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FRANCISCO KORDON AND THIERRY LAMBRE

Abstract. We compute the center and the Lie algebra of outer derivations of a familiy of algebras of differential operators associated to hyperplane arrangements of the affine space $\mathbb{A}^3$. The results are completed for 4-braid arrangements and for reflection arrangements associated to the wreath product of a cyclic group with the symmetric group $\mathfrak{S}_3$. To achieve this we use tools from homological algebra and Lie–Rinehart algebras of differential operators.

Introduction

Let V be a finite-dimensional $\Bbbk$-vector space over a field $\Bbbk$ of characteristic zero, S the algebra of coordinates on V and $\mathcal{A}$ a central hyperplane arrangement in V. We assume throughout the article that $\mathcal{A}$ is a free arrangement in the sense given by K. Saito in [Sai80]: we require that the Lie algebra $\operatorname{Der}\mathcal{A}$ of derivations of S tangent to $\mathcal{A}$ is a free S-module. The algebra $\operatorname{Diff}\mathcal{A}$ of differential operators tangent to $\mathcal{A}$, as seen by F. J. Calderon Moreno in [CM99] and by M. Suárez–Álvarez in [SÁ18], is the subalgebra of $\operatorname{End}(S)$ generated by $\operatorname{Der}\mathcal{A}$ and S. Our results concern the center and the Lie algebra of outer derivations of $\operatorname{Diff}\mathcal{A}$.

The first and simplest example of a free arrangement is the case of a central line arrangement in $V = \Bbbk^2$. This case is studied by the first author and M. Suárez-Álvarez in [KSÁ18] when there are at least 5 lines: a description of the Hochschild cohomology $HH^\bullet(\operatorname{Diff}\mathcal{A})$, including its cup product and Gerstenhaber bracket, is given explicitly in detail through a calculation independent of the methods that we now use. The second and most important family of examples is that of the braid arrangements $\mathcal{B}_n$, given for $n \geq 2$ by the hyperplanes $H_{ij} = \{x \in \Bbbk^n : x_i = x_j\}$ with $i \neq j$: these arrangements are free and have served historically as a proxy to obtain general results, for instance in V. I. Arnold's classical article [Arn69].

In virtue of the freeness of $\mathcal{A}$ the algebra $\operatorname{Diff}\mathcal{A}$ is isomorphic to the enveloping algebra of a Lie–Rinehart algebra (S, L) —see L. Narvaez Macarro's [NM08] and the first author's thesis [Kor19]— and then the spectral sequence introduced by both authors in [KL21] permits the computation of $HH^\bullet(\operatorname{Diff}\mathcal{A})$ in terms of the Hochschild cohomology $H^\bullet(S, \operatorname{Diff}\mathcal{A})$ of S with values on $\operatorname{Diff}\mathcal{A}$ and the Lie–Rinehart cohomology of L. This was successfully applied to arrangements of three lines in [KL21], and, ultimately, to $\mathcal{A} = \mathcal{B}_3$ —see Corollary 1.29.

The homological approach described above allows us to compute the center of $\operatorname{Diff}\mathcal{A}$ under the hypothesis that the Saito's matrix of the arrangement $\mathcal{A}$ is triangular: more generally, we can

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state this result resorting to the hypothesis of triangularizability of Lie–Rinehart algebras that we give in Definition 1.12.

Theorem A (Theorem 3.4)**.** *Let (S, L) be a triangularizable Lie–Rinehart algebra with enveloping algebra U. The center of U is $\Bbbk$.*

Let $\mathcal{A}_r$, $r \geq 1$, be the arrangement in $\mathbb{C}^3$ defined by $0 = xyz(z^r - y^r)(z^r - x^r)(y^r - x^r)$. This arrangement is $\mathcal{B}_4$ when $r = 1$. When $r \geq 2$, it is the reflection arrangement of the wreath product of the cyclic group of order r and the symmetric group $\mathfrak{S}_3$. The homological method yields the following result.

Theorem B (Corollary 5.5)**.** *Let $r \geq 1$. For each hyperplane H in $\mathcal{A}_r$ let f_H be a linear form with kernel H and ∂_H the derivation of $\operatorname{Diff} \mathcal{A}_r$ determined by*

$$\begin{cases} \partial_H(g) = 0 & \text{if } g \in \Bbbk[x_1, x_2, x_3]; \\ \partial_H(\theta) = \theta(f_H)/f_H & \text{if } \theta \in \operatorname{Der} \mathcal{A}_r. \end{cases}$$

The Lie algebra of outer derivations of $\operatorname{Diff} \mathcal{A}_r$ together with the commutator is an abelian Lie algebra of dimension $3r + 3$, the numbers of hyperplanes of $\mathcal{A}_r$, and is generated by the classes of the derivations ∂_H with $H \in \mathcal{A}_r$.

In the pursuit of $HH^\bullet(U)$ a key step is the computation of $H^\bullet(S, U)$. We succeeded in its calculation when $\bullet = 0, 1$ for a family of Lie–Rinehart algebras that generalizes $\operatorname{Der} \mathcal{A}_r$. The result in Corollary 4.6 relates $H^1(S, U)$ to the cokernel of the Saito's matrix — this is an important object of the theory with a rich algebraic structure studied, for instance, by M. Granger, D. Mond and M. Schulze in [GMS11].

There are several ways in which the calculations performed in this article can be continued. In particular, following the methods of J. Alev and M. Chamarie in [AC92] our findings on the algebra of outer derivations of $\operatorname{Diff} \mathcal{A}$ can lead to a description of $\operatorname{Aut}(\operatorname{Diff} \mathcal{A})$ as in [KSÁ18, §7] and M. Suárez-Álvarez and Q. Vivas' [SAV15].

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Unadorned Hom and End are taken over $\Bbbk$. The set of natural numbers $\mathbb{N}$ is that of nonnegative integers. If n and m are positive integers, we denote by $[\![n, m]\!]$ the set of integers k such that $n \leq k \leq m$, and $[\![m]\!] := [\![1, m]\!]$.

1. Generalities

1.1. **Hyperplane arrangements.**

Definition 1.1. *A central* hyperplane arrangement *$\mathcal{A}$ in a finite dimensional vector space V is a finite set $\{H_1, \dots, H_\ell\}$ of subspaces of codimension 1. Choosing a basis of V we may identify the algebra $S(V^*)$ of coordinates of V with $S = \Bbbk[x_1, \dots, x_n]$: for each $i \in [\![\ell]\!]$ let $\lambda_i \in S$ be a linear* *form with kernel* H_i. *Up to a nonzero scalar, the* defining polynomial $Q = \lambda_1 \cdots \lambda_\ell \in S$ *depends only on* $\mathcal{A}$.

Definition 1.2. *The set of* derivations tangent to the arrangement $\mathcal{A}$ *is*

$$\operatorname{Der}\mathcal{A} \coloneqq \{\theta \in \operatorname{Der}(S) \ : \lambda_i \textit{ divides } \theta(\lambda_i) \textit{ for every } i \in [\![\ell]\!]\ \}\,.$$

This is a Lie–subalgebra and a sub-S-module of the Lie algebra of derivations $\operatorname{Der}(S)$ *of* S.

Definition 1.3. *The arrangement* $\mathcal{A}$ *is* free *if* $\operatorname{Der}\mathcal{A}$ *is a free* S*-module.*

Theorem 1.4 (Saito's criterion, [Sai80, Theorem 1.8.ii])**.** *A family of* ℓ *derivations* $(\theta_1, \dots, \theta_\ell)$ *in* $\operatorname{Der}\mathcal{A}$ *is an* S*-basis of* $\operatorname{Der}\mathcal{A}$ *if and only if the determinant of* Saito's matrix

$$M(\theta_1, \dots, \theta_\ell) \coloneqq \begin{pmatrix} \theta_1(x_1) & \cdots & \theta_1(x_\ell) \\ \vdots & & \vdots \\ \theta_\ell(x_1) & \cdots & \theta_\ell(x_\ell) \end{pmatrix}$$

is a nonzero scalar multiple of Q.

The notion of freeness connects arrangements of hyperplanes with commutative algebra, algebraic geometry and combinatorics. While not a property of generic hyperplane arrangements, many of the motivating examples of hyperplane arrangements are free. Saito's criterion in Theorem 1.4 is perhaps the most practical way to prove freeness, though there are other methods to prove this condition —see A. Bigatti, E. Palezzato and M. Torielli's [BPT20] for a discussion on the state of the art.

Example 1.5. Let $n \geq 2$, $E = \mathbb{A}^n$ the affine space with coordinate ring $S = \Bbbk[x_1, \dots, x_n]$. The braid arrangement $\mathcal{B}_n$ in E has hyperplanes H_{ij} with equation $x_i - x_j = 0$, $1 \leq i < j \leq n$, so that the defining polynomial is $Q = \prod_{1 \leq i < j \leq n}(x_j - x_i)$.

Consider the derivations $\theta_1, \dots, \theta_n$ of S defined for $k \in [\![n]\!]$ by

$$\theta_1(x_k) = 1, \qquad\qquad \theta_i(x_k) = (x_k - x_1)\dots(x_k - x_{i-1}) \qquad \text{if } i \geq 2.$$

These derivations satisfy $(x_k - x_j) \mid \theta_i(x_k - x_j)$ for any $i, j, k \in [\![n]\!]$ and therefore belong to $\operatorname{Der}\mathcal{B}_n$. The Saito's matrix $(\theta_i(x_k))$ is triangular and its determinant is Q. By Saito's Criterion, $\operatorname{Der}\mathcal{B}_n$ is a free S-module with basis $\{\theta_1, \dots, \theta_n\}$.

Example 1.6. Let $n \geq 1$ and $E = \mathbb{A}^{n+1}$ be the affine space with coordinate ring $S = \Bbbk[x_0, x_1, \dots, x_n]$. As in Example 1.5 above, the arrangement $\mathcal{B}_{n+1}$ in E has equation $\prod_{0 \leq i < j \leq n}(x_i - x_j)$.

Consider the subspace $V = \{0\} \times \mathbb{A}^n$ of E, defined by the equation $x_0 = 0$, and the hyperplanes $\tilde{H}_{ij}$ of V defined by $x_i - x_j = 0$ for $1 \leq i < j \leq n$. We call $\tilde{\mathcal{B}}_n$ the arrangement formed by these hyperplanes, so that $\tilde{\mathcal{B}}_n$ is defined by equation $x_1 \dots x_n \prod_{0 \leq i < j \leq n}(x_i - x_j) = 0$. The derivations $\alpha_1, \dots, \alpha_n$ of $S = \Bbbk[x_1, \dots, x_n]$ defined for $k \in [\![n]\!]$ by

$$\alpha_1(x_k) = x_k, \qquad\qquad \alpha_i(x_k) = \begin{cases} 0 & \text{if } i > k; \\ x_k \prod_{j<i}(x_k - x_j) & \text{if } i \leq k \end{cases} \qquad \text{if } i \geq 2$$

belong to $\operatorname{Der}\tilde{\mathcal{B}}_n$. Thanks to Saito's Criterion, $(\alpha_1, \dots, \alpha_n)$ is a basis of $\operatorname{Der}\tilde{\mathcal{B}}_n$.

Example 1.7. Let V be a finite-dimensional vector space. We say that $\sigma \in \mathrm{GL}(V)$ is a pseudo-reflection if σ is of finite order and fixes a hyperplane H_σ of V, and it is a reflection if this order is 2. A finite subgroup G of the group of automorphisms of V is a *(pseudo-) reflection group* if it is generated by (pseudo-) reflections, and the set of reflecting hyperplanes $\mathcal{A}(G)$ of a reflection group G is the *reflection arrangement* of G. It is a result by H. Terao in [Ter80] that every reflection arrangement over $\Bbbk = \mathbb{C}$ is free.

Consider the nth braid arrangement $\mathcal{B}_n$ of Example 1.5: identifying the reflection with respect to the plane $x_i - x_j = 0$ with the permutation $(ij) \in \mathfrak{S}_n$ we see that $\mathcal{B}_n = \mathcal{A}(\mathfrak{S}_n)$.

