersions such that :

$$r_N^{-1}(\mathcal{F}) = s_N^{-1}(\mathcal{F}) \text{ and } s_N^{-1}(\mathcal{F}) = C_c^\infty(N; \ker\, ds_N) + C_c^\infty(N; \ker\, dr_N)\ ^{2)}.$$

The *inverse* of (N, r_N, s_N) is (N, s_N, r_N) and if (T, r_T, s_T) is another bi-submersion for $\mathcal{F}$ the *composition* is given by $(N, r_N, s_N) \circ (T, r_T, s_T) := (N \times_{s_N, r_T} T, r_N \circ p_N, s_T \circ p_T)$, where p_N and p_T are the natural projections respectively of $N \times_{s_N, r_T} T$ on N and on T.
A *morphism* from (N, r_N, s_N) to (T, r_T, s_T) is a smooth map $h : N \to T$ such that $s_T \circ h = s_N$ and $r_T \circ h = r_N$ and it is *local* when it is defined only on an open subset of N.
Finally a bi-submersion can be *restricted* : if U is an open subset of N, (U, r_U, s_U) is again a bi-submersion, where r_U and s_U are the restriction of r_N and s_N to U.

For x in M, we define the *fiber* of $\mathcal{F}$ at x to be the quotient $\mathcal{F}_x = \mathcal{F}/I_x\mathcal{F}$. Let $\mathcal{X} = (X_i)_{i \in [\![1,n]\!]} \in \mathcal{F}^n$ be such that $\mathcal{X}_x = ([X_i]_x)_{i \in [\![1,n]\!]}$ is a basis of $\mathcal{F}_x$. For any $\xi = (\xi_i)_{i \in [\![1,n]\!]} \in \mathbb{R}^n$, we consider the vector field $X_\xi := \sum_{i=1}^{n} \xi_i X_i$ and we denote by Ψ_ξ^s its flow at time s. We consider the two smooth submersions from $M \times \mathbb{R}^n$ to M :

$$(s_\mathcal{X}, r_\mathcal{X}) : M \times \mathbb{R}^n \longrightarrow M \times M \ ; \ (x, \xi) \mapsto (x, \Psi_\xi^1(x)) \ .$$

According to Proposition 2.10 and 3.11 of [1], one can find an open neighborhood W of $(x, 0)$ in $M \times \mathbb{R}^n$ such that (W, r_W, s_W) is a bi-submersion, where the map r_W and s_W are the restriction to W of the maps $r_\mathcal{X}$ and $s_\mathcal{X}$ defined above. Such a bi-submersion is called a *path holonomy bi-submersion minimal at* x.

Notice, that any restriction around $(x, 0)$ of a path holonomy bi-submersion minimal at x, is again a path holonomy bi-submersion minimal at x.

Let $\mathcal{U} = (U_i, r_i, s_i)_{i \in I}$ be a family of bi-submersions of $\mathcal{F}$. A bi-submersion (U, r_U, s_U) is *adapted* to $\mathcal{U}$ if for any $u \in U$ there is a local morphism around u from U to a U_i for some i.

A family $\mathcal{U} = (U_i, r_i, s_i)_{i \in I}$ of bi-submersions of $\mathcal{F}$ which satisfies $M = \bigcup_{i \in I} s_i(U_i)$ and the inverse of any element in $\mathcal{U}$ is adapted to $\mathcal{U}$ together with the composition of any two elements of $\mathcal{U}$ is an *atlas*.

The *path holonomy atlas* is the family of bi-submersions of $\mathcal{F}$ generated by the path holonomy bi-submersions.

