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Yachun Li, Yue-Jun Peng, Liang Zhao

CONVERGENCE RATE FROM HYPERBOLIC SYSTEMS OF BALANCE LAWS TO PARABOLIC SYSTEMS

YACHUN LI[1], YUE-JUN PENG[2,*], LIANG ZHAO[3]

[1]School of Mathematical Sciences, MOE-LSC, and SHL-MAC, Shanghai Jiao Tong University
Shanghai 200240, P. R. China

[2]Université Clermont Auvergne, CNRS, Laboratoire de Mathématiques Blaise Pascal
F-63000 Clermont-Ferrand, France

[3]School of Mathematical Sciences, Shanghai Jiao Tong University
Shanghai 200240, P. R. China

Abstract. It is proved recently that partially dissipative hyperbolic systems converge globally-in-time to parabolic systems in a slow time scaling, when initial data are smooth and sufficiently close to constant equilibrium states. Based on this result, we establish error estimates between the smooth solutions of the hyperbolic systems of balance laws and those of the parabolic limit systems in one space dimension. The proof of the error estimates uses a stream function technique together with energy estimates. As applications of the results, we give five examples arising from physical models.

Keywords: Convergence rate, hyperbolic system of balance laws, partial dissipation, parabolic system, stream function

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1. Introduction

Consider a system of balance laws of the form

$$\partial_{t'}U + \sum_{j=1}^{d} \partial_{x_j} F_j(U) = -\frac{Q(U)}{\varepsilon}, \tag{1.1}$$

with initial condition

$$U(0,x) = U_0^\varepsilon(x), \quad x \in \mathbb{R}^d. \tag{1.2}$$

Here $U : \mathbb{R}^+_{t'} \times \mathbb{R}^d_x \to S \subset \mathbb{R}^n$ is the unknown variable, $\varepsilon \in (0,1]$ is a small parameter standing often for the relaxation time in physical models (see Section 5), $t' > 0$ is the usual

∗ Corresponding author

E-mail address: yue-jun.peng@uca.fr

time and $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$ is the space variable. The vector $Q : S \to \mathbb{R}^n$ and the flux function $F_j : S \to \mathbb{R}^n$ are smooth for all $1 \le j \le d$. The set S is called the state space. We assume (1.1) is symmetrizable hyperbolic, i.e., there exists a symmetric positive definite matrix $A_0(U)$, called symmetrizer, such that $A_0(U)F_j'(U)$ is symmetric for all $1 \le j \le d$ and all $U \in S$.

Usually, the source term $Q(U)$ is of the form

$$Q(U) = \begin{bmatrix} 0 \\ q(U) \end{bmatrix},$$

where $q : S \to \mathbb{R}^r$ is a smooth function, $1 \le r \le n$. With the same partition, we denote

$$U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad u \in \mathbb{R}^{n-r}, \quad v \in \mathbb{R}^r,$$

as well as the initial data

$$U_0^\varepsilon(x) = \begin{bmatrix} u_0^\varepsilon(x) \\ v_0^\varepsilon(x) \end{bmatrix}, \quad u_0^\varepsilon(x) \in \mathbb{R}^{n-r}, \quad v_0^\varepsilon(x) \in \mathbb{R}^r.$$

We also denote the flux function as $F_j = \begin{bmatrix} f_j \\ g_j \end{bmatrix}$, where $f_j : S \to \mathbb{R}^{n-r}$ and $g_j : S \to \mathbb{R}^r$ are both smooth functions. More generally, under the same partition, a vector $V \in \mathbb{R}^n$ and an $n \times n$ matrix M will be denoted by $V = \begin{bmatrix} V^I \\ V^{II} \end{bmatrix}$ and $M = \begin{bmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{bmatrix}$, respectively.

A large number of physical models of the form (1.1) can be found in [29, 20]. The local existence of smooth solutions to (1.1)-(1.2) is well-known due to Lax [13] and Kato [11], see also [17]. Under sub-characteristic conditions [16], (1.1) converges formally to a hyperbolic system as $\varepsilon \to 0$. We refer the reader to [16, 2, 9, 1, 30, 24] and references therein for the mathematical analysis and results.

It is well-known that, generally speaking, smooth solutions of the Cauchy problem (1.1)-(1.2) exist only locally in time and singularities may appear in finite time. However, the dissipative structure of the system may prevent the formation of the singularity and lead to global smooth solutions in a neighborhood of an equilibrium state $U_e = \begin{bmatrix} u_e \\ v_e \end{bmatrix} \in S$ for (1.1), i.e., $q(U_e) = 0$. The global existence for (1.1)-(1.2) was proved by Hanouzet-Natalini in one space dimension [6] and was extended by Yong [31] to the case of several space dimensions. In the proof of these results, they need essentially two main conditions. The first one is a partially dissipative condition and the second one is the Shizuta-Kawashima condition **(SK)** (see [25]) at an equilibrium state.

When the slow time $t = \varepsilon t'$ is introduced and the following change of variables is made

$$u^\varepsilon(t,x) = u(t',x), \quad v^\varepsilon(t,x) = \frac{1}{\varepsilon} v(t',x), \tag{1.3}$$

(1.1)-(1.2) becomes

$$\begin{cases} \partial_t u^\varepsilon + \dfrac{1}{\varepsilon}\displaystyle\sum_{j=1}^{d} \partial_{x_j} f_j(u^\varepsilon, \varepsilon v^\varepsilon) = 0, \\ \varepsilon^2 \partial_t v^\varepsilon + \displaystyle\sum_{j=1}^{d} \partial_{x_j} g_j(u^\varepsilon, \varepsilon v^\varepsilon) = -\dfrac{q(u^\varepsilon, \varepsilon v^\varepsilon)}{\varepsilon}, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad v^\varepsilon(0,x) = \dfrac{v_0^\varepsilon(x)}{\varepsilon}. \end{cases} \tag{1.4}$$

Under reasonable conditions on f_j and q, system in (1.4) converges formally to parabolic-type equations as $\varepsilon \to 0$ (see [19]). The justification of the local-in-time convergence was proved in [21]. See also [4] and [12] for a study close to semilinear case. Moreover, in a neighborhood of an equilibrium state, the uniform global existence with respect to ε and the global-in-time convergence of the system were obtained in [22].

In the local-in-time convergence result, the convergence rate was well-known shown and it depends on the local existence time [21]. For large time, it was obtained by Junca-Rascle [10] for damped Euler system with compactly supported initial data or in the isothermal case, and by Goudon-Lin [5] for the so-called M1-model in the radiative transfer theory. Both results hold in one space dimension. However, the convergence rate is a quite open problem for large time for general systems. In this paper, we study this problem in one space dimension based on the result in [22]. When the initial data are sufficiently close to the equilibrium state, we establish the convergence rate for all time by means of energy estimates. More precisely, for $d = 1$, (1.4) becomes

$$\begin{cases} \partial_t u^\varepsilon + \dfrac{1}{\varepsilon}\partial_x f(u^\varepsilon, \varepsilon v^\varepsilon) = 0, \\ \varepsilon^2 \partial_t v^\varepsilon + \partial_x g(u^\varepsilon, \varepsilon v^\varepsilon) = -\dfrac{q(u^\varepsilon, \varepsilon v^\varepsilon)}{\varepsilon}, \end{cases} \tag{1.5}$$

with initial condition

$$u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad v^\varepsilon(0,x) = \frac{v_0^\varepsilon(x)}{\varepsilon}. \tag{1.6}$$

We first recall results on the uniform global existence and the global-in-time convergence of the system established in [22]. Let $G_e \subset S$ and Ω_e be open sets satisfying $U_e \in G_e$ and $\Omega_e \times \{0\} \subset G_e$. For the Cauchy problem (1.5)-(1.6), the assumptions needed in [22] are as follows.