Example 1.8. Let $r, n \geq 1$ and consider the arrangement $\mathcal{A}_r^n$ in $V = \Bbbk^n$ defined by

$$0 = x_1 \dots x_n \prod_{1 \leq i < j \leq n} (x_j^r - x_i^r).$$

Taking $r = 1$ we see that $\mathcal{A}_1^n = \tilde{\mathcal{B}}_n$ for every n. When $r \geq 2$, let $G = C_r \wr \mathfrak{S}_n$ be the wreath product of the cyclic group C_r of order r and the symmetric group $\mathfrak{S}_n$. We see that $\mathcal{A}_r^n$ is the reflection arrangement of G, this is, $\mathcal{A}_r^n = \mathcal{A}(C_r \wr \mathfrak{S}_n)$. There is a well-known basis of $\operatorname{Der}\mathcal{A}_r^n$ in [OT92, §B] that consists of the derivations $\theta_1, \dots, \theta_n$ of $S = \Bbbk[x_1, \dots, x_n]$ defined for $1 \leq k, m \leq n$ by $\theta_m(x_k) = x_k^{(m-1)r+1}$. Consider the derivations $\alpha_1, \dots, \alpha_n$ of S defined for $1 \leq k \leq n$ and $2 \leq m \leq n$ by

$$\alpha_1(x_k) = x_k, \qquad \alpha_m(x_k) = x_k \prod_{i=1}^{m-1} (x_k^r - x_i^r). \tag{1}$$

These derivations belong to $\operatorname{Der}\mathcal{A}_r^n$: evidently $\alpha_1 = \theta_1$, and if $m \geq 2$ then

$$\alpha_m = \theta_m - s_1\theta_{m-1} + \dots + (-1)^{m-1} s_{m-1}\theta_1,$$

where $s_j = \sum_{1 \leq i_1 < \dots < i_j \leq m-1} x_{i_1}^r \cdots x_{i_j}^r$ is the jth elementary symmetric polynomial in variables $x_1^r, \dots, x_{m-1}^r$ for $1 \leq j \leq m-1$. For $1 \leq k \leq n$

$$\begin{aligned}
&\left(\theta_m - s_1\theta_{m-1} + \dots + (-1)^{m-1} s_{m-1}\theta_1\right)(x_k) \\
&\qquad = x_k^{(m-1)r+1} - s_1 x_k^{(m-2)r+1} + \dots + (-1)^{m-1} s_{m-1} x^k = x_k \prod_{i=1}^{m-1} (x_k^r - x_i^r),
\end{aligned}$$

which equals $\alpha_m(x_k)$. Saito's matrix $(\alpha_m(x_k))$ is diagonal and its determinant is

$$\prod_{k=1}^{n} \alpha_k(x_k) = \prod_{k=1}^{n} x_k \prod_{i=1}^{k-1} (x_k^r - x_i^r) = x_1 \dots x_n \prod_{1 \leq i < k \leq n} (x_k^r - x_i^r).$$

It follows from Saito's criterion that $\alpha_1, \dots, \alpha_n$ is a basis of $\operatorname{Der}\mathcal{A}_r^n$.

Example 1.9. Let $r \geq 1$. The arrangement $\mathcal{A}_r \coloneqq \mathcal{A}_r^3$ is defined by the nullity of

$$Q(\mathcal{A}_r) \coloneqq x_1 x_2 x_3 (x_2^r - x_1^r)(x_3^r - x_1^r)(x_3^r - x_2^r).$$

The basis of $\operatorname{Der}\mathcal{A}_r$ in (1) consist in this case of the derivations $\alpha_1, \alpha_2, \alpha_3$ of $S = \Bbbk[x_1, x_2, x_3]$ with Saito's matrix

$$\begin{pmatrix}
x_1 & x_2 & x_3 \\
0 & x_2(x_2^r - x_1^r) & x_3(x_3^r - x_1^r) \\
0 & 0 & x_3(x_3^r - x_2^r)(x_3^r - x_1^r)
\end{pmatrix}$$

1.2. **Lie–Rinehart algebras.**

Definition 1.10. *Let S and $(L, [-,-])$ be, respectively, a commutative and a Lie algebra endowed with a morphism of Lie algebras $L \to \mathrm{Der}(S)$ that we write $\alpha \mapsto \alpha_S$ and a left S-module structure on L which we simply denote by juxtaposition. We say that the pair (S, L) is a* Lie–Rinehart algebra*, or that L is a Lie–Rinehart algebra over S, if the equalities*

$$(s\alpha)_S(t) = s\alpha_S(t), \qquad [\alpha, s\beta] = s[\alpha,\beta] + \alpha_S(s)\beta$$

hold whenever $s, t \in S$ and $\alpha, \beta \in L$.

If S is a commutative algebra and L is a Lie-subalgebra of the Lie algebra of derivations $\mathrm{Der}\, S$ that is at the same time an S-submodule then L is an Lie–Rinehart algebra over S. This applies to our situation of interest:

Proposition 1.11. *Let $\mathcal{A}$ be a hyperplane arrangement in a vector space V. The Lie algebra of derivations $\mathrm{Der}\,\mathcal{A}$ of $\mathcal{A}$ is a Lie–Rinehart algebra over the algebra of coordinates of V.*

Definition 1.12. *Let $n \geq 1$, $S = \Bbbk[x_1, \dots, x_n]$ and L be a subset of derivations of S such that (S, L) is a Lie-Rinehart algebra.*

(*i*) *We call L* triangularizable *if L is a free S-module that admits a basis given by derivations $\alpha_1, \dots, \alpha_n$ satisfying the two conditions*

$$\alpha_i(x_j) = 0 \quad \text{if } i > j, \qquad \alpha_1(x_1) \cdots \alpha_n(x_n) \neq 0.$$

(*ii*) *We say that L satisfies the* Bézout condition *if in addition for each k in $[\![n-1]\!]$, the element $\alpha_k(x_k)$ of S is coprime with the determinant of the matrix*

$$\begin{pmatrix} \alpha_k(x_{k+1}) & \alpha_{k+1}(x_{k+1}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \alpha_k(x_{n-2}) & \alpha_{k+1}(x_{n-2}) & \cdots & \alpha_{n-2}(x_{n-2}) & 0 \\ \alpha_k(x_{n-1}) & \alpha_{k+1}(x_{n-1}) & \cdots & \cdots & \alpha_{n-1}(x_{n-1}) \\ \alpha_k(x_n) & \alpha_{k+1}(x_n) & \cdots & \cdots & \alpha_{n-1}(x_n) \end{pmatrix}$$

Example 1.13. For any $n \geq 2$ the Lie–Rinehart algebra $\mathrm{Der}\,\mathcal{B}_n$ is triangular and satisfies the Bézout condition with the basis $\{\theta_1, \dots, \theta_n\}$ given in Example 1.5. The same goes to $\mathrm{Der}\,\tilde{\mathcal{B}}_n$ with the basis given in Example 1.6.

Example 1.14. Let $r, n \geq 1$. The Lie–Rinehart algebra associated to $\mathcal{A}(C_r \wr \mathfrak{S}_n)$ is triangularizable, as follows immediately from Example 1.8.

Example 1.15. Let $r \geq 1$. The arrangement $\mathcal{A}_r = \mathcal{A}(C_r \wr \mathfrak{S}_3)$ from Example 1.9 is triangularizable thanks to Example 1.8. Moreover, it satisfies the Bézout condition: indeed, $\alpha_2(x_2) = x_2(x_2^r - x_1^r)$ is coprime with $\alpha_2(x_3) = x_3(x_3^r - x_1^r)$, and the determinant

$$\det \begin{pmatrix} \alpha_1(x_2) & \alpha_1(x_3) \\ \alpha_2(x_2) & \alpha_2(x_3) \end{pmatrix} = \det \begin{pmatrix} x_2 & x_3 \\ x_2(x_2^r - x_1^r) & x_3(x_3^r - x_1^r) \end{pmatrix} = x_2 x_3 \left(x_3^r - x_2^r\right)$$

is coprime with $\alpha_1(x_1) = x_1$.

1.3. **Differential operators associated to an arrangement.** Remember from J. C. McConnell and J. C. Robson's [MR01, §15] that the algebra $\operatorname{Diff} S$ of differential operators on $S = \Bbbk[x_1, \dots, x_n]$ is the subalgebra of $\operatorname{End} S$ generated by $\operatorname{Der} S$ and the set of maps given by left multiplication by elements of S. Recall as well from [MR01, §5] that if R is an algebra and $I \subset R$ is a right ideal, the largest subalgebra $\mathbb{I}_R(I)$ of R that contains I as an ideal —the *idealizer* of I in R— is $\{r \in R : rI \subset I\}$.

Definition 1.16. *Let $\mathcal{A}$ be a central arrangement of hyperplanes with defining polynomial Q. The* algebra of differential operators tangent to the arrangement $\mathcal{A}$ *is*

$$\operatorname{Diff}(\mathcal{A}) = \bigcap_{n \geq 1} \mathbb{I}_{\operatorname{Diff}(S)}(Q^n \operatorname{Diff}(S)).$$

As seen in [CM99] for $\Bbbk = \mathbb{C}$ or in [SÁ18] for $\Bbbk$ of characteristic zero, if $\mathcal{A}$ is free then the algebra $\operatorname{Diff} \mathcal{A}$ coincides with the sub-associative algebra of $\operatorname{End}(S)$ generated by $\operatorname{Der} \mathcal{A}$ and the set of maps given by left multiplication by elements of S.

Example 1.17. The arrangement $\mathcal{A} = \tilde{\mathcal{B}}_2$ in $\Bbbk^2$ with equation $0 = xy(y - x)$ admits, by [KL21, §5], a presentation of $\operatorname{Diff} \mathcal{A}$ adapted from [KSÁ18]: the two derivations

$$E = x\partial_x + y\partial_y, \qquad D = y(y - x)\partial_y$$

of $\Bbbk[x, y]$ form a basis of $\operatorname{Der} \mathcal{A}$, and the algebra $\operatorname{Diff} \mathcal{A}$ is generated by the symbols x, y, D and E subject to the relations

$$\begin{aligned} &[y, x] = 0, \\ &[D, x] = 0, \qquad &&[D, y] = y(y - x), \\ &[E, x] = x, \qquad &&[E, y] = y, \qquad &&[E, D] = D. \end{aligned}$$

Given a Lie-Rinehart algebra (S, L), a *Lie–Rinehart module* —or (S, L)-module— is a vector space M which is at the same time an S-module and an L-Lie module in such a way that if $s \in S$, $\alpha \in L$ and $m \in M$ then

$$(s\alpha) \cdot m = s \cdot (\alpha \cdot m), \qquad \alpha \cdot (s \cdot m) = (s\alpha) \cdot m + \alpha_S(s) \cdot m.$$

Theorem 1.18 ([Hue90, §1])**.** *Let (S, L) be a Lie-Rinehart algebra.*

(*i*) *There exists an associative algebra $U = U(S, L)$, the* universal enveloping algebra *of (S, L), endowed with a morphism of algebras $i : S \to U$ and a morphism of Lie algebras $j : L \to U$ satisfying, for $s \in S$ and $\alpha \in L$,*

$$i(s)j(\alpha) = j(s\alpha), \qquad j(\alpha)i(s) - i(s)j(\alpha) = i(\alpha_S(s)).$$

(*ii*) *The algebra U is universal with these properties.*

(*iii*) *The category of U-modules is isomorphic to the category of (S, L)-modules.*

Example 1.19. If $S = \Bbbk[x_1, \dots, x_n]$ then the full Lie algebra of derivations $L = \operatorname{Der} S$ is a Lie–Rinehart algebra and its enveloping algebra is isomorphic to the algebra of differential operators $\operatorname{Diff}(S) = A_n$, the nth Weyl algebra.

The following result —[NM08, §12] when $\Bbbk = \mathbb{C}$ and [Kor19, Theorem 2.19] for $\Bbbk$ of characteristic zero— is our motivation to consider Lie–Rinehart algebras in the algebraic aspects of hyperplane arrangements.

Theorem 1.20. *Let $\mathcal{A}$ be a free hyperplane arrangement on a vector space V and let S be the algebra of coordinate functions on V. There is a canonical isomorphism of algebras*

$$U(S, \operatorname{Der}\mathcal{A}) \cong \operatorname{Diff}(\mathcal{A}).$$

Proposition 1.21. *For $n \geq 1$ there is an isomorphism of algebras*

$$\operatorname{Diff}\mathcal{B}_{n+1} \cong A_1 \otimes \operatorname{Diff}\tilde{\mathcal{B}}_n.$$

Proof. Let $n \in \mathbb{N}$, $S = \Bbbk[x_0, x_1, \ldots, x_n]$, $T = \Bbbk[y_1, \ldots, y_n]$ and observe that the unique morphism of algebras $\Bbbk[z] \otimes T \to S$ given by $z \mapsto x_0$ and $y_k \mapsto x_k - x_0$ if $k \geq 1$ is an isomorphism —we are identifying z with $z \otimes 1$ and y_k with $1 \otimes y_k$.