The *groupoid of the atlas* $\mathcal{U}$ is the quotient $G(\mathcal{U}) = \sqcup_{i \in I} U_i / \sim$ where $U_i \ni u \sim v \in U_j$ if and only if there is a local morphism from U_i to U_j sending u on v. When (U, r_U, s_U) belongs to $\mathcal{U}$ and $u \in U$, let us denote by $[U, r_U, s_U]_u$ its image in $G(\mathcal{U})$. The structural morphisms of $G(\mathcal{U})$ are given by :

source and range : $\mathbf{s}([U, r_U, s_U]_u) = s_U(u), \mathbf{r}([U, r_U, s_U]_u) = r_U(u)$,

[2] If $h : N \to M$ is a smooth submersion $h^{-1}(\mathcal{F})$ is the vector space generated by tangent vector fields fZ where $f \in C_c^\infty(N)$ and Z is a smooth tangent vector field on N which is projectable by dh and such that $dh(Z)$ belongs to $\mathcal{F}$.

inverse : $[U, r_U, s_U]_u^{-1} = [U, s_U, r_U]_u$,

product : $[U, r_U, s_U]_u \cdot [V, r_V, s_V]_v = [U \times_{s_U, r_V} V, r_U \circ p_U, s_V \circ p_V]_{(u,v)}$ when $s_U(u) = r_V(v)$.

The groupoid of an atlas is endowed with the quotient topology which is quite bad, in particular the dimension of the fibers may change.

The *holonomy groupoid* of $\mathcal{F}$ is the groupoid of the path holonomy atlas.

4.2 Subfoliations

A *subfoliation* $\mathcal{F}_1$ of a foliation $\mathcal{F}_2$ is a submodule of $\mathcal{F}_2$ which is a foliation *i.e.* it is locally finitely generated and stable by Lie brackets.

In this section, we fix a foliation $\mathcal{F}_2$ and a subfoliation $\mathcal{F}_1$ of $\mathcal{F}_2$.

4.2.1 The atlas of compatible bi-submersions

Definition 10. A bi-submersion (U, r, s) of $\mathcal{F}_1$ is said to be *compatible* with $\mathcal{F}_2$ if $r^{-1}(\mathcal{F}_2) = s^{-1}(\mathcal{F}_2)$.

Proposition 11. *1. For every $x_0 \in M$, there is a bi-submersion $(\mathcal{W}, r, s)$ of $\mathcal{F}_1$ compatible with $\mathcal{F}_2$ such that $x_0 \in s(\mathcal{W})$.*

2. If $f : (U, r_U, s_U) \to (V, r_V, s_V)$ is a morphism of bi-submersions for $\mathcal{F}_1$ and (V, r_V, s_V) is compatible with $\mathcal{F}_2$, then (U, r_V, s_V) is compatible with $\mathcal{F}_1$.

3. The bi-submersions of $\mathcal{F}_1$ compatible with $\mathcal{F}_2$ form an atlas.[3]

Proof. 1. Fix vector fields $X_1, \dots, X_n$ which generate $\mathcal{F}_1$ in a neighborhood of x_0. For $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, put $\varphi_y = \exp(\sum y_i X_i) \in \exp \mathcal{F}$. On $\mathbb{R}^n \times M$, put $s_0(y, x) = x$ and $t_0(y, x) = \varphi_y(x)$. It is proved in [1, section 2.3] that there is a neighborhood $\mathcal{W}$ of $(0, x_0)$ in $\mathbb{R}^n \times M$ such that $(\mathcal{W}, t, s)$ is a bi-submersion for $\mathcal{F}_1$ where s and t are the restrictions of s_0 and t_0.

Let Y be the vector field on $\mathbb{R}^n \times M$ given by $Y(y, x) = (0, \sum y_i X_i) \in \mathbb{R}^n \times T_x M = T_{(y,x)}(\mathbb{R}^n \times M)$. Since $Y \in s^{-1}\mathcal{F}_1 \subset s^{-1}\mathcal{F}_2$, it follows that $s^{-1}\mathcal{F}_2$ is invariant under $\exp Y$. We find that $s^{-1}(\mathcal{F}_2) = (s \circ \exp Y)^{-1}(\mathcal{F}_2)$ as desired.

2. As $r_U = r_V \circ f$ and $s_U = s_V \circ f$, we find $r_U^{-1}(\mathcal{F}) = f^{-1}(r_V^{-1}(\mathcal{F})) = f^{-1}(s_V^{-1}(\mathcal{F})) = s_U^{-1}(\mathcal{F})$.

