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CONVERGENCE RATES IN ZERO-RELAXATION LIMITS FOR EULER-MAXWELL AND EULER-POISSON SYSTEMS

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Abstract. It was proved that Euler-Maxwell systems converge globally-in-time to drift-diffusion systems in a slow time scaling, as the relaxation time goes to zero. The convergence was established to the Cauchy problem with smooth periodic initial data sufficiently close to constant equilibrium states. In this paper, we establish error estimates between smooth periodic solutions of Euler-Maxwell systems and those of drift-diffusion systems. Similar error estimates are also obtained for Euler-Poisson systems in place of Euler-Maxwell systems. The proof of these results uses stream function techniques together with energy estimates.

Résumé. Il a été prouvé que des systèmes d'Euler-Maxwell convergent globalement en temps vers des systèmes de dérive-diffusion dans une échelle de temps lent, lorsque le temps de relaxation tend vers zéro. La convergence a éte établie au problème de Cauchy avec données initiales régulières périodiques suffisamment proches d'états d'équilibres constants. Dans cet article, nous établissons des estimations d'erreur entre les solutions régulières périodiques des systèmes d'Euler-Maxwell et celles des systèmes de dérive-diffusion. Des estimations d'erreur similaires sont aussi obtenues pour des systèmes d'Euler-Poisson au lieu des systèmes d'Euler-Maxwell. Nous utilisons des techniques de fonctions de courant et des estimations d'energies pour la démonstration de ces résultats.

Keywords: Convergence rate, Euler-Maxwell system, Euler-Poisson system, drift-diffusion system, stream function

AMS Subject Classification (2010) : 35B25, 35K45, 35L45, 35Q35

1. Introduction

In this paper, we consider the convergence rate problems in the zero-relaxation-time limits for Euler-Maxwell and Euler-Poisson systems, which are both important fluid models in plasma physics. We study these problems in the case of periodic solutions. We establish the global error estimates between the smooth solutions of the Euler-Maxwell system and those of the drift-diffusion system. Similar error estimates are also established for the Euler-Poisson system. In the proof, stream function techniques are used for both Euler-Maxwell and Euler-Poisson systems.

We first study the global-in-time convergence rate problem for the three-dimensional one-fluid Euler-Maxwell system for electrons. The Euler-Maxwell system describes the motions of particles in plasmas, which can be written under the form (see [1, 2, 3])

$$\begin{cases} \partial_{t'}\rho + \operatorname{div}(\rho v) = 0, \\ \partial_{t'}(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P(\rho) = -\rho(E + v \times B) - \dfrac{\rho v}{\varepsilon}, \\ \partial_{t'}E - \nabla \times B = \rho v, \quad \operatorname{div}E = 1 - \rho, \\ \partial_{t'}B + \nabla \times E = 0, \quad \operatorname{div}B = 0, \quad t > 0, \quad x \in \mathbb{T}^3, \end{cases} \tag{1.1}$$

with initial conditions

$$t' = 0 : \quad (\rho, v, E, B) = (\rho_0^\varepsilon, v_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)(x), \quad x \in \mathbb{T}^3, \tag{1.2}$$

where $\mathbb{T}^3$ is a torus in $\mathbb{R}^3$, $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the space variable, $t' > 0$ is the usual time and $\varepsilon \in (0, 1]$ is the relaxation time which means the average time required for the state being relaxed to the equilibrium. Here $\rho > 0$ is the mass density, v is the velocity field, E is the electric field and B is the magnetic field, which are all functions of (t', x). The pressure function P is supposed to be smooth and strictly increasing, namely

$$P'(\rho) > 0, \quad \forall \rho > 0. \tag{1.3}$$

System (1.1) admits an equilibrium state

$$(\rho, v, E, B) = (1, 0, 0, B_e),$$

where $B_e \in \mathbb{R}^3$ is an arbitrary constant vector.

There are a lot of physical models with relaxation time [22]. The zero relaxation time limit is a well-known problem in asymptotic analysis and singular perturbation theories. We refer to [14, 18] and references therein for the study of this limit from first-order hyperbolic systems to hyperbolic systems. However, if a slow time scaling is used, the limiting system may become parabolic (see [16]).

Here, the slow time is $t = \varepsilon t'$. if we introduce

$$\rho^\varepsilon(t,x) = \rho\left(\frac{t}{\varepsilon}, x\right), \quad v^\varepsilon(t,x) = \frac{1}{\varepsilon} v\left(\frac{t}{\varepsilon}, x\right), \quad E^\varepsilon(t,x) = E\left(\frac{t}{\varepsilon}, x\right), \quad B^\varepsilon(t,x) = B\left(\frac{t}{\varepsilon}, x\right),$$

then (1.1)-(1.2) becomes

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon v^\varepsilon) = 0, \\ \varepsilon^2\big(\partial_t(\rho^\varepsilon v^\varepsilon) + \mathrm{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon)\big) + \nabla P(\rho^\varepsilon) = -\rho^\varepsilon(E^\varepsilon + \varepsilon v^\varepsilon \times B^\varepsilon) - \rho^\varepsilon v^\varepsilon, \\ \varepsilon \partial_t E^\varepsilon - \nabla \times B^\varepsilon = \varepsilon \rho^\varepsilon v^\varepsilon, \quad \mathrm{div} E^\varepsilon = 1 - \rho^\varepsilon, \\ \varepsilon \partial_t B^\varepsilon + \nabla \times E^\varepsilon = 0, \quad \mathrm{div} B^\varepsilon = 0, \end{cases} \tag{1.4}$$

with initial conditions

$$\rho^\varepsilon(0,x) = \rho_0^\varepsilon(x), \quad v^\varepsilon(0,x) = \frac{v_0^\varepsilon(x)}{\varepsilon}, \quad E^\varepsilon(0,x) = E_0^\varepsilon(x), \quad B^\varepsilon(0,x) = B_0^\varepsilon(x), \tag{1.5}$$

in which the following compatibility conditions should hold:

$$\mathrm{div} E_0^\varepsilon = 1 - \rho_0^\varepsilon, \quad \mathrm{div} B_0^\varepsilon = 0. \tag{1.6}$$

Formally, as $\varepsilon \to 0$, if denoting the limits of $(\rho^\varepsilon, v^\varepsilon, E^\varepsilon, B^\varepsilon)$ as $(\bar\rho, \bar v, \bar E, \bar B)$, one gets that the formal limiting equations of (1.4) are

$$\begin{cases} \partial_t \bar\rho + \mathrm{div}(\bar\rho \bar v) = 0, \\ \bar v = -\bar E - \nabla h(\bar\rho), \\ \nabla \times \bar B = 0, \quad \mathrm{div} \bar B = 0, \\ \mathrm{div} \bar E = 1 - \bar\rho, \quad \nabla \times \bar E = 0, \end{cases} \tag{1.7}$$

where h is the enthalpy function satisfying

$$h'(\rho) = \frac{P'(\rho)}{\rho}.$$

Consequently, it follows from the third equation in (1.7) that $\bar B$ is a constant vector. And since $\nabla \times \bar E = 0$, there exists a potential function $\bar\phi$ satisfying $\bar E = \nabla \bar\phi$. As a result, (1.7) yields the classical drift-diffusion model

$$\begin{cases} \partial_t \bar\rho - \Delta P(\bar\rho) - \mathrm{div}(\bar\rho \nabla \bar\phi) = 0, \\ \Delta \bar\phi = 1 - \bar\rho, \\ \bar\rho(0,x) = \bar\rho_0, \end{cases} \tag{1.8}$$

as well as

$$\bar E = \nabla \bar\phi, \quad \bar v = -\nabla\big(h(\bar\rho) + \bar\phi\big). \tag{1.9}$$

In order to ensure the uniqueness of $\bar\phi$, we impose

$$m_{\bar\phi}(t) \overset{def}{=} \int_{\mathbb{T}^3} \bar\phi(t,x)dx = 0, \quad \forall\, t \geq 0.$$

The study of the global-in-time error estimates between the smooth solutions to (1.4)-(1.5) to those of the drift-diffusion model (1.8) is based on the existence results of the global smooth solutions to the Euler-Maxwell systems. The Euler-Maxwell system (1.4) is symmetrizable hyperbolic for $\rho > 0$, then the problem (1.1)-(1.2) admits a local smooth solution according to Lax [12] and Kato [9]. We point out that the Euler-Maxwell system and the Euler-Poisson system are different due to the coupling terms and to the difference between the Poisson equation and the Maxwell equations. In Chen-Jerome-Wang [3], a global existence result of weak solutions in a one-dimensional Euler-Maxwell system with extra relaxation terms was established by using the fractional step Godunov scheme together with a compensated compactness argument. For smooth initial data, the global existence of smooth solutions, which are sufficiently close to the equilibrium state, to (1.1)-(1.2) was proved in [20, 4, 24]. For the relaxation limit problems, the local-in-time and the global-in-time convergence of (1.1) as $\varepsilon \to 0$ were obtained by Hajjej-Peng [6] and Wasiolek [21], respectively. These results were established for smooth solutions in a torus or in the whole space. In the local-in-time convergence result, the convergence rate was clearly shown and it depends on the local existence time [20]. For large time, as we know, no result has been reached so far.