The derivations in the basis of $\operatorname{Der}\mathcal{B}_{n+1}$ given in Example 1.5 induce derivations $\tilde{\theta}_1, \ldots, \tilde{\theta}_{n+1}$ on $\Bbbk[z] \otimes T$. For $1 \leq i \leq n+1$ and $1 \leq k \leq n$ these derivations satisfy

$$\tilde{\theta}_1 : \begin{cases} z \mapsto 1; \\ y_k \mapsto 0; \end{cases} \qquad \tilde{\theta}_2 : \begin{cases} z \mapsto 0; \\ y_k \mapsto y_k; \end{cases} \qquad \tilde{\theta}_i : \begin{cases} z \mapsto 0; \\ y_k \mapsto y_k \prod_{j=1}^{i-1}(y_k - y_j) \end{cases} \quad \text{if } i \geq 3.$$

The Lie algebra $\operatorname{Der} S$ is isomorphic to the Lie algebra product $\operatorname{Der}\Bbbk[z] \times \operatorname{Der}\tilde{\mathcal{B}}_n$: the derivations $\tilde{\theta}_i$ with $i \geq 2$ correspond to the α_i's in Example 1.6. It follows that the enveloping algebra of the Lie–Rinehart pair $(S, \operatorname{Der}\mathcal{B}_{n+1})$ is isomorphic to the product $U(\Bbbk[z], \operatorname{Der}\Bbbk[z]) \times U(T, \operatorname{Der}\tilde{\mathcal{B}}_n)$. The result is now a consequence of Theorem 1.20 and Example 1.19. ☐

Definition 1.22. *Let $n \geq 1$, $S = \Bbbk[x_1, \ldots, x_n]$ and L a triangularizable Lie–Rinehart algebra over S with basis $(\alpha_1, \ldots, \alpha_n)$. We say that L satisfies the* orthogonality condition *if there exists a family $(u_1, \ldots, u_n)$ of elements of U and $f_k^i \in S$ for $1 \leq k \leq n$ and $1 \leq i \leq n-1$ such that*

$$u_k = \alpha_n + \sum_{i=1}^{n-1} f_k^i \alpha_i, \qquad\qquad [u_k, x_l] = 0 \quad \text{if } k \neq l.$$

Example 1.23. Consider for $n \geq 2$ the Lie–Rinehart algebra $\operatorname{Der}\mathcal{B}_n$ from Example 1.5. The family $(u_1, \ldots, u_n)$ of elements of U defined for $k \in [\![n]\!]$ by

$$u_k = \sum_{i=k}^{n} (-1)^{n-i} \prod_{j=i+1}^{n} (x_j - x_k)\, \theta_i$$

is such that $[u_k, x_l] = 0$, whence the orthogonality condition is satisfied. The Lie–Rinehart algebra $\operatorname{Der}\tilde{\mathcal{B}}_n$ from Example 1.6 also satisfies this condition with a similar choice of orthogonal elements.

Let $r \geq 1$ and $\mathcal{A}_r = \mathcal{A}(C_r \wr \mathfrak{S}_3)$. Let $S = \Bbbk[x_1, x_2, x_3]$ and $L = \operatorname{Der}\mathcal{A}_r$ be the Lie-Rinehart algebra associated to $\mathcal{A}_r$. The derivations $\{\alpha_1, \alpha_2, \alpha_3\}$ given in Example 1.9 make of L a triangular Lie algebra that satisfies the Bézout condition. We identify the universal enveloping algebra of L with $\operatorname{Diff}\mathcal{A}_r$.

Proposition 1.24. *The Lie–Rinehart algebra associated to $\mathcal{A}_r$ together with the family $\{u_1, u_2, u_3\}$ of elements of* $\operatorname{Diff} \mathcal{A}_r$ *defined by*

$$u_1 = \alpha_3 - (x_3^r - x_1^r)\alpha_2 + (x_3^r - x_1^r)(x_2^r - x_1^r)\alpha_1, \qquad u_2 = \alpha_3 - (x_3^r - x_2^r)\alpha_2, \qquad u_3 = \alpha_3$$

satisfies the orthogonality condition.

Proof. The condition $[u_k, x_l] = 0$ if $k, l \in [\![3]\!]$ and $l \neq k$ holds true whenever $l < k$, so we suppose that $l > k$. If $k = 3$ there is nothing to see; the case $k = 2$ amounts to the verification that

$$\begin{aligned}[u_2, x_3] &= \alpha_3(x_3) - (x_3^r - x_2^r)\alpha_2(x_3) \\ &= x_3(x_3^r - x_2^r)(x_3^r - x_1^r) - (x_3^r - x_2^r)x_3(x_3^r - x_1^r) = 0\end{aligned}$$

and for or $k = 1$ we have

$$\begin{aligned}[u_1, x_2] &= \alpha_3(x_2) - (x_3^r - x_1^r)\alpha_2(x_2) + (x_3^r - x_1^r)(x_2^r - x_1^r)\alpha_1(x_2) \\ &= -(x_3^r - x_1^r)x_2(x_2^r - x_1^r) + (x_3^r - x_1^r)(x_2^r - x_1^r)x_2 = 0\end{aligned}$$

and

$$\begin{aligned}[u_1, x_3] &= \alpha_3(x_3) - (x_3^r - x_1^r)\alpha_2(x_3) + (x_3^r - x_1^r)(x_2^r - x_1^r)\alpha_1(x_3) \\ &= x_3(x_3^r - x_2^r)(x_3^r - x_2^r) - (x_3^r - x_1^r)x_3(x_3^r - x_1^r) + (x_3^r - x_1^r)(x_2^r - x_1^r)x_3 = 0.\end{aligned}$$

□

1.4. **Cohomology.** Given an associative algebra A the (associative) enveloping algebra A^e is the vector space $A \otimes A$ endowed with the product $\cdot$ defined by $(a_1 \otimes a_2) \cdot (b_1 \otimes b_2) = a_1 b_1 \otimes b_2 a_2$, so that the category of left A^e-modules is equivalent to that of A-bimodules. The *Hochschild cohomology* of A with values on an A^e-module M is

$$H^\bullet(A, M) \coloneqq \operatorname{Ext}^\bullet_{A^e}(A, M).$$

When $M = A$ we write $HH^\bullet(A) \coloneqq H^\bullet(A, M)$. C. Weibel's book [Wei94] may serve as general reference on this subject.

Definition 1.25 ([Rin63])**.** *Let (S, L) be a Lie–Rinehart algebra with enveloping algebra U and let N be an U-module. The* Lie–Rinehart cohomology of (S, L) with values on N *is*

$$H^\bullet_S(L, N) \coloneqq \operatorname{Ext}^\bullet_U(S, N).$$

Remark 1.26 ([Rin63]). In the setting of Definition 1.25 above, suppose that L is S-projective and let $\Lambda^\bullet_S L$ denote the exterior algebra of L over S. The complex $\operatorname{Hom}_S(\Lambda^\bullet_S L, N)$ with Chevalley–Eilenberg differentials computes $H^\bullet_S(L, N)$.

Theorem 1.27 ([KL21])**.** *Let (S, L) be a Lie–Rinehart algebra with enveloping algebra U, and suppose that L is an S-projective module. There exist a U-module structure on $H^\bullet(S, U)$ and a first-quadrant spectral sequence $E_\bullet$ converging to $HH^\bullet(U)$ with second page*

$$E_2^{p,q} = H^p_S(L, H^q(S, U)).$$

Proposition 1.28. *There are isomorphisms $HH^\bullet(\operatorname{Diff} \mathcal{B}_{n+1}) \cong HH^\bullet(\operatorname{Diff} \tilde{\mathcal{B}}_n)$ for any $n \geq 1$.*

Proof. This is a consequence of applying the Künneth's formula for Hochschild cohomology as in H. Cartan and S. Eilenberg's [CE56, XI.3.I] to the isomorphism $\operatorname{Diff}\mathcal{B}_{n+1} \cong A_1 \otimes \operatorname{Diff}\tilde{\mathcal{B}}_n$ in Proposition 1.21 and the observation that $HH^0(A_1) \cong \Bbbk$ and $HH^i(A_1) \cong \Bbbk$ if $i \neq 0$. $\square$

Corollary 1.29. *The Hilbert series of the Hochschild cohomology of* $\operatorname{Diff}\mathcal{B}_3$ *is*

$$h(t) = 1 + 3t + 6t^2 + 4t^3.$$

Proof. Proposition 1.28 particularizes to $HH^\bullet(\operatorname{Diff}\mathcal{B}_3) \cong HH^\bullet(\operatorname{Diff}\tilde{\mathcal{B}}_2)$, and then [KL21, Corollary 5.8] reads $h_{HH^*(\operatorname{Diff}\mathcal{B}_3)} = h_{HH^*(\operatorname{Diff}\tilde{\mathcal{B}}_2)} = 1 + 3t + 6t^2 + 4t^3$. $\square$

2. Combinatorics of the Koszul complex

We let $n \geq 1$ and assume throughout this section that (S, L) is a Lie–Rinehart algebra with $S = \Bbbk[x_1, \dots, x_n]$ and L a free S-module with basis $(\alpha_1, \dots, \alpha_n)$. Let $U = U(S, L)$ be its Lie–Rinehart enveloping algebra. To compute the Hochschild cohomology of S we use the Koszul resolution of S available in [Wei94, §4.5].

Lemma 2.1. *Let W be the subspace of S with basis $(x_1, \dots, x_n)$. The complex $P_\bullet = S^e \otimes \Lambda^\bullet W$ with differentials $b_\bullet : P_\bullet \to P_{\bullet-1}$ defined for $s, t \in S$, $q \in [\![n]\!]$ and $1 \leq i_1 < \cdots < i_q \leq n$ by*

$$b_q(s|t \otimes x_{i_1} \wedge \cdots \wedge x_{i_q}) = \sum_{j=1}^{q} (-1)^{j+1} [(s x_{i_j}|t) - (s|x_{i_j} t)] \otimes x_{i_1} \wedge \cdots \wedge \check{x}_{i_j} \wedge \cdots \wedge x_{i_q}$$

and augmentation $\varepsilon : S^e \to S$ given by $\varepsilon(s|t) = st$ is a resolution of S by free S^e-modules. The notation is the usual one: the symbol $|$ denotes the tensor product inside S^e and $\check{x}_{i_j}$ means that x_{i_j} is omitted.

Through a classical adjunction, the complex $\operatorname{Hom}_{S^e}(P_\bullet, U)$ is isomorphic to

$$\operatorname{Hom}(\Lambda^\bullet W, U) \cong U \otimes \operatorname{Hom}(\Lambda^\bullet W, \Bbbk) =: \mathfrak{X}^\bullet. \tag{2}$$

We compute the Hochschild cohomology $H^\bullet(S, U)$ from the complex $(\mathfrak{X}^\bullet, d^\bullet)$. For each q in $[\![0, n]\!]$ the basis $\{\hat{x}_{k_1} \wedge \ldots \wedge \hat{x}_{k_q} : 1 \leq k_1 < \ldots < k_q \leq n\}$ of $\operatorname{Hom}(\Lambda^q W, \Bbbk)$ dual to the basis $\{x_{k_1} \wedge \ldots \wedge x_{k_q}\}$ of $\Lambda^q W$ induces a basis of $\mathfrak{X}^q$ as a U-module.

Write $\alpha^I := \alpha_n^{i_n} \ldots \alpha_1^{i_1}$ for each n-tuple of nonnegative integers $I = (i_n, \dots, i_1)$, and call $|I| = i_n + \ldots + i_1$ the *order* of I. A result *à la* Poincaré-Birkhoff-Witt in [Rin63, §3] assures that the set

$$\{\alpha^I : I \in \mathbb{N}^n\} \tag{3}$$

is an S-basis of U. Moreover, U is a filtered algebra, with filtration $(F_p U : p \geq 0)$ given by *the order of differential operators*: $F_p U = \langle f\alpha^I : f \in S, |I| \leq p \rangle$ for each $p \geq 0$.

Proposition 2.2. *Let $q \in \{0, \dots, n\}$.*

(*i*) *The set formed by $\alpha^I \hat{x}_{k_1} \wedge \cdots \wedge \hat{x}_{k_q}$ with $I \in \mathbb{N}^n$ and $1 \leq k_1 < \ldots < k_q \leq n$ is an S-basis of $\mathfrak{X}^q$.*

(*ii*) *There is a filtration $(F_p\mathfrak{X}^q : p \geq 0)$ of vector spaces on $\mathfrak{X}^q$ determined for each $p \geq 0$ by*

$$F_p\mathfrak{X}^q = \langle f\alpha^I \hat{x}_{k_1} \wedge \cdots \wedge \hat{x}_{k_q} : f \in S, 1 \leq k_1 < \ldots < k_q \leq n, I \in \mathbb{N}^n \text{ such that } |I| \leq p \rangle.$$

Proof. In view of (2), for each q the U-module $\mathfrak{X}^q$ admits $\{\hat{x}_{k_1} \wedge \cdots \wedge \hat{x}_{k_q} : 1 \le k_1 < \ldots < k_q \le n\}$ as a basis. The claim follows from this and the S-basis of U in (3) above. $\square$

The differentials $d^q : \mathfrak{X}^q \to \mathfrak{X}^{q+1}$ induced by $b_\bullet : P_\bullet \to P_{\bullet-1}$ satisfy for $q = 0, 1$

$$d^0 : u \mapsto \sum_{k=1}^{n} [u, x_k]\hat{x}_k, \qquad d^1 : \sum_{k=1}^{n} u_k \hat{x}_k \mapsto \sum_{1\le k<l\le n} ([u_k, x_l] - [u_l, x_k])\, \hat{x}_k \wedge \hat{x}_l.$$

Given $m \in [\![n]\!]$ we denote by e_m the n-tuple whose components are all zero except for the $(n-m)$th, where there is a 1.