3. Let (U, r_U, s_U) and (V, r_V, s_V) be bi-submersions (for $\mathcal{F}_1$) compatible with $\mathcal{F}_2$.

   (a) Obviously, the inverse (U, s_U, r_U) of (U, r_U, s_U) is compatible with $\mathcal{F}_2$.

   (b) Recall that the composition (W, r_W, s_W) of (U, r_U, s_U) and (V, r_V, s_V) is constructed as follows: define $W = \{(u, v) \in U \times V;\ s_U(u) = r_V(v)\}$ and for

[3] To make it a set, take only bi-submersions defined on open subsets of $\mathbb{R}^N$ for all N. Note that a bi-submersion (U, r, s) of $\mathcal{F}_1$ is then adapted to this atlas if and only if $r^{-1}\mathcal{F}_2 = s^{-1}\mathcal{F}_2$.

$(u,v) \in W$, set $s_W(u,v) = s_V(v)$ and $r_W(u,v) = r_U(u)$. As $p_U : (u,v) \mapsto u$ and $p_V(u,v) \mapsto v$ are submersions on W, and $s_U \circ p_U = r_V \circ p_V$, we find

$$\begin{aligned} r_W^{-1}(\mathcal{F}_2) &= p_U^{-1}(r_U^{-1}(\mathcal{F}_2)) = p_U^{-1}(s_U^{-1}(\mathcal{F}_2)) \\ &= p_V^{-1}(r_V^{-1}(\mathcal{F}_2)) = p_V^{-1}(s_V^{-1}(\mathcal{F}_2)) \\ &= s_W^{-1}(\mathcal{F}_2). \end{aligned}$$

□

Proposition 12. *The composition of a bi-submersion of $\mathcal{F}_1$ compatible with $\mathcal{F}_2$ with a bi-submersion of $\mathcal{F}_2$ is a bi-submersion of $\mathcal{F}_2$.*

Proof. Let (U, r_U, s_U) be a bi-submersion of $\mathcal{F}_1$ compatible with $\mathcal{F}_2$ and (V, r_V, s_V) be a bi-submersion for $\mathcal{F}_2$. As previously let $W = \{(u,v) \in U \times V;\ s_U(u) = r_V(v)\}$ and denote by p_U and p_V the projections on U and V. We set $s_W = s_V \circ p_V$, $r_W = r_U \circ p_U$ and $\alpha = s_U \circ p_U = r_V \circ p_V$.

Since (U, r_U, s_U) is compatible with $\mathcal{F}_2$ we have :

$$r_W^{-1}(\mathcal{F}_2) = \alpha^{-1}(\mathcal{F}_2) = s_W^{-1}(\mathcal{F}_2)$$

In particular $C_c^\infty(W; \ker dr_W) + C_c^\infty(W; \ker ds_W) \subset s_W^{-1}(\mathcal{F}_2)$.

On the other hand

$$C_c^\infty(W; \ker d\alpha) = C_c^\infty(W; \ker dp_U) \oplus C_c^\infty(W; \ker dp_V)$$

$$\begin{aligned} s_W^{-1}(\mathcal{F}_2) &= p_V^{-1}(C_c^\infty(V; \ker ds_V) + C_c^\infty(V; \ker dr_V)) \\ &= C_c^\infty(W; \ker ds_W) + C_c^\infty(W; \ker d\alpha) \\ &= C_c^\infty(W; \ker dp_U) + C_c^\infty(W; \ker dp_V) + C_c^\infty(W; \ker ds_W) \\ &\subset C_c^\infty(W; \ker dr_W) + C_c^\infty(W; \ker ds_W) + C_c^\infty(W; \ker ds_W) \end{aligned}$$

□

Remark 13. Given bi-submersions (U, r_U, s_U) and (U', r'_U, s'_U) of $\mathcal{F}_1$ and (V, r_V, s_V) and (V', r'_V, s'_V) of $\mathcal{F}_2$ and morphisms $f : U \to U'$ and $g : V \to V'$, we obtain a morphism $(u,v) \mapsto (f(u), g(v))$ of bi-submersions $U \circ V \to U' \circ V'$.