(**A1**) For all $U = \begin{bmatrix} u \\ v \end{bmatrix} \in G_e$,

$$q(u,v) = 0 \Longleftrightarrow v = 0.$$

(**A2**) The matrix $\partial_v q(u_e, 0)$ is invertible.

(**A3**) For all $u \in \Omega_e$, $\partial_u f(u,0) = 0$.

(**A4**) For all $u \in \Omega_e$, $\partial_v A_0^{11}(u,0) = 0$.

(**A5**) There exists a constant $c_0 > 0$ such that

$$\xi^T A_0(u,0)Q'(u,0)\xi \geq c_0|\xi^{II}|^2, \quad \forall\, \xi = \begin{bmatrix} \xi^I \\ \xi^{II} \end{bmatrix} \in \mathbb{R}^n, \quad \forall\, u \in \Omega_e,$$

where $|\cdot|$ denotes the usual Euclidean norm and "T" denotes the transpose of a vector or a matrix.

(**SK**) The kernel of the Jacobian $Q'(U_e)$, $\mathrm{Ker}(Q'(U_e))$, contains no eigenvector of the matrix $F'(U_e)$, where $F = \begin{bmatrix} f \\ g \end{bmatrix}$.

Remark 1.1.
*(i) (**A1**) means that the equilibrium state is of the form $(u_e, 0)$, i.e., $v_e = 0$.*
*(ii) (**A2**)-(**A3**) are necessary to derive the limit system as second-order partial differential equations (see* [19, 22]*). If we denote $A(u,v) = F'(u,v)$, (**A3**) means that $A^{11}(u,0) = 0$.*
*(iii) (**A4**) is a technical assumption, which is satisfied when the system is semilinear or $n - r = 1$. The latter is the case for gas dynamics equations.*
*(iv) (**SK**) is a dissipative condition, which implies a time dissipative estimate of ∇U.*

Remark 1.2. *(**A5**) is the partially dissipative condition, which implies a time dissipation estimate of v. Under this condition, it is proved in* [30] *(see also Proposition 2.1 in* [22]*) that there is a neighbourhood $\Omega_1 \subset \Omega_e$ of u_e, such that*

$$A_0^{12}(u,0) = 0, \quad \forall\, u \in \Omega_1.$$

It follows that, for all $u \in \Omega_1$,

$$A_0(u,0)Q'(u,0) = \begin{bmatrix} A_0^{11}(u,0) & 0 \\ 0 & A_0^{22}(u,0) \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ 0 & \partial_v q(u,0) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & A_0^{22}(u,0)\partial_v q(u,0) \end{bmatrix}.$$

*Thus (**A5**) shows that $A_0^{22}(u,0)\partial_v q(u,0)$ is a positive definite matrix in Ω_1 and*

$$(\xi^{II})^T A_0^{22}(u,0)\partial_v q(u,0)\xi^{II} \geq c_0|\xi^{II}|^2, \quad \forall\, \xi^{II} \in \mathbb{R}^r, \quad \forall\, u \in \Omega_1. \tag{1.7}$$

Under these assumptions, in a neighbourhood of the equilibrium state, the uniform global existence of solutions to (1.5)-(1.6) was proved. The solution $(u^\varepsilon, v^\varepsilon)$ satisfies the following estimate

$$\begin{aligned} & \|u^\varepsilon(t) - u_e\|_k^2 + \varepsilon^2\|v^\varepsilon(t)\|_k^2 + \int_0^t \big(\|v^\varepsilon(\tau)\|_k^2 + \|\partial_x u^\varepsilon(\tau)\|_{k-1}^2\big)d\tau \\ \leq\ & C\big(\|u_0^\varepsilon - u_e\|_k^2 + \|v_0^\varepsilon\|_k^2\big), \quad \forall\, t \geq 0, \end{aligned} \tag{1.8}$$

where $\|\cdot\|_k$ denotes the usual norm of the Sobolev space $H^k \overset{def}{=} H^k(\mathbb{R})$ with integer $k \geq 2$ and $C > 0$ is a generic constant independent of ε and $t > 0$. Moreover, let $\bar{u}_0 \in H^k$. If

$$u_0^\varepsilon \rightharpoonup \bar{u}_0, \quad \text{weakly in } H^k,$$

then

$$u^\varepsilon \rightharpoonup \bar{u}, \quad \text{weakly-}* \text{ in } L^\infty(\mathbb{R}^+; H^k),$$

$$v^\varepsilon \rightharpoonup \bar{v}, \quad \text{weakly in } L^2(\mathbb{R}^+; H^k),$$

where $\bar{u} - u_e \in L^\infty(\mathbb{R}^+; H^k)$ and $\bar{u}$ is the unique solution to the Cauchy problem for a system of second-order partial differential equations

$$\begin{cases} \partial_t \bar{u} - \partial_x \{\partial_v f(\bar{u}, 0)[\partial_v q(\bar{u}, 0)]^{-1} \partial_x g(\bar{u}, 0)\} = 0, \\ \bar{u}(0, x) = \bar{u}_0(x), \end{cases} \tag{1.9}$$

and

$$\bar{v} = -[\partial_v q(\bar{u}, 0)]^{-1} \partial_x g(\bar{u}, 0). \tag{1.10}$$

In addition, the system in (1.9) is parabolic (see [22] and Lemma 3.1 below). It follows that the following estimate holds for $\bar{u}$:

$$\|\bar{u}(t) - u_e\|_k^2 + \|\partial_t \bar{u}(t)\|_{k-2}^2 + \int_0^t \|\partial_x \bar{u}(\tau)\|_k^2 \, d\tau \leq C\|\bar{u}_0 - u_e\|_k^2, \quad \forall\, t \geq 0, \tag{1.11}$$

provided that $\|\bar{u}_0 - u_e\|_k$ is sufficiently small. This together with (1.10) yields

$$\|\bar{v}(t)\|_{k-1}^2 + \int_0^t \left(\|\bar{v}(\tau)\|_k^2 + \|\partial_t \bar{v}(\tau)\|_{k-2}^2\right) d\tau \leq C\|\bar{u}_0 - u_e\|_k^2, \quad \forall\, t \geq 0. \tag{1.12}$$

Estimates (1.8) and (1.11)-(1.12) are useful in Sections 3 and 4 for the proof of the convergence rate. We point out that the above results are also valid in several space dimensions. Moreover, from (1.10) and the initial condition of v^ε in (1.6), in general there are initial layers on variable v in the limit process from (1.5)-(1.6) to (1.9)-(1.10).

The aim of the present paper is to establish error estimates for $u^\varepsilon - \bar{u}$ and $v^\varepsilon - \bar{v}$. For this purpose, we introduce an additional condition which links functions f and q (see **(H)** in Section 2). The main technique in the proof is the use of a stream function derived from the first conservation equations in (1.5) and (1.9). This technique is efficient for the one-dimensional system of conservation laws. It has been successfully used in [10, 5] where error estimates for the density were proved in $L^2(\mathbb{R}^+ \times \mathbb{R})$. The particularity of the results in [10, 5] is that, for the non compactly supported initial data, their limit systems are linear (see examples given in the last section). Our limit system in (1.9) may not be linear. Moreover, for the initial data in H^k our error estimates are showed in $L^2(\mathbb{R}^+; H^k) \cap L^\infty(\mathbb{R}^+; H^{k-1})$ for u and in $L^2(\mathbb{R}^+; H^{k-2})$ for v. The estimates are uniform for all $\varepsilon \in (0, 1]$, contrarily to the local convergence case where ε should be small [21].

