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LOCAL NULL-CONTROLLABILITY OF A PARABOLIC SYSTEM WITH COUPLED NONLINEAR BOUNDARY CONDITIONS

KUNTAL BHANDARI* AND FRANCK BOYER†

Abstract. In this article, we study the boundary local null-controllability of a one-dimensional parabolic system with coupled nonlinear boundary conditions and only one single control. The significant point is that the state components are interacting only at the boundary points in terms of some nonlinear functions. The control function is acting either through a *mixed nonlinear boundary condition on the first component* or through a *Neumann condition on the second component*. The results are slightly different in the two cases.

To study the controllability properties, we first consider the associated linear systems where the boundary nonlinearities are linearized around $(0,0)$. The method of moments helps us to prove the controllability and obtain a suitable control cost namely $Ce^{C/T}$ for the linearized systems. Then applying the source term method developed in [25], followed by the Banach fixed point argument, we obtain the small-time local boundary null-controllability of the system.

Key words. Parabolic systems, boundary local null-controllability, method of moments, source term method, fixed-point argument.

AMS subject classifications. 35K20 - 35K58 - 93B05 - 93B60.

1. Introduction.

1.1. The system under study. This paper is concerned with the boundary null-controllability of some parabolic system where the state components are coupled through the boundary via some nonlinear functions. The systems with boundary interactions often represent several biological or chemical models, see for instance [30] where this kind of models are appeared. Let us consider such a parabolic system (without any control for the moment), given by

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (0,T)\times(0,1), \\ y_1(0,\cdot) = y_{0,1}(\cdot) & \text{in } (0,1), \\ y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1), \end{cases} \tag{1.1}$$

with the boundary conditions at $x=0$

$$\begin{cases} \partial_x y_1(t,0) + f(y_2(t,0)) = 0 & \text{in } (0,T), \\ \partial_x y_2(t,0) = 0 & \text{in } (0,T), \end{cases} \tag{1.2}$$

and at $x=1$

$$\begin{cases} \partial_x y_1(t,1) = 0 & \text{in } (0,T), \\ \partial_x y_2(t,1) - f(y_1(t,1)) = 0 & \text{in } (0,T), \end{cases} \tag{1.3}$$

where the nonlinear functions f are chosen as follows

$$\begin{cases} f(\nu) = \beta\nu + g(\nu), \quad \text{with} \quad g\in\mathcal{C}^1(\mathbb{R}),\ g(0)=g'(0)=0, \text{ satisfying} \\ |g(\nu)-g(\tilde\nu)| \le C|\nu-\tilde\nu|(|\nu|+|\tilde\nu|), \quad \text{for any } \nu,\tilde\nu\in\mathbb{R}, \quad \text{for some constant } C>0, \\ f'(0)=\beta\neq 0 \ \text{ is some real number.} \end{cases} \tag{1.4}$$

In the above model, $\alpha_1,\alpha_2\in\mathbb{R}$ are some parameters and $y_0 := (y_{0,1},y_{0,2})\in(L^2(0,1))^2$ is given initial data.

Remark 1.1. *An immediate example of the nonlinearity f is*

$$f(\nu) = \beta\nu + \nu^2,$$

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and it is clear that $g(\nu) = \nu^2$ *satisfies the required conditions* (1.4).

A second example can be considered as follows,

$$f(\nu) = \frac{\beta\nu}{1+\nu^2}, \quad i.e., \quad g(\nu) = -\frac{\beta\nu^3}{1+\nu^2},$$

which often arises in several biological or chemical models, see for instance [30]. One can check that the above g *satisfies the required properties given by* (1.4).

In our system, we observe that the couplings are made only on the boundary points, there is no internal coupling in the system. Now, while dealing with the boundary controllability of the above model, the following natural question arises.

Question: Is there a (scalar) control $v \in L^2(0,T)$ *acting on the boundary point* $x = 0$*, either through the mixed condition of* y_1, y_2 *or, the Neumann condition of* y_2 *in* (1.2)*, such that both the components will vanish (at least locally) at a given time* $T > 0$*, that is to say,* $y_1(T,x) = y_2(T,x) = 0$ *for all* $x \in (0,1)$?

To be more precise, our goal is to study the controllability property of the system (1.1) by treating the following two boundary control cases: either

$$\begin{cases} \partial_x y_1(t,0) + f(y_2(t,0)) = v(t) & \text{in } (0,T), \\ \partial_x y_2(t,0) = 0 & \text{in } (0,T), \end{cases} \tag{1.5}$$

or,

$$\begin{cases} \partial_x y_1(t,0) + f(y_2(t,0)) = 0 & \text{in } (0,T), \\ \partial_x y_2(t,0) = v(t) & \text{in } (0,T), \end{cases} \tag{1.6}$$

along with the conditions (1.3) at $x = 1$.

From the application point of view, the more interesting situation is the case when we exert a control only on the component y_2, namely (1.6).

1.2. Bibliographic comments and motivations. The controllability of a system of partial differential equations with less number of control(s) than equations is gaining genuine interests to the control community. In light of this, we first refer some pioneer works. The authors in [1,2] established some generalized Kalman rank conditions which are necessary and sufficient for the distributed null-controllability of a class of parabolic systems. The boundary controllability of a system is much more intricate and most of the results are restricted to the 1-D case since the very powerful Carleman technique is often inefficient in this context. Indeed, the boundary controllability of a system of PDEs is no more equivalent to the distributed one unlike the scalar case, see for instance [17]. The authors in [17] also proved a necessary and sufficient condition for the boundary null-controllability of a 2×2 coupled parabolic system with a scalar Dirichlet control. A generalization of this result for the system of n parabolic equations with $m < n$ controls has been achieved in [3].

The above cases mainly dealt with internal coupling. Concerning the controllability with boundary coupling, we mention the book [11] and the survey paper [4] where the authors studied the controllability of some wave, heat and Schrödinger systems on metric graphs. We also refer [8] where the boundary null-controllability of some coupled parabolic systems has been addressed where the boundary coupling is chosen by means of a Kirchhoff-type condition. In most of the known cases, the boundary conditions are linearly posed. Thus, dealing with the systems (1.1)–(1.3)–(1.5)/(1.6) is naturally more interesting since the concerned boundary couplings are in terms of some nonlinear functions.

Now, in the context of controllability of nonlinear systems, first we mention [19, Chapter I, Sec. 4] by Fursikov and Imanuvilov where a small-time local null-controllability of semilinear heat equations has been proved using a perturbation argument. In 2000, Barbu [5], independently Fernández-Cara and Zuazua [18] proved the small-time global null-controllability of semilinear heat equation where the growth of nonlinearities is slower than $|s| \ln^{3/2}(1+|s|)$. In fact, the large-time global null-controllability of semilinear heat equations has recently been obtained in [23] for nonlinearities f that grow slower than $|s| \ln^2(1+|s|)$ with a sign condition: $f(s) > 0$ for $s > 0$ and $f(s) < 0$ for $s < 0$.

In the present work, we deal with the local null-controllability of a parabolic system where the couplings arise on the boundary points through some nonlinear functions. The typical form of the nonlinearities has been given by (1.4). Let us now start with the following linearized control systems.

1.3. The linearized control systems. It is clear that the linearized (around $(0,0)$) control systems associated to the nonlinear systems (1.1)–(1.3)–(1.5)/(1.6) are the set of PDEs (1.1) along with the boundary conditions at $x=1$

$$\begin{cases} \partial_x y_1(t,1) = 0 & \text{in } (0,T), \\ \partial_x y_2(t,1) - f'(0) y_1(t,1) = 0 & \text{in } (0,T), \end{cases} \tag{1.7}$$

and at $x=0$, we have one of the following two situations: either

$$\begin{cases} \partial_x y_1(t,0) + f'(0) y_2(t,0) = v(t) & \text{in } (0,T), \\ \partial_x y_2(t,0) = 0 & \text{in } (0,T), \end{cases} \tag{1.8}$$

or,

$$\begin{cases} \partial_x y_1(t,0) + f'(0) y_2(t,0) = 0 & \text{in } (0,T), \\ \partial_x y_2(t,0) = v(t) & \text{in } (0,T), \end{cases} \tag{1.9}$$

with $f'(0) = \beta \neq 0$.

Our goal is to first derive the global boundary null-controllability of the above linear models and then deduce some local (boundary) null-controllability results by the source term method together with a fixed-point argument, thanks to the pioneer work [25] by Liu, Takahashi and Tucsnak.

In this context, we must mention some recent works on the local null-controllability based on the source term and fixed-point approach. For instance, in [29], the local null-controllability of a Kuramoto-Sivashinsky equation (in 1-D and 2-D) has been studied with a single boundary control. A local controllability result of a non-linear phase-field model has been established in [20] with a Dirichlet boundary control; in particular, the authors used the moments method to prove the global null-controllability of their linearized model. Finally, it is worth mentioning that the authors in [22] proved an internal local null-controllability result of a nonlocal semilinear heat equation using the mentioned approach where a Carleman inequality is established to prove the controllability of their linearized system.

Let us come back to our problems; before stating the main results, we introduce the functional framework associated to the linearized models.

Functional framework. Introduce the space

$$X := (L^2(0,1))^2,$$

and write the following elliptic operator $\mathcal{A}$ associated to the system (1.1)–(1.7) with (1.8) or (1.9),

$$\mathcal{A} = \begin{pmatrix} -\partial_x^2 + \alpha_1 & 0 \\ 0 & -\partial_x^2 + \alpha_2 \end{pmatrix}, \tag{1.10a}$$

with its domain

$$D(\mathcal{A}) := \Big\{ \phi := (\phi_1, \phi_2) \in (H^2(0,1))^2 \,\Big|\, \phi_1'(0) + \beta\phi_2(0) = 0,\ \phi_2'(0) = 0, \\ \phi_1'(1) = 0,\ \phi_2'(1) - \beta\phi_1(1) = 0 \Big\}. \tag{1.10b}$$

It is clear that $(\mathcal{A}, D(\mathcal{A}))$ is a non-self-adjoint operator due to the presence of such coupled boundary conditions. More precisely, we find that, the adjoint operator $\mathcal{A}^*$ of $\mathcal{A}$ has the same formal expression as (1.10a), yet with a different domain given by

$$D(\mathcal{A}^*) = \Big\{ u := (u_1, u_2) \in (H^2(0,1))^2 \,\Big|\, u_1'(0) = 0,\ u_2'(0) + \beta u_1(0) = 0, \\ u_1'(1) - \beta u_2(1) = 0,\ u_2'(1) = 0 \Big\}. \tag{1.11}$$

Notations. Throughout the paper, $C > 0$ denotes the generic constant that may vary line to line and may depend on $\alpha_1, \alpha_2, \beta$ but does not depend on T or y_0. By the notation

$$\gamma_1(\cdot) = O(\gamma_2(\cdot)),$$

for some real valued functions γ_1, γ_2, we mean

$$|\gamma_1(\cdot)| \leq C|\gamma_2(\cdot)|, \quad \text{for some constant } C := C(\alpha_1, \alpha_2, \beta).$$

The set of all non-zero real numbers is denoted by $\mathbb{R}^*$. For any $z \in \mathbb{C}$, the real and complex parts are denoted by $\Re(z)$ and $\Im(z)$ respectively.

1.4. Main results.

1.4.1. Controllability results for the nonlinear systems. With the boundary nonlinearities prescribed in (1.4) we have the following local null-controllability results for our systems.

Theorem 1.2. *Let be f given by (1.4) and assume that $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ be given parameters. Then, we have the following local controllability results.*

1. *The system (1.1)–(1.3)–(1.5) is small time locally null-controllable around the equilibrium, that is to say, for any given time $T > 0$, there is a $\delta > 0$ such that for chosen initial data y_0 with $\|y_0\|_X \leq \delta$, there exists a control $v \in L^2(0,T)$ such that the associated solution y satisfies*

$$y(T,x) = 0, \qquad \forall x \in (0,1).$$

2. *There exists a non-empty strict subset $\mathcal{R} \subset \mathbb{R}$ such that if $f'(0) \notin \mathcal{R}$, then the system (1.1)–(1.3)–(1.6) is small time locally null-controllable around the equilibrium, that is to say, for any given time $T > 0$, there is a $\delta > 0$ such that for chosen initial data y_0 with $\|y_0\|_X \leq \delta$, there exists a control $v \in L^2(0,T)$ such that the associated solution y satisfies*

$$y(T,x) = 0, \qquad \forall x \in (0,1).$$

Our strategy to prove Theorem 1.2 is the following:

- First, we shall prove the global boundary null-controllability results of the associated linear models (1.1)–(1.7) with (1.8) or (1.9) using the so-called method of moments (initially developed by Fattorini and Russell [15, 16]).
  A proper estimate of the control cost (precisely $Ce^{C/T}$) is crucial to deduce the controllability results for the non-linear models.
- Next, we apply the source term method introduced in [25]; more precisely, we prove the null-controllability of our linearized models with additional boundary terms in $L^2(0,T)$ (exponentially decreasing while $t \to T^-$) on the boundary points where the non-linearities appear.
- Thereafter, we use the Banach fixed-point argument to obtain the local (boundary) null-controllability for the non-linear models.

1.4.2. Controllability results for the associated linear systems. Let us present the main theorem concerning the null-controllability of the linearized models.

Theorem 1.3. *Let any $y_0 \in X$ and $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ be given. Then for any time $T > 0$, we have the following.*

1. *There exists a control $v \in L^2(0,T)$ such that the system (1.1)–(1.7)–(1.8) is null-controllable at time T.*
2. *There exists a non-empty strict subset $\mathcal{R} \subset \mathbb{R}$ such that if $f'(0) \notin \mathcal{R}$, then there is a control $v \in L^2(0,T)$ such that the system (1.1)–(1.7)–(1.9) is null-controllable at time T.*

In both cases, the controls satisfy the following estimate

$$\|v\|_{L^2(0,T)} \leq Ce^{C/T}\|y_0\|_X,$$

where the constants $C > 0$ neither depend on T nor on y_0 but depends on α_1, α_2 and $f'(0)$.

The set $\mathcal{R}$ will be specified later, namely in Lemma 4.1.

Remark 1.4. *Some remarks are in order.*

- *In the case when $f'(0) = \beta \in \mathcal{R}$, the linearized system (1.1)–(1.7)–(1.9) is not even approximately controllable in X. Therefore, we cannot say anything about the local null-controllability of our nonlinear systems at least with the technique used by us (the source term method as per [25]), since the control cost obtained for the linearized problems will be extensively used to study the nonlinear systems.*
- *If $f'(0) = \beta = 0$, then it is clear that the linearized models (1.1)–(1.7)–(1.8)/(1.9) are completely decoupled. Thus with a single control function, there is no hope to control the whole 2×2 system and again this will prevent us to study the local null-controllability of the associated nonlinear models with the source term method.*
- *Maybe some other techniques (for instance, the Return method [10, Chapter 6]) could be applied to study the local controllability of the nonlinear systems when we do not have the null-controllability of the linearized models, but this needs further investigations.*

2. Well-posedness and formulation of control problems. In this section, we shall discuss about the existence of the weak solutions to our linearized systems and formulate the associated null-control problems.

2.1. Existence of semigroup. First, we show the existence of the analytic semigroup defined by the operator $(-\mathcal{A}^*, D(\mathcal{A}^*))$ introduced in (1.10a)–(1.11). The same is true for the operator $(-\mathcal{A}, D(\mathcal{A}))$ (see (1.10a)–(1.10b)).