Lemma 2.3. *Let $a = \sum_{|I|=p} f^I \alpha^I$ for $f^I \in S$ with $I \in \mathbb{N}^n$. If $k \in [\![n]\!]$ and $J = (j_n, \ldots, j_1) \in \mathbb{N}^n$ has order $p-1$ then the component of $[a, x_k]$ in α^J is*

$$\sum_{m=1}^{n} (j_m + 1)\alpha_m(x_k) f^{J+e_m}.$$

Proof. If $I = (i_n, \ldots, i_1) \in \mathbb{N}^n$ has order p then

$$[f^I \alpha^I, x_k] \equiv i_n f^I \alpha_n(x_k)\alpha^{I-e_n} + \ldots + i_1 f^I \alpha_1(x_k)\alpha^{I-e_1} \mod F_{p-2}U.$$

If there exists a monomial in this expression belonging to $S\alpha^J$ then there exists $m \in [\![n]\!]$ such that $I - e_m = J$. This happens when the component of $[f^I\alpha^I, x_k]$ in α^J is

$$i_m \alpha_m(x_k) f^I = (j_m + 1)\alpha_m(x_k) f^{J+e_m},$$

and therefore $[a, x_k] \equiv \sum_{|J|=p-1} \sum_{m=1}^{n} (j_m + 1)\alpha_m(x_k) f^{J+e_m} \alpha^J$ modulo $F_{p-2}U$. $\square$

For $g^1, \ldots, g^n \in S$ and $f^1 = (f^1_1, \ldots, f^1_n), \ldots, f^n = (f^n_1, \ldots, f^n_n) \in S^{\times n}$ we let

$$\Omega^0(g^n, \ldots, g^1) \coloneqq \sum_{i=1}^{n} g^i \alpha_i \in F_1 U, \qquad \Omega^1(f^n, \ldots, f^1) \coloneqq \sum_{l=1}^{n}\sum_{i=1}^{n} f^i_k \alpha_i \hat{x}_k \in F_1\mathfrak{X}^1.$$

Proposition 2.4. *Let $p \ge 0$, $u \in F_pU$ and $\omega \in F_p\mathfrak{X}^1$.*

(i) If $\{f^I : I \in \mathbb{N}^n, |I| = p\} \subset S$ is such that $u \equiv \sum_{|I|=p} f^I\alpha^I \mod F_{p-1}U$ then

$$d^0(u) \equiv \sum_{|J|=p-1} d^0\left(\Omega^0\left((j_n+1)f^{J+e_n}, (j_{n-1}+1)f^{J+e_{n-1}}, \ldots, (j_1+1)f^{J+e_1}\right)\right)\alpha^J$$

modulo $F_{p-2}\mathfrak{X}^1$.

(ii) If $\omega \equiv \sum_{l=1}^{n}\sum_{|I|=p} f^I_l \alpha^I \hat{x}_i \mod F_{p-1}\mathfrak{X}^1$ for $\{f^I_l : I \in \mathbb{N}^n, |I| = p, l \in [\![n]\!]\} \subset S$ then

$$d^1(\omega) \equiv \sum_{|J|=p-1} d^1\left(\Omega^1\left((j_n+1)f^{J+e_n}, (j_{n-1}+1)f^{J+e_{n-1}}, \ldots, (j_1+1)f^{J+e_1}\right)\right)\alpha^J$$

modulo $F_{p-2}\mathfrak{X}^2$.

Proof. To prove *(i)* it suffices to see that the desired equality holds in each coefficient of the S-basis $\left(\alpha^J\hat{x}_k : J \in \mathbb{N}^n, k \in [\![n]\!]\right)$ of $\mathfrak{X}^1$ given in Proposition 2.2. Let then $J = (j_n, \ldots, j_1) \in \mathbb{N}^n$ of order $p-1$ and $k \in [\![n]\!]$. Thanks to Lemma 2.3 the component in $\alpha^J\hat{x}_k$ of $d^0(u) = \sum_{l=1}^{n}[u, x_l]\hat{x}_l$ is

$$\sum_{m=1}^{n} (j_m + 1)\alpha_m(x_k) f^{J+e_m}. \tag{4}$$

On the other hand, given $f^n,\dots,f^1 \in S$ a direct calculation shows that the component in $\hat{x}_k$ of $d^0\left(\Omega^0\left(f^n,\dots,f^1\right)\right)$ is $\sum_{i=1}^n f^i\alpha_i(x_k)$. It follows that the component of

$$d^0\left(\Omega^0\left((j_n+1)f^{J+e_n},\dots,(j_1+1)f^{J+e_1}\right)\right)$$

in $\hat{x}_k$ is equal to (4), which is tantamount to what we wanted to see. The proof of *(ii)* is completely analogous. □

3. Cohomologies in degree zero and centers

In this section (S,L) is a triangularizable Lie algebra: $S=\Bbbk[x_1,\dots,x_n]$ for some $n\geq 1$ and L is a sub-S-module of derivations of S with a basis given by derivations $\alpha_1,\dots,\alpha_n$ that satisfy $\alpha_i(x_j)=0$ if $i>j$ and $\alpha_i(x_i)\neq 0$ for every $i\in[\![n]\!]$. Let U be the enveloping algebra of (S,L).

3.1. **The cohomology of S with values on U.** The Hochschild cohomology $H^\bullet(S,U)$ is the cohomology of the complex $(\mathfrak{X}^\bullet,d^\bullet)$ of (2).

Lemma 3.1. *The restriction of $d^0:\mathfrak{X}^0\to\mathfrak{X}^1$ to $F_1\mathfrak{X}^0$ has kernel $F_0\mathfrak{X}^0$.*

Proof. It is evident that $F_0\mathfrak{X}^0=S$ is contained in $\ker d^0$. Let $u\in F_1U$ and $f^1,\dots,f^n\in S$ such that $u\equiv\sum_{i=1}^n f^i\alpha_i$ modulo S. We examine the equations $d^0(u)(1|x_l|1)=0$, that is, $[u,x_l]=0$ for each $1\leq l\leq n$. We first observe that

$$0=[u,x_1]=\sum_{i=1}^n f^i\alpha_i(x_1)=f^1\alpha_1(x_1),$$

and then $f^1=0$. Proceeding inductively on k, we assume that $u=\sum_{i=k}^n f^i\alpha_i$ and compute

$$0=[u,x_k]=f^k\alpha_k(x_k)+\ldots+f^n\alpha_n(x_k)=f^k\alpha_k(x_k).$$

We deduce that $f^k=0$ and conclude that $u\in S$. □

Proposition 3.2. *If $p>0$ and $u\in F_pU$ are such that $d^0(u)\equiv 0$ modulo $F_{p-2}\mathfrak{X}^1$ then $u\in F_{p-1}U$.*

Proof. Let $\{f^I : I\in\mathbb{N}^n, |I|=p\}\subset S$ be such that $u\equiv\sum_{|I|=p}f^I\alpha^I$ modulo $F_{p-1}U$: thanks to Proposition 2.4 we have that

$$d^0(u)\equiv\sum_{|J=(j_n,\dots,j_1)|=p-1} d\left(\Omega^0\left((j_n+1)f^{J+e_n},\dots,(j_1+1)f^{J+e_1}\right)\right)\alpha^J \mod F_{p-2}\mathfrak{X}^1.$$

We deduce that $0=d^0\left(\Omega^0\left((j_n+1)f^{J+e_n},\dots,(j_1+1)f^{J+e_1}\right)\right)$ for each J with $|J|=p-1$, provided that $p-1\geq 0$. Thanks to Lemma 3.1 we deduce that $0=f^{J+e_{N-2}}=\ldots=f^{J+e_{-1}}$. Since we can write every $I\in\mathbb{N}$ with $|I|=p$ as $I=J+e_m$ for some $m\in[\![n]\!]$ we conclude that if $p-1\geq 0$ then $f^I=0$ for every I with $|I|=p$. □

Proposition 3.3. *The inclusion $S\hookrightarrow U=\mathfrak{X}^0$ induces an isomorphism of graded U-modules $H^0(S,U)=S$*

Proof. Let us write $u=u_0+\ldots+u_p$ with $u_q\in F_qU\setminus F_{q-1}U$ and $p\geq 0$ maximal among those q such that $u_q\neq 0$. As $d^0(u)=0$ and $d^0(u_q)\in F_{q-1}\mathfrak{X}^1$ for every $q\in[\![0,p]\!]$ we have that $d(u_p)\equiv 0$ mod $F_{p-2}X^1$, and we may use Proposition 3.2 to see that if $p>0$ then $u_p=0$. We conclude then that $p=0$, so that actually $u\in S$. We obtain the result with the evident observation that every element of S is a 0-cocycle in $\mathfrak{X}^\bullet$. □

3.2. **The cohomology of U.** Our recent calculation of $H^0(S,U)$ leaves us just one step away from the zeroth Hochschild cohomology space of U.

Theorem 3.4. *Let (S,L) be a triangularizable Lie–Rinehart algebra with enveloping algebra U. There is an isomorphism of vector spaces $HH^0(U) \cong \Bbbk$.*

Proof. As a consequence of the immediate degeneracy of the spectral sequence of Theorem 1.27 there is an isomorphism of vector spaces $HH^0(U) \cong H^0_S(L, H^0(S,U))$. In view of Proposition 3.3, this isomorphism amounts to

$$HH^0(U) \cong H^0_S(L,S) \cong \{f \in S : \alpha_i(f) = 0 \text{ if } i \in [\![n]\!]\}$$

Since $\alpha_i(f) = \sum_{j=1}^n \alpha_i(x_j)\partial_j f$ for $f \in S$, the condition that $\alpha_i(f) = 0$ if $i \in [\![n]\!]$ means that $(\partial_1 f, \dots, \partial_n f)$ belongs to the kernel of the Saito's matrix $M = (\alpha_i(x_j))_{i,j=1}^n$. As this matrix is triangular and its determinant is nonzero, the condition $\alpha_i(f) = 0$ for all $i \in [\![n]\!]$ is equivalent to $\partial_j f = 0$ for all $j \in [\![n]\!]$, which is to say that $f \in \Bbbk$. □

Corollary 3.5. *Let $r, n \geq 1$. The centers of $\operatorname{Diff}(\mathcal{A}(C_r \wr \mathfrak{S}_n))$ and of $\operatorname{Diff}\mathcal{B}_n$ are $\Bbbk$.*

Proof. The algebras considered have been shown to satisfy the hypotheses of Theorem 3.4 in Examples 1.13 and 1.8. □

4. The first cohomology space $H^1(S,U)$

We now restrict our attention to the case in which $n = 3$. Let then (S,L) be a Lie-Rinehart algebra with $S = \Bbbk[x_1,x_2,x_3]$ and L the free S-module generated by the subset of derivations $\{\alpha_1,\alpha_2,\alpha_3\}$ in $\operatorname{Der} S$. We suppose that (S,L) is triangularizable, this is, $\alpha_i(x_j) = 0$ if $i > j$ and $\alpha_1(x_1)\alpha_2(x_2)\alpha_3(x_3) \neq 0$, and that (S,L) satisfies the Bézout condition:

- the polynomials $\alpha_2(x_2)$ and $\alpha_2(x_3)$ are coprime;
- the polynomials $\alpha_1(x_1)$ and $\det\left(\begin{smallmatrix} \alpha_1(x_2) & \alpha_1(x_3) \\ \alpha_2(x_2) & \alpha_2(x_2) \end{smallmatrix}\right)$ are coprime.