The paper is organized as follows. In the next section, we state the main results of this paper. Sections 3 and 4 are devoted to the proof of the main results by energy estimates. In the last section, we give examples of systems which are included in [10, 5].

2. Main results

From (**A3**), we see that $f(u,0)$ is constant. It follows that

$$\partial_x f(u,v) = \partial_x\big[f(u,v) - f(u,0)\big].$$

Hence, we may suppose $f(u,0) = 0$ (otherwise, we replace $f(u,v)$ by $f(u,v) - f(u,0)$). Together with (**A1**), there are smooth matrix functions $f_1(u,v)$ of order $(n-r)\times r$ and $q_1(u,v)$ of order $r\times r$ such that

$$\begin{cases} f(u,v) = f_1(u,v)v, \\ q(u,v) = q_1(u,v)v, \end{cases} \tag{2.1}$$

with

$$\begin{cases} f_1(u,0) = \partial_v f(u,0), \\ q_1(u,0) = \partial_v q(u,0). \end{cases} \tag{2.2}$$

By (2.1), (1.5)-(1.6) can be written as

$$\begin{cases} \partial_t u^\varepsilon + \partial_x(f_1(u^\varepsilon, \varepsilon v^\varepsilon)v^\varepsilon) = 0, \\ \varepsilon^2\partial_t v^\varepsilon + \partial_x g(u^\varepsilon, \varepsilon v^\varepsilon) = -q_1(u^\varepsilon, \varepsilon v^\varepsilon)v^\varepsilon, \\ u^\varepsilon(0,x) = u_0^\varepsilon(x), \quad v^\varepsilon(0,x) = \dfrac{v_0^\varepsilon(x)}{\varepsilon}. \end{cases} \tag{2.3}$$

It is easy to see that the formal limit of (2.3) as $\varepsilon \to 0$ is given by

$$\begin{cases} \partial_t \bar{u} + \partial_x(f_1(\bar{u},0)\bar{v}) = 0, \\ \partial_x g(\bar{u},0) = -q_1(\bar{u},0)\bar{v}, \\ \bar{u}(0,x) = \bar{u}_0(x). \end{cases} \tag{2.4}$$

By (**A2**), $q_1(\bar{u},0)$ is invertible when $\bar{u} - u_e$ is sufficiently small. Then we obtain (1.9)-(1.10).

In the study of the convergence rate, we need an additional assumption.

(**H**) There exists a constant matrix D, of order $(n-r)\times r$, such that

$$f_1(u,v) = Dq_1(u,v), \quad \forall\,(u,v) \in G_e.$$

Under this condition, by (2.2), the limit equation in (1.9) for $\bar{u}$ becomes

$$\partial_t \bar{u} - D\partial_{xx} g(\bar{u},0) = 0. \tag{2.5}$$

It is clear that condition (**H**) implies that $f(u,v) = Dq(u,v)$. Since $f(u,0) = 0$ and $q(u,0) = 0$, from the Taylor formula, we have

$$f_1(u,v) = \int_0^1 \partial_v f(u,sv)\,ds, \quad q_1(u,v) = \int_0^1 \partial_v q(u,sv)\,ds.$$

Therefore, the condition in (**H**) is equivalent to

$$\int_0^1 \partial_v f(u,sv)\,ds = D\int_0^1 \partial_v q(u,sv)\,ds.$$

The main results of this paper are stated as folows.

Theorem 2.1. *(Convergence rate for u^ε) Let (**A1**)-(**A5**), (**SK**) and (**H**) hold. Let $k \geq 2$ be an integer, $(u^\varepsilon, v^\varepsilon)$ be the unique solution to* (1.5)-(1.6) *and $\bar{u}$ be the unique solution to* (1.9) *with initial data $\bar{u}_0$ being the weak limit of u_0^ε in H^k. We define*

$$\phi_0^\varepsilon(x) = \int_0^x \big(u_0^\varepsilon(y) - \bar{u}_0(y)\big)\,dy.$$

There exist positive constants δ, C_1 and C_2, independent of ε, such that if

$$\|u_0^\varepsilon - u_e\|_k + \|v_0^\varepsilon\|_k \leq \delta, \tag{2.6}$$

and

$$\|\phi_0^\varepsilon\|_k \leq C_1\varepsilon^\alpha, \tag{2.7}$$

then for all $\varepsilon \in (0,1]$, we have the following estimate

$$\sup_{t\in\mathbb{R}^+} \|u^\varepsilon(t) - \bar{u}(t)\|_{k-1}^2 + \int_0^{+\infty} \|u^\varepsilon(t) - \bar{u}(t)\|_k^2 dt \leq C_2\varepsilon^{\alpha_1}, \tag{2.8}$$

where $\alpha > 0$ is a constant independent of ε and $\alpha_1 = \min(1,\alpha)$.

Theorem 2.2. *(Convergence rate for v^ε) Let the assumptions of Theorem 2.1 hold and $\bar{v}$ be given by* (1.10). *We further assume*

$$\|v_0^\varepsilon\|_{k-2} \leq C_1\varepsilon^\beta, \tag{2.9}$$

with $\beta > 0$ being a constant independent of ε. Then for all $\varepsilon \in (0,1]$, we have the following estimate

$$\sup_{t\in\mathbb{R}^+} \varepsilon^2\|v^\varepsilon(t) - \bar{v}(t)\|_{k-2}^2 + \int_0^{+\infty} \|v^\varepsilon(t) - \bar{v}(t)\|_{k-2}^2\,dt \leq C_2\varepsilon^{2\gamma}. \tag{2.10}$$

where $\gamma = \min(1,\alpha,\beta)$.

3. Proof of Theorem 2.1

In what follows, $C > 0$ is a generic constant independent of ε and any time. We assume that the conditions in Theorem 2.1 hold. Then $k \geq 2$ is an integer. We denote by $\|\cdot\|$ the usual norm of $L^2 \overset{def}{=} L^2(\mathbb{R})$. In the proof of Theorems 2.1-2.2, we use the fact that the embedding from $H^l(\mathbb{R})$ to $L^\infty(\mathbb{R})$ is continuous for all integer $l \geq 1$.

We start with a preliminary result, which is proved in Lemma 2.2 in [22].