4.2.2 Morphism on holonomy groupoids

Let G_i be the (path) holonomy groupoids of $\mathcal{F}_i$ ([1, Example 3.4.3 and Def. 3.5]). Let $\check{G}_1$ be the groupoid associated with the atlas of bi-submersions of $\mathcal{F}_1$ compatible with $\mathcal{F}_2$. Let $\widehat{G}_2$ be the maximal holonomy groupoid of $\mathcal{F}_2$ associated to the "maximal atlas" of all possible bi-submersions ([1, Ex. 3.4.1]).

Let (U, r_U, s_U) in $\mathcal{U}(\mathcal{F}_1)$, $u \in U$ and $x = s_U(u)$. Choose a bi-submersion (V, r_V, s_V) and $v \in V$ representing the unit $x \in G_2^{(0)}$ that is :

- $v \in V$ satisfies $s_V(v) = r_V(v) = x$ and
- There is an identity bisection through v. This means that there is an open subset $W \subset M$ and a smooth map $j : W \to V$ which is a section for both s and r.

According to proposition 12, the composition of (U, r_U, s_U) and (V, r_V, s_V) is a bi-submersion for $\mathcal{F}_2$. Let then U' be the open subset $s_U^{-1}(W)$ of U and define $\psi : U' \to U \circ V$ by $\psi(u) = (u, j \circ s(u))$.

According to remark 13, the class of $\psi(u)$ in $\widehat{G}_2$ only depends on the class of u in $\check{G}_1$. Note also, that taking a bi-submersion (V'', r''_V, s''_V) and $v'' \in V''$ representing the unit $r_U(u) \in G_2^{(0)}$, we find a bi-submersion $V'' \circ U$ for G_2. As the class of v (*resp.* v'') is the identity at x (*resp.* $r_U(u)$), the classes of $(v'', u) \in V'' \circ U$ (*resp.* $(u, v) \in U \circ V$) and $(v'', u, v) \in V'' \circ U \circ V$ coincide.

Finally, for any bi-submersion V_1 for $\mathcal{F}_1$, bi-submersion V_2 for $\mathcal{F}_2$ the class $[(v_1, v_2)]$ of $(v_1, v_2) \in V_1 \circ V_2$ is $\varphi([v_1])[v_2]$.

We thus constructed a "smooth" morphism of groupoids $\varphi : \check{G}_1 \to \widehat{G}_2$. Smoothness means that for every $\gamma_1 \in G_1$ there is a bi-submersion U associated with $\check{G}_1$ such that γ_1 is the class of some element in U and a smooth map ψ from U is a bi-submersion V for $\mathcal{F}_2$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\psi} & V \\ \downarrow & & \downarrow \\ \check{G}_1 & \xrightarrow{\varphi} & \widehat{G}_2 \end{array}$$

commutes, where the vertical maps associate to elements of the bi-submersions their class in the corresponding groupoids.

According to [2] and remark 3.13 of [1], the s-fibers $\check{G}_{1,x}$ and $\widehat{G}_{2,x}$ are smooth manifolds. The restriction of φ defines in particular a smooth map $\check{G}_{1,x} \to \widehat{G}_{2,x}$.

By definition of the path holonomy atlas, the corresponding groupoid G_1 is s-connected, since it is generated by any small neighborhood of the identity. More precisely, and thanks to [2], the s-fibers $G_{1,x}$ are connected manifolds. It follows that $\varphi(G_1) \subset G_2$.

It follows that if (U, r_U, s_U) is a bi-submersion for $\mathcal{F}_1$ adapted to the path holonomy atlas of $\mathcal{F}_1$ and (V, r_V, s_V) is a bi-submersion for $\mathcal{F}_2$ adapted to the path holonomy atlas of $\mathcal{F}_2$, then $U \circ V$ is a adapted to the path holonomy atlas of $\mathcal{F}_2$.