Now we turn to the global error estimates for the Euler-Poisson system. The Euler-Poisson system is also an important model that describes the motions of unmagnetized particles in plasmas or semiconductors. The three-dimensional one-fluid Euler-Poisson system on a torus $\mathbb{T}^3$ can be written as (see [2, 17])

$$\begin{cases} \partial_{t'}\rho + \operatorname{div}(\rho v) = 0, \\ \partial_{t'}(\rho v) + \operatorname{div}(\rho v \otimes v) + \nabla P(\rho) = -\rho\nabla\phi - \dfrac{\rho v}{\varepsilon}, \\ \Delta\phi = 1 - \rho, \end{cases} \tag{1.10}$$

with initial conditions

$$t' = 0: \quad (\rho, v) = (\rho_0^\varepsilon(x), v_0^\varepsilon(x)), \quad x \in \mathbb{T}^3. \tag{1.11}$$

Here ρ, v and P are defined in the same way as in (1.1), ϕ is the scaled electric potential satisfying

$$m_\phi(t) \overset{def}{=} \int_{\mathbb{T}^3} \phi(t,x)dx = 0, \quad \forall\, t \geq 0. \tag{1.12}$$

We define ϕ_0^ε by

$$\Delta\phi_0^\varepsilon = 1 - \rho_0^\varepsilon.$$

The relaxation limit problem as $\varepsilon \to 0$ for smooth solutions of (1.10) has been widely analyzed by many authors. See [10, 11, 25] for the local-in-time convergence and [19] for the global-in-time convergence, respectively. The derivation of the limiting system is similar

to that of the Euler-Maxwell system, see the details in Section 3. However, similarly to the Euler-Maxwell system, so far there is no result on the global convergence rate for (1.10). We mention that the global convergence rate was obtained in some other one-dimensional models. See Junca-Rascle [8] for the isothermal Euler system with damping, Goudon-Lin [5] for the so-called M1-model in the radiative transfer theory and Li-Peng-Zhao [13] for the general hyperbolic systems of balance laws. For multi-dimensional models, no result has been reached by now.

The aim of this paper is to establish error estimates for both the Euler-Maxwell and Euler-Poisson systems. The main results are given in Theorem 2.1 and Theorem 3.1 for Euler-Maxwell system and Euler-Poisson system, respectively. We employ stream function techniques to provide an error estimate for $\rho^\varepsilon - \bar{\rho}$ in $L^2(\mathbb{R}^+; L^2(\mathbb{T}^3))$. This estimate is not trivial since it cannot be derived from the classical energy estimates for the Euler-Maxwell or Euler-Poisson system. For a conservative equation

$$\partial_t z + \mathrm{div} w = 0, \tag{1.13}$$

we call φ a stream function associated to this equation if φ satisfies

$$\partial_t \varphi = w, \quad \mathrm{div}\varphi = -z. \tag{1.14}$$

The existence of stream functions is clear. Indeed, define

$$\varphi(t,x) = \varphi(0,x) + \int_0^t w(\tau,x)d\tau,$$

with

$$\mathrm{div}\varphi(0,x) = -z(0,x).$$

Then it is easy to check that φ is a stream function.

In one space dimension, the stream function technique was successfully used for obtaining convergence rates in [8, 5, 13]. In this paper, we need to find an appropriate stream function for the Euler-Maxwell system. By subtracting the mass equation of the limiting system (1.7) from that of the original system (1.4), we obtain the following conservative equation

$$\partial_t(\rho^\varepsilon - \bar{\rho}) + \mathrm{div}(\rho^\varepsilon v^\varepsilon - \bar{\rho}\bar{v}) = 0.$$

Hence, the stream function φ satisfies

$$\begin{cases} \partial_t \varphi = \rho^\varepsilon v^\varepsilon - \bar{\rho}\bar{v}, \\ \mathrm{div}\varphi = -(\rho^\varepsilon - \bar{\rho}). \end{cases}$$

A natural candidate of the stream function is the error of the electric field $\varphi = E^\varepsilon - \bar{E}$, since

$$\mathrm{div}(E^\varepsilon - \bar{E}) = -(\rho^\varepsilon - \bar{\rho}).$$

However, since we lose the information of $\partial_t \bar{E}$ after taking the limit $\varepsilon \to 0$, we can only get the information of $\mathrm{div}(\partial_t \bar{E})$ directly from the limiting system, and as a result, $\partial_t \varphi$ is not equal to $\rho^\varepsilon v^\varepsilon - \bar{\rho}\bar{v}$, but takes the following form with some additional divergence free term K :

$$\partial_t \varphi = \rho^\varepsilon v^\varepsilon - \bar{\rho}\bar{v} + K.$$

Here this K may not be uniquely determined. Consequently, we can regard $E^\varepsilon - \bar{E}$ as the stream function of the following modified conservative equation

$$\partial_t(\rho^\varepsilon - \bar{\rho}) + \mathrm{div}(\rho^\varepsilon v^\varepsilon - \bar{\rho}\bar{v} + K) = 0. \tag{1.15}$$

Similar situation occurs for the Euler-Poisson system. It is worth mentioning that the right choice of K and the energy estimates for the terms containing K are one of the main difficulties in the proof. This is very different from the classical stream function techniques in one-dimensional case.

Given an integer $k \geq 3$, we want to establish the following error estimates.

(i) For the Euler-Maxwell system,
- for ρ and εv in $L^\infty(\mathbb{R}^+; H^{k-2}(\mathbb{T}^3))$,
- for E and B in $L^\infty(\mathbb{R}^+; H^{k-1}(\mathbb{T}^3))$,
- for ρ and E in $L^2(\mathbb{R}^+; H^{k-1}(\mathbb{T}^3))$,
- for v and ∇B in $L^2(\mathbb{R}^+; H^{k-2}(\mathbb{T}^3))$.

(ii) For the Euler-Poisson system,
- for ρ and εv in $L^\infty(\mathbb{R}^+; H^{k-2}(\mathbb{T}^3))$,
- for $\nabla\phi$ in $L^\infty(\mathbb{R}^+; H^{k-1}(\mathbb{T}^3))$,
- for ρ and $\nabla\phi$ in $L^2(\mathbb{R}^+; H^{k-1}(\mathbb{T}^3))$,
- for v in $L^2(\mathbb{R}^+; H^{k-2}(\mathbb{T}^3))$.

It is worth mentioning that these estimates are uniform for all $\varepsilon \in (0, 1]$, contrarily to the local convergence case where ε should be small [6]. For simplicity, we denote by $\|\cdot\|$, $\|\cdot\|_\infty$ and $\|\cdot\|_l$ the usual norms of $L^2 \overset{def}{=} L^2(\mathbb{T}^3)$, $L^\infty \overset{def}{=} L^\infty(\mathbb{T}^3)$ and $H^l \overset{def}{=} H^l(\mathbb{T}^3)$ for all integers $l \geq 1$, respectively. We will repeatedly use the continuous embedding $H^2 \hookrightarrow L^\infty$.

This paper is organized as follows. Sections 2 is devoted to the convergence rates in the zero-relaxation-time limit for the Euler-Maxwell system. In Section 3, similar estimates are established for the Euler-Poisson system.

2. Convergence Rate for Euler-Maxwell systems

In what follows, we denote by $k \geq 3$ an integer and $C > 0$ a generic constant independent of ε and any time.

The study of the error estimates is based on the results on the uniform global existence and the global-in-time convergence from (1.4)-(1.5) to (1.8)-(1.9) established in [21]. These results can be stated as follows. Let $U_e = (1, 0, 0, B_e)^\top$ be an equilibrium state. We denote

$$U^\varepsilon = (\rho^\varepsilon, \varepsilon v^\varepsilon, E^\varepsilon, B^\varepsilon)^\top, \quad U_0^\varepsilon = (\rho_0^\varepsilon, v_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon)^\top,$$

where $\top$ denotes the transpose of a vector or a matrix. Then there exists a constant $\delta_0 > 0$, which is independent of ε, such that if $\|U_0^\varepsilon - U_e\|_k \le \delta_0$, then the periodic problem (1.4)-(1.5) admits a unique global solution U^ε satisfying

$$\|U^\varepsilon(t) - U_e\|_k^2 + \int_0^t \big(\|\rho^\varepsilon(\tau) - 1\|_k^2 + \|v^\varepsilon(\tau)\|_k^2\big) d\tau \le C\|U_0^\varepsilon - U_e\|_k^2, \quad \forall\, t > 0. \tag{2.1}$$

Moreover, if $(\bar\rho_0 - 1, \bar E_0) \in H^k \times H^k$ and

$$(\rho_0^\varepsilon, E_0^\varepsilon, B_0^\varepsilon) \rightharpoonup (\bar\rho_0, \bar E_0, B_e), \quad \text{weakly in } H^k,$$

then

$$(\rho^\varepsilon, E^\varepsilon, B^\varepsilon) \rightharpoonup (\bar\rho, \bar E, B_e), \quad \text{weakly-}* \text{ in } L^\infty(\mathbb{R}^+; H^k),$$

$$v^\varepsilon \rightharpoonup \bar v, \quad \text{weakly in } L^2(\mathbb{R}^+; H^k),$$

where $(\bar\rho, \bar\phi)$ is the solution to the periodic problem of the classical drift-diffusion model (1.8) and $(\bar E, \bar v)$ satisfy (1.9).

The result on the convergence rate for (1.4)-(1.5) is stated as follows.

Theorem 2.1. *Let $k \ge 3$ be an integer and (1.3) hold. Let $(\rho^\varepsilon, v^\varepsilon, E^\varepsilon, B^\varepsilon)$ and $(\bar\rho, \bar v, \bar E)$ be the unique smooth solutions to (1.4)-(1.5) and (1.8)-(1.9), respectively. Denote $\bar E_0 = \bar E(0, \cdot)$. There exists a positive constant $\delta > 0$, independent of ε, such that if*

$$\|\rho_0^\varepsilon - 1\|_k + \|v_0^\varepsilon\|_k + \|E_0^\varepsilon\|_k + \|B_0^\varepsilon - B_e\|_k \le \delta, \tag{2.2}$$

and for any given positive constants p and C_1 independent of ε satisfying

$$\|v_0^\varepsilon\|_{k-2} + \|E_0^\varepsilon - \bar E_0\|_{k-1} + \|B_0^\varepsilon - B_e\|_{k-1} \le C_1\varepsilon^p, \tag{2.3}$$

then for all $\varepsilon \in (0, 1]$, there exists a positive constant C_2 independent of ε, such that we have the following estimate

$$\begin{aligned} &\sup_{t\in\mathbb{R}^+} \big(\|\rho^\varepsilon(t) - \bar\rho(t)\|_{k-2}^2 + \varepsilon^2\|v^\varepsilon(t) - \bar v(t)\|_{k-2}^2 + \|E^\varepsilon(t) - \bar E(t)\|_{k-1}^2 + \|B^\varepsilon(t) - B_e\|_{k-1}^2\big) \\ + &\int_0^{+\infty} \big(\|\rho^\varepsilon(\tau) - \bar\rho(\tau)\|_{k-1}^2 + \|v^\varepsilon(\tau) - \bar v(\tau)\|_{k-2}^2 + \|E^\varepsilon(\tau) - \bar E(\tau)\|_{k-1}^2 + \|\nabla B^\varepsilon(\tau)\|_{k-2}^2\big)\, d\tau \\ \le\ & C_2\varepsilon^{2p_1}, \end{aligned}$$

where $p_1 = \min(p, 1)$.