Lemma 3.1. *Let (**A1**)-(**A3**) and (**SK**) hold. There is a neighbourhood $\Omega_2 \subset \Omega_1$ of u_e, such that* (1.9) *is parabolic in the sense that*

$$\partial_v f(u,0)[\partial_v q(u,0)]^{-1}\partial_u g(u,0)$$

is a positive definite matrix for all $u \in \Omega_2$, namely, there exists a constant $c_1 > 0$ such that

$$\eta^T \partial_v f(u,0)[\partial_v q(u,0)]^{-1}\partial_u g(u,0)\eta \geq c_1|\eta|^2, \quad \forall\, \eta \in \mathbb{R}^{n-r}, \quad \forall\, u \in \Omega_2.$$

We remark that under condition (**H**), the above inequality is equivalent to

$$\eta^T D\partial_u g(u,0)\eta \geq c_1|\eta|^2, \quad \forall\, \eta \in \mathbb{R}^{n-r}, \quad \forall\, u \in \Omega_2. \tag{3.1}$$

Subtracting (2.5) from the first equation in (2.3), we have

$$\partial_t(u^\varepsilon - \bar{u}) + \partial_x\left[f_1(u^\varepsilon, \varepsilon v^\varepsilon)v^\varepsilon + D\partial_x g(\bar{u},0)\right] = 0.$$

Then there exists a stream function ϕ^ε satisfying

$$\begin{cases} \partial_t \phi^\varepsilon = -\big[f_1(u^\varepsilon, \varepsilon v^\varepsilon)v^\varepsilon + D\partial_x g(\bar{u},0)\big], \\ \partial_x \phi^\varepsilon = u^\varepsilon - \bar{u}, \\ \phi^\varepsilon(0,x) = \displaystyle\int_0^x \big(u_0^\varepsilon(y) - \bar{u}_0(y)\big)\, dy. \end{cases}$$

For all integer l with $0 \leq l \leq k$, we introduce

$$(u^l, v^l) = (\partial_x^l u^\varepsilon, \partial_x^l v^\varepsilon), \quad (\bar{u}^l, \bar{v}^l) = (\partial_x^l \bar{u}, \partial_x^l \bar{v}), \quad \phi^l = \partial_x^l \phi^\varepsilon.$$

Then

$$\begin{cases} \partial_t \phi^l = -\partial_x^l\big[f_1(u^\varepsilon, \varepsilon v^\varepsilon)v^\varepsilon + D\partial_x g(\bar{u},0)\big], \\ \partial_x \phi^l = u^l - \bar{u}^l, \\ \phi^l(0,x) = \partial_x^l \phi^\varepsilon(0,x). \end{cases} \tag{3.2}$$

Lemma 3.2. *For all $T > 0$, we have*

$$\|\phi^l(T)\|^2 + \int_0^T \langle u^l - \bar{u}^l, D\partial_x^l[g(u^\varepsilon, \varepsilon v^\varepsilon) - g(\bar{u},0)]\rangle dt \leq C\varepsilon^{2\alpha_1}, \quad \forall\, 0 \leq l \leq k, \tag{3.3}$$

where α_1 is defined in Theorem 2.1 and $\langle\cdot,\cdot\rangle$ stands for the inner product in $L^2(\mathbb{R})$.

Proof. Since $\delta > 0$ is sufficiently small, (1.8) together with (2.6) implies that $(u^\varepsilon, \varepsilon v^\varepsilon) \in G_e$ and $(u^\varepsilon - u_e, \varepsilon v^\varepsilon)$ is uniformly small with respect to ε in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$. In addition, the weak convergence of u_0^ε to $\bar{u}_0$ in H^k implies that $\|\bar{u}_0 - u_e\|_k \le \delta$. By (1.11), we have $\bar{u} \in \Omega_2$ and $\bar{u} - u_e$ is sufficiently small.

Let $0 \le l \le k$. Applying ∂_x^l to the second equation in (2.3), we have

$$\varepsilon^2 \partial_t v^l + \partial_x^{l+1} g(u^\varepsilon, \varepsilon v^\varepsilon) = -\partial_x^l \big[q_1(u^\varepsilon, \varepsilon v^\varepsilon) v^\varepsilon\big].$$

Taking the inner product of this equality with $D^T \phi^l$ in $L^2(\mathbb{R})$ and integrating over $[0, T]$, we get

$$\begin{aligned} 0 &= \int_0^T \langle \phi^l, \varepsilon^2 D \partial_t v^l + D \partial_x^{l+1} g(u^\varepsilon, \varepsilon v^\varepsilon) + \partial_x^l (D q_1(u^\varepsilon, \varepsilon v^\varepsilon) v^\varepsilon) \rangle dt \\ &:= I_1 + I_2 + I_3, \end{aligned} \tag{3.4}$$

with the natural correspondence of I_1, I_2 and I_3, which are treated term by term as follows. First, since D is a constant matrix,

$$\begin{aligned} I_1 &= \varepsilon^2 \int_0^T \langle \phi^l, D \partial_t v^l \rangle dt \\ &= \varepsilon^2 \int_0^T \frac{d}{dt} \langle \phi^l, D v^l \rangle dt - \varepsilon^2 \int_0^T \langle \partial_t \phi^l, D v^l \rangle dt. \end{aligned}$$

For all $\varepsilon \in (0, 1]$, we use (1.8), (1.11), (2.7) and (3.2) to obtain

$$\begin{aligned} \left| \varepsilon^2 \int_0^T \frac{d}{dt} \langle \phi^l, D v^l \rangle dt \right| &= \varepsilon^2 \big| \langle \phi^l(T), D v^l(T) \rangle - \langle \phi^l(0), D v^l(0) \rangle \big| \\ &\le \frac{1}{4} \|\phi^l(T)\|^2 + \frac{1}{4} \|\phi^l(0)\|^2 + C\varepsilon^2 \\ &\le \frac{1}{4} \|\phi^l(T)\|^2 + C\varepsilon^{2\alpha_1} \end{aligned}$$

and

$$\begin{aligned} \varepsilon^2 \left| \int_0^T \langle \partial_t \phi^l, D v^l \rangle dt \right| &\le \varepsilon^2 \int_0^T \big| \langle \partial_x^l \big(f_1(u^\varepsilon, \varepsilon v^\varepsilon) v^\varepsilon + D \partial_x g(\bar{u}, 0) \big), D v^l \rangle \big| dt \\ &\le C\varepsilon^2. \end{aligned}$$

These estimates imply that

$$|I_1| \le \frac{1}{4} \|\phi^l(T)\|^2 + C\varepsilon^{2\alpha_1}. \tag{3.5}$$

Similarly,

$$\begin{aligned} I_2 &= \int_0^T \langle \phi^l, D \partial_x^{l+1} g(u^\varepsilon, \varepsilon v^\varepsilon) \rangle dt \\ &= -\int_0^T \langle \partial_x \phi^l, D \partial_x^l g(u^\varepsilon, \varepsilon v^\varepsilon) \rangle dt \\ &= -\int_0^T \langle u^l - \bar{u}^l, D \partial_x^l g(u^\varepsilon, \varepsilon v^\varepsilon) \rangle dt. \end{aligned} \tag{3.6}$$

For I_3, we use again (3.2) to obtain

$$\begin{aligned} I_3 &= \int_0^T \langle \phi^l, \partial_x^l \big(f_1(u^\varepsilon, \varepsilon v^\varepsilon) v^\varepsilon \big) \rangle dt \\ &= -\int_0^T \langle \phi^l, \partial_t \phi^l + D\partial_x^{l+1} g(\bar{u}, 0) \rangle dt \\ &= \frac{1}{2}\|\phi^l(0)\|^2 - \frac{1}{2}\|\phi^l(T)\|^2 + \int_0^T \langle u^l - \bar{u}^l, D\partial_x^l g(\bar{u}, 0) \rangle dt. \end{aligned} \tag{3.7}$$

Thus, since $-I_2 - I_3 = I_1$, combining (3.4)-(3.7) and using (2.7) and (3.2), we get (3.3). □

Lemma 3.3. *(L^2-estimate) It holds*

$$\int_0^{+\infty} \|u^\varepsilon(t) - \bar{u}(t)\|^2 \, dt \le C\varepsilon^{2\alpha_1}. \tag{3.8}$$