4.2.3 The morphism of foliation C*-algebras

For every bi-submersion (U, r, s) of $\mathcal{F}_i$ $(i = 1, 2)$, we denote by $\Omega^{1/2}U$ the bundle over U of half densities on the bundle $\ker dr \oplus \ker ds$.

Let (U_1, r_1, s_1) be a bi-submersion for $\mathcal{F}_1$ and (U_2, r_2, s_2) a bi-submersion for $\mathcal{F}_2$. Put $(W, r_W, s_W) = (U_1, r_1, s_1) \circ (U_2, r_2, s_2)$. Denote by $p_i : W \to U_i$ the projections. For $w = (u_1, u_2) \in W$, since $r_W = r_1 \circ p_1$, and $\ker(dp_1)_w = \ker(dr_2)_{u_2})$, we find exact sequences

$$0 \to \ker(dr_2)_{u_2} \to \ker(dr_W)_w \to \ker(dr_1)_{u_1} \to 0$$

and in the same way,

$$0 \to \ker(ds_1)_{u_1} \to \ker(ds_W)_w \to \ker(ds_2)_{u_2} \to 0.$$

It follows that $\Omega^{1/2}W$ identifies with $\Omega^{1/2}U_1 \otimes \Omega^{1/2}U_2$.

The algebra $C^*(M, \mathcal{F}_i)$ is constructed in [1, section 4.4]. Recall that for every bi-submersion U in the path holonomy atlas of $\mathcal{F}_i$ we have a linear map $\Theta_i^U : C_c^\infty(U; \Omega^{1/2}) \to C^*(M, \mathcal{F}_i)$, and that given an atlas $(U_j)_{j \in J}$ of bi-submersions adapted to the path holonomy atlas, the vector span of $\Theta_i^{U_j} : C_c^\infty(U_j; \Omega^{1/2}),\ j \in J$ is dense in $C^*(M, \mathcal{F}_i)$.

Proposition 14. *1. There is a morphism $\Phi : C^*(M,\mathcal{F}_1) \to \mathcal{M}(C^*(M,\mathcal{F}_2))$ characterized by the equality $\Phi(\Theta_1^{U_1}(f_1))\Theta_2^{U_2}(f_2) = \Theta_2^{U_1\circ U_2}(f_1 \bullet f_2)$ for every bi-submersion U_1 for $\mathcal{F}_1$, every bi-submersion U_2 for $\mathcal{F}_2$, every $f_1 \in C_c^\infty(U_1;\Omega^{1/2}U_1)$, $f_2 \in C_c^\infty(U_2;\Omega^{1/2}U_2)$ where $f_1 \bullet f_2 \in C_c^\infty(U_1\circ U_2;\Omega^{1/2}(U_1\circ U_2))$ is the restriction to $U_1\circ U_2 \subset U_1\times U_2$ of $f_1\otimes f_2 \in C_c^\infty(U_1\times U_2;\Omega^{1/2}U_1\otimes\Omega^{1/2}U_2)$.*

2. The morphism Φ is non degenerate.

Proof. 1. We prove that, given a (faithful) representation ϖ_2 of $C^*(M,\mathcal{F}_2)$, there is a representation ϖ_1 of $C^*(M,\mathcal{F}_1)$ such that $\varpi_1(\Theta_1^{U_1}(f_1))\varpi_2(\Theta_1^{U_2}(f_2)) = \varpi_2(\Theta_2^{U_1\circ U_2}(f_1 \bullet f_2))$.

Recall from [1, section 5] that the representation ϖ_2 corresponds to a triple (μ, H, π_2) where μ is a measure on M which is quasi invariant by G_2, $H = (H_x)_{x\in M}$ is a μ measurable field of Hilbert spaces and π_2 is a representation of G_2 on the field (μ, H).

Fix a a bi-submersion (U_1, r_1, s_1) for $\mathcal{F}_1$. Let (U_2, r_2, s_2) be a bi-submersion for $\mathcal{F}_2$, denote by (W, r_W, s_W) the composition $U_1\circ U_2$ and let $p_i : W \to U_i$ be the projections. Put also $q = s_1\circ p_1 = r_2\circ p_2$.