Remark 2.1. *It is not necessary to impose an initial error condition on* $\rho_0^\varepsilon - \bar{\rho}_0$. *Indeed,* (2.3) *together with* (1.6)-(1.7) *implies that*

$$\|\rho_0^\varepsilon - \bar{\rho}_0\|_{k-2} \le C_1 \varepsilon^p.$$

2.1. **Proof of Theorem 2.1.** Let $(\rho^\varepsilon, v^\varepsilon, E^\varepsilon, B^\varepsilon)$ be the unique smooth solution to (1.4)-(1.5), and $(\bar{\rho}, \bar{v}, \bar{E})$ be the unique solution to the drift-diffusion model (1.8). For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3$, we denote

$$\partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \partial x_3^{\alpha_3}} \quad \text{with} \quad |\alpha| = \alpha_1 + \alpha_2 + \alpha_3,$$

and

$$\begin{cases} (\rho_\alpha, v_\alpha, E_\alpha, B_\alpha) = (\partial^\alpha \rho^\varepsilon, \partial^\alpha v^\varepsilon, \partial^\alpha E^\varepsilon, \partial^\alpha B^\varepsilon), \\ (\bar{\rho}_\alpha, \bar{v}_\alpha, \bar{E}_\alpha, \bar{B}_\alpha) = (\partial^\alpha \bar{\rho}, \partial^\alpha \bar{v}, \partial^\alpha \bar{E}, \partial^\alpha \bar{B}). \end{cases}$$

We first prove the following result.

Lemma 2.1. *Assume* $\|\bar{\rho}_0 - 1\|_k$ *is sufficiently small. Then the solution* $(\bar{\rho}, \bar{v}, \bar{E})$ *to* (1.8)-(1.9) *satisfies*

$$\|\bar{\rho}(t) - 1\|_k^2 + \int_0^t \|\bar{\rho}(\tau) - 1\|_{k+1}^2 \, d\tau \le C\|\bar{\rho}_0 - 1\|_k^2, \quad \forall\, t > 0, \tag{2.4}$$

$$\|\bar{v}(t)\|_{k-1}^2 + \|\partial_t \bar{v}(t)\|_{k-3}^2 + \int_0^t \big(\|\bar{v}(\tau)\|_k^2 + \|\partial_t \bar{v}(\tau)\|_{k-2}^2\big) d\tau \le C\|\bar{\rho}_0 - 1\|_k^2, \quad \forall\, t > 0. \tag{2.5}$$

and

$$\|\bar{E}(t)\|_k^2 + \|\partial_t \bar{E}(t)\|_{k-1}^2 + \int_0^t \left(\|\bar{E}(\tau)\|_{k+1}^2 + \|\partial_t \bar{E}(\tau)\|_k^2\right) d\tau \le C\|\bar{\rho}_0 - 1\|_k^2, \quad \forall\, t > 0. \tag{2.6}$$

Proof. Let $\bar{n} = \bar{\rho} - 1$ and $\alpha \in \mathbb{N}^3$ be a multi-index with $|\alpha| \le k$. From (1.8), for any $t > 0$, we have

$$\begin{cases} \partial_t \bar{n} - \operatorname{div}\big(P'(\bar{\rho})\nabla \bar{n}\big) - \operatorname{div}(\bar{\rho}\nabla \bar{\phi}) = 0, \\ \Delta \bar{\phi} = -\bar{n}, \quad m_{\bar{\phi}}(t) = 0. \end{cases} \tag{2.7}$$

Applying ∂^α to the first equation in (2.7) and taking the inner product with $\partial^\alpha \bar{n}$ in L^2, it holds

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\|\partial^\alpha \bar{n}\|^2 + \big\langle P'(\bar{\rho})\partial^\alpha(\nabla \bar{n}), \partial^\alpha(\nabla \bar{n})\big\rangle + \langle \partial^\alpha(\nabla \bar{\phi}), \nabla(\partial^\alpha \bar{n})\rangle \\ = \; & \big\langle P'(\bar{\rho})\partial^\alpha(\nabla \bar{n}) - \partial^\alpha(P'(\bar{\rho})\nabla \bar{n}), \partial^\alpha(\nabla \bar{n})\big\rangle - \langle \partial^\alpha(\bar{n}\nabla \bar{\phi}), \nabla(\partial^\alpha \bar{n})\rangle, \end{aligned} \tag{2.8}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in L^2. By the Poisson equation in (2.7), we have

$$\langle \partial^\alpha(\nabla \bar{\phi}), \nabla(\partial^\alpha \bar{n})\rangle = \|\partial^\alpha \bar{n}\|^2.$$

Since P is strictly increasing and $\bar{\rho}$ is close to 1, there is a constant $P_1 > 0$ such that

$$\big\langle P'(\bar{\rho})\partial^\alpha(\nabla \bar{n}), \partial^\alpha(\nabla \bar{n})\big\rangle \ge P_1 \|\partial^\alpha(\nabla \bar{n})\|^2.$$

Then applying the Poincaré inequality to the Poisson equation in (2.7) yields

$$\|\nabla\partial^\alpha\bar{\phi}\| \le C\|\partial^\alpha\bar{n}\|. \tag{2.9}$$

By the Moser-type inequalities [15], we obtain

$$\big\langle P'(\bar{\rho})\partial^\alpha(\nabla\bar{n}) - \partial^\alpha(P'(\bar{\rho})\nabla\bar{n}), \partial^\alpha(\nabla\bar{n})\big\rangle - \langle\partial^\alpha(\bar{n}\nabla\bar{\phi}), \nabla(\partial^\alpha\bar{n})\rangle \le C\|\bar{n}\|_k\|\bar{n}\|_{k+1}^2.$$

Thus, adding (2.8) for all $\alpha \in \mathbb{N}^3$ with $|\alpha| \le k$ yields

$$\frac{1}{2}\frac{d}{dt}\|\bar{n}\|_k^2 + \|\bar{n}\|_k^2 + p_1\|\nabla\bar{n}\|_k^2 \le C\|\bar{n}\|_k\|\bar{n}\|_{k+1}^2.$$

Since $\|\bar{n}\|_k$ is sufficiently small, integrating the above inequality over $[0,t]$ yields (2.4).

As a consequence of (2.4) together with (2.9), we obtain

$$\|\bar{E}(t)\|_k^2 + \int_0^t \|\bar{E}(\tau)\|_{k+1}^2 d\tau \le C\|\bar{\rho}_0 - 1\|_k^2, \quad \forall\ t > 0. \tag{2.10}$$

On the other hand, for a multi-index β with $|\beta| \le k-1$, (1.9) gives

$$\|\partial^\beta\bar{v}\| \le C\|\bar{n}\|_{|\beta|+1}. \tag{2.11}$$

Applying $\partial_t\partial^\beta$ to both sides of the Poisson equation in (2.7) leads to

$$\Delta\partial_t\partial^\beta\bar{\phi} = \mathrm{div}\partial^\beta(\bar{\rho}\bar{v}).$$

It follows from (1.9) and (2.11) that

$$\|\partial_t\partial^\beta\bar{E}\| \le C\|\partial^\beta(\bar{\rho}\bar{v})\| \le C\|\bar{n}\|_{|\beta|+1}. \tag{2.12}$$

By (2.4), this implies that

$$\|\partial_t\bar{E}(t)\|_{k-1}^2 + \int_0^t \|\partial_t\bar{E}(\tau)\|_k^2 d\tau \le C\|\bar{\rho}_0 - 1\|_k^2, \quad \forall\ t > 0.$$

Combining this inequality with (2.10) yields (2.6).

Similarly, (2.4) together with (2.11) implies

$$\|\bar{v}(t)\|_{k-1}^2 + \int_0^t \|\bar{v}(\tau)\|_k^2 d\tau \le C\|\bar{\rho}_0 - 1\|_k^2, \quad \forall\ t > 0. \tag{2.13}$$

For a multi-index γ with $|\gamma| \le k-3$, applying $\partial_t\partial^\gamma$ to the equation for $\bar{v}$ in (1.9) yields

$$\partial_t\partial^\gamma\bar{v} = -\partial^\gamma(h''(\bar{\rho})\nabla\bar{\rho}\partial_t\bar{\rho} + h'(\bar{\rho})\nabla\partial_t\bar{\rho} + \partial_t\bar{E}).$$

Since $\partial_t\bar{\rho} = -\mathrm{div}(\bar{\rho}\bar{v})$, by (2.4) and (2.11), we obtain

$$\|\partial_t\partial^\gamma\bar{v}\| \le C\|\bar{v}\|_{|\gamma|+2} + \|\partial_t\bar{E}\|_{|\gamma|},$$

which, together with (2.13), implies (2.5). □

From (1.8)-(1.9), we have

$$\mathrm{div}\partial_t\nabla\bar{\phi} = -\partial_t\bar{\rho} = \mathrm{div}(\bar{\rho}\bar{v}),$$

which implies that there exists a function $\bar{H}$ such that

$$\begin{cases} \partial_t \nabla \bar{\phi} - \bar{\rho}\bar{v} = \nabla \times \bar{H}, \\ \mathrm{div}\bar{H} = 0. \end{cases} \tag{2.14}$$

In order to make $\bar{H}$ uniquely determined, we add a restriction condition

$$m_{\bar{H}}(t) = \int_{\mathbb{T}^3} \bar{H}(t,x)dx = 0, \quad \forall\, t \geq 0. \tag{2.15}$$

The estimate for $\bar{H}$ is as follows.