Lemma 4.1. *Let $\{f^i_l : i \in \{1,2\}, l \in [\![3]\!]\} \subset S$ and write $\omega = \sum_{l=1}^3 \left(f^2_l \alpha_2 + f^1_l \alpha_1\right)\hat{x}_l \in \mathfrak{X}^1$. If ω is a cocycle then there exist unique elements g_{11}, g_{12}, g_{22} of S such that $g_{11}\alpha_1(x_1) = f^1_1$, $g_{12}\alpha_1(x_1) = f^2_1$ and $g_{22}\alpha_2(x_2) = f^2_2 - g_{12}\alpha_1(x_2)$. These elements satisfy*

$$\omega \equiv d(\tfrac{1}{2}g_{11}\alpha_1^2 + g_{12}\alpha_2\alpha_1 + \tfrac{1}{2}g_{22}\alpha_2^2) \mod F_0\mathfrak{X}^1.$$

Proof. The components in $\hat{x}_1 \wedge \hat{x}_2$ and $\hat{x}_1 \wedge \hat{x}_3$ of $d\omega = 0$ tell us that $\alpha_1(x_j)f^1_1 + \alpha_2(x_j)f^j_1 = \alpha_1(x_1)f^1_j$ for $j \in \{2,3\}$. We can arrange these two equations as

$$\begin{pmatrix} \alpha_1(x_2) & \alpha_2(x_2) \\ \alpha_1(x_3) & \alpha_2(x_3) \end{pmatrix} \begin{pmatrix} f^1_1 \\ f^2_1 \end{pmatrix} = \alpha_1(x_1) \begin{pmatrix} f^1_2 \\ f^1_3 \end{pmatrix} \tag{5}$$

and then Cramer's rule tells us that if $i \in \{1,2\}$ then $f^i_1 \det \tilde{M} = \alpha_1(x_1) \det \tilde{M}_i$, where $\tilde{M}$ is the matrix on the left hand of (5) and $\tilde{M}_i$ is the matrix obtained by replacing the ith column of $\tilde{M}$ by $\left(\begin{smallmatrix} f^1_2 \\ f^1_3 \end{smallmatrix}\right)$. It follows that $\alpha_1(x_1)$ divides f^i_1 —because it is coprime with $\det \tilde{M}$ in view of the Bézout hypothesis— and then there exist g_{11} and g_{12} in S such that $g_{1i}\alpha_1(x_1) = f^i_1$.

Let $u_1 \coloneqq \frac{1}{2}g_{11}\alpha_1^2 + g_{12}\alpha_2\alpha_1$ and $\tilde\omega \coloneqq \omega - d(u_1)$, and write $\tilde\omega = \sum_{l=1}^3 \left(\tilde f_l^2\alpha_2 + \tilde f_l^1\alpha_1\right)\hat x_l$. Since

$$\begin{aligned} d(u_1) : x_1 \mapsto [u, x_1] &\equiv g_{11}\alpha(x_1)\alpha_1 + g_{12}\alpha_1(x_1)\alpha_2 \mod S \\ &= f_1^2\alpha_1 + f_1^2\alpha_2, \\ x_2 \mapsto [u, x_2] &\equiv g_{11}\alpha_1(x_2)\alpha_1 + g_{12}\alpha_2(x_2)\alpha_1 + g_{12}\alpha_1(x_2)\alpha_2 \mod S \end{aligned} \tag{6}$$

we have $\tilde f_1^1 = \tilde f_1^2 = 0$. Now, the equation $d\tilde\omega = 0$ in $\hat x_1 \wedge \hat x_2$ and $\hat x_1 \wedge \hat x_3$ tells us, as in (5), that $\tilde f_2^1 = \tilde f_3^1 = 0$, and in $\hat x_2 \wedge \hat x_3$ that $\alpha_2(x_2)\tilde f_3^2 = \alpha_2(x_3)\tilde f_2^2$. Thanks to the Bézout condition there exists $g_{22} \in S$ such that $g_{22}\alpha_2(x_2) = \tilde f_2^2$; in view of (6), $\tilde f_2^2$ is equal to $f_2^2 - g_{12}\alpha_1(x_2)$. Put $u_2 \coloneqq \frac{1}{2}g_{22}\alpha_2^2$. We see that $d(u_2)(x_1) = 0$ and that

$$\begin{aligned} d(u_2)(x_2) = [u_2, x_2] &\equiv g_{22}\alpha_2(x_2)\alpha_2 \mod S \\ &= \tilde f_2^2\alpha_2. \end{aligned}$$

The difference $\bar\omega \coloneqq \tilde\omega - d(u_2)$ is therefore a coboundary with no component modulo S in $\hat x_1$ nor in $\hat x_2$, so we can write $\bar\omega \equiv \left(f^1\alpha_1 + f^2\alpha_2\right)\hat x_3 \mod F_0\mathfrak{X}^1$. Now, the equations that come from $\bar\omega$ being a coboundary are $0 = f^1\alpha_1(x_1)$ in $\hat x_1 \wedge \hat x_3$, from which $f^1 = 0$, and $0 = f^2\alpha_2(x_2)$ in $\hat x_2 \wedge \hat x_3$, whence finally $\bar\omega \in F_0\mathfrak{X}^1$. We have in this way obtained that $\omega \equiv d(u_1 + u_2) \mod F_0\mathfrak{X}^1$, as desired. □

Proposition 4.2. *Let $p \geq 0$ and $\omega \in F_p\mathfrak{X}^1$, and let $\left\{f_l^{(i_3,i_2,i_1)} : l \in [\![3]\!], i_3, i_2, i_1 \geq 0\right\} \subset S$ such that*

$$\omega \equiv \sum_{l=1}^{3} \sum_{i_1+i_2+i_3=p} f_l^{(i_3,i_2,i_1)}\alpha^{(i_3,i_2,i_1)}\hat x_l \mod F_{p-1}\mathfrak{X}^1. \tag{7}$$

If ω is a cocycle and $f_1^{(p,0,0)} = f_2^{(p,0,0)} = f_3^{(p,0,0)} = 0$ then $\omega \in F_{p-1}\mathfrak{X}^1$.

Proof. Let us prove by descending induction on i from p to 0 that

the cocycle ω is cohomologous modulo $F_{p-1}\mathfrak{X}^1$ to a cocycle of the form (7) with $f_l^{(i_3,i_2,i_1)} = 0$ if $l \in [\![3]\!]$ and $i_3 \geq i$. (8)

Our hypotheses give us the truth of (8) for $i = p$. Suppose now that (8) is true for $p, \ldots, i$ and assume, without loss of generality, that ω *is* of the form (7) with $f_l^{(i_3,i_2,i_1)} = 0$ if $l \in [\![3]\!]$, $i_1 + i_2 + i_3 = p$ and $i_3 \geq i$.

Lemma 4.3. *Let $q \in \{0, \ldots, p-i+1\}$. The cocycle ω is cohomologous modulo $F_{p-1}\mathfrak{X}^1$ to a cocycle of the form (7) with $f_l^{(i-1,p-i+1,0)} = \ldots = f_l^{(i-1,p-i+1-q,q)} = 0$ and $f_l^{(i_3,i_2,i_1)} = 0$ if $i_1 + i_2 + i_3 = p$ and $i_3 \geq i$ for every $l \in [\![3]\!]$.*

The auxiliary result above implies at once the truth of the inductive step of the proof of (8), thus demonstrating Proposition 4.2. □

Proof of Lemma 4.3. Suppose that $q = 0$. Equation $d\omega = 0$ in its component $(i-1, p-i, 0)$ reads, thanks to Proposition 2.4,

$$0 = d\left(\Omega^1(if^{(i,p-i,0)}, (p-i+1)f^{(i-1,p-i+1,0)}, f^{(i-1,p-i,1)})\right)$$

and the inductive hypothesis (8) tells us that $f^{(i,p-i,0)} = 0$. Applying now Lemma 4.1 in we obtain that there are $g_{11}, g_{12}, g_{22} \in S$ such that

$$
\begin{aligned}
&g_{11}\alpha_1(x_1) = f_1^{(i-1,p-i,1)}, \qquad g_{12}\alpha_1(x_1) = (p-i+1)f_1^{(i-1,p-i+1,0)},\\
&g_{22}\alpha_2(x_2) = (p-i+1)f_2^{(i-1,p-i+1,0)} - g_{12}\alpha_1(x_2)
\end{aligned}
$$

Let $v = (\frac{1}{2}g_{11}\alpha_1^2 + \frac{1}{(p-i+1)}g_{12}\alpha_2\alpha_1)\alpha^{(i-1,p-i,0)}$ and write $\tilde{\omega} = \omega - d(v)$, so that there exists $\left\{\tilde{f}_l^{(i_3,i_2,i_1)}\right\} \subset S$ such that

$$
\tilde{\omega} \equiv \sum_{l=3}^{n} \sum_{i_1+i_2+i_3=p} \tilde{f}_l^{(i_3,i_2,i_1)}\alpha^{(i_3,i_2,i_1)}\hat{x}_l \mod F_{p-1}\mathfrak{X}^1.
$$

Recall that $d(v) : x_l \mapsto [v, x_l]$ for $l \in [\![3]\!]$. Since

$$
\begin{aligned}
[v, x_1] &\equiv (g_{11}\alpha_1(x_1)\alpha_1 + \tfrac{1}{(p-i+1)}g_{12}\alpha_1(x_1)\alpha_2)\alpha^{(i-1,p-i,0)} \mod F_{p-1}U\\
&= f_1^{(i-1,p-i,1)}\alpha^{(i-1,p-i,1)} + f_1^{(i-1,p-i+1,0)}\alpha^{(i-1,p-i+1,0)},
\end{aligned}
$$

we have that $\tilde{f}_1^{(i-1,p-i,1)} = \tilde{f}_1^{(i-1,p-i+1,0)} = 0$. Moreover, as $[v, x_2], [v, x_3] \in \bigoplus_{i_3<i} S\alpha^{(i_3,i_2,i_1)}$ the coefficients $\tilde{f}_l^{(i_3,i_2,i_1)}$ are equal to $f_l^{(i_3,i_2,i_1)}$ and therefore to zero if $i_1 + i_2 + i_3 = p$, $i_3 \geq i$ and $l \in [\![3]\!]$.

We now look at equation $d\tilde{\omega} = 0$, again in its coefficient of $\alpha^{(i-1,p-i,0)}$ to obtain that $0 = d\left(\Omega^1(0, (p-i+1)\tilde{f}^{(i-1,p-i+1,0)}, \tilde{f}^{(i-1,p-i,1)})\right)$. This equation in its component in $\hat{x}_1 \wedge \hat{x}_2$ tells us, thanks to (5), that $\tilde{f}_2^{(i-1,p-i,1)} = \tilde{f}_3^{(i-1,p-i,1)} = 0$. On the other hand, applying Lemma 4.1 we get $g \in S$ such that $g\alpha_2(x_2) = (p-i+1)\tilde{f}_2^{(i-1,p-i+1,0)}$. Let now $\lambda = 1/(p-i+2)(p-i+1)$ and $\tilde{v} = \lambda g\alpha^{(i-1,p-i+2,0)}$. Since $[\tilde{v}, x_1] = 0$,

$$
\begin{aligned}
[\tilde{v}, x_2] &\equiv \lambda g(p-i+2)\alpha_2(x_2)\alpha^{(i-1,p-i+1,0)} \mod F_{p-1}U\\
&= \lambda(p-i+1)\tilde{f}_2^{(i-1,p-i+1,0)}(p-i+2)\alpha^{(i-1,p-i+1,0)}\\
&= \tilde{f}_2^{(i-1,p-i+1,0)}\alpha^{(i-1,p-i+1,0)}
\end{aligned}
$$

and $[\tilde{v}, x_3] \in \bigoplus_{i_3<i} S\alpha^{(i_3,i_2,i_1)}$, the difference $\tilde{\omega} - d(\tilde{v})$ is a cohomologous modulo $F_{p-1}\mathfrak{X}^1$ to a cocycle $\eta = \sum_{l=3}^{n}\sum_{i_1+i_2+i_3=p} h_l^{(i_3,i_2,i_1)}\alpha^{(i_3,i_2,i_1)}\hat{x}_l$ with $h_l^{(i_3,i_2,i_1)} = 0$ if $i_1 + i_2 + i_3 = p$, $i_3 \geq i$ and $l \in [\![3]\!]$ and $h_l^{(i-1,p-i,1)} = h_l^{(i-1,p-i+1,0)} = 0$ if $l \in \{1, 2\}$. Applying Lemma 4.1 one final time we obtain $\tilde{g}_{11}, \tilde{g}_{12}, \tilde{g}_{22} \in S$ such that $\tilde{g}_{11}\alpha_1(x_1) = (p-i+1)h_1^{(i-1,p-i+1,0)}$, $\tilde{g}_{12}\alpha_1(x_1) = h_1^{(i-1,p-i,1)}$ and $\tilde{g}_{22}\alpha_2(x_2) = (p-i+1)h_2^{(i-1,p-i,1)} - \tilde{g}_{12}\alpha_1(x_2)$ —and therefore $\tilde{g}_{11}$, $\tilde{g}_{12}$ and $\tilde{g}_{22}$ must be equal to 0— and that satisfy

$$
\Omega^1(0, (p-i+1)h^{(i-1,p-i+1,0)}, h^{(i-1,p-i,1)}) \equiv d(\tfrac{1}{2}\tilde{g}_{11}\alpha_1^2 + \tilde{g}_{12}\alpha_2\alpha_1 + \tfrac{1}{2}\tilde{g}_{22}\alpha_2^2) \mod F_0\mathfrak{X}^1.
$$

It follows that $h^{(i-1,p-i+1,0)} = h^{(i-1,p-i,1)} = 0$, and therefore that $\eta \equiv 0 \mod F_{p-1}\mathfrak{X}^1$. This finishes the proof of the base step of Lemma 4.3.