(a) *We show that μ is quasi-invariant by the groupoid G_1.* Choose positive smooth sections of the bundles of 1-densities of the bundles $\ker ds_1$, $\ker ds_2$, $\ker dr_1$, $\ker dr_2$. Thanks to these choices we construct measures $\mu\circ\lambda^{r_1}$ and $\mu\circ\lambda^{s_1}$ on U_1, measures $\mu\circ\lambda^{r_2}$ and $\mu\circ\lambda^{s_2}$ on U_2 and measures $\mu\circ\lambda^{r_W}$ and $\mu\circ\lambda^{s_W}$ on W. Consider also the measure $\mu\circ\lambda^q$. As (W, r_W, s_W) is a bi-submersion for $\mathcal{F}_2$ as well as (W, q, s_W), it follows that the measures $\mu\circ\lambda^{r_W}$ and $\mu\circ\lambda^q$ are equivalent. Using a cover of $s_1(U_1)$ by open sets of the form $r(U_2)$, we deduce that $\mu\circ\lambda^{r_1}$ and $\mu\circ\lambda^{s_1}$ are equivalent. This being true for every bi-submersion U_1, it follows that the measure μ is quasi-invariant by the groupoid G_1.

(b) *We construct a representation π_1 of the groupoid G_1.* Associated with the bi-submersions (U_2, r_2, s_2) and (W, r_W, s_W) for $\mathcal{F}_2$, we have measurable families of isomorphisms $\pi_{U_2}(u_2) : H_{s_2(u_2)} \to H_{r_2(u_2)}$ and $\pi_W(u_1,u_2) : H_{s_2(u_2)} \to H_{r_1(u_1)}$. We wish to define $\pi_{U_1}(u_1) : H_{s_1(u_1)} \to H_{r_1(u_1)}$ by putting (almost everywhere) $\pi_{U_1}(u_1) = \pi_W(u_1,u_2)\circ\pi_{U_2}(u_2)^{-1}$. To do so, we define the measurable section $\tilde{\pi}_{U_1}$ over W by setting $\tilde{\pi}_{U_1}(u_1,u_2) = \pi_W(u_1,u_2)\circ\pi_{U_2}(u_2)^{-1}$ and show it is constant (almost everywhere) along the fibers of $p_1 : W \to U_1$. Let (U_2', r_2', s_2') be another bi-submersion for $\mathcal{F}_2$. Since the identity of $U_2'\circ U_1\circ U_2$ is an isomorphism of bi-submersions between the composions $U_2'\circ(U_1\circ U_2)$ and $(U_2'\circ U_1)\circ U_2$, it follows that, (almost everywhere in $U_2'\circ U_1\circ U_2$) we have

$$\pi_{U_2'}(u_2')\pi_{U_1\circ U_2}(u_1,u_2) = \pi_{U_2'\circ U_1}(u_2',u_1)\pi_{U_2}(u_2)$$

whence

$$\pi_{U_1\circ U_2}(u_1,u_2)\circ\pi_{U_2}(u_2)^{-1} = \pi_{U_2'}(u_2')^{-1}\circ\pi_{U_2'\circ U_1}(u_2',u_1).$$

This shows that $\tilde{\pi}_{U_1}(u_1,u_2)$ is independent on u_2, *i.e.* is constant along the fibers of p_1.

The triple (μ, H, π_1) yields now a representation ϖ_1 of $C^*(M,\mathcal{F}_1)$ on the Hilbert space $L^2(M,\mu,H)$ of L^2 sections of the bundle H.