Lemma 2.2. *The solution $\bar{H}$ to* (2.14)-(2.15) *satisfies*

$$\bar{H} \in L^\infty(\mathbb{R}^+; H^k) \quad \text{and} \quad \partial_t \bar{H} \in L^2(\mathbb{R}^+; H^{k-1}).$$

Proof. Taking the rotation operator to both sides of the first equation in (2.14), we have

$$\Delta \bar{H} = \nabla \times (\bar{\rho}\bar{v}), \quad \forall\, t \geq 0. \tag{2.16}$$

which, by Lemma 2.1, implies $\nabla \bar{H} \in L^\infty(\mathbb{R}^+; H^{k-1})$. Besides, by using (1.7) and (1.9), it is easy to see that

$$\partial_t(\bar{\rho}\bar{v}) = (\partial_t \bar{\rho})\bar{v} + \bar{\rho}\partial_t \bar{v} = -\mathrm{div}(\bar{\rho}\bar{v})\bar{v} + \bar{\rho}\partial_t \bar{v},$$

which implies that $\partial_t(\bar{\rho}\bar{v}) \in L^2(\mathbb{R}^+; H^{k-2})$. Taking the time derivative to both sides of the equation (2.16), we obtain

$$\Delta \partial_t \bar{H} = \nabla \times \partial_t(\bar{\rho}\bar{v}).$$

For a multi-index $\beta \in \mathbb{N}^3$ with $|\beta| \leq k-2$, applying ∂^β to both sides of the above equation, and making the inner product of the resulting equation with $\partial^\beta \partial_t \bar{H}$, we have

$$\begin{aligned} \|\nabla \partial^\beta \partial_t \bar{H}\|^2 \leq C|\left\langle \partial^\beta \partial_t \bar{H}, \nabla \times \partial_t \partial^\beta(\bar{\rho}\bar{v})\right\rangle| &\leq C|\left\langle \nabla \times \partial^\beta \partial_t \bar{H}, \partial_t \partial^\beta(\bar{\rho}\bar{v})\right\rangle| \\ &\leq \frac{1}{2}\|\nabla \partial^\beta \partial_t \bar{H}\|^2 + C\|\partial_t \partial^\beta(\bar{\rho}\bar{v})\|^2, \end{aligned}$$

here we have used the Young's inequality and the fact $\mathrm{div}\partial_t \bar{H} = 0$. This implies that $\partial_t \nabla \bar{H} \in L^2(\mathbb{R}^+; H^{k-2})$. Then applying the Poincaré inequality to $\bar{H}$ and $\partial_t \bar{H}$ ends the proof. □

Now we have the following uniform estimate with respect to ε.

Lemma 2.3. *It holds*

$$\int_0^t \left(\|E^\varepsilon(\tau)\|_{k-1}^2 + \|\nabla G^\varepsilon(\tau)\|_{k-2}^2\right) d\tau \leq C\|U_0^\varepsilon - U_e\|_k^2, \quad \forall\, t \geq 0, \tag{2.17}$$

where $G^\varepsilon = B^\varepsilon - B_e$.

Proof. Let $t > 0$ and

$$E^\varepsilon = -\varepsilon^2 \partial_t v^\varepsilon - f^\varepsilon, \tag{2.18}$$

in which

$$f^\varepsilon = \varepsilon^2 (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla h(\rho^\varepsilon) + v^\varepsilon + \varepsilon (v^\varepsilon \times B^\varepsilon).$$

Since $\varepsilon \in (0, 1]$, from (2.1), we obtain

$$\int_0^t \|f^\varepsilon(\tau)\|_{k-1}^2 d\tau \le C \|U_0^\varepsilon - U_e\|_k^2. \tag{2.19}$$

For a multi-index $\beta \in \mathbb{N}^3$ with $|\beta| \le k-1$, applying ∂^β to both sides of (2.18), we obtain

$$E_\beta = -\varepsilon^2 \partial_t v_\beta - \partial^\beta f^\varepsilon.$$

Taking the inner product of the above equation with E_β in L^2 yields

$$\|E_\beta\|^2 = -\left\langle E_\beta, \partial^\beta f^\varepsilon \right\rangle - \varepsilon^2 \left\langle \partial_t v_\beta, E_\beta \right\rangle,$$

in which by the Young's inequality,

$$\left|\left\langle E_\beta, \partial^\beta f^\varepsilon \right\rangle\right| \le \frac{1}{2} \|E_\beta\|^2 + C \|\partial^\beta f^\varepsilon\|^2,$$

and by (1.4) and the Moser-type inequalities,

$$\begin{aligned}
\varepsilon^2 \left\langle \partial_t v_\beta, E_\beta \right\rangle &= \varepsilon^2 \frac{d}{dt} \left\langle v_\beta, E_\beta \right\rangle - \varepsilon^2 \left\langle v_\beta, \partial_t E_\beta \right\rangle \\
&= \varepsilon^2 \frac{d}{dt} \left\langle v_\beta, E_\beta \right\rangle - \varepsilon \left\langle v_\beta, \nabla \times G_\beta + \varepsilon \partial^\beta (\rho^\varepsilon v^\varepsilon) \right\rangle \\
&\ge \varepsilon^2 \frac{d}{dt} \left\langle v_\beta, E_\beta \right\rangle - C \|v^\varepsilon\|_k^2 - \nu \|\nabla G^\varepsilon\|_{k-2}^2,
\end{aligned}$$

where $\nu > 0$ is a sufficiently small constant to be determined later. Hence,

$$2\varepsilon^2 \frac{d}{dt} \left\langle v_\beta, E_\beta \right\rangle + \|E_\beta\|^2 \le C\nu \|\nabla G^\varepsilon\|_{k-2}^2 + C \|\partial^\beta f^\varepsilon\|^2 + C \|v^\varepsilon\|_k^2.$$

Summing up all $|\beta| \le k-1$, integrating over $[0, t]$, and noticing (2.1) and (2.19), we have

$$\begin{aligned}
\int_0^t \|E(\tau)\|_{k-1}^2 d\tau &\le -2\varepsilon^2 \sum_{|\beta| \le k-1} \left\langle v_\beta(T), E_\beta(T) \right\rangle + C \|U_0^\varepsilon - U_e\|_k^2 + C\nu \int_0^t \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau \\
&\le C \|U_0^\varepsilon - U_e\|_k^2 + C\nu \int_0^t \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau.
\end{aligned} \tag{2.20}$$

For a multi-index $\gamma \in \mathbb{N}^3$ with $|\gamma| \le k-2$, applying ∂^γ to the equation for $\partial_t E^\varepsilon$ in (1.4), we obtain

$$\nabla \times G_\gamma = \varepsilon \partial_t E_\gamma - \varepsilon \partial^\gamma (\rho^\varepsilon v^\varepsilon),$$

where $G_\gamma = \partial^\gamma G^\varepsilon$. Taking inner product with $\nabla \times G_\gamma$ in L^2, and using the equations for E^ε and B^ε in (1.4), the Cauchy-Schwarz inequality and the Moser-type inequalities, we have

$$\begin{aligned}\|\nabla \times G_\gamma\|^2 &= \frac{d}{dt}\langle \varepsilon E_\gamma, \nabla \times G_\gamma\rangle - \langle E_\gamma, \varepsilon \nabla \times \partial_t B_\gamma\rangle - \langle \varepsilon \partial^\gamma(\rho^\varepsilon v^\varepsilon), \nabla \times G_\gamma\rangle \\ &\le \frac{d}{dt}\langle \varepsilon E_\gamma, \nabla \times G_\gamma\rangle - \langle \nabla \times E_\gamma, \varepsilon \partial_t B_\gamma\rangle + \frac{1}{2}\|\nabla \times G_\gamma\|^2 + C\|v^\varepsilon\|_k^2 \\ &\le \frac{d}{dt}\langle \varepsilon E_\gamma, \nabla \times G_\gamma\rangle + \|\nabla \times E_\gamma\|^2 + \frac{1}{2}\|\nabla \times G_\gamma\|^2 + C\|v^\varepsilon\|_k^2,\end{aligned}$$

which implies

$$\|\nabla \times G_\gamma\|^2 \le 2\frac{d}{dt}\langle \varepsilon E_\gamma, \nabla \times G_\gamma\rangle + 2\|\nabla \times E_\gamma\|^2 + C\|v^\varepsilon\|_k^2.$$

Since $\mathrm{div} G^\varepsilon = 0$, summing up all $|\gamma| \le k-2$, integrating over $[0,t]$ and using (2.1), we have

$$\begin{aligned}\int_0^t \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau &\le 2\sum_{|\gamma|\le k-2}\langle \varepsilon E_\gamma(T), \nabla \times G_\gamma(T)\rangle + C\int_0^t \|E^\varepsilon(\tau)\|_{k-1}^2 d\tau + C\|U_0^\varepsilon - U_e\|_k^2 \\ &\le C\int_0^t \|E^\varepsilon(\tau)\|_{k-1}^2 d\tau + C\|U_0^\varepsilon - U_e\|_k^2.\end{aligned}$$

Substituting this inequality into (2.20), we have

$$\int_0^t \left(\|E^\varepsilon(\tau)\|_{k-1}^2 + \|\nabla G^\varepsilon(\tau)\|_{k-2}^2\right) d\tau \le C\|U_0^\varepsilon - U_e\|_k^2,$$

provided that ν is sufficiently small. This proves (2.17). □

For convenience, we denote

$$(N^\varepsilon, w^\varepsilon, F^\varepsilon, G^\varepsilon) = (\rho^\varepsilon - \bar\rho, v^\varepsilon - \bar v, E^\varepsilon - \bar E, B^\varepsilon - B_e),$$

and

$$(N_\alpha, w_\alpha, F_\alpha, G_\alpha) = (\partial^\alpha N^\varepsilon, \partial^\alpha w^\varepsilon, \partial^\alpha F^\varepsilon, \partial^\alpha G^\varepsilon).$$

Subtracting the first equation in (1.7) from the mass equation in (1.4), we have

$$\partial_t N^\varepsilon + \mathrm{div}(\rho^\varepsilon v^\varepsilon - \bar\rho \bar v) = 0.$$

From (1.4), (1.8)-(1.9) and (2.14), we have

$$\mathrm{div} F^\varepsilon = \mathrm{div} E^\varepsilon - \mathrm{div}\bar E = -N^\varepsilon, \tag{2.21}$$

and

$$\partial_t F^\varepsilon = \partial_t E^\varepsilon - \partial_t \bar E = (\rho^\varepsilon v^\varepsilon - \bar\rho \bar v) + \frac{1}{\varepsilon}\nabla \times G^\varepsilon - \nabla \times \bar H. \tag{2.22}$$

According to the definition in the introduction (see (1.13)-(1.14)), F^ε is a stream function of (1.15) with the corresponding $K = \varepsilon^{-1}\nabla \times G^\varepsilon - \nabla \times \bar H$.