We finally deal with the inductive step of Lemma 4.3. Let q, i and ω be as in the statement. The component in $\alpha^{(i-1,p-i-q,q)}$ of equation $d\omega = 0$ yields

$$
0 = d\left(\Omega^1\left(if^{(i,p-i-q,q)}, (p-i-q+1)f^{(i-1,p-i-q+1,q)}, (q+1)f^{(i-1,p-i-q,q+1)}\right)\right).
$$

Now, our inductive hypotheses of (8) and of Lemma 4.3 tell us, respectively, that $f^{(i,p-i-q,q)}=0$ and that $f^{(i-1,p-i-q+1,q)}=0$, and therefore our equation above reduces to

$$0=d\left(\Omega^1\left(0,0,f^{(i-1,p-i-q,q+1)}\right)\right).$$

Applying to this situation Lemma 4.1 we obtain $g\in S$ such that $g\alpha_1(x_l)=f_l^{(i-1,p-i-q,q+1)}$ for $l\in[\![3]\!]$. Let $v=\frac{1}{(q+2)}g\alpha^{(i-1,p-i-q,q+2)}$ and write $\tilde\omega=\omega-d(v)$: let $\left\{\tilde f_l^{(i_3,i_2,i_1)}\right\}\subset S$ such that $\tilde\omega\equiv\sum_{l=3}^n\sum_{i_1+i_2+i_3=p}\tilde f_l^{(i_3,i_2,i_1)}\alpha^{(i_3,i_2,i_1)}\hat x_l$ modulo $F_{p-1}\mathfrak{X}^1$. As

$$[v,x_1]\equiv g\alpha_1(x_1)\alpha^{(i-1,p-i-q,q+1)}=f_1^{(i-1,p-i-q,q+1)}\alpha^{(i-1,p-i-q,q+1)}\mod F_{p-1}U$$

and if $j\in\{2,3\}$ then

$$[v,x_j]\in S\alpha^{(i-2,p-i-q,q+2)}\oplus S\alpha^{(i-1,p-i-q-1,q+2)}\oplus S\alpha^{(i-1,p-i-q,q+1)}\oplus F_{p-1}U,$$

we obtain that $\tilde f_l^{(i_3,i_2,i_1)}=0$ whenever $i_3\geq i$ and $\tilde f_l^{(i-1,p-i+1,0)}=\ldots=\tilde f_l^{(i-1,p-i+1-q,q)}=0$ for every $l\in[\![3]\!]$ and, in addition, that $\tilde f_1^{(i-1,p-i-q,q+1)}=0$. As a consequence of this, the component in $\alpha^{(i-1,p-i-q,q)}$ of equation $d\tilde\omega=0$ reduces to $0=d\left(\Omega^1\left(0,0,\tilde f^{(i-1,p-i-q,q+1)}\right)\right)$. The element $\tilde g$ that is provided for this situation by Lemma 4.1 satisfies $\tilde g\alpha_1(x_l)=\tilde f_l^{(i-1,p-i-q,q+1)}$ for $l\in[\![3]\!]$: it follows that $g=0$ and hence $\tilde f_l^{(i-1,p-i-q,q+1)}=0$ for $l\in[\![3]\!]$ and $\tilde\omega\equiv0\mod F_{p-1}\mathfrak{X}^1$. This finishes the proof of Lemma 4.3. □

From this point on we demand to (S,L) that in addition it satisfy the orthogonality condition: that there be a family u_1,u_2,u_3 of elements of U that can be written as $u_k=\alpha_3+h_k^2\alpha_2+h_k^1\alpha_1$ for some $\left\{h_k^i:k\in[\![3]\!],i\in[\![2]\!]\right\}\subset S$ and such that $[u_k,x_l]=0$ whenever $k\neq l$. The idea is that that we can add to any cocycle in $F_p\mathfrak{X}^1$ an S-linear combination of $u_k^p\hat x_k$ to remove its components in the maximum power of α_3 and in this way obtain a cocycle that falls in the hypotheses of Proposition 4.2.

Corollary 4.4. *Let $\{u_1,u_2,u_3\}$ be the family that makes (S,L) satisfy the orthogonality condition.*

- *(i) The cochains $\eta_k^p=u_k^p\hat x_k\in F_p\mathfrak{X}^1$ defined for $p\geq0$ and $k\in[\![3]\!]$ are cocycles.*
- *(ii) Every cocycle in $\mathfrak{X}^1$ is cohomologous to one in the S-submodule of $\mathfrak{X}^1$ generated by $\{\eta_k^p:k\in[\![3]\!],p\geq0\}$.*

Proof. Let us denote by Z^1 the S-module generated by $\{\eta_l^p:l\in[\![3]\!],p\geq0\}$. We prove by induction on $p\geq0$ that if $\omega\in F_p\mathfrak{X}^1$ is a cocycle then there exist $z\in Z^1$ and $u\in U$ such that $\omega=d^0(u)+z$. We first observe that $F_0(\mathfrak{X}^1)=F_0(Z^1)$ because $\hat x_l=\eta_l^0$, and then for $p=0$ we have that $\omega\in F_0(\mathfrak{X}^1)\subset Z^1$.

Assume now that $p>0$ and let $\left\{f_l^I:l\in[\![3]\!],I\in\mathbb{N}^3\right\}\subset S$ such that $\omega\equiv\sum_{l=1}^3\sum_{|I|=p}f_l^I\alpha^I\hat x_l$ mod $F_{p-1}\mathfrak{X}^1$. Defining $z=\sum_{l=1}^3 f_l^{(p,0,0)}\eta_l^p$ we see that the cocycle $\tilde\omega:=\omega-z$ has its components in $\alpha^p\hat x_1,\alpha^p\hat x_2,\alpha^p\hat x_3$ equal to zero, and applying Proposition 4.2 we deduce that $\tilde\omega$ is a coboundary modulo $F_{p-1}\mathfrak{X}^1$: let $u\in U$ and $\omega'\in F_{p-1}\mathfrak{X}^1$ be such that $\tilde\omega=d^0(u)+\omega'$. The inductive hypothesis tells us that there exist $u'\in U$ and $z'\in Z^1$ such that $\omega'=d^0(u')+z'$, and thus $\omega=\tilde\omega+z=d^0(u+u')+(z+z')$, as we wanted. □

Proposition 4.5. *Let $p\geq0$.*

(i) *Let $\omega \in F_p\mathfrak{X}^1$ be a cocycle, so that there exist $\{f_1, f_2, f_3\} \subset S$ and $u \in U$ such that $\omega \equiv \sum_{l=1}^3 f_l\eta_l^p + du$ modulo $F_{p-1}\mathfrak{X}^1$. The cocycle ω is equivalent to a coboundary modulo $F_{p-1}\mathfrak{X}^1$ if and only if $\sum_{l=1}^3 f_l\hat{x}_l$ is a coboundary.*

(ii) *The unique S-linear map $\gamma_p : F_0\mathfrak{X}^1 \to F_pX^1$ such that $\hat{x}_l \mapsto \eta_l^p$ if $1 \le l \le 3$ induces an isomorphism of S-modules*

$$F_pH^1(S,U)/F_{p-1}H^1(S,U) \cong F_0H^1(S,U).$$

Proof. Suppose that $\omega_0 = \sum_{l=1}^n f_l\hat{x}_l$ is a coboundary and let $v \in U$ such that $d^0(v) = \omega_0$. Thanks to Proposition 3.2 we may assume that $v \in F_1\mathfrak{X}^1$ and write $v \equiv g^3\alpha_3 + g^2\alpha_2 + g^1\alpha_1 \mod S$ for some $g^3, g^2, g^1 \in S$. In view of Proposition 2.4 there exist $f_0^I \in F_0\mathfrak{X}^1$ such that we may write

$$d^0\left(\frac{g^3}{p+1}\alpha_3^{p+1} + g^2\alpha_3^p\alpha_2 + g^1\alpha_3^p\alpha_1\right) \equiv d^0(v)\,\alpha_3^p + \sum_{i_3<p} f_0^{(i_3,i_2,i_1)}\alpha^{(i_3,i_2,i_1)} \mod F_{p-1}\mathfrak{X}^1.$$

It follows that the difference $\omega - d^0\left(\frac{g^3}{p+1}\alpha_3^{p+1} + g^2\alpha_3^p\alpha_2 + g^1\alpha_3^p\alpha_1\right)$ is a cochain whose components in $\alpha_3^p\hat{x}_1, \alpha_3^p\hat{x}_2, \alpha_3^p\hat{x}_N$ are zero. Applying Proposition 4.2 we see that ω is equivalent to a coboundary modulo $F_{p-1}\mathfrak{X}^1$.

Reciprocally, let $u \in U$ such that $d^0(u) = \omega$. Thanks to Proposition 3.2 we know that $u \in F_{p+1}U$: let us write $u \equiv \sum_{|K|=p+1} h^K\alpha^K$ with $\{h^K : K \in \mathbb{N}^3, |K| = p+1\} \subset S$. Taking into account Proposition 2.4 again we see that

$$d^0(u) \equiv d\left(\Omega^0\left((p+1)h^{(p+1,0,0)}, h^{(p,1,0)}, h^{(p,0,1)}\right)\right)\alpha_3^p + \sum_{i_3<p} f_0^{(i_3,i_2,i_1)}\alpha^{(i_3,i_2,i_1)}$$

modulo $F_{p-1}\mathfrak{X}^1$ for some $f_0^I \in F_0\mathfrak{X}^1$. The equality of this to ω implies, looking at the components in $\alpha_3^p\hat{x}_1, \alpha_3^p\hat{x}_2, \alpha_3^p\hat{x}_3$, that $d\left(\Omega^0\left((p+1)h^{(p+1,0,0)}, h^{(p,1,0)}, h^{(p,0,1)}\right)\right) = \sum_{l=1}^3 f_l\hat{x}_l$. This completes the proof of the first item.

Now, the truth of the first item implies two things: first, that the composition $F_0\mathfrak{X}^1 \to F_p\mathfrak{X}^1/F_{p-1}\mathfrak{X}^1$ of γ_p with the projection to the quotient descends to cohomology and, second, that the map induced in cohomology by this composition is a monomorphism. It is also surjective thanks to Corollary 4.4(ii). □

Recall that the filtered S-module $F_\bullet H^1(S,U)$ has a graded associated S-module $\operatorname{Gr}_\bullet H^1(S,U) = \bigoplus_{p\ge 0}\operatorname{Gr}_p H^1(S,U)$ given by $\operatorname{Gr}_p H^1(S,U) := F_pH^1(S,U)/F_{p-1}H^1(S,U)$. We have just seen that $\operatorname{Gr}_p H^1(S,U)$ is isomorphic as an S-module to $F_0H^1(S,U)$ for any $p \ge 0$: we claim that we can make it an isomorphism of *graded* S-modules.

Given $p \ge 0$, the map $\gamma_p : F_0\mathfrak{X}^1 \to F_pX^1$ induces an isomorphism of S-modules $F_0H^1(S,U) \cong \operatorname{Gr}_p H^1(S,U)$ that shifts the polynomial degree in $3(p-1)$: indeed, for each $l \in [\![3]\!]$ the class of $\eta_l \in \mathfrak{X}^1$, which has polynomial degree 2, is sent to the class of η_l^p, which has polynomial degree $3p-1$. On the other hand, the morphism of S-modules $\gamma : F_0(H^1(S,U)) \otimes \Bbbk[\alpha_3] \to \operatorname{Gr} H^1(S,U)$ such that $[\eta_l] \otimes \alpha_3^p \mapsto [\eta_l^p]$ for $l \in [\![3]\!]$ and $p \ge 0$ does respect the graduation and is an isomorphism because so is each γ_p. In addition to this, we observe that

$$F_0(H^1(S,U)) = \frac{S \otimes \langle \hat{x}_1, \hat{x}_2, \hat{x}_3\rangle}{Sd^0(\alpha_1) + Sd^0(\alpha_2) + Sd^0(\alpha_3)} \cong \operatorname{coker} M.$$

We summarize our findings in the following statement.

Corollary 4.6. *Let $S=\Bbbk[x_1,x_2,x_3]$ and L a free S-submodule of* $\operatorname{Der} S$ *generated by derivations $\alpha_1,\alpha_2,\alpha_3$ in such a way that (S,L) is a triangularizable Lie–Rinehart algebra that satisfies the Bézout and orthogonality conditions. Let U be the Lie–Rinehart enveloping algebra of L. There is an isomorphism of S-graded modules*

$$H^1(S,U)\cong \operatorname{coker} M\otimes \Bbbk[\alpha_3],$$

where M is the Saito's matrix of (S,L).