Recall that the representations ϖ_1 and ϖ_2 are defined on any bi-submersion (V, r_V, s_V) of $\mathcal{F}_i$ by setting $\varpi_i \circ \Theta_i^V = \hat{\pi}_i^V$ where, for $f \in C_c^\infty(V; \Omega^{1/2}V)$ and $\xi \in L^2(M, \mu, H)$, we have a formula:

$$\hat{\pi}_i^V(f)(\xi)(x) = \int_{V^x} (1 \otimes \rho^V(v))(f(v))\pi_i^V(v)(\xi(s_V(v)))D^V(v)^{1/2}.$$

Recall that:

- $V^x = r_V^{-1}(x)$;
- ρ^V is an isomorphism between $\Omega^{1/2} \ker ds_V$ and $\Omega^{1/2} \ker dr_V$ corresponding to the choice of a Riemannian metric on M, thus $(1 \otimes \rho^V(v))(f(v))$ is a 1-density on V^x and thus can be integrated;
- the isomorphism ρ^V is used to compare Lebesgue measures on the fibers of s_V and of r^V; using it we obtain the Radon Nikodym derivative D^V between the measures $\mu \circ \lambda^{s_V}$ and $\mu \circ \lambda^{r_V}$ which is used in this formula.

It is an easy check that, under the identification $\Omega^{1/2}(U_1 \circ U_2)$ with $\Omega^{1/2}(U_1) \otimes \Omega^{1/2}(U_2)$, we have $\rho^{U_1 \circ U_2}(u_1, u_2) = \rho^{U_1}(u_1) \otimes \rho^{U_2}(u_2)$ and also $D^{U_1 \circ U_2}(u_1, u_2) = D^{U_1}(u_1)D^{U_2}(u_2)$.

One then checks immediately the equality $\hat{\pi}_1^{U_1}(f_1)\hat{\pi}_2^{U_2}(f_2) = \hat{\pi}_2^{U_1 \circ U_2}(f_1 \bullet f_2)$.

2. Take a bi-submersion U_1 for $\mathcal{F}_1$, a bi-submersion U_2 for $\mathcal{F}_2$. Let $(u_1, u_2) \in U_1 \circ U_2$. If the class of u_1 is a unit of G_1 - *i.e.* if U_1 carries an identity bissection through u_1, then the classes of (u_1, u_2) and u_2 coincide. It follows that the bi-submersions of the form $U_1 \circ U_2$ form an atlas of bi-submersions equivalent to the path holonomy atlas of $\mathcal{F}_2$ and therefore the span of $\Theta_2^{U_1 \circ U_2}(f_1 \bullet f_2)$ is dense in $C^*(M, \mathcal{F}_2)$. The result follows. □

4.3 Stability

Since for every $x \in M$ there is $X \in \mathcal{F}$ such that $X_x \neq 0$, we can find, using the construction of section 2.1,

- a locally finite open cover $(W_n)_{n \in \mathbb{N}}$;
- for every $n \in \mathbb{N}$ a submanifold $U_n \subset W_n$ (closed in W_n) and a compact subset $U_n' \subset U_n$;
- a vector field $X_n \in \mathcal{F}$,

such that

- $(t, u) \mapsto \exp_{tX_n}(u)$ is a diffeomorphism $f_n :]-1, 1[\times U_n \to W_n$;
- $\bigcup_{n \in \mathbb{N}} f_n(]-1, 1[\times U_n') = M$.

It then follows from section 2.2, that we can construct a family of open sets V_j satisfying the properties explained in step 2.

The vector field X_n defines a subfoliation $\mathcal{F}_n = \{fX_n;\ f \in C_c^\infty(M)\}$ of $\mathcal{F}$. Of course X_n defines also an action β_n of $\mathbb{R}$ on M and $C^*(M; \mathcal{F}_n)$ is a quotient of $C(M) \rtimes_{\beta_n} \mathbb{R}$ (the holonomy groupoid of $(M, \mathcal{F}_n)$ is a quotient of $M \times_{\beta_n} \mathbb{R}$ since $M \times_{\beta_n} \mathbb{R}$ is an s-connected Lie groupoid

with associated foliation $\mathcal{F}_n$). Now it follows from the construction above that, if $n = n(j)$, the characteristic function q_j of V_j is a multiplier of $C(M) \rtimes_{\beta_n} \mathbb{R}$ and thus of $C^*(M; \mathcal{F}_n)$ and finally, thanks to proposition 14, of $C^*(M, \mathcal{F})$. The image of q_j as a left multiplier of $C(M) \rtimes_{\beta_n} \mathbb{R}$ is $C_0(V_j)\Big(C(M) \rtimes_{\beta_n} \mathbb{R}\Big)$, therefore $q_j C^*(M, \mathcal{F}) = C_0(V_j)C^*(M, \mathcal{F})$.