Now we turn to the estimates for $(N^\varepsilon, w^\varepsilon, F^\varepsilon, G^\varepsilon)$.

Lemma 2.4. *It holds*

$$\sup_{t\in\mathbb{R}^+}\left(\|N^\varepsilon(t)\|_{k-2}^2+\|F^\varepsilon(t)\|_{k-1}^2+\|G^\varepsilon(t)\|_{k-1}^2\right)$$
$$+\int_0^{+\infty}\left(\|N^\varepsilon(\tau)\|_{k-1}^2+\|F^\varepsilon(\tau)\|_{k-1}^2+\|\nabla G^\varepsilon(\tau)\|_{k-2}^2\right)d\tau\le C\varepsilon^{2p_1},$$

where p_1 is defined in Theorem 2.1.

The proof of Lemma 2.4 follows from a series of lemmas that we give the details as follows. Let $T>0$. To begin, we subtract the equation for $\bar v$ in (1.9) from the second equation in (1.4), leading to

$$\begin{aligned}\varepsilon^2\left(\partial_t(\rho^\varepsilon v^\varepsilon)+\operatorname{div}(\rho^\varepsilon v^\varepsilon\otimes v^\varepsilon)\right) &+ \nabla P(\rho^\varepsilon)-\nabla P(\bar\rho)\\ &= -(\rho^\varepsilon E^\varepsilon-\bar\rho\bar E)-\varepsilon\rho^\varepsilon(v^\varepsilon\times B^\varepsilon)-(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v).\end{aligned}$$

For all $\alpha\in\mathbb{N}^3$ with $|\alpha|\le k-1$, applying ∂^α to the above equation, taking the inner product with F_α in L^2 and integrating over $[0,T]$, we have

$$\begin{aligned}0 &= \int_0^T\varepsilon^2\left\langle F_\alpha,\partial^\alpha\partial_t(\rho^\varepsilon v^\varepsilon)\right\rangle dt+\int_0^T\left\langle F_\alpha,\partial^\alpha(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v)\right\rangle dt\\ &\quad+\int_0^T\left\langle F_\alpha,\partial^\alpha\left(\nabla(P(\rho^\varepsilon)-P(\bar\rho))\right)\right\rangle dt+\int_0^T\left\langle F_\alpha,\partial^\alpha(\rho^\varepsilon F^\varepsilon)\right\rangle dt\\ &\quad+\int_0^T\left\langle F_\alpha,\partial^\alpha(N^\varepsilon\bar E)\right\rangle dt+\int_0^T\left\langle F_\alpha,\varepsilon^2\partial^\alpha\left(\operatorname{div}(\rho^\varepsilon v^\varepsilon\otimes v^\varepsilon)\right)+\varepsilon\partial^\alpha(\rho^\varepsilon v^\varepsilon\times B^\varepsilon)\right\rangle dt\\ &\overset{def}{=} I_1+I_2+I_3+I_4+I_5+I_6,\end{aligned}\tag{2.23}$$

with the natural correspondence for I_1, I_2, I_3 ,I_4, I_5 and I_6, of which the estimates are given by a series of lemmas as follows. In lemmas 2.5-2.8, the parameters p_1 and δ are defined in Theorem 2.1, and $\mu>0$ is a sufficiently small constant to be chosen in Lemma 2.9.

Lemma 2.5. *(Estimate of I_1) For all $|\alpha|\le k-1$, it holds*

$$|I_1|\le C\varepsilon^{2p_1}+\frac14\|F_\alpha(T)\|^2+\mu\int_0^T\|\nabla G^\varepsilon(\tau)\|_{k-3}^2d\tau.\tag{2.24}$$

Proof. Recall that

$$I_1=\int_0^T\varepsilon^2\left\langle F_\alpha,\partial^\alpha\partial_t(\rho^\varepsilon v^\varepsilon)\right\rangle dt.$$

Taking the integration by parts with respect to time, we have

$$I_1 = \int_0^T\frac{d}{dt}\varepsilon^2\left\langle F_\alpha,\partial^\alpha(\rho^\varepsilon v^\varepsilon)\right\rangle dt-\int_0^T\varepsilon^2\left\langle\partial_tF_\alpha,\partial^\alpha(\rho^\varepsilon v^\varepsilon)\right\rangle dt,\tag{2.25}$$

in which

$$\begin{aligned}\left|\int_0^T \frac{d}{dt}\varepsilon^2 \langle F_\alpha, \partial^\alpha(\rho^\varepsilon v^\varepsilon)\rangle\, dt\right| &\le \frac{1}{4}\|F_\alpha(T)\|^2 + \frac{1}{4}\|F_\alpha(0)\|^2 + C\varepsilon^4\|v^\varepsilon(T)\|_{k-2}^2 + C\varepsilon^2\|v_0^\varepsilon\|_{k-1}^2 \\ &\le \frac{1}{4}\|F_\alpha(T)\|^2 + C\varepsilon^{2p_1}. \end{aligned} \tag{2.26}$$

By using (2.1), (2.6), (2.22), the Cauchy-Schwarz inequality and the Moser-type inequalities, the last term on the right hand side of (2.25) is estimated as

$$\begin{aligned} &\left|\int_0^T \varepsilon^2 \langle \partial_t F_\alpha, \partial^\alpha(\rho^\varepsilon v^\varepsilon)\rangle\, dt\right| \\ \le\ &\left|\int_0^T \varepsilon \langle \nabla\times G_\alpha, \partial^\alpha(\rho^\varepsilon v^\varepsilon)\rangle\, dt\right| + \varepsilon^2\left|\int_0^T \langle \partial^\alpha(\rho^\varepsilon v^\varepsilon), \partial^\alpha(\rho^\varepsilon v^\varepsilon)\rangle\, dt\right| \\ &+\left|\varepsilon^2\int_0^T \left\langle \partial_t \bar{E}_\alpha, \partial^\alpha(\rho^\varepsilon v^\varepsilon)\right\rangle dt\right| \\ \le\ &\left|\int_0^T \varepsilon \langle \nabla\times G_\alpha, \partial^\alpha(\rho^\varepsilon v^\varepsilon)\rangle\, dt\right| + C\varepsilon^2. \end{aligned}$$

It remains to estimate the first term on the right-hand side of the above inequality. When $|\alpha| = 0$, by (2.1) and the Young's inequality, we have

$$\begin{aligned}\left|\int_0^T \varepsilon \langle \nabla\times G^\varepsilon, \rho^\varepsilon v^\varepsilon\rangle\, dt\right| &\le \mu\int_0^T \|\nabla\times G^\varepsilon(\tau)\|^2 d\tau + C\varepsilon^2 \\ &= \mu\int_0^T \|\nabla G^\varepsilon(\tau)\|^2 d\tau + C\varepsilon^2, \end{aligned}$$

where $\mu > 0$ is sufficiently small and to be determined later. For $1 \le |\alpha| \le k-1$, by using vector formulas, we obtain

$$\begin{aligned}\left|\int_0^T \varepsilon \langle \nabla\times G_\alpha, \partial^\alpha(\rho^\varepsilon v^\varepsilon)\rangle\, dt\right| &= \left|\int_0^T \langle \varepsilon G_\alpha, \nabla\times\partial^\alpha(\rho^\varepsilon v^\varepsilon)\rangle\right| \\ &\le \mu\int_0^T \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau + C\varepsilon^2. \end{aligned}$$

Combining (2.25), (2.26) and these two estimates above yields (2.24). □

Lemma 2.6. *(Estimate of I_2) For all $|\alpha| \le k-1$, it holds*

$$\begin{aligned} I_2 \ge\ &\frac{1}{2}\|F_\alpha(T)\|^2 + \frac{1}{4}\|G_\alpha(T)\|^2 - C\varepsilon^{2p_1} \\ &-C\delta\sup_{0\le t\le T}\|G^\varepsilon(t)\|_{k-1}^2 - C\mu\int_0^T \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau. \end{aligned} \tag{2.27}$$

Proof. Recall that

$$I_2 = \int_0^T \langle F_\alpha, \partial^\alpha(\rho^\varepsilon v^\varepsilon - \bar{\rho}\bar{v})\rangle\, dt.$$

Noticing (2.14) and (2.22), we have

$$\partial^\alpha(\rho^\varepsilon v^\varepsilon - \bar{\rho}\bar{v}) = \partial_t F_\alpha - \frac{1}{\varepsilon}\nabla\times G_\alpha + \nabla\times\partial^\alpha \bar{H}.$$