Proof. With $l = 0$, (3.3) gives

$$\int_0^T \langle u^\varepsilon - \bar{u}, D(g(u^\varepsilon, \varepsilon v^\varepsilon) - g(\bar{u}, 0)) \rangle dt \le C\varepsilon^{2\alpha_1}. \tag{3.9}$$

We use the Taylor formula to obtain

$$g(u^\varepsilon, \varepsilon v^\varepsilon) - g(\bar{u}, 0) = \int_0^1 \big[\partial_u g(\tilde{U}^\varepsilon(s))(u^\varepsilon - \bar{u}) + \partial_v g(\tilde{U}^\varepsilon(s))(\varepsilon v^\varepsilon) \big] ds, \tag{3.10}$$

where

$$\tilde{U}^\varepsilon(s) = \big(\bar{u} + s(u^\varepsilon - \bar{u}), \varepsilon s v^\varepsilon \big).$$

From (1.8) and (1.11), it is easy to show that $\tilde{U}^\varepsilon(s) - U_e$ is sufficiently small in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$, uniformly with respect to $\varepsilon \in (0, 1]$ and $s \in [0, 1]$. Therefore, noticing (3.1), the continuity of $\partial_u g$ implies that there is a constant $c_2 > 0$ independent of ε such that

$$\int_0^T \int_0^1 \langle u^\varepsilon - \bar{u}, D\partial_u g(\tilde{U}^\varepsilon(s))(u^\varepsilon - \bar{u}) \rangle ds dt \ge 2c_2 \int_0^T \|u^\varepsilon(t) - \bar{u}(t)\|^2 dt. \tag{3.11}$$

Moreover, by the Young inequality,

$$\int_0^T \int_0^1 \langle u^\varepsilon - \bar{u}, D\partial_v g(\tilde{U}^\varepsilon(s))(\varepsilon v^\varepsilon) \rangle ds dt \ge -c_2 \int_0^T \|u^\varepsilon(t) - \bar{u}(t)\|^2 dt - C\varepsilon^2.$$

It follows from (3.10) that

$$\int_0^T \langle u^\varepsilon - \bar{u}, D(g(u^\varepsilon, \varepsilon v^\varepsilon) - g(\bar{u}, 0)) \rangle dt \ge c_2 \int_0^T \|u^\varepsilon(t) - \bar{u}(t)\|^2 dt - C\varepsilon^2,$$

which implies (3.8) from (3.9) by letting $T \to +\infty$. □

The proof of Theorem 2.1 follows from the following estimate.

Lemma 3.4. *(Higher order estimates) It holds*

$$\sup_{t \in \mathbb{R}^+} \|u^\varepsilon(t) - \bar{u}(t)\|_{k-1}^2 + \int_0^{+\infty} \|u^\varepsilon(t) - \bar{u}(t)\|_k^2 \, dt \le C\varepsilon^{2\alpha_1}. \tag{3.12}$$

Proof. For all integer l with $1 \le l \le k$, we need to control the integral in (3.3). From (3.10) we obtain

$$\begin{aligned}\partial_x^l\big[g(u^\varepsilon,\varepsilon v^\varepsilon)-g(\bar u,0)\big] &= \int_0^1 \partial_u g(\tilde U^\varepsilon(s))(u^l-\bar u^l)ds\\ &\quad+\int_0^1\big[\partial_x^l\big(\partial_u g(\tilde U^\varepsilon(s))(u^\varepsilon-\bar u^\varepsilon)\big)-\partial_u g(\tilde U^\varepsilon(s))(u^l-\bar u^l)\big]ds\\ &\quad+\int_0^1\partial_x^l\big[\partial_v g(\tilde U^\varepsilon(s)(\varepsilon v^\varepsilon))\big]ds.\end{aligned}$$

Similarly to (3.11), we have

$$\int_0^T\int_0^1\langle u^l-\bar u^l, D\partial_u g(\tilde U^\varepsilon(s))(u^l-\bar u^l)\rangle ds dt \ge 2c_2\int_0^T\|u^l(t)-\bar u^l(t)\|^2dt.$$

By the Moser-type inequalities, we also have

$$\left\|\int_0^1\big[\partial_x^l\big(\partial_u g(\tilde U^\varepsilon(s))(u^\varepsilon-\bar u^\varepsilon)\big)\big)-\partial_u g(\tilde U^\varepsilon(s))(u^l-\bar u^l)\big]ds\right\|\le C\|u^\varepsilon-\bar u^\varepsilon\|_{l-1},$$

$$\left\|\int_0^1\partial_x^l\big[\partial_v g(\tilde U^\varepsilon(s)(\varepsilon v^\varepsilon))\big]ds\right\|\le C\varepsilon\|v^\varepsilon\|_l.$$

Hence, by the Young inequality, these estimates imply that

$$\begin{aligned}&\int_0^T\langle u^l-\bar u^l, D\partial_x^l(g(u^\varepsilon,\varepsilon v^\varepsilon)-g(\bar u,0))\rangle dt\\ \ge\ & 2c_2\int_0^T\|u^l(t)-\bar u^l(t)\|^2dt - C\int_0^T\|u^l(t)-\bar u^l(t)\|\big(\|u^\varepsilon(t)-\bar u^\varepsilon(t)\|_{l-1}+\varepsilon\|v^\varepsilon(t)\|_l\big)dt\\ \ge\ & c_2\int_0^T\|u^l(t)-\bar u^l(t)\|^2dt - C\int_0^T\|u^\varepsilon(t)-\bar u(t)\|_{l-1}^2dt - C\varepsilon^{2\alpha_1}. \qquad (3.13)\end{aligned}$$

It follows from (3.3) that

$$\int_0^T\|u^l(t)-\bar u^l(t)\|^2dt \le C\int_0^T\|u^\varepsilon(t)-\bar u(t)\|_{l-1}^2dt + C\varepsilon^{2\alpha_1}.$$

By Lemma 3.3 and an induction argument on l, we obtain

$$\int_0^T\|u^l(t)-\bar u^l(t)\|^2dt \le C\varepsilon^{2\alpha_1},\quad \forall\, 0\le l\le k.$$

Adding these inequalities for all $0\le l\le k$ and letting $T\to+\infty$ yields

$$\int_0^{+\infty}\|u(t)-\bar u(t)\|_k^2dt \le C\varepsilon^{2\alpha_1}.$$

Therefore, (3.13) implies that

$$\int_0^T\langle u^l-\bar u^l, D\partial_x^l(g(u^\varepsilon,\varepsilon v^\varepsilon)-g(\bar u,0))\rangle dt \ge -C\varepsilon^{2\alpha_1}.$$

Combining the two inequalities above together with (3.3) yields (3.12). □

4. Proof of Theorem 2.2

In this section, we prove the convergence rate of v^ε. Let the assumptions in Theorem 2.2 hold. For convenience, we introduce

$$w^\varepsilon = v^\varepsilon - \bar{v}.$$

From (2.3) and (2.4), we have

$$\varepsilon^2\partial_t v^\varepsilon + q_1(\bar{u},0)v^\varepsilon = -\partial_x g(u^\varepsilon,\varepsilon v^\varepsilon) - [q_1(u^\varepsilon,\varepsilon v^\varepsilon) - q_1(\bar{u},0)]v^\varepsilon,$$

$$\varepsilon^2\partial_t \bar{v} + q_1(\bar{u},0)\bar{v} = -\partial_x g(\bar{u},0) + \varepsilon^2\partial_t\bar{v}.$$