Proposition 2.1. *The operator $(-\mathcal{A}^*, D(\mathcal{A}^*))$ defined by (1.10a)–(1.11), generates an analytic semigroup in X, denoted by $\left(e^{-t\mathcal{A}^*}\right)_{t\geq 0}$.*

Proof. We consider the following densely defined sesquilinear form h; for all $u := (u_1, u_2)$, $\psi := (\psi_1, \psi_2) \in (H^1(0,1))^2$,

$$h(u,\psi) := \int_0^1 \Big(u_1'(x)\overline{\psi_1'(x)} + u_2'(x)\overline{\psi_2'(x)}\Big)\,\mathrm{d}x + \int_0^1 \Big(\alpha_1 u_1(x)\overline{\psi_1(x)} + \alpha_2 u_2(x)\overline{\psi_2(x)}\Big)\,\mathrm{d}x - \beta u_1(0)\overline{\psi_2(0)} - \beta u_2(1)\overline{\psi_1(1)}. \tag{2.1}$$

It is clear that h is continuous in $(H^1(0,1))^2$ and moreover, we have

$$|h(u,\psi)| \leq \kappa \|u\|_{(H^1(0,1))^2}\|\psi\|_{(H^1(0,1))^2},$$

where $\kappa > 0$ depends on the parameters α_1, α_2 and β.

Also, any $u \in (H^1(0,1))^2$ satisfies

$$\Re\,(h(u,u)) \geq \kappa_1 \|u\|^2_{(H^1(0,1))^2} - \kappa_2 \|u\|^2_X,$$

for some constants $\kappa_1, \kappa_2 > 0$.

Then, by [27, Proposition 1.51 and Theorem 1.52], the negative operator associated with h generates an analytic semigroup in X of angle $\pi/2 - \arctan\kappa$. Now, thanks to the following Lemma 2.2, we have that the operator associated with h is indeed $\mathcal{A}^*$ with its domain $D(\mathcal{A}^*)$, which completes the proof. ∎

Lemma 2.2. *Let h be the sesquilinear form in $(H^1(0,1))^2$ as defined by (2.1). Then, the operator associated with h is $(\mathcal{A}^*, D(\mathcal{A}^*))$.*

Proof. Denote by $(\widetilde{\mathcal{A}}, D(\widetilde{\mathcal{A}}))$ the operator associated with the form h, which is by definition given by

$$\begin{cases} D(\widetilde{\mathcal{A}}) = \Big\{u \in (H^1(0,1))^2 \mid \exists\, w \in X \ \text{ s.t. } \ h(u,\psi) = (w,\psi)_X, \ \ \forall \psi \in (H^1(0,1))^2\Big\}, \\ \widetilde{\mathcal{A}}u := w = (w_1, w_2) \in X. \end{cases}$$

- First, we fix some $u := (u_1, u_2) \in D(\mathcal{A}^*)$. Then for all $\psi := (\psi_1, \psi_2) \in (H^1(0,1))^2$, we have

$$\begin{aligned} h(u,\psi) &:= \int_0^1 \Big(u_1'(x)\overline{\psi_1'(x)} + u_2'(x)\overline{\psi_2'(x)}\Big)\,\mathrm{d}x + \int_0^1 \Big(\alpha_1 u_1(x)\overline{\psi_1(x)} + \alpha_2 u_2(x)\overline{\psi_2(x)}\Big)\,\mathrm{d}x \\ &\qquad - \beta u_1(0)\overline{\psi_2(0)} - \beta u_2(1)\overline{\psi_1(1)} \\ &= \int_0^1 \Big(-u_1''(x) + \alpha_1 u_1(x)\Big)\overline{\psi_1(x)}\,\mathrm{d}x + \int_0^1 \Big(-u_2''(x) + \alpha_2 u_2(x)\Big)\overline{\psi_2(x)}\,\mathrm{d}x = (\mathcal{A}^* u, \psi), \end{aligned}$$

where we performed an integration by parts on and also use the boundary conditions satisfied by $u \in D(\mathcal{A}^*)$. Thus, for the chosen $u \in D(\mathcal{A}^*)$, there is a $w = \mathcal{A}^* u \in X$ such that $h(u,\psi) = (w,\psi)_X$, for all $\psi \in (H^1(0,1))^2$, which concludes the inclusion $D(\mathcal{A}^*) \subseteq D(\widetilde{\mathcal{A}})$.

- Conversely, let $u \in D(\widetilde{\mathcal{A}})$. By definition, there exists some $w \in X$ such that $h(u,\psi) = (w,\psi)_X$ with $\widetilde{\mathcal{A}} u = w$, for all $\psi \in (H^1(0,1))^2$, and accordingly

$$\int_0^1 \Big(u_1'(x)\overline{\psi_1'(x)} + u_2'(x)\overline{\psi_2'(x)}\Big)\,\mathrm{d}x + \int_0^1 \Big(\alpha_1 u_1(x)\overline{\psi_1(x)} + \alpha_2 u_2(x)\overline{\psi_2(x)}\Big)\,\mathrm{d}x$$
$$-\beta u_1(0)\overline{\psi_2(0)} - \beta u_2(1)\overline{\psi_1(1)} = \int_0^1 \Big(w_1(x)\overline{\psi_1(x)} + w_2(x)\overline{\psi_2(x)}\Big)\,\mathrm{d}x.$$

In fact, since $w_i \in L^2(0,1)$ $(i = 1,2)$, we have $u_1', u_2' \in H^1(0,1)$ and thus an integration by parts yields

$$\int_0^1 \Big(-u_1''(x) + \alpha_1 u_1(x)\Big)\overline{\psi_1(x)}\,\mathrm{d}x + \int_0^1 \Big(-u_2''(x) + \alpha_2 u_2(x)\Big)\overline{\psi_2(x)}\,\mathrm{d}x$$
$$-\beta u_1(0)\overline{\psi_2(0)} - \beta u_2(1)\overline{\psi_1(1)} + u_1'(1)\overline{\psi_1(1)} - u_1'(0)\overline{\psi_1(0)} + u_2'(1)\overline{\psi_2(1)} - u_2'(0)\overline{\psi_2(0)}$$
$$= \int_0^1 \Big(w_1(x)\overline{\psi_1(x)} + w_2(x)\overline{\psi_2(x)}\Big)\,\mathrm{d}x,$$

for all $\psi \in (H^1(0,1))^2$.
In particular, by considering any $\psi \in (H_0^1(0,1))^2$, we conclude that

$$w_1(x) = -u_1''(x) + \alpha_1 u_1(x), \quad \forall x \in (0,1),$$
$$w_2(x) = -u_2''(x) + \alpha_2 u_2(x), \quad \forall x \in (0,1).$$

Once we have the above information, then choosing any $\psi \in (H^1(0,1))^2$ eventually gives us

$$u_1'(0) = 0, \quad u_2'(0) + \beta u_1(0) = 0,$$
$$u_2'(1) = 0, \quad u_1'(1) - \beta u_2(1) = 0.$$

which are exactly the boundary conditions of an element of $D(\mathcal{A}^*)$ and thus $D(\widetilde{\mathcal{A}}) \subseteq D(\mathcal{A}^*)$. The proof is finished. ∎

2.2. A homogeneous adjoint system backward in time. The adjoint problem (backward in time) of our control systems (1.1)–(1.7) with (1.8) or (1.9) is

$$\begin{cases} -\partial_t q_1 - \partial_x^2 q_1 + \alpha_1 q_1 = \widetilde{w}_1 & \text{in } (0,T)\times(0,1), \\ -\partial_t q_2 - \partial_x^2 q_2 + \alpha_2 q_2 = \widetilde{w}_2 & \text{in } (0,T)\times(0,1), \\ \partial_x q_1(t,0) = 0 & \text{in } (0,T), \\ \partial_x q_2(t,0) + \beta q_1(t,0) = 0 & \text{in } (0,T), \\ \partial_x q_1(t,1) - \beta q_2(t,1) = 0 & \text{in } (0,T), \\ \partial_x q_2(t,1) = 0 & \text{in } (0,T), \\ q_1(T,\cdot) = \zeta_1(\cdot) & \text{in } (0,1), \\ q_2(T,\cdot) = \zeta_2(\cdot) & \text{in } (0,1), \end{cases} \tag{2.2}$$

where $\widetilde{w} := (\widetilde{w}_1, \widetilde{w}_2) \in L^2(0,T;X)$ is some given right hand side.

Proposition 2.3. *For any given $\zeta \in X$ and $\widetilde{w} \in L^2(0,T;X)$, there exists a unique weak solution $q := (q_1,q_2) \in \mathcal{C}^0([0,T];X) \cap L^2(0,T;(H^1(0,1))^2)$ to (2.2) that satisfies the following energy estimate*

$$\|q\|_{\mathcal{C}^0([0,T];X)} + \|q\|_{L^2(0,T;(H^1(0,1))^2)} + \|\partial_t q\|_{L^2(0,T;(H^{-1}(0,1))^2)} \leq C e^{CT}\big(\|\zeta\|_X + \|\widetilde{w}\|_{L^2(0,T;X)}\big),$$

where the constant $C > 0$ does not depend on $T > 0$.

Proof. We just give a short sketch of the proof. Start with $\zeta \in D(\mathcal{A}^*)$ and $\widetilde{w} \in \mathcal{C}^1([0,T];X)$ (which indeed gives the existence of a strong solution $q \in \mathcal{C}^1([0,T];X) \cap \mathcal{C}^0([0,T];D(\mathcal{A}^*))$ to (2.2)) to prove the estimate and then the usual density argument gives the required result.

Let us test the first and second equation of (2.2) by q_1 and q_2 respectively, which provides

$$-\frac{1}{2}\frac{d}{dt}\|q(t)\|_X^2 + \|\partial_x q(t)\|_X^2 + \alpha_1\|q_1(t)\|_{L^2(0,1)}^2 + \alpha_2\|q_2(t)\|_{L^2(0,1)}^2 + \beta q_1(t,0)q_2(t,0) - \beta q_1(t,1)q_2(t,1) \leq \frac{1}{2}\left(\|\widetilde{w}(t)\|_X^2 + \|q(t)\|_X^2\right). \tag{2.3}$$

Using the following trace inequality

$$|\phi(\tau)| \leq \|\phi\|_{L^2(0,1)}^{\frac{1}{2}}\|\phi\|_{H^1(0,1)}^{\frac{1}{2}}, \quad \text{for } \tau \in \{0,1\}, \quad \forall \phi \in H^1(0,1), \tag{2.4}$$

we get from (2.3) that

$$-\frac{1}{2}\frac{d}{dt}\|q(t)\|_X^2 + \|q(t)\|_{(H^1(0,1))^2}^2 \leq C\epsilon\|q(t)\|_{(H^1(0,1))^2}^2 + \frac{C}{\epsilon}\|q(t)\|_X^2 + \|\widetilde{w}(t)\|_X^2, \tag{2.5}$$

for some constant $C > 0$ that depends on the parameters α_1, α_2 and β.

Choosing small enough $\epsilon > 0$ fix and by using the Gronwall's lemma we deduce that

$$\|q\|_{\mathcal{C}^0([0,T];X)}^2 \leq Ce^{CT}\left(\|\zeta\|_X^2 + \|\widetilde{w}\|_{L^2(0,T;X)}^2\right). \tag{2.6}$$

The other estimates can be proved in a standard fashion, more details can be found in [24, Chapter 4] (see also [13, Chapter 7]). ∎

2.3. A non homogeneous forward system. Let us consider the following non homogeneous system

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = F_1 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = F_2 & \text{in } (0,T)\times(0,1), \\ \partial_x y_1(t,0) + \beta y_2(t,0) = G_1(t) & \text{in } (0,T), \\ \partial_x y_2(t,0) = G_2(t) & \text{in } (0,T), \\ \partial_x y_1(t,1) = G_3(t) & \text{in } (0,T), \\ \partial_x y_2(t,1) - \beta y_1(t,1) = G_4(t) & \text{in } (0,T), \\ y_1(0,\cdot) = y_{0,1}(\cdot) & \text{in } (0,1), \\ y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1), \end{cases} \tag{2.7}$$

where $F_1, F_2 \in L^2(0,T;L^2(0,1))$ and $G_j \in L^2(0,T;\mathbb{R})$ for $j = 1,2,3,4$.

Recall that the adjoint operator $(-\mathcal{A}^*, D(\mathcal{A}^*))$ given by (1.10a)–(1.11), defines an analytic semigroup in X, thanks to Proposition 2.1. Therefore, the existence of a unique weak solution to (2.7) can be shown using the method of transposition; see [10, 31]. More precisely, we test the PDEs in (2.7) against the solution q to the adjoint system (2.2) with $\widetilde{w}_1 = \widetilde{w}_2 = 0$. We express this below.

Theorem 2.4. *For any given $y_0 := (y_{0,1}, y_{0,2}) \in X$, $F := (F_1, F_2) \in L^2(0,T;X)$ and $G := (G_1, G_2, G_3, G_4) \in L^2(0,T;\mathbb{R}^4)$, there exists unique weak solution y to (2.7) belonging to the space $\mathcal{C}^0([0,T];X) \cap L^2(0,T;(H^1(0,1))^2)$ in the following sense: for any $t \in [0,T]$ and $\zeta := (\zeta_1, \zeta_2) \in X$, we have*

$$\left(y(t),\zeta\right)_X = (y_0, e^{-t\mathcal{A}^*}\zeta)_X + \int_0^t \left(F(s), e^{-(t-s)\mathcal{A}^*}\zeta\right)_X \mathrm{d}s - \int_0^t \left\langle \begin{pmatrix} G_1(s) \\ G_2(s) \end{pmatrix}, \left(e^{-(t-s)\mathcal{A}^*}\zeta\right)\Big|_{\{x=0\}} \right\rangle_{\mathbb{R}^2} \mathrm{d}s + \int_0^t \left\langle \begin{pmatrix} G_3(s) \\ G_4(s) \end{pmatrix}, \left(e^{-(t-s)\mathcal{A}^*}\zeta\right)\Big|_{\{x=1\}} \right\rangle_{\mathbb{R}^2} \mathrm{d}s.$$

Moreover, we have

$$\|y\|_{\mathcal{C}^0([0,T];X)} + \|y\|_{L^2(0,T;(H^1(0,1))^2)} + \|\partial_t y\|_{L^2(0,T;(H^{-1}(0,1))^2)} \leq Ce^{CT}\left(\|y_0\|_X + \|F\|_{L^2(0,T;X)} + \|G\|_{L^2(0,T;\mathbb{R}^4)}\right).$$

2.4. Formulation of the control problems. First, observe that the existence of a unique weak solutions to our control systems (1.1)–(1.7) with (1.8) or (1.9) is clear from the Theorem 2.4. Now, let us prescribe the formulation of the control problems in both cases.