Then using (1.4), (2.1), (2.3), the Cauchy-Schwarz inequality and Lemma 2.2, we have

$$\begin{aligned} I_2 &= \int_0^T \left(\frac{1}{2}\frac{d}{dt}\|F_\alpha(t)\|^2 - \left\langle \nabla \times F_\alpha, \frac{1}{\varepsilon}G_\alpha - \partial^\alpha \bar{H} \right\rangle \right) dt \\ &= \int_0^T \frac{1}{2}\frac{d}{dt}\|F_\alpha(t)\|^2 dt + \int_0^T \left\langle \varepsilon \partial_t G_\alpha, \frac{1}{\varepsilon}G_\alpha \right\rangle dt \\ &\quad - \int_0^T \left(\frac{d}{dt}\left\langle \varepsilon G_\alpha, \partial^\alpha \bar{H} \right\rangle - \left\langle \varepsilon G_\alpha, \partial_t \partial^\alpha \bar{H} \right\rangle \right) dt \\ &= \frac{1}{2}\|F_\alpha(T)\|^2 + \frac{1}{2}\|G_\alpha(T)\|^2 - \frac{1}{2}\|F_\alpha(0)\|^2 - \frac{1}{2}\|G_\alpha(0)\|^2 \\ &\quad - \int_0^T \frac{d}{dt}\left\langle \varepsilon G_\alpha, \partial^\alpha \bar{H} \right\rangle dt + \int_0^T \left\langle \varepsilon G_\alpha, \partial_t \partial^\alpha \bar{H} \right\rangle dt \\ &\geq \frac{1}{2}\|F_\alpha(T)\|^2 + \frac{1}{2}\|G_\alpha(T)\|^2 - C\varepsilon^{2p_1} - \int_0^T \frac{d}{dt}\left\langle \varepsilon G_\alpha, \partial^\alpha \bar{H} \right\rangle dt + \int_0^T \left\langle \varepsilon G_\alpha, \partial_t \partial^\alpha \bar{H} \right\rangle dt. \end{aligned}$$

Since $\bar{\rho} \in L^\infty(\mathbb{R}^+; H^k)$ and $\bar{v} \in L^\infty(\mathbb{R}^+; H^{k-1})$, we obtain that $\bar{H}$ is continuous at $t = 0$. As a result, by the Young's inequality

$$\begin{aligned} \left| \int_0^T \frac{d}{dt}\left\langle \varepsilon G_\alpha, \partial^\alpha \bar{H} \right\rangle dt \right| &\leq \varepsilon \left| \left\langle G_\alpha(T), \partial^\alpha \bar{H}(T) \right\rangle \right| + \varepsilon \left| \left\langle G_\alpha(0), \partial^\alpha \bar{H}(0) \right\rangle \right| \\ &\leq \frac{1}{4}\|G_\alpha(T)\|^2 + C\varepsilon^{2p_1}. \end{aligned}$$

It suffices to prove the following

$$\left| \int_0^T \left\langle \varepsilon G_\alpha, \partial_t \partial^\alpha \bar{H} \right\rangle dt \right| \leq C\varepsilon^2 + C\delta \sup_{0 \leq t \leq T} \|G^\varepsilon(t)\|_{k-1}^2 + \mu \int_0^T \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau. \tag{2.28}$$

Actually, for $1 \leq |\alpha| \leq k-1$, (2.28) obviously holds by noticing (2.17) and Lemma 2.2. When $\alpha = 0$, it needs a little more calculations. Since $\operatorname{div} G^\varepsilon = 0$, there exists a function χ^ε such that

$$\begin{cases} \nabla \times \chi^\varepsilon = G^\varepsilon, \\ \operatorname{div} \chi^\varepsilon = 0. \end{cases}$$

In order to make χ^ε uniquely determined, we add a restriction condition

$$\int_{\mathbb{T}^3} \chi^\varepsilon(t, x) dx = 0, \quad \forall\, t \geq 0.$$

Noticing $\operatorname{div} \chi^\varepsilon = 0$, using (2.1) and the Poincaré inequality, we immediately have

$$\|\chi^\varepsilon(t)\|_{k-1} \leq C\|\nabla \chi^\varepsilon(t)\|_{k-1} = C\|G^\varepsilon(t)\|_{k-1}, \quad \forall\, t > 0. \tag{2.29}$$

Consequently, in view of the expression for $\bar{H}$ in (2.14), using (1.8)-(1.9) and taking integration by parts, we have

$$\begin{aligned}
\left\langle \varepsilon G^\varepsilon, \partial_t \bar{H} \right\rangle &= \left\langle \varepsilon \nabla \times \chi^\varepsilon, \partial_t \bar{H} \right\rangle \\
&= \left\langle \varepsilon \chi^\varepsilon, \nabla \times \partial_t \bar{H} \right\rangle \\
&= \left\langle \varepsilon \chi^\varepsilon, \partial_t (\partial_t \nabla \bar{\phi} - \bar{\rho}\bar{v}) \right\rangle \\
&= -\left\langle \varepsilon \mathrm{div} \chi^\varepsilon, \partial_{tt} \bar{\phi} \right\rangle - \left\langle \varepsilon \chi^\varepsilon, \partial_t (\bar{\rho}\bar{v}) \right\rangle \\
&= -\left\langle \varepsilon \chi^\varepsilon, (\partial_t \bar{\rho})\bar{v} + \bar{\rho}\partial_t \bar{v} \right\rangle \\
&= \left\langle \varepsilon \chi^\varepsilon, \mathrm{div}(\bar{\rho}\bar{v})\bar{v} \right\rangle + \left\langle \varepsilon \chi^\varepsilon, \bar{\rho}\partial_t (\nabla (h(\bar{\rho}) + \bar{\phi})) \right\rangle \\
&= \left\langle \varepsilon \chi^\varepsilon, \mathrm{div}(\bar{\rho}\bar{v})\bar{v} \right\rangle - \left\langle \varepsilon \chi^\varepsilon, \nabla \bar{\rho} \partial_t (\bar{\phi} + h(\bar{\rho})) \right\rangle \\
&= \left\langle \varepsilon \chi^\varepsilon, \mathrm{div}(\bar{\rho}\bar{v})\bar{v} \right\rangle - \left\langle \varepsilon \chi^\varepsilon, \nabla \bar{\rho} \partial_t \bar{\phi} \right\rangle - \left\langle \varepsilon \chi^\varepsilon, \nabla \bar{\rho} \partial_t h(\bar{\rho}) \right\rangle .
\end{aligned}$$

Now we estimate the three terms on the right-hand side of the above. First,

$$|\varepsilon \left\langle \chi^\varepsilon, \mathrm{div}(\bar{\rho}\bar{v})\bar{v} \right\rangle| \leq C\varepsilon \|G^\varepsilon\|_{k-1} \|\bar{v}\|_{k-1}^2.$$

Noticing the Poisson equation in (2.7) and using the Poincaré inequality and the estimate (2.12), we obtain

$$\left|\varepsilon \left\langle \chi^\varepsilon, \partial_t \bar{\phi} \nabla \bar{\rho} \right\rangle\right| \leq C\varepsilon \|\chi^\varepsilon\|_{k-1} \|\nabla \bar{\rho}\| \|\partial_t \nabla \bar{\phi}\| \leq C\varepsilon \|G^\varepsilon\|_{k-1} \|\bar{\rho} - 1\|_{k-1}^2.$$

Finally, using (1.8) and the Cauchy-Schwarz inequality, we have

$$|\langle \varepsilon \chi^\varepsilon, \nabla \bar{\rho} \partial_t h(\bar{\rho}) \rangle| = |\langle \varepsilon \chi^\varepsilon, \nabla \bar{\rho} h'(\bar{\rho}) \mathrm{div}(\bar{\rho}\bar{v}) \rangle| \leq C\varepsilon \|G^\varepsilon\|_{k-1} (\|\bar{\rho} - 1\|_{k-1}^2 + \|\bar{v}\|_{k-1}^2).$$

Combining all these estimates, we arrive at

$$\left|\left\langle \varepsilon G^\varepsilon, \partial_t \bar{H} \right\rangle\right| \leq C\varepsilon \|G^\varepsilon\|_{k-1} (\|\bar{\rho} - 1\|_{k-1}^2 + \|\bar{v}\|_{k-1}^2).$$

Integrating the above over $[0, T]$, noticing the estimates (2.4), (2.5) and (2.29) and using the Young's inequality, we have

$$\begin{aligned}
\left| \int_0^T \left\langle \varepsilon G^\varepsilon, \partial_t \bar{H} \right\rangle dt \right| &\leq C \int_0^T \|G^\varepsilon\|_{k-1}^2 (\|\bar{\rho} - 1\|_{k-1}^2 + \|\bar{v}\|_{k-1}^2) dt \\
&\quad + C\varepsilon^2 \int_0^T (\|\bar{\rho} - 1\|_{k-1}^2 + \|\bar{v}\|_{k-1}^2) dt \\
&\leq C\delta \sup_{0 \leq t \leq T} \|G^\varepsilon(t)\|_{k-1}^2 + C\varepsilon^2,
\end{aligned}$$

which implies (2.28) and therefore ends the proof. □

Lemma 2.7. *(Estimate of I_3) For all $|\alpha| \le k-1$, there exists a positive constant $c_1 > 0$, such that*

$$I_3 \ge c_1 \int_0^T \|N_\alpha(\tau)\|^2 d\tau - C\delta \int_0^T \|N^\varepsilon(\tau)\|_{k-1}^2 d\tau. \tag{2.30}$$

Proof. Recalling and using (2.21) yield

$$\begin{aligned} I_3 &= \int_0^T \langle F_\alpha, \partial^\alpha \left(\nabla(P(\rho^\varepsilon) - P(\bar\rho))\right)\rangle \, dt \\ &= -\int_0^T \langle \mathrm{div} F_\alpha, \partial^\alpha (P(\rho^\varepsilon) - P(\bar\rho))\rangle \, dt \\ &= \int_0^T \langle N_\alpha, \partial^\alpha (P(\rho^\varepsilon) - P(\bar\rho))\rangle \, dt. \end{aligned}$$