Therefore, w^ε satisfies

$$\varepsilon^2\partial_t w^\varepsilon + q_1(\bar{u},0)w^\varepsilon = R^\varepsilon, \tag{4.1}$$

where, by noting $v^\varepsilon = w^\varepsilon + \bar{v}$,

$$R^\varepsilon = R_1^\varepsilon - [q_1(u^\varepsilon,\varepsilon v^\varepsilon) - q_1(\bar{u},0)]w^\varepsilon,$$

with

$$-R_1^\varepsilon = \partial_x[g(u^\varepsilon,\varepsilon v^\varepsilon) - g(\bar{u},0)] + [q_1(u^\varepsilon,\varepsilon v^\varepsilon) - q_1(\bar{u},0)]\bar{v} + \varepsilon^2\partial_t\bar{v}.$$

Recall that $A_0(\bar{u},0)$ is a symmetric positive definite matrix and $A_0^{12}(\bar{u},0) = 0$. Hence, $A_0^{22}(\bar{u},0)$ is also a symmetric positive definite matrix. It follows that there is a constant $c_3 > 0$ such that

$$c_3\|w\|^2 \le \langle A_0^{22}(\bar{u},0)w, w\rangle \le C\|w\|^2, \quad \forall\, w \in \mathbb{R}^r. \tag{4.2}$$

Similarly, (1.7) implies that

$$\langle A_0^{22}(\bar{u},0)q_1(\bar{u},0)w, w\rangle \ge c_0\|w\|^2, \quad \forall\, w \in \mathbb{R}^r. \tag{4.3}$$

Lemma 4.1. *For all $T > 0$, the solution w^ε of (4.1) satisfies*

$$\begin{aligned} & c_3\varepsilon^2\|w^\varepsilon(T)\|^2 + 2c_0\int_0^T \|w^\varepsilon(t)\|^2 dt \\ \le\ & \int_0^T \big(2\langle A_0^{22}(\bar{u},0)R^\varepsilon, w^\varepsilon\rangle + \varepsilon^2\langle\partial_t A_0^{22}(\bar{u},0)w^\varepsilon, w^\varepsilon\rangle\big)dt + C\varepsilon^2\|w^\varepsilon(0)\|^2. \end{aligned} \tag{4.4}$$

Proof. Taking the inner product of (4.1) with $A_0^{22}(\bar{u},0)w^\varepsilon$ in $L^2(\mathbb{R})$, since $A_0^{22}(\bar{u},0)$ is symmetric, we have

$$\begin{aligned} & \varepsilon^2\frac{d}{dt}\langle A_0^{22}(\bar{u},0)w^\varepsilon, w^\varepsilon\rangle + 2\langle A_0^{22}(\bar{u},0)q_1(\bar{u},0)w^\varepsilon, w^\varepsilon\rangle \\ =\ & 2\langle A_0^{22}(\bar{u},0)R^\varepsilon, w^\varepsilon\rangle + \varepsilon^2\langle\partial_t A_0^{22}(\bar{u},0)w^\varepsilon, w^\varepsilon\rangle. \end{aligned}$$

Integrating this equality over $[0,T]$ and using (4.2)-(4.3), we obtain (4.4). □

Lemma 4.2. *(L^2-estimate) It holds*

$$\varepsilon^2 \sup_{t\in\mathbb{R}^+} \|v^\varepsilon(t)-\bar v(t)\|^2 + \int_0^{+\infty} \|v^\varepsilon(t)-\bar v(t)\|^2 dt \le C\varepsilon^{2\gamma}, \tag{4.5}$$

where $\gamma>0$ is defined in Theorem 2.2.

Proof. We want to apply Lemma 4.1 by estimating the terms on the right-hand side of (4.4). From (1.11) and the weak convergence of u_0^ε to $\bar u_0$ in H^k, we have obviously

$$\|\partial_t A_0^{22}(\bar u,0)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} \le C\|\bar u_0-u_e\|_k \le C\delta,$$

where $\delta>0$ is defined in (2.6). It follows that, for all $\varepsilon\in(0,1]$,

$$\varepsilon^2\langle \partial_t A_0^{22}(\bar u,0)w^\varepsilon, w^\varepsilon\rangle \le C\delta\|w^\varepsilon\|^2. \tag{4.6}$$

For the first term on the right-hand side of (4.4), we use (3.10) to obtain

$$\|\partial_x[g(u^\varepsilon,\varepsilon v^\varepsilon)-g(\bar u,0)]\| \le C\big(\|u^\varepsilon-\bar u\|_1+\varepsilon\|v^\varepsilon\|_1\big).$$

Similarly, by (1.12), we have

$$\|[q_1(u^\varepsilon,\varepsilon v^\varepsilon)-q_1(\bar u,0)]\bar v+\varepsilon^2\partial_t\bar v\| \le C\big(\|u^\varepsilon-\bar u\|+\varepsilon\|v^\varepsilon\|+\varepsilon^2\|\partial_t\bar v\|\big).$$

Therefore, by the definition of R_1^ε,

$$\|R_1^\varepsilon\| \le C\big(\|u^\varepsilon-\bar u\|_1+\varepsilon\|v^\varepsilon\|_1+\varepsilon^2\|\partial_t\bar v\|\big).$$

On the other hand, we also have

$$\|q_1(u^\varepsilon,\varepsilon v^\varepsilon)-q_1(\bar u,0)\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} \le C\big(\|u^\varepsilon-\bar u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}+\varepsilon\|v^\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}\big).$$

By (1.8), (1.11) and

$$u^\varepsilon-\bar u = u^\varepsilon-u_e-(\bar u-u_e),$$

we see that

$$\|u^\varepsilon-\bar u\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})}+\varepsilon\|v^\varepsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R})} \le C\delta.$$

It follows that

$$2\|A_0^{22}(\bar u,0)(q_1(u^\varepsilon,\varepsilon v^\varepsilon)-q_1(\bar u,0))w^\varepsilon\| \le C\delta\|w^\varepsilon\|.$$

Hence,

$$2\|A_0^{22}(\bar u,0)R^\varepsilon\| \le C\delta\|w^\varepsilon\| + C\big(\|u^\varepsilon-\bar u\|_1+\varepsilon\|v^\varepsilon\|_1+\varepsilon^2\|\partial_t\bar v\|\big).$$

Applying the Young inequality together with (4.6) yields

$$2\langle A_0^{22}(\bar u,0)R^\varepsilon, w^\varepsilon\rangle + \varepsilon^2\langle \partial_t A_0^{22}(\bar u,0)w^\varepsilon, w^\varepsilon\rangle \le c_0\|w^\varepsilon\|^2 + C\big(\|u^\varepsilon-\bar u\|^2+\varepsilon^2\|v^\varepsilon\|_1^2+\varepsilon^2\|\partial_t\bar v\|_1^2\big),$$

provided that δ is sufficiently small. By Lemma 4.1 and Theorem 2.1, this implies that

$$\varepsilon^2\|w^\varepsilon(T)\|^2 + \int_0^T \|w^\varepsilon(t)\|^2 dt \le C\varepsilon^{2\alpha_1} + C\varepsilon^2\|w^\varepsilon(0)\|^2, \quad \forall\, T>0.$$

Finally, noting (2.9) and

$$\varepsilon w^\varepsilon(0) = v_0^\varepsilon - \varepsilon \bar{v}_0,$$

we obtain (4.5), since $T > 0$ is arbitrary. □

When $k = 2$, Lemma 4.2 implies the result in Theorem 2.2. For $k \geq 3$, the proof of Theorem 2.2 is completed by the following estimate.