Proposition 2.5. *Let any $y_0 \in X$, $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $T > 0$ be given, and assume that $\beta = f'(0) \neq 0$.*

1. *A function $v \in L^2(0,T)$ is a null-control for the system (1.1) with the boundary conditions (1.7)–(1.8) if and only if it satisfies: for any $\zeta \in X$,*

$$\left(y_0, e^{-T\mathcal{A}^*}\zeta\right)_X = \int_0^T v(t) \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \left(e^{-(T-t)\mathcal{A}^*}\zeta\right)(x)\big|_{x=0} \right\rangle_{\mathbb{R}^2} dt. \tag{2.8}$$

2. *A function $v \in L^2(0,T)$ is a null-control for the system (1.1) with the boundary conditions (1.7)–(1.9) if and only if it satisfies: for any $\zeta \in X$,*

$$\left(y_0, e^{-T\mathcal{A}^*}\zeta\right)_X = \int_0^T v(t) \left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \left(e^{-(T-t)\mathcal{A}^*}\zeta\right)(x)\big|_{x=0} \right\rangle_{\mathbb{R}^2} dt. \tag{2.9}$$

We hereby introduce the observation operators $\mathcal{B}_1^*$ and $\mathcal{B}_2^*$ associated with the control problems (2.8) and (2.9) respectively as follows,

$$\mathcal{B}_1^* = \mathbb{1}_{\{x=0\}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} : (H^1(0,1))^2 \to \mathbb{R}, \tag{2.10a}$$

$$\mathcal{B}_2^* = \mathbb{1}_{\{x=0\}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} : (H^1(0,1))^2 \to \mathbb{R}. \tag{2.10b}$$

3. Spectral analysis of the associated adjoint operator. Let us write the eigenvalue problem $\mathcal{A}^* u = \lambda u$, for $\lambda \in \mathbb{C}$.

$$\begin{cases} -u_1'' + \alpha_1 u_1 = \lambda u_1 & \text{in } (0,1), \\ -u_2'' + \alpha_2 u_2 = \lambda u_2 & \text{in } (0,1), \\ u_1'(0) = 0, \quad u_2'(0) + \beta u_1(0) = 0, \\ u_1'(1) - \beta u_2(1) = 0, \quad u_2'(1) = 0. \end{cases} \tag{3.1}$$

with the parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\beta \neq 0$.

(a) $\mathcal{A}^*$ has compact resolvent. Let us denote the spectrum of $\mathcal{A}^*$ by $\sigma(\mathcal{A}^*)$ and consider the operator $\mathcal{A}^* - \xi \,\mathrm{I_d}$ for any $\xi \in \mathbb{C} \setminus \sigma(\mathcal{A}^*)$. Our claim is to show that $R_\xi := (\mathcal{A}^* - \xi \,\mathrm{I_d})^{-1} : X \to D(\mathcal{A}^*)$ exists and it is compact for any $\xi \in \mathbb{C} \setminus \sigma(\mathcal{A}^*)$.

The sesquilinear form associated with the operator $\mathcal{A}^* - \xi \,\mathrm{I_d}$ for any $\xi \in \mathbb{C} \setminus \sigma(\mathcal{A}^*)$ is

$$\begin{aligned} h(u,\phi) := \int_0^1 \left(u_1'(x)\overline{\phi_1'(x)} + u_2'(x)\overline{\phi_2'(x)}\right)dx + (\alpha_1 - \xi)\int_0^1 u_1(x)\overline{\phi_1(x)}dx \\ + (\alpha_2 - \xi)\int_0^1 u_2(x)\overline{\phi_2(x)}dx - \beta u_1(0)\overline{\phi_2(0)} - \beta u_2(1)\overline{\phi_1(1)}, \end{aligned} \tag{3.2}$$

for all $\phi := (\phi_1, \phi_2) \in (H^1(0,1))^2$.

One can show that h is continuous and elliptic. In particular, there exists constants $\gamma_1, \gamma_2 > 0$ such that we have

$$\Re\left(h(u,u)\right) \geq \gamma_1 \|u\|^2_{(H^1(0,1))^2} + (-\Re(\xi) - \gamma_2)\|u\|^2_X. \tag{3.3}$$

From (3.3), it follows that for $\xi \in \mathbb{C}$ with $\Re(\xi) \leq -\gamma_2 - 1$, the operator $(\mathcal{A}^* - \xi \,\mathrm{I_d})$ is invertible. In particular,

$$(-\infty, -\gamma_2 - 1] \subset \rho(\mathcal{A}^*), \tag{3.4}$$

where $\rho(\mathcal{A}^*)$ denotes the set of resolvents of $\mathcal{A}^*$.

Finally, we have that $D(\mathcal{A}^*) \hookrightarrow (H^1(0,1))^2 \hookrightarrow X$ with compact embedding and so R_ξ is a compact operator in X for any $\xi \in \mathbb{C}$ with $\Re(\xi) \leq -\gamma_2 - 1$ and consequently for any $\xi \in \rho(\mathcal{A}^*)$.

The above analysis confirms that the spectrum $\sigma(\mathcal{A}^*)$ is discrete and contains only the eigenvalues of the operator $\mathcal{A}^*$.

(b) All the eigenvalues have geometric multiplicity 1. Assume that for some $\lambda \in \sigma(\mathcal{A}^*)$, there exists two linearly independent solutions (ϕ_1, ϕ_2) and (ψ_1, ψ_2) of the eigenvalue problem (3.1). We consider

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} C_1\phi_1 + C_2\psi_1 \\ C_1\phi_1 + C_2\psi_2 \end{pmatrix},$$

for some choices of C_1, $C_2 \in \mathbb{R}^*$ such that $u_1(0) = C_1\phi_1(0) + C_2\psi_1(0) = 0$ (recall that $\phi_1(0) \neq 0$ and $\psi_1(0) \neq 0$). But we also have $u_1'(0) = 0$ from the boundary conditions in (3.1) which leads that $u_1 \equiv 0$ in $[0, 1]$.

Once we have this, the second component $u_2 \equiv 0$ follows immediately from the set of equations (3.1).

Hence, (ϕ_1, ϕ_2) and (ψ_1, ψ_2) cannot be linearly independent, which proves that the geometric multiplicity of any eigenvalue is 1.

(c) The set of eigenvalues and eigenfunctions. Let us find the set of eigenvalues of the operator $\mathcal{A}^*$. Hereinafter, we assume that $\alpha_1 \leq \alpha_2$ without loss of generality.

One can first observe that $\lambda = \alpha_1$ or α_2 cannot be an eigenvalue for $\mathcal{A}^*$. Indeed, by putting $\lambda = \alpha_1$ or α_2 in the set of equations (3.1) and then using the boundary conditions, one can show that $u_1 = u_2 = 0$.

Let us now take $\mu \in \mathbb{C}$ such that $\mu^2 = \lambda - \alpha_1$ and so $\lambda - \alpha_2 = \mu^2 - \theta$ with $\theta := \alpha_2 - \alpha_1 \geq 0$. Then, the equations of u_1 and u_2 can be rewritten as

$$\begin{aligned} u_1''(x) + \mu^2 u_1(x) &= 0, \qquad x \in (0,1), \\ u_2''(x) + (\mu^2 - \theta) u_2(x) &= 0, \quad x \in (0,1), \end{aligned}$$

along with the boundary conditions as given in (3.1). We look for the solutions of the forms

$$u_1(x) = K_{1,1}e^{i\mu x} + K_{1,2}e^{-i\mu x}, \qquad x \in [0,1], \tag{3.5}$$

$$u_2(x) = K_{2,1}e^{i\sqrt{\mu^2-\theta}\,x} + K_{2,2}e^{-i\sqrt{\mu^2-\theta}\,x}, \quad x \in [0,1], \tag{3.6}$$

for some constants $K_{i,j} \in \mathbb{C}$, $1 \leq i, j \leq 2$.

Using the boundary conditions at $x = 0$, we have

$$K_{1,1} = K_{1,2}, \quad K_{2,2} = K_{2,1} + \frac{2\beta}{i\sqrt{\mu^2-\theta}} K_{1,1}. \tag{3.7}$$

From the boundary condition $u_1'(1) - \beta u_2(1) = 0$, we get (using also (3.7))

$$K_{1,1}\left(i\mu\big(e^{i\mu} - e^{-i\mu}\big) - \frac{2\beta^2}{i\sqrt{\mu^2-\theta}} e^{-i\sqrt{\mu^2-\theta}} \right) - \beta K_{2,1}\big(e^{i\sqrt{\mu^2-\theta}} + e^{-i\sqrt{\mu^2-\theta}}\big) = 0, \tag{3.8}$$

and from the condition $u_2'(1) = 0$, we get

$$-2\beta K_{1,1} e^{-i\sqrt{\mu^2-\theta}} + iK_{2,1}\sqrt{\mu^2-\theta}\big(e^{i\sqrt{\mu^2-\theta}} - e^{-i\sqrt{\mu^2-\theta}}\big) = 0. \tag{3.9}$$

Combining the two equations (3.8)–(3.9), we have the following system of equations,

$$\begin{pmatrix} i\mu\big(e^{i\mu} - e^{-i\mu}\big) - \frac{2\beta^2}{i\sqrt{\mu^2-\theta}} e^{-i\sqrt{\mu^2-\theta}} & -\beta\big(e^{i\sqrt{\mu^2-\theta}} + e^{-i\sqrt{\mu^2-\theta}}\big) \\ -2\beta e^{-i\sqrt{\mu^2-\theta}} & i\sqrt{\mu^2-\theta}\big(e^{i\sqrt{\mu^2-\theta}} - e^{-i\sqrt{\mu^2-\theta}}\big) \end{pmatrix} \begin{pmatrix} K_{1,1} \\ K_{2,1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.10}$$

We look for the non-trivial $(K_{1,1}, K_{2,1})$ which is possible only when the determinant of the coefficient matrix is zero, which yields the following transcendental equation

$$\frac{1}{\beta^2}\mu\sqrt{\mu^2-\theta}\,\sin(\mu)\sin\left(\sqrt{\mu^2-\theta}\right) - 1 = 0, \quad \mu \in \mathbb{C}, \tag{3.11}$$

where (by definition) $\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$ for any $z \in \mathbb{C}$.

To obtain the eigenvalues of $\mathcal{A}^*$, we need to find all μ that satisfy the equation (3.11). Let us find the set of solutions to (3.1).

In (3.5), we set $K_{1,1} = K_{1,2} = 1/2$ and then from (3.9) and (3.7), we respectively get

$$K_{2,1} = \frac{\beta e^{-i\sqrt{\mu^2-\theta}}}{i\sqrt{\mu^2-\theta}\big(e^{i\sqrt{\mu^2-\theta}} - e^{-i\sqrt{\mu^2-\theta}}\big)}, \quad K_{2,2} = \frac{\beta e^{i\sqrt{\mu^2-\theta}}}{i\sqrt{\mu^2-\theta}\big(e^{i\sqrt{\mu^2-\theta}} - e^{-i\sqrt{\mu^2-\theta}}\big)}.$$

Using the above values of $K_{i,j}$ for $1 \leq i,j \leq 2$ in (3.5)–(3.6), we obtain the solution to (3.1), denoted by Φ_λ as follows

$$\Phi_\lambda(x) := \begin{pmatrix} \cos(\mu x) \\ -\frac{\beta}{\sqrt{\mu^2-\theta}\,\sin\left(\sqrt{\mu^2-\theta}\right)} \cos\left(\sqrt{\mu^2-\theta}(1-x)\right) \end{pmatrix}, \quad x \in [0,1], \tag{3.12}$$

associated with $\lambda = \mu^2 + \alpha_1$, where $\theta = \alpha_2 - \alpha_1$ and $\mu \in \mathbb{C}$ satisfies the equation (3.11). The second component of the function Φ_λ is well-defined since $\mu^2 - \theta = \lambda - \alpha_2 \neq 0$ as $\lambda = \alpha_2$ cannot be an eigenvalue for $\mathcal{A}^*$ (mentioned earlier), also $\sin\left(\sqrt{\mu^2-\theta}\right)$ cannot vanish since μ satisfies (3.11).

Characteristics of the eigenvalues. We begin with the following lemma which ensures the existence of countable number of real eigenvalues of $\mathcal{A}^*$.

Lemma 3.1 (Existence of real eigenvalues). *There is a non-negative integer $k_0 := k_0(\alpha_1, \alpha_2, \beta)$, such that for all $k \geq k_0$, there exist exactly two roots $\mu_{k,1}$ and $\mu_{k,2}$ of the equation (3.11) in $(k\pi, (k+1)\pi)$ where*

$$k\pi < \mu_{k,1} < (k+1/2)\pi, \quad (k+1/2)\pi < \mu_{k,2} < (k+1)\pi.$$

Proof. Let us denote

$$h(\mu) := \frac{1}{\beta^2}\mu\sqrt{\mu^2-\theta}\,\sin(\mu)\sin\left(\sqrt{\mu^2-\theta}\right) - 1, \quad \text{for } \mu \in \mathbb{C}. \tag{3.13}$$

- *Existence.* Observe that

$$h(k\pi) = -1 < 0, \quad h((k+1)\pi) = -1 < 0. \tag{3.14}$$

Also, we compute for large k,

$$\begin{aligned} h\big((k+\frac{1}{2})\pi\big) &= \frac{1}{\beta^2}(k+\frac{1}{2})\pi\sqrt{(k+\frac{1}{2})^2\pi^2-\theta}\,(-1)^k \sin\left(\sqrt{(k+\frac{1}{2})^2\pi^2-\theta}\right) \\ &= \frac{1}{\beta^2}(k+\frac{1}{2})\pi\sqrt{(k+\frac{1}{2})^2\pi^2-\theta}\,\cos\big(O(1/k)\big), \end{aligned} \tag{3.15}$$

thanks to the fact that

$$\sqrt{(k+\frac{1}{2})^2\pi^2-\theta} = (k+\frac{1}{2})\pi\left[1 - \frac{\theta}{2(k+\frac{1}{2})^2\pi^2} + O(1/k^4)\right].$$

Thus, from the expression (3.15), it is easy to observe that there exists some $k_0 \in \mathbb{N}$, depending on $\alpha_1, \alpha_2, \beta$ such that

$$h\big((k+\frac{1}{2})\pi\big) > 0, \quad \forall k \geq k_0. \tag{3.16}$$

Therefore, the properties (3.14)–(3.16) ensures that there exists at least one root of h in $(k\pi, (k+1/2)\pi)$ and one in $((k+1/2)\pi, (k+1)\pi)$ for any $k \geq k_0$.

- *Uniqueness.* Let us choose

$$h_1(\mu) = \frac{1}{\beta^2}\mu\sqrt{\mu^2-\theta}\,\sin(\mu)\sin\left(\sqrt{\mu^2-\theta}\right).$$

The roots of $h_1(\mu)=0$ are $k\pi$ and $\sqrt{k^2\pi^2+\theta}$, for all $k\geq 0$. In fact, there exists some $k_\theta\in\mathbb{N}$ such that for all $k\geq k_\theta$, one has $\sqrt{k^2\pi^2+\theta}\in(k\pi,(k+1)\pi)$.
Now, consider the rectangle

$$R_k:=\left\{z=x+iy\in\mathbb{C}\mid k\pi-\pi/2\leq x\leq(k+1)\pi+\pi/2,\quad -\pi/2\leq y\leq\pi/2\right\}\subset\mathbb{C}. \tag{3.17}$$

There exists some non-negative integer $\widehat{k}\geq k_\theta$ ($\widehat{k}$ depends on the parameters α_1,α_2,β), such that

$$|h_1(z)|=\frac{1}{\beta^2}\,|z|\,|\sin(z)|\left|\sqrt{z^2-\theta}\right|\left|\sin(\sqrt{z^2-\theta})\right|>1,\quad \text{on }\ \partial R_k,\quad \forall k\geq\widehat{k}.$$

As a consequence, we have

$$|h(z)-h_1(z)|=1<|h_1(z)|,\quad \text{on }\ \partial R_k,\quad \forall k\geq\widehat{k}. \tag{3.18}$$

Thus, by Rouche's theorem, the number of roots of h coincides with the number of roots of h_1 inside the rectangle R_k, and that is exactly 2. But we have already shown that for each $k\geq k_0$ (k_0 is defined in Lemma 3.1), that there exist at least two real solutions in $(k\pi,(k+1)\pi)$ of the equation $h(\mu)=0$. Hence, from the previous argument, the real roots of h are the only roots of it inside each rectangle R_k for all $k\geq k_0$, where k_0 can be chosen in such a way that $k_0\geq\widehat{k}$.
We denote those roots by $\mu_{k,1}$ and $\mu_{k,2}$ that satisfy

$$\mu_{k,1}\in(k\pi,(k+1/2)\pi),\quad \mu_{k,2}\in((k+1/2)\pi,(k+1)\pi),\quad \forall k\geq k_0.$$

To complete the analysis, we consider the following vertical strips,

$$\mathcal{V}_k:=\left\{z\in\mathbb{C}\mid k\pi-\frac{\pi}{2}\leq\Re(z)\leq(k+1)\pi+\frac{\pi}{2}\right\},\quad \forall k\geq k_0,$$

and the only roots of h are $\mu_{k,1},\mu_{k,2}\in\mathcal{V}_k$, since any point $\mu\in\mathcal{V}_k\setminus R_k$ satisfies $|h(\mu)|>0$.
The proof is finished. ∎

Let us show that $\mu_{k,1}$ and $\mu_{k,2}$ satisfy some asymptotic properties for large $k\in\mathbb{N}$.