We use the Taylor's formula to obtain

$$P(\rho^\varepsilon) - P(\bar\rho) = \int_0^1 P'(\tilde\rho^\varepsilon(s)) N^\varepsilon ds.$$

where

$$\tilde\rho^\varepsilon(s) = \bar\rho + sN^\varepsilon.$$

From (2.1) and (2.4), it is easy to show that $\tilde\rho^\varepsilon(s) - 1$ is sufficiently small in $L^\infty(\mathbb{R}^+ \times \mathbb{T}^3)$, uniformly with respect to $\varepsilon \in (0,1]$ and $s \in [0,1]$. Besides, we have

$$\begin{aligned} \partial^\alpha (P(\rho^\varepsilon) - P(\bar\rho)) &= \int_0^1 P'(\tilde\rho^\varepsilon(s)) N_\alpha ds \\ &+ \int_0^1 \left(\partial^\alpha \left(P'(\tilde\rho^\varepsilon(s)) N^\varepsilon\right) - P'(\tilde\rho^\varepsilon(s)) N_\alpha\right) ds. \end{aligned}$$

First, noticing (2.1), (2.2), (2.3) and (2.4), it is easy to find that when δ is sufficiently small, $\tilde\rho^\varepsilon$ are uniformly bounded from below for all $\varepsilon \in (0,1]$ and $s \in [0,1]$. Thus the continuity and monotonicity of the pressure function $P(\rho)$ imply that there exists a positive constant $c_1 > 0$, such that

$$\int_0^1 \langle N_\alpha, P'(\tilde\rho^\varepsilon(s)) N_\alpha\rangle \, ds \ge 2c_1 \|N_\alpha\|^2. \tag{2.31}$$

Then, noticing δ is sufficiently small, the weak convergence from $(\rho_0^\varepsilon, E_0^\varepsilon)$ to $(\bar\rho_0, \bar E_0)$ implies that

$$\|\bar\rho_0 - 1\|_k + \|\bar E_0\|_k \le \delta.$$

Hence, by the Moser-type inequalities, we have

$$\|\partial^\alpha \left(P'(\tilde\rho^\varepsilon(s)) N^\varepsilon\right) - P'(\tilde\rho^\varepsilon(s)) N_\alpha\| \le C\delta \|N^\varepsilon\|_{k-1},$$

which, along with the Cauchy-Schwarz inequality, implies

$$\int_0^1 \langle N_\alpha, \partial^\alpha \left(P'(\tilde\rho^\varepsilon(s)) N^\varepsilon\right) - P'(\tilde\rho^\varepsilon(s)) N_\alpha\rangle \, ds \ge -c_1 \|N_\alpha\|^2 - C\delta \|N^\varepsilon\|_{k-1}^2. \tag{2.32}$$

Thus, combining (2.31) and (2.32) yields (2.30). □

Lemma 2.8. *(Estimates of I_4, I_5 and I_6) For all $|\alpha| \le k-1$, there exists a positive constant $c_2 > 0$, such that*

$$I_4 + I_5 + I_6 \ge c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau - C\delta \int_0^T \|N^\varepsilon(\tau)\|_{k-1}^2 d\tau - C\varepsilon^2. \tag{2.33}$$

Proof. Recall that

$$\begin{aligned} I_4 &= \int_0^T \langle F_\alpha, \partial^\alpha(\rho^\varepsilon F^\varepsilon)\rangle \, dt, \\ I_5 &= \int_0^T \left\langle F_\alpha, \partial^\alpha(N^\varepsilon \bar{E})\right\rangle dt, \\ I_6 &= \int_0^T \left\langle F_\alpha, \varepsilon^2\partial^\alpha\left(\mathrm{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon)\right) + \varepsilon\partial^\alpha(\rho^\varepsilon v^\varepsilon \times B^\varepsilon)\right\rangle dt. \end{aligned}$$

First, due to (2.1) and (2.2), we notice that when δ is sufficiently small, ρ^ε are uniformly bounded from below, which implies that there exists a positive constant $c_2 > 0$, such that

$$\langle F_\alpha, \rho^\varepsilon F_\alpha\rangle \ge 4c_2\|F_\alpha\|^2.$$

As a result, by using the Moser-type inequalities, the Young's inequality and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} I_4 = \int_0^T \langle F_\alpha, \partial^\alpha(\rho^\varepsilon F^\varepsilon)\rangle \, dt &= \int_0^T \langle F_\alpha, \rho^\varepsilon F_\alpha\rangle \, dt + \int_0^T \langle F_\alpha, \partial^\alpha(\rho^\varepsilon F^\varepsilon) - \rho^\varepsilon F_\alpha\rangle \, dt \\ &\ge 4c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau - c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau \\ &\quad -C \int_0^T \|\partial^\alpha(\rho^\varepsilon F^\varepsilon) - \rho^\varepsilon F_\alpha\|^2 dt \\ &\ge 3c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau - C\delta \int_0^T \|F^\varepsilon(\tau)\|_{k-1}^2 d\tau. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |I_5| &\le c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau + C\delta \int_0^T \|N^\varepsilon(\tau)\|_{k-1}^2 d\tau, \\ |I_6| &\le c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau + C\varepsilon^2. \end{aligned}$$

These estimates imply (2.33) provided that $\delta > 0$ is sufficiently small. □

Since $I_2 + I_3 + I_4 + I_5 + I_6 = -I_1$, combining (2.21), (2.23) and Lemmas 2.5-2.8, and summing up all $|\alpha| \le k-2$, it follows

$$\begin{aligned} &\|N^\varepsilon(T)\|_{k-2}^2 + \|F^\varepsilon(T)\|_{k-1}^2 + \|G^\varepsilon(T)\|_{k-1}^2 + \int_0^T \left(\|N^\varepsilon(\tau)\|_{k-1}^2 + \|F^\varepsilon(\tau)\|_{k-1}^2\right) d\tau \\ \le\ & C\varepsilon^{2p_1} + C\delta \sup_{0\le t\le T} \|G^\varepsilon(t)\|_{k-1}^2 + C\mu \int_0^T \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau, \end{aligned}$$

provided that δ is sufficiently small. Taking the superior limit with respect to T, we have

$$\begin{aligned}
&\sup_{t\in\mathbb{R}^+}\left(\|N^\varepsilon(t)\|_{k-2}^2+\|F^\varepsilon(t)\|_{k-1}^2+\|G^\varepsilon(t)\|_{k-1}^2\right)+\int_0^{+\infty}\left(\|N^\varepsilon(\tau)\|_{k-1}^2+\|F^\varepsilon(\tau)\|_{k-1}^2\right)d\tau\\
\leq\ & C\varepsilon^{2p_1}+C\mu\int_0^{+\infty}\|\nabla G^\varepsilon(\tau)\|_{k-2}^2d\tau, \qquad (2.34)
\end{aligned}$$

provided that δ is sufficiently small.

The proof of Lemma 2.4 will be completed provided that we prove the following result.

Lemma 2.9. *It holds*

$$\int_0^{+\infty}\|\nabla G^\varepsilon(\tau)\|_{k-2}^2d\tau\leq C\varepsilon^{2p_1}. \qquad (2.35)$$

Proof. Let $T>0$. The Maxwell equations in (1.4) can be rewritten as

$$\begin{cases}\varepsilon\partial_tF^\varepsilon-\nabla\times G^\varepsilon=\varepsilon\rho^\varepsilon v^\varepsilon-\varepsilon\partial_t\bar{E},\\ \varepsilon\partial_tG^\varepsilon+\nabla\times F^\varepsilon=0,\quad \mathrm{div}G^\varepsilon=0.\end{cases}$$

For a multi-index $\gamma\in\mathbb{N}^3$ with $|\gamma|\leq k-2$, applying ∂^γ to the above yields

$$\begin{cases}\varepsilon\partial_tF_\gamma-\nabla\times G_\gamma=\varepsilon\partial^\gamma(\rho^\varepsilon v^\varepsilon)-\varepsilon\partial_t\bar{E}_\gamma,\\ \varepsilon\partial_tG_\gamma+\nabla\times F_\gamma=0,\quad \mathrm{div}G_\gamma=0.\end{cases}$$

Taking the inner product of the first equation with $\nabla\times G_\gamma$ in L^2, and using the Moser-type inequalities, the Young's inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\nabla\times G_\gamma\|^2 &= \langle\varepsilon\partial_tF_\gamma,\nabla\times G_\gamma\rangle-\left\langle\varepsilon\partial^\gamma(\rho^\varepsilon v^\varepsilon-\partial_t\bar{E}),\nabla\times G_\gamma\right\rangle\\
&= \frac{d}{dt}\langle\varepsilon F_\gamma,\nabla\times G_\gamma\rangle-\langle\nabla\times F_\gamma,\varepsilon\partial_tG_\gamma\rangle-\left\langle\varepsilon\partial^\gamma(\rho^\varepsilon v^\varepsilon-\partial_t\bar{E}),\nabla\times G_\gamma\right\rangle\\
&\leq \frac{d}{dt}\langle\varepsilon F_\gamma,\nabla\times G_\gamma\rangle+\|\nabla\times F_\gamma\|^2+\frac{1}{2}\|\nabla\times G_\gamma\|^2+C\varepsilon^2\|v^\varepsilon\|_{k-2}^2+C\varepsilon^2\|\partial_t\bar{E}\|_{k-2}^2,
\end{aligned}$$

which implies

$$\|\nabla\times G_\gamma\|^2\leq 2\frac{d}{dt}\langle\varepsilon F_\gamma,\nabla\times G_\gamma\rangle+2\|\nabla\times F_\gamma\|^2+C\varepsilon^2\|v^\varepsilon\|_{k-2}^2+C\varepsilon^2\|\partial_t\bar{E}\|_{k-2}^2.$$

Summing up all $|\gamma| \le k-2$, integrating over $[0,T]$, and using (2.1), (2.3), (2.6) and (2.34), we obtain that there exists a constant $\tilde{c} > 0$, such that

$$\begin{aligned}
&\int_0^T \|\nabla \times G^\varepsilon(\tau)\|_{k-2}^2 d\tau \\
\le\ & C\varepsilon \sum_{|\gamma|\le k-2} \langle F_\gamma(T), \nabla \times G_\gamma(T)\rangle - C\varepsilon \sum_{|\gamma|\le k-2} \langle F_\gamma(0), \nabla \times G_\gamma(0)\rangle + C\varepsilon^2 \\
&+C\int_0^T \|\nabla \times F^\varepsilon(\tau)\|_{k-2}^2 d\tau \\
\le\ & C\varepsilon^2 \|F^\varepsilon(T)\|_{k-2}^2 + C\|G^\varepsilon(T)\|_{k-2}^2 + C\varepsilon^{2p_1} + \frac{\tilde{c}\mu}{2}\int_0^T \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau \\
\le\ & C\varepsilon^{2p_1} + \frac{\tilde{c}\mu}{2}\int_0^T \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau.
\end{aligned}$$

Since $\operatorname{div} G^\varepsilon = 0$, by choosing $\mu < 1/\tilde{c}$, we arrive at

$$\int_0^T \|\nabla G^\varepsilon(\tau)\|_{k-2}^2 d\tau \le C\varepsilon^{2p_1}.$$

Since T is arbitrary, this proves (2.35). □

The proof of Theorem 2.1 follows from the following estimate.