As $C^*(V_j, \mathcal{F}_n)$ is stable and acts in a non degenerate way on $q_j C^*(M, \mathcal{F})$, it follows that the Hilbert-$C^*(M, \mathcal{F})$ module $q_j C^*(M, \mathcal{F})$ is stable. Finally, putting $q = \sum q_j$, we find that $qC^*(M, \mathcal{F})$ is stable.

To conclude, we prove that the Hilbert-$C^*(M, \mathcal{F})$ module $qC^*(M, \mathcal{F})$ is full. It follows from corollary 6 that the Hilbert-$C^*(M, \mathcal{F})$ module $C^*(M, \mathcal{F})$ is stable, *i.e.* that the C^*-algebra $C^*(M, \mathcal{F})$ is stable.

Proposition 15. *Let $W \subset M$ be open. Assume that W meets all the leaves of $\mathcal{F}$. Then $C_0(W)C^*(M, \mathcal{F})$ is a full Hilbert submodule of $C^*(M, \mathcal{F})$.*

Proof. Note that if $x, y \in M$ are in the same leaf, there is a bi-submersion (U, r, s) in the path holonomy atlas and $u \in U$ such that $r(u) = x$ and $s(u) = y$. It follows that for every $x \in M$, there is a bi-submersion (U, r, s) in the path holonomy atlas such that $x \in s(U)$ and $r(U) \subset W$.

Let $\mathcal{U} = (U_i, r_i, s_i)_{i \in I}$ be a family of bi-submersions defining the path holonomy atlas of $\mathcal{F}$. For $i \in I$, put $U_i' = \{u \in U;\ r_i(u) \in W\}$. Define then $\widetilde{\mathcal{U}} = (\tilde{U}_{i,j}, \tilde{r}_{i,j}, \tilde{s}_{i,j})_{(i,j) \in I^2}$ where $\tilde{U}_{i,j} = (U_i')^{-1} \circ U_j' = \{(u, v) \in U_i \times U_j;\ r_i(u) = r_j(u) \in W\}$, and where we put $\tilde{r}_{i,j}(u, v) = s_i(u)$, and $\tilde{s}_{i,j}(u, v) = s_j(v)$.

Note that since $U_j' \circ \tilde{U}_{k,\ell}$ is adapted to $\mathcal{U}$, it follows that $\tilde{U}_{i,j} \circ \tilde{U}_{k,\ell}$ is adapted to $\widetilde{\mathcal{U}}$. We deduce that $\widetilde{\mathcal{U}}$ is an atlas. It is obviously adapted to $\mathcal{U}$; on the other hand, since $\mathcal{U}$ is minimal, it is adapted to $\widetilde{\mathcal{U}}$.

It follows that the subspaces $C_c^\infty(\tilde{U}_{i,j})$ span a dense subalgebra of $C^*(M, \mathcal{F})$ and therefore $\{f^* g;\ f \in C_c^\infty(U_i'),\ g \in C_c^\infty(U_j')\}$ span a dense subalgebra of $C^*(M, \mathcal{F})$. As the image of $C_c^\infty(U_k')$ is in $C_0(W)C^*(M, F)$, the result follows. □

This ends the proof of theorem 2.

Remarks 16.

1. Note also that since the reduced C^*-algebra of the foliation is a quotient of the full one, it is also stable if $\mathcal{F}$ vanishes nowhere.
2. As $C^*(M, \mathcal{F})$ corresponding to the minimal (path holonomy) atlas is a sub-algebra of the C^*-algebra of any atlas in a non degenerate way, it follows that, under our assumption that $\mathcal{F}$ vanishes nowhere, all these C^*-algebras are stable.

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