Lemma 3.2 (Asymptotic expressions). *Let $\mu_{k,1}$ and $\mu_{k,2}$ be as given by Lemma 3.1. Then, we have the following asymptotic formulas,*

$$\mu_{k,1}=k\pi+\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}}{k\pi}+O\left(\frac{1}{k^3}\right),\quad \textit{for large } k\geq k_0, \tag{3.19}$$

$$\mu_{k,2}=(k+1)\pi-\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}}{(k+1)\pi}+O\left(\frac{1}{k^3}\right),\quad \textit{for large } k\geq k_0, \tag{3.20}$$

where $\theta=\alpha_2-\alpha_1>0$.

Proof. Recall that $\mu_{k,1}\in(k\pi,(k+1/2)\pi)$ for all $k\geq k_0$ and therefore $\mu_{k,1}=k\pi+\delta_k$ for some $\delta_k\in(0,\pi/2)$. Then, the equation $h(\mu_{k,1})=0$ implies

$$\frac{1}{\beta^2}(k\pi+\delta_k)\sqrt{(k\pi+\delta_k)^2-\theta}\,\sin(k\pi+\delta_k)\sin\left(\sqrt{(k\pi+\delta_k)^2-\theta}\right)-1=0,\quad \forall k\geq k_0. \tag{3.21}$$

Now, we compute that

$$\begin{aligned}\sqrt{(k\pi+\delta_k)^2-\theta}&=(k\pi+\delta_k)\left[1-\frac{\theta}{2(k\pi+\delta_k)^2}+O\left(\frac{1}{k^4}\right)\right]\\&=k\pi+\delta_k-\frac{\theta}{2(k\pi+\delta_k)}+O\left(\frac{1}{k^3}\right),\end{aligned}$$

using what, we get from (3.21),

$$\begin{aligned}\beta^2 &= (k\pi+\delta_k)^2\left[1-\frac{\theta}{2(k\pi+\delta_k)^2}+O\left(\frac{1}{k^4}\right)\right](\sin\delta_k)\sin\left(\delta_k-\frac{\theta}{2(k\pi+\delta_k)}+O\left(\frac{1}{k^3}\right)\right)\\ &= (k\pi+\delta_k)^2\left[1-\frac{\theta}{2(k\pi+\delta_k)^2}+O\left(\frac{1}{k^4}\right)\right](\delta_k+O(\delta_k^3))\left(\delta_k-\frac{\theta}{2(k\pi+\delta_k)}+O\left(\frac{1}{k^3}\right)+O(\delta_k^3)\right).\end{aligned}$$

Thus, we have

$$(k\pi)^2\delta_k\left(\delta_k-\frac{\theta}{2k\pi}\right)\sim\beta^2,\quad\text{for large } k.$$

Let $\gamma_k := k\pi\delta_k$ to simplify the computations. Since

$$\gamma_k^2-\frac{\theta}{2}\gamma_k\sim\beta^2,\quad\text{for large } k,$$

we have $(\gamma_k)_{k\geq K}$ for K large, is a bounded sequence. The limits of every possible subsequences $\gamma_{\varphi(k)}\to\gamma$ satisfies: $\gamma\geq 0$ and $\gamma^2-\frac{\theta}{2}\gamma=\beta^2$. Thus we necessarily have $\gamma=\frac{\theta}{4}+\sqrt{\beta^2+\frac{\theta^2}{16}}$, which gives

$$\delta_k\sim\frac{\frac{\theta}{4}+\sqrt{\beta^2+\frac{\theta^2}{16}}}{k\pi},\quad\text{for large } k.$$

In the next step, by expressing $\delta_k=\dfrac{\frac{\theta}{4}+\sqrt{\beta^2+\frac{\theta^2}{16}}}{k\pi}+\widetilde{\delta}_k$, one can find that

$$\widetilde{\delta}_k\sim\frac{C}{k^3},\quad\text{for large enough } k,$$

for some $C>0$ depending on α_1,α_2 and β. This concludes the proof for (3.19).

Similarly, by expressing $\mu_{k,2}=(k+1)\pi-\widehat{\delta}_k$ for some $\widehat{\delta}_k\in(0,\pi/2)$ and proceeding in a same way as before, one can obtain

$$\widehat{\delta}_k=\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}}{(k+1)\pi}+O\left(\frac{1}{k^3}\right),\quad\text{for large enough } k,$$

which gives the second asymptotic formula (3.20). ∎

Corollary 3.3 (Properties of real eigenvalues). *Let k_0 be as given in Lemma 3.1. For each $k\geq k_0$, there exists exactly two eigenvalues, denoted by $\lambda_{k,1}$ and $\lambda_{k,2}$ of $\mathcal{A}^*$ in the interval $\left(k^2\pi^2+\alpha_1,(k+1)^2\pi^2+\alpha_1\right)$ such that*

$$k^2\pi^2+\alpha_1<\lambda_{k,1}<(k+1/2)^2\pi^2+\alpha_1,$$

$$(k+1/2)^2\pi^2+\alpha_1<\lambda_{k,2}<(k+1)^2\pi^2+\alpha_1,$$

and moreover, they enjoy the following asymptotic formulas,

$$\lambda_{k,1}=k^2\pi^2+2\sqrt{\beta^2+\frac{(\alpha_2-\alpha_1)^2}{16}}+\frac{(\alpha_1+\alpha_2)}{2}+O\left(\frac{1}{k^2}\right),\quad\textit{for large } k\geq k_0,\tag{3.22}$$

$$\lambda_{k,2}=(k+1)^2\pi^2-2\sqrt{\beta^2+\frac{(\alpha_2-\alpha_1)^2}{16}}+\frac{(\alpha_1+\alpha_2)}{2}+O\left(\frac{1}{k^2}\right),\quad\textit{for large } k\geq k_0.\tag{3.23}$$

Proof. By our construction, we recall that any eigenvalue of $\mathcal{A}^*$ is of the form $\lambda=\mu^2+\alpha_1$, where μ satisfies the equation (3.11).

- Therefore, we have

$$\lambda_{k,1} = \mu_{k,1}^2 + \alpha_1 \quad \text{and} \quad \lambda_{k,2} = \mu_{k,2}^2 + \alpha_1, \quad \forall k \geq k_0,$$

  where $\mu_{k,i}$ $(i = 1, 2)$ are given by Lemma 3.1.
- The asymptotic formulas of $\lambda_{k,1}$ and $\lambda_{k,2}$ can be deduced from the asymptotics of $\mu_{k,1}$ and $\mu_{k,2}$ obtained in Lemma 3.2.

The proof is complete. □

We finally denote the set of all such real eigenvalues of $\mathcal{A}^*$ by Λ_∞ where

$$\Lambda_\infty := \left\{\lambda_{k,1}, \lambda_{k,2}\right\}_{k\geq k_0}. \tag{3.24}$$

Conclusion on the structure of $\sigma(\mathcal{A}^*)$. Beside the set of real eigenvalues Λ_∞, we also have a finite set of eigenvalues of lower frequencies. To determine this, let us first recall that $(-\infty, -\gamma_2 - 1] \subset \rho(\mathcal{A}^*)$ (see (3.4)). Then, it is clear that there exists some $K \in \mathbb{N}^*$ with $K \geq k_0$ (k_0 is given in Lemma 3.1) and $M > 0$ large enough such that

$$\frac{1}{\beta^2}\,|\mu|\,|\sin(\mu)|\left|\sqrt{\mu^2-\theta}\right|\left|\sin\left(\sqrt{\mu^2-\theta}\right)\right| > 1, \quad \text{for } \mu \in \partial R_0,$$

where R_0 is the rectangle given by

$$R_0 := \left\{z = x + iy \in \mathbb{C} \mid -\gamma_2 - 1 \leq x \leq K\pi + \frac{\pi}{2}, \quad -M \leq y \leq M\right\} \subset \mathbb{C}.$$

Since $\mathcal{A}^*$ has compact resolvent, the above explanation ensures that there exists at most finite number of roots of the function $h(\mu)$ (given by (3.13)) inside R_0. But remember that, we have chosen $K > k_0$ while considering the rectangle R_0 and thus some of the roots of h in R_0 may coincide with some roots inside R_k for $k \geq k_0$ (R_k are defined in (3.17)).

Therefore, we collect all the roots of h inside R_0 which are not coinciding with any roots in R_k for $k \geq k_0$. More precisely, we set

$$\Lambda_0 := \left\{\lambda = \mu^2 + \alpha_1 \mid \mu \in R_0 \setminus \Big(\bigcup_{k\geq k_0}^{K} R_k\Big) \text{ satisfying } h(\mu) = 0\right\}. \tag{3.25}$$

To conclude, the spectrum of $\mathcal{A}^*$ contains a finite set of eigenvalues Λ_0 possibly with some complex entries and a countable set of real eigenvalues Λ_∞, that is to say,

$$\sigma(\mathcal{A}^*) := \Lambda_0 \cup \Lambda_\infty. \tag{3.26}$$

For each eigenvalue $\lambda \in \sigma(\mathcal{A}^*)$, the associated eigenfunction Φ_λ is given by (3.12). Using the fact that $\sqrt{\mu^2-\theta}\sin\left(\sqrt{\mu^2-\theta}\right) = \dfrac{\beta^2}{\mu\sin\mu}$ (thanks to the equation (3.11)), the eigenfunctions can be rewritten as

$$\Phi_\lambda(x) := \begin{pmatrix} \cos(\mu x) \\ -\frac{\mu\sin\mu}{\beta}\cos\left(\sqrt{\mu^2-\theta}(1-x)\right) \end{pmatrix}, \quad x \in [0,1], \tag{3.27}$$

associated with the eigenvalues $\lambda = \mu^2 + \alpha_1$.

(d) Riesz basis property of the family of eigenfunctions. Let us write the following proposition.

Proposition 3.4. *The set of eigenfunctions $\left\{\Phi_\lambda\right\}_{\lambda\in\sigma(\mathcal{A}^*)}$ forms a Riesz basis for the space X.*

To prove the above proposition, let us first find the asymptotics of the eigenfunctions for the real eigenvalues $\lambda \in \Lambda_\infty$.

Asymptotics of the eigenfunctions. Recall the asymptotics of $\mu_{k,1}$ and $\mu_{k,2}$ from (3.19) and (3.20) respectively, so that

$$\sqrt{\mu_{k,1}^2-\theta}=k\pi+\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}}{k\pi}+O\left(\frac{1}{k^3}\right),\quad \text{for large } k\geq k_0, \tag{3.28}$$

$$\sqrt{\mu_{k,2}^2-\theta}=(k+1)\pi-\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}}{(k+1)\pi}+O\left(\frac{1}{k^3}\right),\quad \text{for large } k\geq k_0, \tag{3.29}$$

From the asymptotic of $\mu_{k,1}$ given by (3.19), we obtain for large enough $k\geq k_0$, that

$$\cos\left(\mu_{k,1}x\right)\sim\cos(k\pi x)-\sin(k\pi x)\left[\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}}{k\pi}+O\left(\frac{1}{k^3}\right)\right]x, \tag{3.30}$$

and using (3.28), one has

$$\begin{aligned}\cos\left(\sqrt{\mu_{k,1}^2-\theta}\,(1-x)\right)&\sim(-1)^k\cos(k\pi x)\\&\quad+(-1)^k\sin(k\pi x)\left[\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}}{k\pi}+O\left(\frac{1}{k^3}\right)\right](1-x).\end{aligned} \tag{3.31}$$

Also, note that

$$\mu_{k,1}\sin\mu_{k,1}\sim(-1)^k\left[\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}\right]+O\Big(\frac{1}{k^2}\Big),\quad \text{for large } k\geq k_0. \tag{3.32}$$

Then, using the above three items (3.30)–(3.31)–(3.32) in (3.27) for $\lambda=\lambda_{k,1}$, we have

$$\Phi_{\lambda_{k,1}}(x)\sim\begin{pmatrix}\cos(k\pi x)-\sin(k\pi x)\,\frac{\left(\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}\right)x}{k\pi}+O\left(\frac{1}{k^3}\right)\\ -\frac{1}{\beta}\left[\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}\right]\left(\cos(k\pi x)+\sin(k\pi x)\,\frac{\left(\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}\right)(1-x)}{k\pi}+O\left(\frac{1}{k^3}\right)\right)\end{pmatrix}, \tag{3.33}$$

for large $k\geq k_0$.

Next, we find a similar asymptotic expression for $\Phi_{\lambda_{k,2}}$ when k is large enough. From the asymptotic of $\mu_{k,2}$ given by (3.20), we have, when $k\geq k_0$ is large, that

$$\cos\left(\mu_{k,2}x\right)\sim\cos((k+1)\pi x)+\sin((k+1)\pi x)\left[\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}}{k\pi}+O\left(\frac{1}{k^3}\right)\right]x, \tag{3.34}$$

and then, using (3.29), we get

$$\begin{aligned}\cos\left(\sqrt{\mu_{k,2}^2-\theta}\,(1-x)\right)&\sim(-1)^{k+1}\cos((k+1)\pi x)\\&\quad-(-1)^{k+1}\sin((k+1)\pi x)\left[\frac{\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}}{k\pi}+O\left(\frac{1}{k^3}\right)\right](1-x).\end{aligned} \tag{3.35}$$

Beside that, we observe that

$$\mu_{k,2}\sin\mu_{k,2}\sim(-1)^k\left[\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}\right]+O\Big(\frac{1}{k^2}\Big),\quad \text{for large } k\geq k_0. \tag{3.36}$$

Then, using (3.34), (3.35) and (3.36) in (3.27) for $\lambda = \lambda_{k,2}$, we get

(3.37)
$$\Phi_{\lambda_{k,2}}(x) \sim \begin{pmatrix} \cos((k+1)\pi x) + \sin((k+1)\pi x)\,\frac{\left(\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}\right)x}{(k+1)\pi} + O\left(\frac{1}{k^3}\right) \\ \frac{1}{\beta}\left[\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}\right]\left(\cos((k+1)\pi x) - \sin((k+1)\pi x)\,\frac{\left(\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}\right)(1-x)}{(k+1)\pi} + O\left(\frac{1}{k^3}\right)\right) \end{pmatrix},$$

for large $k \geq k_0$.