Recall that the cokernel of M has a rich algebraic structure — see M. Granger, D. Mond and M. Schulze's [GMS11].

5. Computation of $HH^1(U)$

The spectral sequence in Theorem 1.27, regardless of its degeneracy, gives us an strategy to obtain the first Hochschild cohomology space $HH^\bullet(U)$ of the enveloping algebra U of a Lie–Rinehart algebra (S,L): indeed, $HH^1(U)$ is isomophic to $H^1_S(L,H^0(S,U))\oplus H^0_S(L,H^1(S,U))$. In Sections 3 and 4 we computed $H^0(S,U)$ and $H^1(S,U)$ when (S,L) is as in Corollary 4.6, which is the conclusion of Section 4. In this section we describe their L-module structure and use it to compute their respective Lie–Rinehart cohomology spaces for the case in which (S,L) is associated to a hyperplane arrangement of the form $\mathcal{A}_r=\mathcal{A}(C_r\wr\mathfrak{S}_3)$ as in Example 1.9.

5.1. **The L-module structure on $H^\bullet(S,U)$.** Let (S,L) be a Lie-Rinehart pair with enveloping algebra U. Let us describe the construction in [KL21] that gives an L-module structure to the Hochschild cohomology $H^\bullet(S,U)$ of S with values on U.

Fix $\alpha\in L$ and an S^e-projective resolution $\varepsilon:P_\bullet\to S$. Let $\alpha_\bullet$ be an α_S^e-lifting of $\alpha_S:S\to S$ to $P_\bullet$, that is, a morphism of complexes $\alpha_\bullet=(\alpha_q:P_q\to P_q)_{q\geq 0}$ such that $\varepsilon\circ\alpha_0=\alpha_S\circ\varepsilon$ and for each $q\geq 0$, s, $t\in S$ and $p\in P_q$

$$\alpha_q((s\otimes t)\cdot p)=(\alpha_S(s)\otimes t+s\otimes\alpha_S(t))\cdot p+(s\otimes t)\cdot p.$$

The endomorphism $\alpha_\bullet^\sharp$ of $\operatorname{Hom}_{S^e}(P_\bullet,U)$, defined for each $q\geq 0$ to be

$$\alpha_q^\sharp(\phi):p\mapsto[\alpha,\phi(p)]-\phi\circ\alpha_q(p)\quad\text{whenever } \phi\in\operatorname{Hom}_{S^e}(P_q,U)\text{ and } p\in P_q,$$

allows us to define the map $\nabla_\alpha^\bullet:H^\bullet(S,U)\to H^\bullet(S,U)$ as the unique graded endomorphism such that

$$\nabla_\alpha^q([\phi])=[\alpha_q^\sharp(\phi)],$$

where [-] denotes class in cohomology. The final result is that $\alpha\mapsto\nabla_\alpha^q$ defines an L-module structure on $H^q(S,U)$ for each $q\geq 0$.

5.2. **The liftings.** From now on we work on the Lie–Rinehart algebra associated to $\mathcal{A}_r$ and put $E:=\alpha_1$, $D:=\alpha_2$ and $C:=\alpha_3$. The commuting relations in L are determined by the rules

$$\begin{aligned}&[E,C]=(2r+1)C, &&[E,D]=(r+1)D,\\ &[D,C]=r(x_3^r+x_2^r-x_1^r),\end{aligned}\tag{9}$$

as a straightforward calculation shows.

Proposition 5.1. *The rules*

$$\begin{aligned}
&D_1(1|x_1|1) = 0,\\
&D_1(1|x_k|1) = \sum_{s+t=r} x_k^s|x_k|x_k^t - \sum_{s+t=r-1} x_k^s|x_1|x_1^t x_k - x_1^r|x_k|1 \quad \text{if } k = 2,3
\end{aligned}$$

define a D^e-lifting of $D : S \to S$.

Proof. It is evident that $d_1 \circ D_1$ and $D_0 \circ d_1$ coincide at $1|x_1|1$; if $k = 2,3$ then $d_1 \circ D_1(1|x_k|1)$ is

$$\begin{aligned}
\sum_{s+t=r} x_k^s(x_k|1 - 1|x_k)x_k^t - \sum_{s+t=r-1} x_k^s(x_1|1 - 1|x_1)x_1^t x_k - x_1^r x_k|1 + x_1^r|x_k\\
= \left(x_k^{r+1} - x_k x_1^r\right)|1 - 1|\left(x_k^{r+1} - x_k x_1^r\right),
\end{aligned}$$

which equals $D_0 \circ d_1(1|x_k|1) = D_0(x_k|1 - 1|x_k) = D(x_k)|1 - 1|D(x_k)$ because $x_k(x_k^r - x_1^r)$ is $D(x_k)$. □

Proposition 5.2. *For every $p \geq 0$ we have that*

$$\begin{aligned}
&D_1^\sharp(\eta_1^p) \equiv pr(x_3^r + x_2^r - x_1^r)\eta_1^p + rx_1^{r-1}x_2\eta_2^p + rx_1^{r-1}x_3\eta_3^p \mod F_{p-1}\mathfrak{X}^1 \mod \operatorname{Im} d^0\\
&D_1^\sharp(\eta_2^p) \equiv ((1-p)x_1^r + (p-r-1)x_2^r + px_3^r)\,\eta_2^p \mod F_{p-1}\mathfrak{X}^1,\\
&D_1^\sharp(\eta_3^p) \equiv ((1-p)x_1^r + px_2^r + (p-r-1)x_3^r)\,\eta_3^p \mod F_{p-1}\mathfrak{X}^1.
\end{aligned}$$

Proof. Recall from Corollary 4.4 the cocycle $\eta_l^p = u_l^p\hat{x}_l$ for each $l \in [\![3]\!]$ that is such that u_l commutes with every x_j with $j \neq l$. The commuting relations (9) in L give

$$[D, u_3] = [D, C] = r(x_3^r + x_2^r - x_1^r)C = r(x_3^r + x_2^r - x_1^r)u_3$$

and

$$\begin{aligned}
[D, u_2] &= [D, C - (x_3^r - x_2^r)D]\\
&= r(x_3^r + x_2^r - x_1^r)C - (rx_3^r(x_3^r - x_1^r) - rx_2^r(x_2^r - x_1^r))\,D\\
&= r(x_3^r + x_2^r - x_1^r)\,(C - (x_3^r - x_2^r)D) = r(x_3^r + x_2^r - x_1^r)u_2.
\end{aligned}$$

It follows that if $l = 2,3$ then $[D, u_l^p] \equiv p(x_3^r + x_2^r - x_1^r)u_l^p$ modulo $F_{p-1}U$. We can now compute

$$\begin{aligned}
&D_1^\sharp(\eta_l^p)(1|x_1|1) = [D, \eta_l^p(1|x_1|1)] - \eta_l^p \circ D_1(1|x_1|1) = [D, 0] - \eta_l^p(0) = 0;\\
&D_1^\sharp(\eta_l^p)(1|x_l|1) = [D, \eta_l^p(1|x_l|1)] - \eta_l^p \circ D_1(1|x_l|1)\\
&\quad = [D, u_l^p] - \eta_l^p\left(\sum_{s+t=r} x_l^s|x_l|x_l^t - \sum_{s+t=r-1} x_l^s|x_1|x_1^t x_l - x_1^r|x_l|1\right)\\
&\quad \equiv p(x_3^r + x_2^r - x_1^r)u_l^p - ((r+1)x_l^r - x_1^r)\,u_l^p \mod F_{p-1}U
\end{aligned}$$

and for $m \neq 1, l$

$$\begin{aligned}
&D_1^\sharp(\eta_l^p)(1|x_m|1) = [D, \eta_l^p(1|x_m|1)] - \eta_l^p \circ D_1(1|x_m|1)\\
&\quad = [D, 0] - \eta_l^p\left(\sum_{s+t=r} x_m^s|x_m|x_m^t - \sum_{s+t=r-1} x_1^s|x_1|x_1^t x_m - x_1^r|x_m|1\right) = 0.
\end{aligned}$$

With this information at hand we are able to see that

$$D_1^\sharp(\eta_2^p) \equiv ((1-p)x_1^r + (p-r-1)x_2^r + px_3^r)\,\eta_2^p \mod F_{p-1}\mathfrak{X}^1,$$
$$D_1^\sharp(\eta_3^p) \equiv ((1-p)x_1^r + px_2^r + (p-r-1)x_3^r)\,\eta_3^p \mod F_{p-1}\mathfrak{X}^1.$$

Let us now consider the action of D on η_1^p. To begin with, we have

$$\begin{aligned}
[D,u_1^p] &\equiv pu_1^{p-1}[D,u_1] \mod F_{p-1}U \\
&= pu_1^{p-1}[D,(C-(x_3^r-x_1^r)D+(x_3^r-x_1^r)(x_2^r-x_1^r)E)] \\
&= pu_1^{p-1}\Big(r(x_3^r+x_2^r-x_1^r)C - rx_3^r(x_3^r-x_1^r)D \\
&\qquad + D\left((x_3^r-x_1^r)(x_2^r-x_1^r)\right)E - (r+1)(x_3^r-x_1^r)(x_2^r-x_1^r)D\Big)
\end{aligned}$$

and we observe that $D_1^\sharp(\eta_1^p)(1|x_1|1) = [D,u_1^p] - \eta_1^p(D_1(1|x_1|1)) = [D,u_1^p]$. On the other hand, if $m \in \{2,3\}$ then

$$\begin{aligned}
D_1^\sharp(\eta_1^p)(1|x_m|1) &= [D,\eta_1^p(1|x_m|1)] - \eta_1^p(D_1(1|x_m|1)) \\
&= [D,0] - \eta_1^p\left(\sum_{s+t=r} x_m^s|x_m|x_m^t - \sum_{s+t=r-1} x_1^s|x_1|x_1^t x_m - x_1^r|x_m|1\right) \\
&\equiv rx_1^{r-1}x_m u_1^p.
\end{aligned}$$

From these computations we see that the cocycle

$$D_1^\sharp(\eta_1^p) - pr(x_3^r+x_2^r-x_1^r)\eta_1^p - rx_1^{r-1}x_2\eta_2^p - rx_1^{r-1}x_3\eta_3^p$$

has component zero in $C^p\hat{x}_1$, $C^p\hat{x}_2$ and $C^p\hat{x}_3$, and then Proposition 4.2 tells us that $D_1^\sharp(\eta_1^p)$ is cohomologous modulo $F_{p-1}\mathfrak{X}^1$ to $pr(x_3^r+x_2^r-x_1^r)\eta_1^p + rx_1^{r-1}x_2\eta_2^p + rx_1^{r-1}x_3\eta_3^p$. □

5.3. **Invariants of $H^1(S,U)$ by the action of L.** We already have explicit descriptions of $H^1(S,U)$, in Section 4, and of the action of L thereon, in Subsection 5.1 above: the next step is to calculate the intersection of the kernels of the actions of E, D and C on $H^1(S,U)$.

Proposition 5.3. $H^0_S(L,H^1(S,U)) = 0$.