Lemma 4.3. *(Higher order estimates) For $k \geq 3$, it holds*

$$\varepsilon^2 \sup_{t\in\mathbb{R}^+} \|v^\varepsilon(t) - \bar{v}(t)\|_{k-2}^2 + \int_0^{+\infty} \|v^\varepsilon(t) - \bar{v}(t)\|_{k-2}^2 dt \leq C\varepsilon^{2\gamma}. \tag{4.7}$$

Proof. Let l be an integer with $1 \leq l \leq k-2$. Applying ∂_x^l to (4.1) yields

$$\varepsilon^2 \partial_t \partial_x^l w^\varepsilon + q_1(\bar{u}, 0)\partial_x^l w^\varepsilon = \partial_x^l R^\varepsilon + J_l^\varepsilon, \tag{4.8}$$

where

$$\begin{aligned} \partial_x^l R^\varepsilon &= \partial_x^l R_1^\varepsilon - \partial_x^l[(q_1(u^\varepsilon, \varepsilon v^\varepsilon) - q_1(\bar{u}, 0))w^\varepsilon], \\ J_l^\varepsilon &= q_1(\bar{u}, 0)\partial_x^l w^\varepsilon - \partial_x^l[q_1(\bar{u}, 0)w^\varepsilon]. \end{aligned}$$

We want to apply Lemma 4.1 to (4.8) by replacing w^ε by $\partial_x^l w^\varepsilon$ and R^ε by $\partial_x^l R^\varepsilon + J_l^\varepsilon$ in (4.1).

By the definition of R_1^ε, we have, as in the proof of Lemma 4.2,

$$\|\partial_x^l R_1^\varepsilon\| \leq C\big(\|u^\varepsilon - \bar{u}\|_{k-1} + \varepsilon\|v^\varepsilon\|_{k-1} + \varepsilon^2\|\partial_t \bar{v}\|_{k-2}\big).$$

Now

$$\begin{aligned} & -\partial_x^l[(q_1(u^\varepsilon, \varepsilon v^\varepsilon) - q_1(\bar{u}, 0))w^\varepsilon] \\ = \; & -(q_1(u^\varepsilon, \varepsilon v^\varepsilon) - q_1(\bar{u}, 0))\partial_x^l w^\varepsilon \\ & + (q_1(u^\varepsilon, \varepsilon v^\varepsilon) - q_1(\bar{u}, 0))\partial_x^l w^\varepsilon - \partial_x^l[(q_1(u^\varepsilon, \varepsilon v^\varepsilon) - q_1(\bar{u}, 0))w^\varepsilon]. \end{aligned}$$

Similarly to the proof in Lemma 4.2, by the regularity of $\bar{u}$, we have

$$\|(q_1(u^\varepsilon, \varepsilon v^\varepsilon) - q_1(\bar{u}, 0))\partial_x^l w^\varepsilon\| \leq C\delta\|\partial_x^l w^\varepsilon\|.$$

By the Moser-type inequalities, it is clear that

$$\|(q_1(u^\varepsilon, \varepsilon v^\varepsilon) - q_1(\bar{u}, 0))\partial_x^l w^\varepsilon - \partial_x^l[(q_1(u^\varepsilon, \varepsilon v^\varepsilon) - q_1(\bar{u}, 0))w^\varepsilon]\| \leq C\|w^\varepsilon\|_{l-1},$$

$$\|J_l^\varepsilon\| \leq C\|w^\varepsilon\|_{l-1}.$$

Hence,

$$\begin{aligned} 2\|A_0^{22}(\bar{u}, 0)(\partial_x^l R^\varepsilon + J_l^\varepsilon)\| \;\leq\; & C\big(\|u^\varepsilon - \bar{u}\|_{k-1} + \varepsilon\|v^\varepsilon\|_{k-1} + \varepsilon^2\|\partial_t \bar{v}\|_{k-2}\big) \\ & + C\delta\|\partial_x^l w^\varepsilon\| + C\|w^\varepsilon\|_{l-1}. \end{aligned}$$

It follows from the Young inequality that

$$\begin{aligned} 2|\langle A_0^{22}(\bar{u},0)(\partial_x^l R^\varepsilon + J_l^\varepsilon), \partial_x^l w^\varepsilon\rangle| \;\leq\; & C\big(\|u^\varepsilon-\bar{u}\|_{k-1}^2+\varepsilon^2\|v^\varepsilon\|_{k-1}^2+\varepsilon^2\|\partial_t\bar{v}\|_{k-2}^2\big) \\ & + C\delta\|\partial_x^l w^\varepsilon\|^2 + C\|w^\varepsilon\|_{l-1}^2. \end{aligned}$$

By (4.6), we also have

$$\varepsilon^2\langle\partial_t A_0^{22}(\bar{u},0)\partial_x^l w^\varepsilon, \partial_x^l w^\varepsilon\rangle \leq C\delta\|\partial_x^l w^\varepsilon\|^2.$$

Applying Lemma 4.1 to (4.8), we have

$$\begin{aligned} & c_3\varepsilon^2\|\partial_x^l w^\varepsilon(T)\|^2 + c_0\int_0^T \|\partial_x^l w^\varepsilon(t)\|^2 dt \\ \leq\; & C\int_0^T \big(\|u^\varepsilon-\bar{u}\|_{k-1}^2+\varepsilon^2\|v^\varepsilon\|_{k-1}^2+\varepsilon^2\|\partial_t\bar{v}\|_{k-2}^2+\|w^\varepsilon\|_{l-1}^2\big)dt + C\varepsilon^2\|w^\varepsilon(0)\|_{k-2}^2, \end{aligned}$$

provided that δ is sufficiently small. This together with the result in Theorem 2.1 yields

$$\varepsilon^2\|\partial_x^l w^\varepsilon(T)\|^2+\int_0^T \|\partial_x^l w^\varepsilon(t)\|^2 dt \leq C\int_0^T \|w^\varepsilon\|_{l-1}^2 dt+C\varepsilon^2\|w^\varepsilon(0)\|_{k-2}^2+C\varepsilon^{2\alpha_1},\ \ \forall\, 1\leq l\leq k-2.$$

By Lemma 4.2 and an induction argument on l, we obtain

$$\varepsilon^2\|\partial_x^l w^\varepsilon(T)\|^2 + \int_0^T \|\partial_x^l w^\varepsilon(t)\|^2 dt \leq C\varepsilon^2\|w^\varepsilon(0)\|_{k-2}^2 + C\varepsilon^{2\alpha_1},$$

which is true for all $0\leq l\leq k-2$. Since T is arbitrary, (4.7) follows from the above inequality together with (2.9). □

5. Examples

In this section, we give five examples of systems which fulfill all conditions (**A1**)-(**A5**), (**SK**) and (**H**). Then Theorems 2.1-2.2 can be applied. In these examples, $t>0$ is the slow time and the systems are expressed after scaling (1.3). For the convergence rate, Example 5.3 with compactly supported initial data or in the isothermal case and Example 5.5 were studied in [10] and [5], respectively.