A known Riesz basis for X. Let us consider the family

$$\mathcal{G} := \left\{\widetilde{\Phi}_k\right\}_{k\geq 0} \cup \left\{\widetilde{\Psi}_{-1}\right\} \cup \left\{\widetilde{\Psi}_k\right\}_{k\geq 0}, \tag{3.38}$$

where

$$\widetilde{\Phi}_0 := \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \widetilde{\Phi}_k := \begin{pmatrix} \cos(k\pi x) \\ -\frac{1}{\beta}\left[\sqrt{\beta^2+\frac{\theta^2}{16}}+\frac{\theta}{4}\right]\cos(k\pi x) \end{pmatrix}, \quad \forall k \geq 1, \tag{3.39a}$$

$$\widetilde{\Psi}_{-1} := \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \widetilde{\Psi}_k := \begin{pmatrix} \cos((k+1)\pi x) \\ \frac{1}{\beta}\left[\sqrt{\beta^2+\frac{\theta^2}{16}}-\frac{\theta}{4}\right]\cos((k+1)\pi x) \end{pmatrix}, \quad \forall k \geq 0. \tag{3.39b}$$

It can be shown that the family $\mathcal{G}$ given by (3.38)–(3.39) forms a Riesz basis for X.

Below, we write a result which is borrowed from [28, Corollary 11.4, Chapter II–§11] (see also [21, Lemma 6.2]).

Lemma 3.5. *Let $\{\phi_n\}_{n\geq 0}$ be a Riesz basis in a Hilbert space H and $\{\psi_n\}_{n\geq N}$ (for $N \geq 0$) be another sequence in H such that*

$$\sum_{n\geq N} \|\phi_n - \psi_n\|_H^2 < +\infty.$$

Then, there exists an $M \geq N$ such that the family $\{\phi_n\}_{n=1}^{M} \cup \{\psi_n\}_{n\geq M+1}$ forms a Riesz basis in H.

We are now ready to prove the Riesz basis property of the set of eigenfunction of $\mathcal{A}^*$ in X.

Proof of Proposition 3.4. The proof is made of two steps.

Step 1. Recall the known Riesz basis $\mathcal{G}$ given by (3.38). Then, observe that the family of real eigenfunctions $\left\{\Phi_{\lambda_{k,1}}, \Phi_{\lambda_{k,2}}\right\}_{k\geq k_0}$ of $\mathcal{A}^*$ satisfies

$$\sum_{k\geq k_0}\left(\|\Phi_{\lambda_{k,1}} - \widetilde{\Phi}_k\|_X^2 + \|\Phi_{\lambda_{k,2}} - \widetilde{\Psi}_k\|_X^2\right) \leq C\sum_{k\geq k_0}\frac{1}{k^2} < +\infty.$$

This can be deduced from the asymptotics of the eigenfunctions $\Phi_{\lambda_{k,1}}$ and $\Phi_{\lambda_{k,2}}$ given by (3.33) and (3.37) respectively (this is the so-called *quadratically closeness* property).

Let us consider, for some $M \geq 0$, the finite subset of $\mathcal{G}$, given by

$$\widehat{\mathcal{G}} = \left\{\widetilde{\Phi}_k \mid 0 \leq k \leq M\right\} \cup \left\{\widetilde{\Psi}_{-1}\right\} \cup \left\{\widetilde{\Psi}_k \mid 0 \leq k \leq M\right\}.$$

Then, according to Lemma 3.5, there exists some $M \geq k_0$ such that the family

$$\widehat{\mathcal{G}} \cup \left\{\Phi_{\lambda_{k,1}}, \Phi_{\lambda_{k,2}}\right\}_{k\geq M+1}$$

forms a Riesz basis for X.

Step 2. The finite set given by

$$E_0 := \left\{\Phi_\lambda\right\}_{\lambda\in\sigma(\mathcal{A}^*)} \setminus \left\{\Phi_{\lambda_{k,1}}, \Phi_{\lambda_{k,2}}\right\}_{k\geq M+1},$$

is linearly independent since they are eigenfunctions of $\mathcal{A}^*$ associated with distinct eigenvalues. Therefore, the number of elements in E_0 cannot exceed $2M+3$ since $\widehat{\mathcal{G}}$ has exactly $2M+3$ number of linearly independent elements and $E_0 \subset \operatorname{Span}\widehat{\mathcal{G}}$. Our goal is to show that E_0 contains exactly $2M+3$ eigenfunctions.

Now, recall that $\mathcal{A}^*$ is a densely defined discrete operator (that is to say, $\mathcal{A}^*$ has compact resolvent) and so $\mathcal{A}$ is. Then, we define the linear manifold $\mathfrak{S}_\infty(\mathcal{A})$ by

$$\mathfrak{S}_\infty(\mathcal{A}) := \big\{\phi \mid P(\lambda)\phi = 0,\ \forall \lambda \in \sigma(\mathcal{A})\big\};$$

$P(\lambda)$ is the projection operator given by

$$P(\lambda) = -\frac{1}{2\pi i}\int_{\mathcal{C}_\lambda} (\xi I - \mathcal{A})^{-1} d\xi, \quad \forall \lambda \in \sigma(\mathcal{A}),$$

where $\mathcal{C}_\lambda$ is a closed curve containing only the eigenvalue $\lambda \in \sigma(\mathcal{A})$ and traversed once in the positive sense, $\sigma(\mathcal{A})$ is the spectrum of $\mathcal{A}$ (which is the same as $\sigma(\mathcal{A}^*)$). Since $\mathcal{A}^*$ is densely defined discrete operator, by using [12, Lemma 5, XIX.5.5] we have

$$\overline{\operatorname{Span}\big\{\Phi_\lambda \mid \lambda \in \sigma(\mathcal{A}^*)\big\}} = \mathfrak{S}_\infty(\mathcal{A})^\perp.$$

Therefore, one can write

$$X = \mathfrak{S}_\infty(\mathcal{A}) \oplus \overline{\operatorname{Span}\big\{\Phi_\lambda \mid \lambda \in \sigma(\mathcal{A}^*)\big\}}.$$

On the other hand, by [12, Lemma 5, XIX.2.5], $\mathfrak{S}_\infty(\mathcal{A})$ is either trivial or infinite dimensional. But, according to the previous step, it is clear that $\overline{\operatorname{Span}\big\{\Phi_\lambda \mid \lambda \in \sigma(\mathcal{A}^*)\big\}}$ has at most finite codimension and thus the only possibility is $\mathfrak{S}_\infty(\mathcal{A}) = \{0\}$, which yields

$$X = \overline{\operatorname{Span}\big\{\Phi_\lambda \mid \lambda \in \sigma(\mathcal{A}^*)\big\}}.$$

In other words, $\big\{\Phi_\lambda\big\}_{\lambda\in\sigma(\mathcal{A}^*)}$ spans the space X. So, the number of linearly independent elements in E_0 have to be exactly $2M+3$ and moreover, $\big\{\Phi_\lambda\big\}_{\lambda\in\sigma(\mathcal{A}^*)}$ forms a Riesz basis for X.

The proof is achieved. ∎

Remark 3.6. *To prove the Riesz basis property of the set of eigenfunctions of $\mathcal{A}^*$, we cannot directly apply the so-called Bari's theorem [6] (see also [32, Chapter 1, Theorem 15]) since, initially we did not have any precise information about the number of eigenfunctions for the lower frequencies. So, we use a slightly different approach to prove our result. A more general result is available in [21].*

Remark 3.7. *The proof of Proposition 3.4 ensures that the number of eigenvalues in the finite set Λ_0 (given by (3.25)) is exactly $2k_0+1$. If needed, one can introduce the index j and write the set*

$$\Lambda_0 = \big\{\lambda_j \mid 1 \le j \le 2k_0+1\big\}. \tag{3.40}$$

4. Boundary controllability of the linearized systems. This section is devoted to prove the boundary null-controllability of the linear systems associated to our non-linear models. We recall here that $f'(0) = \beta \neq 0$ where f is the nonlinear function appearing on the boundary conditions of our nonlinear models and is given by (1.4).

4.1. Approximate controllability. Let us first discuss about the approximate controllability of the linearized systems in the space X at any time $T > 0$. We write the following lemma.

Lemma 4.1. *Let any $y_0 \in X$ and parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ be given. Then we have the following.*

1. *The system (1.1)–(1.7)–(1.8) is approximately controllable in X at any given time $T > 0$ and the observation terms satisfy*

$$|\mathcal{B}_1^*\Phi_\lambda| = 1, \quad \forall \lambda \in \sigma(\mathcal{A}^*). \tag{4.1}$$

2. *Assume without loss of generality that $\alpha_1 \le \alpha_2$ with $\theta = \alpha_2 - \alpha_1$ and we introduce the set*

$$\mathcal{R} := \left\{ \pm \left((-1)^k \big(k+\frac{1}{2}\big)\pi \sqrt{\big(k+\frac{1}{2}\big)^2\pi^2 + \theta}\, \sin\left(\sqrt{\big(k+\frac{1}{2}\big)^2\pi^2 + \theta} \right) \right)^{\frac{1}{2}} ;\ k \ge 0 \right\} \cap \mathbb{R}^*. \tag{4.2}$$

Then, the system (1.1)–(1.7)–(1.9) *is approximately controllable in* X *at any time* $T > 0$ *if and only if* $\beta \notin \mathcal{R}$. *In that case, there exists a constant* $\gamma_0 > 0$ *such that the observation terms satisfy*

$$|\mathcal{B}_2^*\Phi_\lambda| \geq \gamma_0, \quad \forall \lambda \in \sigma(\mathcal{A}^*). \tag{4.3}$$

Proof. Recall the observation operators $\mathcal{B}_1^*$, $\mathcal{B}_2^*$ respectively defined by (2.10a), (2.10b) and the explicit expressions of the eigenfunctions Φ_λ from (3.27).

1. It is easy to see that

$$\mathcal{B}_1^*\Phi_\lambda = 1, \quad \forall \lambda \in \sigma(\mathcal{A}^*).$$

So, by using Fattorini-Hautus test (see [14], [26]), the linear control system (1.1)–(1.7)–(1.8) is approximately controllable in X at any time $T > 0$.

2. For any parameters $\alpha_1 \leq \alpha_2$ we write $\theta = \alpha_2 - \alpha_1$ as earlier. Let us compute that

$$\mathcal{B}_2^*\Phi_\lambda = -\frac{\mu \sin \mu}{\beta} \cos\left(\sqrt{\mu^2 - \theta}\right), \quad \forall \lambda \in \sigma(\mathcal{A}^*),$$

where, by construction $\lambda = \mu^2 + \alpha_1$.
If possible, assume that $\mathcal{B}_2^*\Phi_\lambda = 0$ for some eigenvalue λ. Since $\mu \sin \mu \neq 0$ (if so, then the eigenvalue equation (3.11) is invalid), this yields

$$\begin{aligned} &\cos\left(\sqrt{\mu^2 - \theta}\right) = 0, \\ \Longleftrightarrow\ &\mu^2 = \big(k + \frac{1}{2}\big)^2\pi^2 + \theta, \quad \text{for some } k \geq 0. \end{aligned}$$

But we need to show that the above μ satisfies the eigenvalue equation (3.11) also. Indeed, this is possible only when

$$\frac{1}{\beta^2}\big(k + \frac{1}{2}\big)\pi\sqrt{\big(k + \frac{1}{2}\big)^2\pi^2 - \theta}\, \sin\left(\big(k + \frac{1}{2}\big)\pi\right) \sin\left(\sqrt{\big(k + \frac{1}{2}\big)^2\pi^2 - \theta}\right) = 1.$$

The above equality tells that when

$$\beta^2 = (-1)^k\big(k + \frac{1}{2}\big)\pi\sqrt{\big(k + \frac{1}{2}\big)^2\pi^2 - \theta}\, \sin\left(\sqrt{\big(k + \frac{1}{2}\big)^2\pi^2 - \theta}\right), \tag{4.4}$$

for some $k \geq 0$, then the system is not approximately controllable in X. In fact, for any (α_1, α_2), it is clear that there exist countable number of k such that the quantity in the right hand side of (4.4) is positive. This leads to introduce the set

$$\mathcal{R} := \left\{ \pm\left((-1)^k\big(k + \frac{1}{2}\big)\pi\sqrt{\big(k + \frac{1}{2}\big)^2\pi^2 + \theta}\, \sin\left(\sqrt{\big(k + \frac{1}{2}\big)^2\pi^2 + \theta}\right)\right)^{\frac{1}{2}};\ k \geq 0 \right\} \cap \mathbb{R}^*.$$

Then, from the above analysis, it is clear that $\mathcal{B}_2^*\Phi_\lambda \neq 0$ if and only if $\beta \notin \mathcal{R}$ and so the Fattorini-Hautus criterion confirms the approximate controllability of the system (1.1)–(1.7)–(1.9).
Finally, to find the required lower bound (4.3), it is enough to consider the eigenfunctions for large frequencies. In fact, we see

$$\mathcal{B}_2^*\Phi_{\lambda_{k,j}} = -\frac{\mu_{k,j} \sin \mu_{k,j}}{\beta} \cos\left(\sqrt{\mu_{k,j}^2 - \theta}\right), \quad \text{for all } k \geq k_0,\ j = 1, 2.$$

Then, using the asymptotic information (3.28) and (3.32), we get

$$\mathcal{B}_2^*\Phi_{\lambda_{k,1}} \sim -1 + O\Big(\frac{1}{k^2}\Big), \quad \text{for large } k.$$

On the other hand, using (3.29) and (3.36), one can get

$$\mathcal{B}_2^*\Phi_{\lambda_{k,2}} \sim 1 + O\Big(\frac{1}{k^2}\Big), \quad \text{for large } k.$$

So, there exists some constant $\gamma_0 > 0$, independent in k, and some $K_1 \in \mathbb{N}^*$ depending on $\alpha_1, \alpha_2, \beta$ such that

$$|\mathcal{B}_2^*\Phi_{\lambda_{k,j}}| \geq \gamma_0, \quad \forall k \geq K_1,\ j = 1, 2.$$

But, for the choices of $\beta \notin \mathcal{R}$, we have proved that $\mathcal{B}_2^*\Phi_\lambda \neq 0$ for all $\lambda \in \sigma(\mathcal{A}^*)$ and therefore, the lower bound (4.3) follows, possibly with some smaller $\gamma_0 > 0$.

This completes the proof. ∎

4.2. Bounds on the eigenfunctions. We have the following result.

Lemma 4.2. *There exists some constant $C > 0$ depending on the parameters $\alpha_1, \alpha_2, \beta$ but independent in $\lambda \in \sigma(\mathcal{A}^*)$ such that the eigenfunctions of $\mathcal{A}^*$ satisfy the following bounds*

$$\|\Phi_\lambda\|_X \leq C, \quad \forall \lambda \in \sigma(\mathcal{A}^*). \tag{4.5}$$

Proof. Recall the formal expression of Φ_λ from (3.27), one has

$$\|\Phi_\lambda\|_X \leq C\left(1 + \frac{|\mu \sin \mu|}{|\beta|}\right),$$

for any $\lambda \in \sigma(\mathcal{A}^*)$ where $\lambda = \mu^2 + \alpha_1$.