Proof. Recall that the polynomial grading on S induces a grading on S, on U and on the cohomology $H^\bullet(S,U)$. Since the derivation E induces the linear endomorphism ∇_E^1 of $H^1(S,U)$ that sends the class of an homogeneous element a of degree $|a|$ to the class of $|a|a$ it follows that $\ker\nabla_E^1 = H^1(S,U)_0$, where $H^1(S,U)_0$ is the subspace of $H^1(S,U)$ formed by elements of degree zero. Remember that if $k \in [\![3]\!]$ then $|u_k| = |\alpha_3| = 2r$, and therefore $|\eta_k| = 2r-1$. In view of our calculation in Corollary 4.6 this means that

$$\ker\nabla_E^1 = H^1(S,U)_0 \cong \begin{cases} \frac{S_1\otimes\langle\hat{x}_1,\hat{x}_2,\hat{x}_3\rangle}{\Bbbk(x_1\hat{x}+x_2\hat{x}_2+x_3\hat{x}_3)} \oplus \langle\eta_1,\eta_2,\eta_3\rangle & \text{if } r=1; \\ \frac{S_1\otimes\langle\hat{x}_1,\hat{x}_2,\hat{x}_3\rangle}{\Bbbk(x_1\hat{x}+x_2\hat{x}_2+x_3\hat{x}_3)} & \text{if } r\geq 2. \end{cases}$$

We begin by supposing that $r \geq 2$. We observe that if $f_1, f_2, f_3 \in S_1$ then

$$\begin{aligned} D_1^\sharp\left(\sum f_i\hat{x}_i\right) &= \sum\left(D(f_i)\hat{x}_i + f_iD_1^\sharp(\hat{x}_i)\right) \\ &= \sum D(f_i)\hat{x}_i + f_1r\left(x_1^{r-1}x_2\hat{x}_2 + x_1^{r-1}x_3\hat{x}_3\right) + f_2(x_1^r-(r+1)x_2^r)\hat{x}_2 \\ &\qquad\qquad + f_3(x_1^r-(r+1)x_3^r)\hat{x}_3 \\ &= D(f_1)\hat{x}_1 + \left(D(f_2) + f_1rx_1^{r-1}x_2 + f_2(x_1^r-(r+1)x_2^r)\right)\hat{x}_2 \\ &\qquad\qquad + \left(D(f_3) + f_1rx_1^{r-1}x_3 + f_3(x_1^r-(r+1)x_3^r)\right)\hat{x}_3 \end{aligned}$$

belongs to the homogeneous component of degree r of $F_0H^1(S,U)$, which is precisely

$$\left(\frac{S\otimes\langle\hat{x}_1,\hat{x}_2,\hat{x}_3\rangle}{Sd^0(E)+Sd^0(D)+Sd^0(C)}\right)_r = \frac{S_{r+1}\otimes\langle\hat{x}_1,\hat{x}_2,\hat{x}_3\rangle}{S_rd^0(E)+\Bbbk d^0(D)}.$$

It follows that if $\nabla_D^1\left([\sum f_i\hat{x}_i]\right) = \left[D_1^\sharp\left(\sum f_i\hat{x}_i\right)\right]$ is zero in cohomology there must exist $g \in S_r$ and $\mu \in \Bbbk$ such that

$$D_1^\sharp\left(\sum f_i\hat{x}_i\right) = g(x_1\hat{x}_1+x_2\hat{x}_2+x_3\hat{x}_3) + \mu\left(x_2(x_2^r-x_1^r)\hat{x}_2 - x_3(x_3^r-x_1^r)\hat{x}_3\right). \tag{10}$$

Let us write $f_i = f_{i,1}x_1 + f_{i,2}x_2 + f_{i,3}x_3$ with $f_{i,j} \in \Bbbk$ for $i,j \in [\![3]\!]$. Up to the addition of coboundary that is a scalar multiple of $d^0(E) = x_1\hat{x}_1+x_2\hat{x}_2+x_3\hat{x}_3$ we may suppose that $f_{1,1} = 0$. In $\hat{x}_1$ we have $D(f_1) = gx_1$, or, in other words,

$$f_{1,2}(x_2^{r+1}-x_1x_2^r) + f_{1,3}(x_3^{r+1}-x_1x_3^r) = gx_1.$$

The components in x_2^{r+1} and in x_3^{r+1} of this equality read $f_{1,2} = 0$ and $f_{1,3} = 0$: this implies that $g = 0$, and of course that $f_1 = 0$. Next, equation (10) in $\hat{x}_2$ yields the equality in S_{r+1}

$$D(f_2) + f_2(x_1^r-(r+1)x_2^r) = \mu(x_2(x_2^r-x_1^r)).$$

In x_1^{r+1} and x_3^{r+1} we have $f_{2,1} = 0$ and $f_{2,3} = 0$, and what remains is $-rf_{2,2}x_2^{r+1} = \mu(x_2(x_2^r - x_1^r)$. It follows that $\mu = 0$ and therefore $f_2 = 0$; analogously, $f_3 = 0$. We conclude that $\ker\nabla_D^1|_{H^1(S,U)_0} = 0$ when $r \geq 2$.

Let us now suppose that $r = 1$ and compute the kernel of the restriction of ∇_D^1 to $H^1(S,U)_0$. Let then $f_1, f_2, f_3 \in S$ and $\lambda_1, \lambda_2, \lambda_3$ be such that $\nabla_D^1\left([\sum f_i\hat{x}_i + \lambda_i\eta_i]\right)$ is zero in cohomology. Since

$$\begin{aligned} H^1(S,U)_1 \cong{}& \frac{S_2\otimes\langle\hat{x}_1,\hat{x}_2,\hat{x}_3\rangle}{S_1(x_1\hat{x}+x_2\hat{x}_2+x_3\hat{x}_3)+\Bbbk(x_2(x_2-x_1)\hat{x}_2+x_3(x_3-x_1)\hat{x}_3)} \\ &\qquad\qquad \oplus \frac{S_1\langle\eta_1,\eta_2,\eta_3\rangle}{\Bbbk(x_1\eta_1+x_2\eta_2+x_3\eta_3)} \oplus \langle\eta_1^2,\eta_2^2,\eta_3^2\rangle \end{aligned}$$

there exist $\mu_1, \mu_2 \in \Bbbk$ and $g \in S_1$ such that

$$\begin{aligned} D_1^\sharp\left(\sum f_i\hat{x}_i + \lambda_i\eta_i\right) = g(x_1\hat{x}+x_2\hat{x}_2+x_3\hat{x}_3) + \mu_2(x_2(x_2-x_1)\hat{x}_2 + x_3(x_3-x_1)\hat{x}_3) \\ + \mu_1(x_1\eta_1+x_2\eta_2+x_3\eta_3) \end{aligned} \tag{11}$$

We know from Proposition 5.2 that modulo S $D_1^\sharp(\eta_1) \equiv (x_3+x_2-x_1)\eta_1 + x_2\eta_2 + x_3\eta_3$, $D_1^\sharp(\eta_2) \equiv (-x_2+x_3)\eta_2$ and $D_1^\sharp(\eta_3) \equiv (x_2-x_3)\eta_3$ and since $D_1^\sharp(\sum S\hat{x}_i) \subset S$ the equality (11) implies that

$$\lambda_1(x_3+x_2-x_1)\eta_1 + (\lambda_1 x_2 + \lambda_2(-x_2+x_3))\,\eta_2 + (\lambda_1 x_3 + \lambda_3(x_2-x_3))\,\eta_3 = \mu_1(x_1\eta_1 + x_2\eta_2 + x_3\eta_3)$$

This is an equality in $\bigoplus_{i=1}^3 S_1\eta_i$. In $S_1\eta_3$ we have $\lambda_3 x_2 + (\lambda_1-\lambda_3)x_3 = \mu_1 x_3$, so $\lambda_3 = 0$ and $\lambda_1 = \mu_1$. In $S_1\eta_2$ an analogous argument shows that $\lambda_2 = 0$ and $\lambda_1 = \mu_1$ again, and finally in $S_1\eta_1$ we have

$$\lambda_1(x_3+x_2-x_1) = \lambda_1 x_1.$$

It follows that $\lambda_1 = \mu_1 = 0$. Consider now what is left of (11): it is precisely (10) replacing r by 1. The same argument, therefore, allows us to see that $\ker \nabla_D^1|_{H^1(S,U)_0} = 0$ when $r = 1$.

We conclude that $H_S^0(L, H^0(S,L)) \subset \ker\left(\nabla_D^1 : H^1(S,U)_0 \to H^1(S,U)_1\right) = 0$, from which $H_S^0(L, H^1(S,L)) = 0$ independently of $r \geq 1$. ☐

Corollary 5.4. *Let $r \geq 1$ and $A_r = \mathcal{A}(C_r \wr \mathfrak{S}_3)$. If (S,L) is its associated Lie–Rinehart algebra and U its enveloping algebra then $HH^1(U) \cong H_S^1(L,S)$. In particular, the dimension of $HH^1(U)$ is $3r+3$, the number of hyperplanes of $\mathcal{A}_r$.*

Proof. Thanks to Theorem 1.27 $HH^1(U) \cong H_S^1(L, H^0(S,U)) \oplus H_S^0(L, H^1(S,U))$; Proposition 3.3 tells us that $H^0(S,U) = S$ and Proposition 5.3 above that the second summand is zero. ☐

Let $f \in S_1$ be a linear form whose kernel is one of the hyperplanes in $\mathcal{A}_3$. It is a direct verification that there is a unique derivation $\partial_f : U \to U$ such that

$$\begin{cases} \partial_f(g) = 0 & \text{if } g \in S; \\ \partial_f(\theta) = \theta(f)/f & \text{if } \theta \in \operatorname{Der} \mathcal{A}_r. \end{cases}$$

Fix as well $\Bbbk = \mathbb{C}$ and factorize the defining polynomial $Q(\mathcal{A}_r) = x_1x_2x_3\prod_{1\leq i<j\leq 3}(x_j^r - x_i^r)$ as

$$Q(\mathcal{A}_r) = x_1x_2x_3\prod_{j=0}^{r-1}(x_2 - e^{2j\pi i/r}x_1)(x_3 - e^{2j\pi i/r}x_1)(x_3 - e^{2j\pi i/r}x_2) \tag{12}$$

Corollary 5.5. *The Lie algebra of outer derivations of* $\operatorname{Diff}\mathcal{A}_r$ *together with the commutator is an abelian Lie algebra of dimension $3r+3$ generated by the classes of the derivations ∂_f with f in a linear factor of (12).*

Proof. We claim that the classes of ∂_f, with f one of the linear factors in (12), are linearly independent in $\operatorname{OutDer}(U)$. Indeed, let $u \in U$ and $\lambda_f \in \Bbbk$ be such that

$$\sum \lambda_f \partial_f(v) = [u,v] \qquad \text{for every } v \in U. \tag{13}$$

Evaluating (13) on each $v = g \in S$ we obtain that the left side vanishes and therefore $u \in H^0(S,U)$, which is equal to S in view of Proposition 3.3. Write $u = \sum_{j\geq 0} u_j$ with $u_j \in S_j$. Evaluating now (13) on E we obtain that $\sum_{f\in\mathbb{A}}\lambda_f = -\sum_{j\geq 0} ju_j$. In each homogeneous component S_j with $j \neq 0$ we have $ju_j = 0$ and therefore $u \in S_0 = \Bbbk$ and, when $j = 0$, $\sum_f \lambda_f = 0$.

Evaluating the left hand side of (13) on C gives $\sum_f \lambda_f\partial_f(C)$. Now, if $\partial_f(C) = C(f)/f = \partial_3(f)C(x_3)/f$ is nonzero then $\partial_3(f) \neq 0$ and thus f is a factor of $C(x_3)$: let us, then, factor

$C(x_3)$ by x_3 and $f_{l,j} = x_3 - e^{2j\pi i/r}x_l$ for $l = 1,2$ and $j \in [\![0, r-1]\!]$, and in this way reformulate the evaluation of (13) at C as the nullity of

$$\sum_{f\in\mathbb{A}} \partial_3(f)C(x_3)/f = \lambda_{x_3}(x_3^r - x_2^r)(x_3^r - x_1^r) + \sum_{l=1,2}\sum_{j=0}^{r-1} \lambda_{f_{l,j}} x_3(x_3^r - x_2^r)(x_3^r - x_1^r)/f_{l,j}.$$

Fix now $l \in [\![2]\!]$ and $j \in [\![0, r-1]\!]$ and apply the morphism of algebras $\varepsilon_{l,j} : S \to \Bbbk[x_1, x_2]$ that sends x_3 to $e^{2k\pi i/r}x_l$: since $\varepsilon_{l,j}\left((x_3^r - x_{l'}^r)/f_{j',l'}\right) = 0$ whenever $l \neq l'$ and $j \neq j'$ we obtain that

$$\varepsilon_{l,j} : \sum_{f\in\mathbb{A}} \partial_3(f)C(x_3)/f \mapsto \lambda_{f_{l,j}} x_3(x_3^r - x_2^r)(x_3^r - x_1^r)/f_{l,j}.$$

As the expression at which we evaluated $\varepsilon_{l,j}$ was zero, it follows that $\lambda_{f_{l,j}} = 0$ and, immediately, that also $\lambda_{x_3} = 0$.

We observe that the indexes that survive in the sum $\sum \lambda_f \partial_f$ are x_1, x_2 and $f_j = x_2 - e^{2j\pi i/r}x_1$ with $j \in [\![0, r-1]\!]$; evaluating at D we obtain

$$\sum \lambda_f \partial_f(D) = \lambda_{x_2}(x_2^r - x_1^r) + \sum_{j=0}^{r-1} \lambda_{f_j} x_2(x_2^r - x_1^r)/f_j.$$

Reasoning as above we get that $\lambda_{x_2} = \lambda_{f_j} = 0$ for every j. Recalling now that $\sum_{f\in\mathbb{A}} \lambda_f = 0$ we see that $\lambda_{x_1} = 0$ as well.

The classes of ∂_f with f a linear factor in (12) span $\operatorname{OutDer} U$ because the dimension of $\operatorname{OutDer} U \cong HH^1(U)$ is, thanks to Corollary 5.4, precisely $|\mathcal{A}|$. The composition $\partial_f \circ \partial_g : U \to U$ is evidently equal to zero for any $f, g \in \mathbb{A}$, as a straightforward calculation shows, and therefore the Lie algebra structure in $\operatorname{OutDer} U$ vanishes. □

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