Example 5.1. *(Wave equation of heat conduction) This concerns a linear equation which is written as (see* [8, 21] *and references therein)*

$$\varepsilon^2\partial_{tt}^2 w - \partial_{xx} w + \partial_t w = 0, \quad t>0,\ x\in\mathbb{R}.$$

Let us introduce

$$u^\varepsilon = \partial_x w, \quad v^\varepsilon = -\partial_t w,$$

then the equation becomes

$$\begin{cases} \partial_t u^\varepsilon + \partial_x v^\varepsilon = 0, \\ \varepsilon^2\partial_t v^\varepsilon + \partial_x u^\varepsilon = -v^\varepsilon, \end{cases} \tag{5.1}$$

which is of the form (1.5) *with*

$$f(u,v)=q(u,v)=v,\quad g(u,v)=u,\quad D=1.$$

Its corresponding parabolic limit equation is

$$\partial_t\bar{u}-\partial_{xx}\bar{u}=0, \tag{5.2}$$

*which is a classical heat equation. It is easily checked that conditions (**A1**)-(**A5**), (**SK**) and (**H**) are satisfied. Then Theorems 2.1-2.2 can be applied.*

Example 5.2. *(A generalized discrete two-velocity model) The model is written as (see for instance* [27, 23, 15, 22]*) :*

$$\begin{cases}\varepsilon\partial_t f+\partial_x f=\varepsilon^{-1}(f+g)^{\nu}(g-f),\\ \varepsilon\partial_t g-\partial_x g=\varepsilon^{-1}(f+g)^{\nu}(f-g),\quad t>0,\ x\in\mathbb{R},\end{cases}$$

where ν is a real number and $f+g>0$. We show that this system fulfills all conditions in Theorem 2.1 for $\nu=-1$. To see this, we introduce

$$u^\varepsilon=f+g,\quad v^\varepsilon=\frac{1}{\varepsilon}(f^2-g^2).$$

The equivalent system for $\nu=-1$ is

$$\begin{cases}\partial_t u^\varepsilon+\partial_x\Big(\dfrac{v^\varepsilon}{u^\varepsilon}\Big)=0,\\ \varepsilon^2\partial_t v^\varepsilon+\dfrac{1}{2}\partial_x\Big((u^\varepsilon)^2+\varepsilon^2\Big(\dfrac{v^\varepsilon}{u^\varepsilon}\Big)^2\Big)=-\dfrac{2v^\varepsilon}{u^\varepsilon},\end{cases} \tag{5.3}$$

which is of the form (1.5) *with*

$$f(u,v)=\frac{1}{2}q(u,v)=\frac{v}{u},\ \ g(u,v)=\frac{1}{2}\Big((u^2+\varepsilon^2\Big(\frac{v}{u}\Big)^2\Big),\ \ D=\frac{1}{2}.$$

The corresponding limit equation is

$$\partial_t\bar{u}-\frac{1}{4}\partial_{xx}\big(\bar{u}^2\big)=0, \tag{5.4}$$

*which is a nonlinear parabolic equation for $\bar{u}>0$. It is easily checked that conditions (**A1**)-(**A5**), (**SK**) and (**H**) are satisfied. Then Theorems 2.1-2.2 can be applied.*

Example 5.3. *(Euler equations with damping) The equations in one space variable are of the form (see* [18, 26, 3, 14, 22]*)*

$$\begin{cases}\partial_t\rho+\partial_x(\rho w)=0,\\ \varepsilon^2\big(\partial_t(\rho w)+\partial_x(\rho w^2)\big)+\partial_x p(\rho)=-\rho w.\end{cases} \tag{5.5}$$

Here $\varepsilon>0$ is the relaxation time, $\rho>0$ is the density, w is the velocity and $p(\rho)$ is the pressure function satisfying

$$p'(\rho)>0,\quad \forall\,\rho>0. \tag{5.6}$$

System (5.5) *is of the form* (1.5) *with*

$$u=\rho,\ \ v=\rho w,\ \ f(u,v)=q(u,v)=v,\ \ g(u,v)=\frac{v^2}{u}+p(u),\ \ D=1.$$

Its corresponding limit equation is

$$\partial_t\bar{\rho}-\partial_{xx}p(\bar{\rho})=0, \tag{5.7}$$

which is in general a nonlinear parabolic equation under condition (5.6)*. It is easily checked that conditions* (***A1***)*-*(***A5***)*,* (***SK***) *and* (***H***) *are satisfied. Then Theorems 2.1-2.2 can be applied. For weak solutions, an error estimate for* ρ *in* $L^2(\mathbb{R}^+\times\mathbb{R})$ *was obtained in* [10] *when the initial data are of compact support or in the isothermal case, i.e.,* $p(\rho)=\rho$*, which implies that* (5.7) *is a linear heat equation.*

Example 5.4. *(The Euler equations with damping in Lagrangian coordinates) In this example (see* [7, 22]*), let* $dy=\rho dx-\rho v dt$*. Then the Lagrangian coordinates* (t,y) *are well defined for* $\rho>0$ *and the Euler equations* (5.3) *are equivalent to (see* [28]*)*

$$\begin{cases}\partial_t\tau-\partial_y v=0,\\ \varepsilon^2\partial_t v+\partial_y\tilde{p}(\tau)=-v,\quad t>0,\quad y\in\mathbb{R},\end{cases} \tag{5.8}$$

where

$$\tau=1/\rho,\quad \tilde{p}(\tau)=p(1/\tau).$$

System (5.8) *is still of the form* (1.5) *with*

$$u=\tau,\ \ f(u,v)=-q(u,v)=-v,\ \ g(u,v)=\tilde{p}(u),\ \ D=-1.$$

Condition (5.6) *is equivalent to*

$$\tilde{p}'(\tau)<0,\quad \forall\,\tau>0. \tag{5.9}$$

The corresponding limit equation is

$$\partial_t\bar{\tau}+\partial_{yy}p(1/\bar{\tau})=0, \tag{5.10}$$

which is a nonlinear parabolic equation under condition (5.9)*. It is easily checked that conditions* (***A1***)*-*(***A5***)*,* (***SK***) *and* (***H***) *are satisfied. Then Theorems 2.1-2.2 can be applied. We remark that* (5.10) *is a nonlinear equation even if* p *is a linear function.*

Example 5.5. *(The M1 model) This model in one space variable is written as*

$$\begin{cases}\partial_t\rho+\partial_x(\rho w)=0,\\ \varepsilon^2\partial_t(\rho w)+\partial_x\big(\rho B(\varepsilon w)\big)=-\rho w,\end{cases} \tag{5.11}$$

where $\rho > 0$ is the density, w is the velocity and $B(w)$ is defined by

$$B(w) = \frac{1}{3} + \frac{2w^2}{2 + \Delta(w)}, \quad \Delta(w) = \sqrt{4 - 3w^2}.$$

This system is of the form (1.5) *with*

$$u = \rho, \quad v = \rho w, \quad f(u, v) = q(u, v) = v, \quad g(u, v) = uB(v/u), \quad D = 1.$$

Its corresponding limit equation is

$$\partial_t \bar{\rho} - \frac{1}{3}\partial_{xx}\bar{\rho} = 0, \tag{5.12}$$

*which is a linear heat equation. It is easily checked that conditions (**A1**)-(**A5**), (**SK**) and (**H**) are satisfied. Then Theorems 2.1-2.2 can be applied. Remark that for smooth solutions an error estimate for ρ in $L^2(\mathbb{R}^+ \times \mathbb{R})$ was obtained in* [5].

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