Using the asymptotics of $\mu \sin \mu$ given by (3.32) and (3.36) for large modulus of eigenvalues, one can obtain the required bound (4.5) of the eigenfunctions. For rest of the finite eigenmodes, the bound follows immediately. ∎

4.3. The moments problem. Recall that the set of eigenfunctions $\{\Phi_\lambda\}_{\lambda \in \sigma(\mathcal{A}^*)}$ forms a Riesz basis in X. Thus, it is enough to check the control problems (2.8) and (2.9) with Φ_λ for each $\lambda \in \sigma(\mathcal{A}^*)$.

This gives us the following.

- For any $y_0 \in X$ and parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, a function $v \in L^2(0, T)$ is a boundary null-control for the system (1.1)–(1.7)–(1.8) if and only if, we have

$$\frac{\overline{e^{-T\lambda}}\big(y_0, \Phi_\lambda\big)_X}{\overline{\mathcal{B}_1^*\Phi_\lambda}} = \int_0^T v(t)\overline{e^{-(T-t)\lambda}}\,\mathrm{d}t, \quad \forall \lambda \in \sigma(\mathcal{A}^*). \tag{4.6}$$

- Similarly, for any $y_0 \in X$ and parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, if $\beta \notin \mathcal{R}$, then a function $v \in L^2(0, T)$ is a boundary null-control for the system (1.1)–(1.7)–(1.9) if and only if, we have

$$\frac{\overline{e^{-T\lambda}}\big(y_0, \Phi_\lambda\big)_X}{\overline{\mathcal{B}_2^*\Phi_\lambda}} = \int_0^T v(t)\overline{e^{-(T-t)\lambda}}\,\mathrm{d}t, \quad \forall \lambda \in \sigma(\mathcal{A}^*). \tag{4.7}$$

Here, we have used the fact that $e^{-t\mathcal{A}^*}\Phi_\lambda = e^{-t\lambda}\Phi_\lambda$, for any $\lambda \in \sigma(\mathcal{A}^*)$.

The above set of equations are the moments problems for our linear models and we shall solve these in the next paragraphs.

4.4. Existence of a bi-orthogonal family. The study of existence and estimates of biorthogonal families to exponential functions in $L^2(0, T)$ is a long story starting from the pioneering work [16] in the framework of control theory, up to very recent developments. In this paper, we will make use of [9, Theorem IV.1.10] which is similar to [7, Theorem 1.5] but with a simplified set of assumptions.

In order to use this theorem, we need to show that the set of eigenvalues $\sigma(\mathcal{A}^*)$, defined by (3.26), belongs to some sector of the complex half-plane and satisfies the uniform spectral gap property and some conditions on the counting function.

Remark 4.3. *Without loss of generality, we assume that all the eigenvalues of $\mathcal{A}^*$ have positive real parts. If not, one could choose some $\alpha_0 > 0$ such that $\lambda + \alpha_0 > 0$ for any $\lambda \in \sigma(\mathcal{A}^*)$. In that case, the factor $e^{T\alpha_0}$ would appear in the estimation of control.*

The sector condition. For any $\eta > 0$, we define the sector

$$\mathcal{S}_\eta := \{z \in \mathbb{C} \mid \Re(z) > 0, \text{ and } |\Im(z)| < (\sinh \eta)\, \Re(z)\}.$$

Recall that, our set of eigenvalues $\sigma(\mathcal{A}^*)$ has at most finite number of complex eigenvalues lying inside Λ_0. This, along with Remark 4.3, it is clear that there exists some $\eta > 0$ such that

$$\sigma(\mathcal{A}^*) \subset \mathcal{S}_\eta.$$

The gap condition. We have that Using the asymptotic expressions (3.22) and (3.23) of the real set of eigenvalues Λ_∞, one has the following: there exists some $\rho_1 > 0$ and some $k_0 \geq 1$, depending only on α_1, α_2 and β, such that

$$\begin{aligned}
|\lambda_{k+1,1} - \lambda_{k,1}| &\geq \rho_1 k, \quad \text{for } \ k \geq k_0,\\
|\lambda_{k+1,2} - \lambda_{k,2}| &\geq \rho_1 k, \quad \text{for } \ k \geq k_0,\\
|\lambda_{k,2} - \lambda_{k,1}| &\geq \rho_1 k \quad \text{for } \ k \geq k_0,\\
|\lambda_{k,2} - \lambda_{k+1,1}| &\geq \rho_1 \quad \text{for } \ k \geq k_0.
\end{aligned}$$

Note that the gap between $\lambda_{k,2}$ and $\lambda_{k+1,1}$ tends to a finite positive number as k goes to infinity but does not tend to infinity like for the other cases. This is the reason why we needed to compute the precise asymptotic expansions of the eigenvalues.

Using above and the fact that the spectrum is discrete, we can say that there is some $\rho > 0$ such that

$$|\lambda - \widetilde{\lambda}| \geq \rho, \quad \lambda, \widetilde{\lambda} \in \sigma(\mathcal{A}^*) \text{ with } \lambda \neq \widetilde{\lambda},$$

which is the uniform spectral gap property.

The condition on counting function. Let us re-denote the real set of eigenvalues Λ_∞ given by (3.24) as $\{\widehat{\lambda}_{2k+i}\}_{k\geq k_0, i=2,3}$, where

$$\widehat{\lambda}_{2k+2} = \lambda_{k,1}, \quad \widehat{\lambda}_{2k+3} = \lambda_{k,2}, \quad \forall k \geq k_0.$$

Here we start with the index $2k_0 + 2$ since the set Λ_0 of lower frequencies contains exactly $2k_0 + 1$ number of elements, see (3.40).

Let $\mathcal{N}$ be the counting function associated with the set of eigenvalues $\sigma(\mathcal{A}^*)$, defined by

$$\mathcal{N}(r) = \#\{\lambda \in \sigma(\mathcal{A}^*) \mid |\lambda| \leq r\}, \quad \forall r > 0.$$

We have that the function $\mathcal{N}$ is piecewise constant and non-decreasing in the interval $[0, +\infty)$. Also for every $r \in (0, +\infty)$, we have $\mathcal{N}(r) < +\infty$ and $\lim_{r\to+\infty} \mathcal{N}(r) = +\infty$.

Our goal is to show that there exists some $\kappa_0 > 0$ independent in the set of eigenvalues such that

$$\mathcal{N}(r) \leq \kappa_0 r^{1/2}, \quad \forall r > 0, \tag{4.8a}$$

$$|\mathcal{N}(r) - \mathcal{N}(s)| \leq \kappa_0 \left(1 + |r - s|^{1/2}\right), \quad \forall r, s > 0. \tag{4.8b}$$

It is enough to start with large r so that $\mathcal{N}(r) > 2k_0 + 1$. In such situation, we can play with the real set of eigenvalues $\{\widehat{\lambda}_{2k+i}\}_{k\geq k_0, i=2,3}$.

Assuming that $\mathcal{N}(r)$ is even, we see

$$\mathcal{N}(r) = 2k + 2 \Longleftrightarrow \widehat{\lambda}_{2k+2} = \lambda_{k,1} \leq r, \quad \text{for all } r > 0 \text{ large enough.}$$

But we have (since $\alpha_1 > 0$)

$$k^2\pi^2 < k^2\pi^2 + \alpha_1 < \lambda_{k,1},$$

which gives

$$\mathcal{N}(r) \leq \frac{2}{\pi}\sqrt{r} + 2, \quad \text{for large } r > 0.$$

This is the first required condition (4.8a). Similar property can be achieved with odd $\mathcal{N}(r)$.

Let us consider $r > s > 0$ large enough and $\mathcal{N}(r) = 2k+2$, $\mathcal{N}(s) = 2m+2$. Obviously, one has $k > m$ since $r > s$. Now, using the characteristics of real eigenvalues given by Corollary 3.3, we have

$$k^2\pi^2 + \alpha_1 < \widehat{\lambda}_{2k+2} = \lambda_{k,1} \leq r, \qquad s < \widehat{\lambda}_{2m+3} = \lambda_{m,2} < (m+1)^2\pi^2 + \alpha_1,$$

which gives

$$\begin{aligned} r - s &\geq k^2\pi^2 - (m+1)^2\pi^2, \\ \text{i.e.,}\quad k^2\pi^2 &\leq (r-s) + (m+1)^2\pi^2, \\ \text{i.e.,}\quad k\pi &\leq (r-s)^{1/2} + (m+1)\pi. \end{aligned}$$

Replacing $k = \frac{\mathcal{N}(r)-2}{2}$ and $m = \frac{\mathcal{N}(s)-2}{2}$, we get

$$\begin{aligned} \frac{\mathcal{N}(r)-2}{2} &\leq \frac{1}{\pi}(r-s)^{1/2} + \frac{\mathcal{N}(s)}{2}, \\ \text{or,}\quad \mathcal{N}(r) - \mathcal{N}(s) &\leq 2\left(1 + \frac{1}{\pi}(r-s)^{1/2}\right), \end{aligned}$$

which is the second required condition (4.8b). By a similar approach we can obtain this condition for other possibilities of $\mathcal{N}(r)$ and $\mathcal{N}(s)$.

Therefore, according to [9, Theorem IV.1.10], there exists a family $\{q_\lambda\}_{\lambda\in\sigma(\mathcal{A}^*)} \subset L^2(0,T)$ such that

$$\int_0^T q_\lambda(t)\overline{e^{-(T-t)\widetilde{\lambda}}}\mathrm{d}t = \delta_\lambda\delta_{\widetilde{\lambda}}, \quad \forall \lambda, \widetilde{\lambda} \in \sigma(\mathcal{A}^*),$$

with the following estimate

$$\|q_\lambda\|_{L^2(0,T)} \leq Ce^{CT\frac{\Re(\lambda)}{2} + C\sqrt{\Re(\lambda)} + \frac{C}{T}}, \quad \forall \lambda \in \sigma(\mathcal{A}^*), \tag{4.9}$$

where the constant $C > 0$ depends on η, ρ, κ_0 but independent in λ or T.

4.5. Existence of control. In this section, we shall solve the set of moments equations (4.6) and (4.7) to construct boundary controls for our linear systems.

Proof of Theorem 1.3.

1. *Construction of a control for the system* (1.1)–(1.7)–(1.8). Let us start with any given parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\beta \neq 0$, initial data $y_0 \in X$ and consider

$$v(t) = \sum_{\lambda\in\sigma(\mathcal{A}^*)} v_\lambda(t), \quad \forall t \in [0,T], \quad \text{with} \tag{4.10a}$$

$$v_\lambda(t) = \frac{\overline{e^{-T\lambda}}}{\overline{\mathcal{B}_1^*\Phi_\lambda}}\big(y_0, \Phi_\lambda\big)_X q_\lambda(t), \quad \forall t \in [0,T], \text{ and } \lambda \in \sigma(\mathcal{A}^*). \tag{4.10b}$$

Observe that, this choice of v is well-defined (as $\mathcal{B}_1^*\Phi_\lambda \neq 0$ for all $\lambda \in \sigma(\mathcal{A}^*)$) and formally satisfies the moments equation (4.6).

Using the bi-orthogonal estimate (4.9), lower bounds of the observation terms (4.1) and the upper bounds of the eigenfunctions (4.5), we get

$$\|v_\lambda\|_{L^2(0,T)} \leq Ce^{C/T}e^{-T\frac{\Re(\lambda)}{2} + C\sqrt{\Re(\lambda)}}\|y_0\|_X, \quad \forall \lambda \in \sigma(\mathcal{A}^*). \tag{4.11}$$

Applying Cauchy-Schwarz inequality, we get

$$C\sqrt{\Re(\lambda)} \leq \frac{T}{4}\Re(\lambda) + \frac{C^2}{T}, \quad \forall \lambda \in \sigma(\mathcal{A}^*),$$

using what in (4.11) and taking sum over $\lambda \in \sigma(\mathcal{A}^*)$, we obtain the required estimate for v,

$$\|v\|_{L^2(0,T)} = \sum_{\lambda\in\sigma(\mathcal{A}^*)} \|v_\lambda\|_{L^2(0,T)} \leq Ce^{C/T}\|y_0\|_X, \tag{4.12}$$

here we have used the properties of the eigenvalues for large modulus, given by Corollary 3.3.

2. *Construction of a control for the system* (1.1)–(1.7)–(1.9). Let any $y_0 \in X$ be given. Also, for any given parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, choose $\beta \notin \mathcal{R}$. Then, we consider

$$v(t) = \sum_{\lambda \in \Lambda} v_\lambda(t), \quad \forall t \in [0,T], \quad \text{with} \tag{4.13a}$$

$$v_\lambda(t) = \frac{\overline{e^{-T\lambda}}}{\overline{\mathcal{B}_2^* \Phi_\lambda}} \big(y_0, \Phi_\lambda\big)_X q_\lambda(t), \quad \forall t \in [0,T], \text{ and } \lambda \in \sigma(\mathcal{A}^*). \tag{4.13b}$$

This v is well-defined since $\mathcal{B}_2^* \Phi_\lambda \neq 0$ for all $\lambda \in \sigma(\mathcal{A}^*)$ as we have chosen $\beta \notin \mathcal{R}$ (see Lemma 4.1). Moreover, the above v formally solves (4.7).

Then, following the same strategy as previous, one can obtain the required estimate for the control v in (4.13a)–(4.13b).

The proof is complete. ∎

5. The nonlinear system: a source term and fixed point argument. In the previous section, we obtained that the control cost for the linear systems is $Ce^{C/T}\|y_0\|_X$. Using this we shall deal with some nonlinear systems in the current section. As indicated in Section 1.1, the boundary nonlinearities are given by

$$\begin{cases} f(\nu) = \beta\nu + g(\nu), \quad \text{with} \quad g \in \mathcal{C}^1(\mathbb{R}),\ g(0) = g'(0) = 0, \text{ satisfying} \\ |g(\nu) - g(\tilde{\nu})| \leq C|\nu - \tilde{\nu}|(|\nu| + |\tilde{\nu}|), \quad \text{for any } \nu, \tilde{\nu} \in \mathbb{R}, \quad \text{for some constant } C > 0, \\ f'(0) = \beta \neq 0 \quad \text{is some real number.} \end{cases} \tag{5.1}$$

5.1. Source term method. Let us discuss the source term method (see [25]) for our case. We assume the constants $p > 0$, $q > 1$ in such a way that

$$1 < q < \sqrt{2}, \quad \text{and} \quad p > \frac{q^2}{2 - q^2}. \tag{5.2}$$

We also redenote the constant appearing in the control estimate (4.12) of the linearized models by M, more precisely we write the control cost by $Me^{M/T}$ (just to make a difference with the generic constant C). We now define the functions

$$\begin{cases} \rho_0(t) = e^{-\frac{pM}{(q-1)(T-t)}}, \\ \rho_{\mathcal{S}}(t) = e^{-\frac{(1+p)q^2 M}{(q-1)(T-t)}}, \end{cases} \qquad \forall t \in \left[T\left(1 - \frac{1}{q^2}\right), T\right], \tag{5.3}$$

extended in $\big[0, T(1-1/q^2)\big]$ in a constant way such that the functions ρ_0 and $\rho_{\mathcal{S}}$ are continuous and non-increasing in $[0,T]$ with $\rho_0(T) = \rho_{\mathcal{S}}(T) = 0$.

Remark 5.1. *We compute that*

$$\frac{\rho_0^2(t)}{\rho_{\mathcal{S}}(t)} = e^{\frac{q^2 M + pM(q^2-2)}{(q-1)(T-t)}}, \quad \forall t \in \left[T\left(1 - \frac{1}{q^2}\right), T\right],$$

Due to the choices of p, q in (5.2), we have $M\big(q^2 + p(q^2-2)\big) < 0$, $(q-1) > 0$ and therefore we can conclude that

$$\frac{\rho_0^2(t)}{\rho_{\mathcal{S}}(t)} \leq 1, \quad \forall t \in [0,T].$$

With these functions, we define the following weighted spaces (recall that $X = (L^2(0,1))^2$),

$$\mathcal{S} := \left\{ S \in L^2(0,T) \mid \frac{S}{\rho_{\mathcal{S}}} \in L^2(0,T) \right\}, \tag{5.4a}$$

$$\mathcal{Y} := \left\{ y = (y_1, y_2) \in L^2(0,T;X) \mid \frac{y}{\rho_0} \in L^2(0,T;X) \right\}, \tag{5.4b}$$

$$\mathcal{V} := \left\{ v \in L^2(0,T) \mid \frac{v}{\rho_0} \in L^2(0,T) \right\}. \tag{5.4c}$$

We introduce the inner products in the spaces $\mathcal{S}$ and $\mathcal{V}$ respectively by

$$\left\langle S, \widetilde{S} \right\rangle_{\mathcal{S}} := \int_0^T \rho_{\mathcal{S}}^{-2} S(t)\widetilde{S}(t)\, dt \quad \text{and} \quad \left\langle v, \widetilde{v} \right\rangle_{\mathcal{V}} := \int_0^T \rho_0^{-2} v(t)\widetilde{v}(t)\, dt,$$

for any $S, \widetilde{S} \in \mathcal{S}$ and $v, \widetilde{v} \in \mathcal{V}$. The corresponding norms in those spaces are

$$\|S\|_{\mathcal{S}} := \left(\int_0^T \left| \frac{S(t)}{\rho_{\mathcal{S}}(t)} \right|^2 \mathrm{d}t \right)^{1/2}, \quad \text{and} \quad \|v\|_{\mathcal{V}} := \left(\int_0^T \left| \frac{v(t)}{\rho_0(t)} \right|^2 \mathrm{d}t \right)^{1/2}. \tag{5.5}$$

Now, let us consider the following system

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (0,T)\times(0,1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (0,T)\times(0,1), \\ y_1(0,\cdot) = y_{0,1}(\cdot) & \text{in } (0,1), \\ y_2(0,\cdot) = y_{0,2}(\cdot) & \text{in } (0,1), \end{cases} \tag{5.6}$$

along with the boundary conditions at $x = 1$ as

$$\begin{cases} \partial_x y_1(t,1) = 0 & \text{in } (0,T), \\ \partial_x y_2(t,1) - \beta y_1(t,1) = S_2(t) & \text{in } (0,T), \end{cases} \tag{5.7}$$

and at $x = 0$, we consider either

$$\begin{cases} \partial_x y_1(t,0) + \beta y_2(t,0) = v(t) + S_1(t) & \text{in } (0,T), \\ \partial_x y_2(t,0) = 0 & \text{in } (0,T), \end{cases} \tag{5.8}$$

or,

$$\begin{cases} \partial_x y_1(t,0) + \beta y_2(t,0) = S_1(t) & \text{in } (0,T), \\ \partial_x y_2(t,0) = v(t) & \text{in } (0,T), \end{cases} \tag{5.9}$$

for any given $y_0 \in X$, $S_1, S_2 \in L^2(0,T)$ and $v \in L^2(0,T)$.

Then, our goal is to prove the following result using the technique developed in [25].

Proposition 5.2. *Let any $S_1, S_2 \in L^2(0,T)$, $y_0 \in X$ and parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ be given.*

1. *Then, there exists a linear map*

$$(y_0, S_1, S_2) \in X \times L^2(0,T) \times L^2(0,T) \mapsto (y, v) \in \mathcal{Y} \times \mathcal{V},$$

such that (y, v) solves the set of equations (5.6)–(5.7)–(5.8).

2. *On the other hand, if $\beta \notin \mathcal{R}$, then there exists a linear map*

$$(y_0, S_1, S_2) \in X \times L^2(0,T) \times L^2(0,T) \mapsto (y, v) \in \mathcal{Y} \times \mathcal{V},$$

such that (y, v) solves the set of equations (5.6)–(5.7)–(5.9), where the set $\mathcal{R}$ is given by (4.2).

In both cases, we have the following estimate

$$\left\| \frac{y}{\rho_0} \right\|_{C^0(0,T;X)} + \left\| \frac{y}{\rho_0} \right\|_{L^2(0,T;(H^1(0,1))^2)} + \left\| \frac{v}{\rho_0} \right\|_{L^2(0,T)} \leq C e^{C\left(T + \frac{1}{T}\right)} \left(\|y_0\|_X + \|S_1\|_{\mathcal{S}} + \|S_2\|_{\mathcal{S}} \right), \tag{5.10}$$

where the constant $C > 0$ neither depends on y_0 nor on T.

Proof. Let us define the sequence $\{T_k\}_{k\geq 0}$ with

$$T_k := T - \frac{T}{q^k}, \quad \forall k \geq 0,$$

for given $T > 0$. With this T_k, we have the following relation between ρ_0 and $\rho_{\mathcal{S}}$

$$\rho_0(T_{k+2}) = \rho_{\mathcal{S}}(T_k) e^{\frac{M}{T_{k+2} - T_{k+1}}}, \quad \forall k \geq 0. \tag{5.11}$$

We also define a sequence $\{a_k\}_{k\geq 0}$ with

$$a_0 = y_0 \in X, \quad a_{k+1} = \widetilde{y}(T_{k+1}^-), \quad \forall k \geq 0, \tag{5.12}$$

where $\widetilde{y} := (\widetilde{y}_1, \widetilde{y}_2) \in \mathcal{C}^0([T_k, T_{k+1}]; X) \cap L^2(T_k, T_{k+1}; (H^1(0,1))^2)$ is the unique weak solution to the following system for every $k \geq 0$,

$$\begin{cases} \partial_t \widetilde{y}_1 - \partial_x^2 \widetilde{y}_1 + \alpha_1 \widetilde{y}_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \partial_t \widetilde{y}_2 - \partial_x^2 \widetilde{y}_2 + \alpha_2 \widetilde{y}_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \partial_x \widetilde{y}_1(t,0) + \beta \widetilde{y}_2(t,0) = S_1(t) & \text{in } (T_k, T_{k+1}), \\ \partial_x \widetilde{y}_2(t,0) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \widetilde{y}_1(t,1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \widetilde{y}_2(t,1) - \beta \widetilde{y}_1(t,1) = S_2(t) & \text{in } (T_k, T_{k+1}), \\ \widetilde{y}_1(T_k^+, \cdot) = 0 & \text{in } (0,1), \\ \widetilde{y}_2(T_k^+, \cdot) = 0 & \text{in } (0,1). \end{cases} \tag{5.13}$$

Moreover, using Theorem 2.4, we have

$$\|\widetilde{y}\|_{\mathcal{C}^0([T_k,T_{k+1}];X)} + \|\widetilde{y}\|_{L^2(T_k,T_{k+1};(H^1(0,1))^2)} \leq Ce^{CT} \left(\|S_1\|_{L^2(T_k,T_{k+1})} + \|S_2\|_{L^2(T_k,T_{k+1})} \right).$$

In particular,

$$\|a_{k+1}\|_X \leq Ce^{CT} \left(\|S_1\|_{L^2(T_k,T_{k+1})} + \|S_2\|_{L^2(T_k,T_{k+1})} \right), \quad \forall k \geq 0, \tag{5.14}$$

recall the definition (5.12).

Weighted estimate of the control. For every $k \geq 0$, we consider the following system (introduce $\hat{y} := (\hat{y}_1, \hat{y}_2)$)

$$\begin{cases} \partial_t \hat{y}_1 - \partial_x^2 \hat{y}_1 + \alpha_1 \hat{y}_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \partial_t \hat{y}_2 - \partial_x^2 \hat{y}_2 + \alpha_2 \hat{y}_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \partial_x \hat{y}_1(t,1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(t,1) - \beta \hat{y}_1(t,1) = 0 & \text{in } (T_k, T_{k+1}), \\ \big(\hat{y}_1(T_k^+, \cdot), \hat{y}_2(T_k^+, \cdot)\big) = a_k & \text{in } (0,1), \end{cases} \tag{5.15}$$

with a control v_k acting through either

$$\begin{cases} \partial_x \hat{y}_1(t,0) + \beta \hat{y}_2(t,0) = v_k(t) & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(t,0) = 0 & \text{in } (T_k, T_{k+1}), \end{cases} \tag{5.16}$$

or, we consider

$$\begin{cases} \partial_x \hat{y}_1(t,0) + \beta \hat{y}_2(t,0) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x \hat{y}_2(t,0) = v_k(t) & \text{in } (T_k, T_{k+1}). \end{cases} \tag{5.17}$$

- In the first case (5.15)–(5.16), we have the existence of a null-control $v_k \in L^2(T_k, T_{k+1})$ for any given set of parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$ and $\beta \neq 0$ by Theorem 1.3–Item 1.
- In the second case (5.15)–(5.17), for any given set of parameters $(\alpha_1, \alpha_2) \in \mathbb{R}^2$, if $\beta \notin \mathcal{R}$, then there exists a null-control $v_k \in L^2(T_k, T_{k+1})$ for the concerned system as per Theorem 1.3–Item 2.

In both cases, we have the following estimate

$$\|v_k\|_{L^2(T_k,T_{k+1})} \leq Me^{\frac{M}{T_{k+1}-T_k}} \|a_k\|_X, \quad \forall k \geq 0, \tag{5.18}$$

and the associated solutions satisfy

$$\hat{y}(T_{k+1}^-, x) = 0, \quad \forall x \in (0,1), \quad \forall k \geq 0. \tag{5.19}$$

Now, combining (5.14) and (5.18), we have for every $k \geq 0$

$$\begin{aligned}\|v_{k+1}\|_{L^2(T_{k+1},T_{k+2})} &\leq Me^{\frac{M}{T_{k+2}-T_{k+1}}}\|a_{k+1}\|_X \\ &\leq Ce^{CT}Me^{\frac{M}{T_{k+2}-T_{k+1}}}\rho_{\mathcal{S}}(T_k)\left(\left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(T_k,T_{k+1})}+\left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(T_k,T_{k+1})}\right),\end{aligned}$$

since $\rho_{\mathcal{S}}$ is an non-increasing function in (T_k, T_{k+1}).

Using the relation (5.11), we obtain for every $k \geq 0$

$$\|v_{k+1}\|_{L^2(T_{k+1},T_{k+2})} \leq Ce^{CT}\rho_0(T_{k+2})\left(\left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(T_k,T_{k+1})}+\left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(T_k,T_{k+1})}\right),$$

for some $C > 0$. But the function ρ_0 being non-increasing, we deduce that

$$\left\|\frac{v_{k+1}}{\rho_0}\right\|_{L^2(T_{k+1},T_{k+2})} \leq Ce^{CT}\left(\left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(T_k,T_{k+1})}+\left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(T_k,T_{k+1})}\right), \quad \forall k \geq 0. \tag{5.20}$$

Let us define

$$v := \sum_{k\geq 0} v_k \mathbb{1}_{(T_k,T_{k+1})}. \tag{5.21}$$

From the estimate (5.18), we have (since $\rho_0(T_1) = e^{-\frac{pM}{T}\frac{q}{(q-1)}}$)

$$\begin{aligned}\|v_0\|_{L^2(0,T_1)} &\leq Me^{\frac{M}{T_1}}\|a_0\|_X \\ &= Me^{\frac{1}{T}\frac{q(1+p)M}{(q-1)}}\rho_0(T_1)\|y_0\|_X \\ &\leq Ce^{\frac{C}{T}}\rho_0(T_1)\|y_0\|_X,\end{aligned} \tag{5.22}$$

where $C = \frac{M(1+p)q}{(q-1)} > M$. But we know that ρ_0 is non-increasing function in $(0, T_1)$ which yields

$$\left\|\frac{v_0}{\rho_0}\right\|_{L^2(0,T_1)} \leq Ce^{C/T}\|y_0\|_X. \tag{5.23}$$

Now, using the estimates (5.23) and (5.20), we have

$$\left\|\frac{v}{\rho_0}\right\|_{L^2(0,T)} \leq Ce^{C(T+\frac{1}{T})}\left(\|y_0\|_X+\left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(0,T)}+\left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(0,T)}\right). \tag{5.24}$$

Weighted estimate of the solution. Let us set $y = \widetilde{y} + \hat{y}$. Then, for every $k \geq 0$, y satisfies

$$\begin{cases} \partial_t y_1 - \partial_x^2 y_1 + \alpha_1 y_1 = 0 & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \partial_t y_2 - \partial_x^2 y_2 + \alpha_2 y_2 = 0 & \text{in } (T_k, T_{k+1}) \times (0,1), \\ \partial_x y_1(t,0) + \beta y_2(t,0) = S_1(t) + v_k(t) & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(t,0) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x y_1(t,1) = 0 & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(t,1) - \beta y_1(t,1) = S_2(t) & \text{in } (T_k, T_{k+1}), \\ \big(y_1(T_k,\cdot), y_2(T_k,\cdot)\big) = a_k & \text{in } (0,1), \end{cases} \tag{5.25}$$

instead, the control v_k acts through the condition,

$$\begin{cases} \partial_x y_1(t,0) + \beta y_2(t,0) = S_1(t) & \text{in } (T_k, T_{k+1}), \\ \partial_x y_2(t,0) = v_k(t) & \text{in } (T_k, T_{k+1}). \end{cases} \tag{5.26}$$

Here, $y(T_0) = a_0 = y_0$, and we have

$$y(T_k^-) = \widetilde{y}(T_k^-) + \hat{y}(T_k^-) = a_k + 0 = \hat{y}(T_k^+) + \widetilde{y}(T_k^+) = y(T_k^+), \quad \forall k \geq 1,$$

thanks to the facts (5.12), (5.19) and the chosen initial data in the systems (5.13) and (5.15). Thus, y is continuous at T_k for all $k \geq 0$.

Using Theorem 2.4, we have the following estimate for y (for every $k \geq 0$):

$$\|y\|_{\mathcal{C}^0([T_k,T_{k+1}];X)} + \|y\|_{L^2(T_k,T_{k+1};(H^1(0,1))^2)} \leq Ce^{CT}\Big(\|a_k\|_X + \|v_k\|_{L^2(T_k,T_{k+1})} + \|S_1\|_{L^2(T_k,T_{k+1})} + \|S_2\|_{L^2(T_k,T_{k+1})}\Big).$$

Start with $k \geq 1$; using the estimates of a_k and v_k, respectively from (5.14) and (5.18), we deduce that

$$\begin{aligned}\|y\|_{\mathcal{C}^0([T_k,T_{k+1}];X)} + \|y\|_{L^2(T_k,T_{k+1};(H^1(0,1))^2)} &\leq Ce^{CT}e^{\frac{M}{T_{k+1}-T_k}}\left(\|S_1\|_{L^2(T_{k-1},T_k)} + \|S_2\|_{L^2(T_{k-1},T_k)}\right)\\ &\quad + Ce^{CT}\left(\|S_1\|_{L^2(T_k,T_{k+1})} + \|S_2\|_{L^2(T_k,T_{k+1})}\right)\\ &\leq Ce^{CT}e^{\frac{M}{T_{k+1}-T_k}}\rho_{\mathcal{S}}(T_{k-1})\left(\left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(T_{k-1},T_{k+1})} + \left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(T_{k-1},T_{k+1})}\right) \quad \forall k \geq 1,\end{aligned}$$

since $\rho_{\mathcal{S}}$ is a non-increasing function. Now, thanks to the relation (5.11), one has

$$\|y\|_{\mathcal{C}^0([T_k,T_{k+1}];X)} + \|y\|_{L^2(T_k,T_{k+1};(H^1(0,1))^2)} \leq Ce^{CT}\rho_0(T_{k+1})\left(\left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(T_{k-1},T_{k+1})} + \left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(T_{k-1},T_{k+1})}\right), \quad \forall k \geq 1.$$

Now, using the fact that ρ_0 is a non-increasing function, we deduce

$$\left\|\frac{y}{\rho_0}\right\|_{\mathcal{C}^0([T_k,T_{k+1}];X)} + \left\|\frac{y}{\rho_0}\right\|_{L^2(T_k,T_{k+1};(H^1(0,1))^2)} \leq Ce^{CT}\left(\left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(T_{k-1},T_{k+1})} + \left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(T_{k-1},T_{k+1})}\right), \quad \forall k \geq 1. \tag{5.27}$$

For $k = 0$, we use the estimate of v_0 from (5.22) to deduce

$$\begin{aligned}\|y\|_{\mathcal{C}^0([0,T_1];X)} + \|y\|_{L^2(0,T_1;(H^1(0,1))^2} &\leq Ce^{CT}\left(\|a_0\|_X + \|v_0\|_{L^2(0,T_1)} + \|S_1\|_{L^2(0,T_1)} + \|S_2\|_{L^2(0,T_1)}\right)\\ &\leq Ce^{C(T+\frac{1}{T})}\rho_0(T_1)\left(\|y_0\|_X + \|S_1\|_{L^2(0,T_1)} + \|S_2\|_{L^2(0,T_1)}\right).\end{aligned}$$

Using an easy observation: $\|S\|_{L^2(0,T)} \leq \left\|\frac{S}{\rho_{\mathcal{S}}}\right\|_{L^2(0,T)}$, it gives us

$$\left\|\frac{y}{\rho_0}\right\|_{\mathcal{C}^0([0,T_1];X)} + \left\|\frac{y}{\rho_0}\right\|_{L^2(0,T_1;(H^1(0,1))^2)} \leq Ce^{C(T+\frac{1}{T})}\left(\|y_0\|_X + \left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(0,T_1)} + \left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(0,T_1)}\right). \tag{5.28}$$

So, the estimates (5.27) and (5.28) conclude that

$$\left\|\frac{y}{\rho_0}\right\|_{\mathcal{C}^0([0,T];X)} + \left\|\frac{y}{\rho_0}\right\|_{L^2(0,T;(H^1(0,1))^2)} \leq Ce^{C(T+\frac{1}{T})}\left(\|y_0\|_X + \left\|\frac{S_1}{\rho_{\mathcal{S}}}\right\|_{L^2(0,T)} + \left\|\frac{S_2}{\rho_{\mathcal{S}}}\right\|_{L^2(0,T)}\right). \tag{5.29}$$

Finally, using the definitions of norms (5.5) (of the weighted spaces) in the estimates (5.24) and (5.29), we have the required estimate (5.10) of our proposition. ∎

5.2. Application of the fixed point argument. In this section, we prove the main theorem regarding the local null-controllability of our systems.

We assume the initial data $y_0 \in X$ such that $\|y_0\|_X \leq \delta$ for some $\delta > 0$ which will be determined later. We also introduce the set

$$\boldsymbol{\mathcal{S}}_\delta := \{S \in \mathcal{S} \times \mathcal{S} \mid \|S\|_{\mathcal{S}\times\mathcal{S}} \leq \delta\},$$

where the space $\mathcal{S}$ is defined in (5.4a).

Now, recall from Proposition 5.2 that for any given source term $S := (S_1, S_2) \in \mathcal{S} \times \mathcal{S}$, there exists a control v such that the trajectory $y = (y_1, y_2)$ of (5.6)–(5.7) with (5.8) or (5.9) (depending on the choices of the parameters as per Proposition 5.2) satisfies the estimate (5.10). In what follows, we define an operator $\mathcal{N}$ acting on $\boldsymbol{\mathcal{S}}_\delta$ by

$$\mathcal{N}(S)(t) = \begin{pmatrix} -g(y_2(t,0)) \\ g(y_1(t,1)) \end{pmatrix}. \tag{5.30}$$

Proof of Theorem 1.2. We start by showing that $\mathcal{N}$ is a contraction map from $\boldsymbol{\mathcal{S}}_\delta$ onto itself for some suitable choice of $\delta > 0$.

Step 1. Let us show that $\boldsymbol{\mathcal{S}}_\delta$ is invariant under the map $\mathcal{N}$ provided $\delta > 0$ is small enough.

Thanks to the choices of g given by (5.1) we have $|g(\nu)| \leq C|\nu|^2$, which yields, by the trace theorem,

$$\begin{aligned}\left|\frac{\mathcal{N}(S)(t)}{\rho_{\mathcal{S}}(t)}\right| &\leq \frac{C}{|\rho_{\mathcal{S}}(t)|}\left(|y_2(t,0)|^2 + |y_1(t,1)|^2\right) \\ &\leq \frac{C}{|\rho_{\mathcal{S}}(t)|}\left(\|y_2(t)\|_{L^2(0,1)}\|y_2(t)\|_{H^1(0,1)} + \|y_1(t)\|_{L^2(0,1)}\|y_1(t)\|_{H^1(0,1)}\right),\end{aligned}$$

and it follows that

$$\begin{aligned}\left|\frac{\mathcal{N}(S)(t)}{\rho_{\mathcal{S}}(t)}\right|^2 &\leq \frac{C}{|\rho_{\mathcal{S}}(t)|^2}\|y(t)\|_X^2\|y(t)\|^2_{(H^1(0,1))^2} \\ &\leq C\frac{|\rho_0(t)|^4}{|\rho_{\mathcal{S}}(t)|^2}\left\|\frac{y(t)}{\rho_0(t)}\right\|_X^2\left\|\frac{y(t)}{\rho_0(t)}\right\|^2_{(H^1(0,1))^2}.\end{aligned} \tag{5.31}$$

– Now, using Remark 5.1 we have that the quantity $\dfrac{\rho_0^2(t)}{\rho_{\mathcal{S}}(t)}$ is bounded by 1 irrespective on t.

– On the other hand, since we have $\|y_0\|_X \leq \delta$ and $S = (S_1, S_2) \in \boldsymbol{\mathcal{S}}_\delta$, using the result (5.10), we deduce that

$$\left\|\frac{y}{\rho_0}\right\|_{\mathcal{C}^0([0,T];X)} + \left\|\frac{y}{\rho_0}\right\|_{L^2(0,T;(H^1(0,1))^2)} \leq Ce^{C\left(T+\frac{1}{T}\right)}\delta. \tag{5.32}$$

Using the bound (5.32) in (5.31), we get

$$\left\|\frac{\mathcal{N}(S)}{\rho_{\mathcal{S}}}\right\|_{L^2(0,T)} \leq C\left\|\frac{y}{\rho_0}\right\|_{\mathcal{C}^0([0,T];X)}\left\|\frac{y}{\rho_0}\right\|_{L^2(0,T;(H^1(0,1))^2)} \leq Ce^{C\left(T+\frac{1}{T}\right)}\delta^2. \tag{5.33}$$

Thus, for $\delta > 0$ small enough, the map $\mathcal{N}$ stabilizes $\boldsymbol{\mathcal{S}}_\delta$.

Step 2. In this step, we prove that $\mathcal{N}$ is a contraction map for $\delta > 0$ small enough.

For any $S, \widetilde{S} \in \boldsymbol{\mathcal{S}}_\delta$, we denote the trajectories respectively by $y = (y_1, y_2)$ and $\widetilde{y} = (\widetilde{y}_1, \widetilde{y}_2)$ associated with the controls v and $\widetilde{v}$ of the system (5.6)–(5.7)–(5.8) (or (5.9)), by means of Proposition 5.2.

Then, using the properties of g from (5.1), it follows that

$$\begin{aligned}(5.34)\quad \left|\frac{\mathcal{N}(S)(t)-\mathcal{N}(\widetilde{S})(t)}{\rho_{\mathcal{S}}(t)}\right| &\le \frac{C}{|\rho_{\mathcal{S}}(t)|}\Big(|y_2(t,0)-\widetilde{y}_2(t,0)|\left(|y_2(t,0)|+|\widetilde{y}_2(t,0)|\right)\\ &\qquad +|y_1(t,1)-\widetilde{y}_1(t,1)|\left(|y_1(t,1)|+|\widetilde{y}_1(t,1)|\right)\Big)\\ &\le \frac{C}{|\rho_{\mathcal{S}}(t)|}\|y(t)-\widetilde{y}(t)\|_X^{\frac12}\|y(t)-\widetilde{y}(t)\|_{(H^1(0,1))^2}^{\frac12}\\ &\quad\times\left(\|y(t)\|_X^{\frac12}\|y(t)\|_{(H^1(0,1))^2}^{\frac12}+\|\widetilde{y}(t)\|_X^{\frac12}\|\widetilde{y}(t)\|_{(H^1(0,1))^2}^{\frac12}\right)\\ &\le \frac{C\rho_0^2(t)}{\rho_{\mathcal{S}}(t)}\left\|\frac{y-\widetilde{y}}{\rho_0}\right\|_{\mathcal{C}^0([0,T];X)}^{\frac12}\left\|\frac{y(t)-\widetilde{y}(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}^{\frac12}\times\Big(\left\|\frac{y}{\rho_0}\right\|_{\mathcal{C}^0([0,T];X)}^{\frac12}\left\|\frac{y(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}^{\frac12}\\ &\qquad+\left\|\frac{\widetilde{y}}{\rho_0}\right\|_{\mathcal{C}^0([0,T];X)}^{\frac12}\left\|\frac{\widetilde{y}(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}^{\frac12}\Big).\end{aligned}$$

Now, as per Proposition 5.2 the solution map is linear and thus, one has $y-\widetilde{y}$ is the trajectory of the set of equations (5.6)–(5.7)–(5.8) (instead, (5.6)–(5.7)–(5.9)) with the source terms $S-\widetilde{S}$ (where $S:=(S_1,S_2)$ and $\widetilde{S}:=(\widetilde{S}_1,\widetilde{S}_2)$), control $v-\widetilde{v}$ and initial data $y(0,\cdot)-\widetilde{y}(0,\cdot)=0$. Then, thanks to the estimate (5.10), $y-\widetilde{y}$ satisfies the following:

$$\begin{aligned}(5.35)\quad \left\|\frac{y-\widetilde{y}}{\rho_0}\right\|_{\mathcal{C}^0([0,T];X)}+\left\|\frac{y-\widetilde{y}}{\rho_0}\right\|_{L^2(0,T;(H^1(0,1))^2)} &\le Ce^{C\left(T+\frac1T\right)}\left(\|S_1-\widetilde{S}_1\|_{\mathcal{S}}+\|S_2-\widetilde{S}_2\|_{\mathcal{S}}\right)\\ &\le Ce^{C\left(T+\frac1T\right)}\|S-\widetilde{S}\|_{\mathcal{S}\times\mathcal{S}}.\end{aligned}$$

Also, we recall the estimate (5.32) for y (similarly for $\widetilde{y}$) since we started with initial data $\|y_0\|\le\delta$ and S (or, $\widetilde{S}$) in $\boldsymbol{\mathcal{S}}_\delta$. Beside this, we have by Remark 5.1 that $\dfrac{\rho_0^2(t)}{\rho_{\mathcal{S}}(t)}\le 1$, $\forall t\in[0,T]$. As a consequence, the inequality (5.34) yields to

$$\begin{aligned}\left|\frac{\mathcal{N}(S)(t)-\mathcal{N}(\widetilde{S})(t)}{\rho_{\mathcal{S}}(t)}\right| &\le C_T\delta^{\frac12}\|S-\widetilde{S}\|_{\mathcal{S}\times\mathcal{S}}^{\frac12}\left\|\frac{y(t)-\widetilde{y}(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}^{\frac12}\\ &\quad\times\left(\left\|\frac{y(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}^{\frac12}+\left\|\frac{\widetilde{y}(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}^{\frac12}\right),\end{aligned}$$

with $C_T:=Ce^{C\left(T+\frac1T\right)}$. This implies

$$\begin{aligned}\left\|\frac{\mathcal{N}(S)-\mathcal{N}(\widetilde{S})}{\rho_{\mathcal{S}}}\right\|_{(L^2(0,T))^2}^2 &\le C_T\delta\,\|S-\widetilde{S}\|_{\mathcal{S}\times\mathcal{S}}\\ &\quad\times\int_0^T\left\|\frac{y(t)-\widetilde{y}(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}\left(\left\|\frac{y(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}+\left\|\frac{\widetilde{y}(t)}{\rho_0(t)}\right\|_{(H^1(0,1))^2}\right)\\ &\le C_T\delta\,\|S-\widetilde{S}\|_{\mathcal{S}\times\mathcal{S}}\left\|\frac{y-\widetilde{y}}{\rho_0}\right\|_{L^2(0,T;(H^1(0,1))^2)}\left(\left\|\frac{y}{\rho_0}\right\|_{L^2(0,T;(H^1(0,1))^2)}+\left\|\frac{\widetilde{y}}{\rho_0}\right\|_{L^2(0,T;(H^1(0,1))^2)}\right)\\ &\le C_T\delta^2\|S-\widetilde{S}\|_{\mathcal{S}\times\mathcal{S}}^2,\end{aligned}$$

where we have used the estimates (5.32) and (5.35).

So, eventually we proved that

$$(5.36)\qquad \|\mathcal{N}(S)-\mathcal{N}(\widetilde{S})\|_{\mathcal{S}\times\mathcal{S}}\le C_T\delta\,\|S-\widetilde{S}\|_{\mathcal{S}\times\mathcal{S}},$$

for $\delta>0$ chosen small enough and this ensures that the map $\mathcal{N}$ is a contraction in the closed ball $\boldsymbol{\mathcal{S}}_\delta$.

Hence, by applying Banach fixed-point argument, there exists unique fixed point of the map $\mathcal{N}$ denoted by $S_0 := (S_{0,1}, S_{0,2})$ in the ball $\boldsymbol{\mathcal{S}_\delta}$.

Thanks to Proposition 5.2, for the above $S_0 \in \boldsymbol{\mathcal{S}}_\delta$ and initial data $y_0 \in X$ with $\|y_0\|_X \leq \delta$, there exists a control $v \in \mathcal{V}$ such that the trajectory $y := (y_1, y_2)$ and the control v to (5.6)–(5.7)–(5.8) or (5.6)–(5.7)–(5.9) (depending on the choices of $f'(0)$, i.e. β, as given by Proposition 5.2) satisfy the estimates (5.10). Then, the property $\lim_{t\to T^-} \rho_0(t) = 0$ forces that $y(T, x) = 0$ for all $x \in (0, 1)$ and that the local null-controllability holds for the concerned nonlinear systems (1.1)–(1.3) with (1.5) or (1.6).

This completes the proof of Theorem 1.2. ∎

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