Lemma 2.10. *It holds*

$$\sup_{t\in\mathbb{R}^+} \varepsilon^2 \|w^\varepsilon(t)\|_{k-2}^2 + \int_0^{+\infty} \|w^\varepsilon(t)\|_{k-2}^2 dt \le C\varepsilon^{2p_1}. \tag{2.36}$$

Proof. The second equation in (1.4) is equivalent to

$$\varepsilon^2 \partial_t v^\varepsilon + \varepsilon^2 (v^\varepsilon \cdot \nabla) v^\varepsilon + \nabla h(\rho^\varepsilon) = -E^\varepsilon - \varepsilon v^\varepsilon \times B^\varepsilon - v^\varepsilon.$$

Subtracting the equation for $\bar{v}$ in (1.9) from the above equation, we have

$$\varepsilon^2 \partial_t w^\varepsilon + w^\varepsilon = R_1^\varepsilon, \tag{2.37}$$

where

$$R_1^\varepsilon = -\varepsilon^2 (v^\varepsilon \cdot \nabla) v^\varepsilon - \nabla(h(\rho^\varepsilon) - h(\bar{\rho})) - F^\varepsilon - \varepsilon v^\varepsilon \times B^\varepsilon - \varepsilon^2 \partial_t \bar{v}.$$

Let $T > 0$. For all $\alpha \in \mathbb{N}^3$ with $|\alpha| \le k-2$, applying ∂^α to (2.37), taking the inner product with w_α in L^2 and integrating over $[0,T]$, we get

$$\|\varepsilon w_\alpha(T)\|^2 + \int_0^T \|w_\alpha(\tau)\|^2 d\tau \le C\int_0^T \|\partial_x^\alpha R_1^\varepsilon(\tau)\|^2 d\tau + \|\varepsilon w_\alpha(0)\|^2. \tag{2.38}$$

The Taylor's formula gives

$$h(\rho^\varepsilon) - h(\bar{\rho}) = \int_0^1 h'(\bar{\rho} + sN^\varepsilon) N^\varepsilon ds.$$

We write the term $\varepsilon v^\varepsilon \times B^\varepsilon$ as

$$\varepsilon v^\varepsilon \times B^\varepsilon = \varepsilon v^\varepsilon \times G^\varepsilon + \varepsilon v^\varepsilon \times B_e.$$

Since $k \geq 3$, we use the Moser-type inequality for the quadratic term

$$\|\partial^\alpha (uv)\| \leq C\|u\|_{k-1}\|v\|_{k-1}, \quad \forall\, u, v \in H^k, \ \ |\alpha| \leq k-1.$$

Therefore, for $|\alpha| \leq k-2$, we have

$$\|\partial^\alpha (v^\varepsilon \times G^\varepsilon)\| \leq C\|v^\varepsilon\|_k \|G^\varepsilon\|_k,$$

$$\varepsilon\|\partial^\alpha (v^\varepsilon \cdot \nabla) v^\varepsilon\| \leq C(\varepsilon\|v^\varepsilon\|_k)\|v^\varepsilon\|_k,$$

and by (2.1) and (2.4),

$$\|\partial^\alpha \nabla (h(\rho^\varepsilon) - h(\bar\rho))\| \leq C\|N^\varepsilon\|_{k-1}.$$

Thus, using (2.1) and (2.5) together with Lemma 2.4, we obtain

$$\int_0^T \|\partial^\alpha R_1^\varepsilon(\tau)\|^2 d\tau \leq C\varepsilon^{2p_1}.$$

Besides, (2.3) implies

$$\|\varepsilon w_\alpha(0)\|^2 \leq C\varepsilon^{2p_1}.$$

Substituting these estimates into (2.38) and summing up all $|\alpha| \leq k-2$, we arrive at

$$\|\varepsilon w^\varepsilon(T)\|_{k-2}^2 + \int_0^T \|w^\varepsilon(\tau)\|_{k-2}^2 d\tau \leq C\varepsilon^{2p_1},$$

which implies (2.36) since $T > 0$ is arbitrary. □

3. Convergence rate for Euler-Poisson system

In what follows, we denote by $k \geq 3$ an integer and $C > 0$ a generic constant independent of ε and any time.

3.1. **Statement of the problem.** Similarly to section 2, we introduce a slow time $t = \varepsilon t'$ and the following parabolic scaling,

$$\rho^\varepsilon(t,x) = \rho\left(\frac{t}{\varepsilon}, x\right), \qquad v^\varepsilon(t,x) = \frac{1}{\varepsilon} v\left(\frac{t}{\varepsilon}, x\right), \qquad \phi^\varepsilon(t,x) = \phi\left(\frac{t}{\varepsilon}, x\right),$$

then (1.10)-(1.11) becomes

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon v^\varepsilon) = 0, \\ \varepsilon^2 \left(\partial_t(\rho^\varepsilon v^\varepsilon) + \operatorname{div}(\rho^\varepsilon v^\varepsilon \otimes v^\varepsilon)\right) + \nabla P(\rho^\varepsilon) = -\rho^\varepsilon \nabla \phi^\varepsilon - \rho^\varepsilon v^\varepsilon, \\ \Delta \phi^\varepsilon = 1 - \rho^\varepsilon, \quad m_{\phi^\varepsilon}(t) = 0, \end{cases} \tag{3.1}$$

with initial conditions

$$\rho^\varepsilon(0,x)=\rho_0^\varepsilon(x),\quad v^\varepsilon(0,x)=\frac{v_0^\varepsilon(x)}{\varepsilon},\quad \phi^\varepsilon(0,x)=\phi_0^\varepsilon(x), \tag{3.2}$$

in which ϕ_0^ε is defined by

$$\Delta\phi_0^\varepsilon=1-\rho_0^\varepsilon.$$

Let $(\bar\rho,\bar v,\bar\phi)$ be the formal limit of $(\rho^\varepsilon,v^\varepsilon,\phi^\varepsilon)$ as $\varepsilon\to 0$. It is easy to see that it is also governed by the the drift-diffusion model (1.8) with

$$\bar v=-\nabla h(\bar\rho)-\nabla\bar\phi. \tag{3.3}$$

The study of the error estimates is based on the results about the uniform global existence and the global-in-time convergence of the system established in [19, 7, 23]. These results can be stated as follows. Let $V_e=(1,0,0)^\top$ be an equilibrium state. We denote

$$V^\varepsilon=(\rho^\varepsilon,\varepsilon v^\varepsilon,\nabla\phi^\varepsilon)^\top,\quad V_0^\varepsilon=(\rho_0^\varepsilon,v_0^\varepsilon,\nabla\phi_0^\varepsilon)^\top.$$

There exists a constant $\delta_0>0$, which is independent of ε, such that if $\|V_0^\varepsilon-V_e\|_k\le\delta_0$, then the periodic problem (3.1)-(3.2) admits a unique global solution V^ε satisfying

$$\|V^\varepsilon(t)-V_e\|_k^2+\int_0^t\|v^\varepsilon(\tau)\|_k^2d\tau\le C\|V_0^\varepsilon-V_e\|_k^2,\quad\forall\, t>0. \tag{3.4}$$

Moreover, if

$$\rho_0^\varepsilon\rightharpoonup\bar\rho_0,\qquad\text{weakly in } H^k,$$

then

$$(\rho^\varepsilon,\nabla\phi^\varepsilon)\rightharpoonup(\bar\rho,\nabla\bar\phi),\quad\text{weakly-}*\text{ in } L^\infty(\mathbb{R}^+;H^k),$$

$$v^\varepsilon\rightharpoonup\bar v,\quad\text{weakly in } L^2(\mathbb{R}^+;H^k).$$

Moreover, by the Poincaré inequality, the estimate below for $\bar\phi$ holds

$$\|\bar\phi(t)\|_{k+1}^2+\|\partial_t\bar\phi(t)\|_k^2+\int_0^t\left(\|\bar\phi(\tau)\|_{k+2}^2+\|\partial_t\bar\phi(\tau)\|_{k+1}^2\right)d\tau\le C\|\bar\rho_0-1\|_k^2,\quad\forall\, t>0. \tag{3.5}$$

The result on the convergence rate for (3.1)-(3.2) is stated as follows.

Theorem 3.1. *Let $k\ge 3$ be an integer,* (1.3) *and* (1.12) *hold. Let $(\rho^\varepsilon,v^\varepsilon,\phi^\varepsilon)$ and $(\bar\rho,\bar v,\bar\phi)$ be the unique smooth solutions to* (3.1)-(3.2) *and* (1.8)-(1.9)*, respectively. Assume*

$$\|\rho_0^\varepsilon-1\|_k+\|v_0^\varepsilon\|_k\le\delta, \tag{3.6}$$

and

$$\|v_0^\varepsilon\|_{k-2}+\|\nabla(\phi_0^\varepsilon-\bar\phi_0)\|_k\le C_3\varepsilon^q, \tag{3.7}$$

for some positive constants δ, q and C_3 independent of ε, where δ is sufficiently small. Then for all $\varepsilon \in (0,1]$, there exists a positive constant C_4 independent of ε, such that the following estimate holds

$$\begin{aligned}&\sup_{t\in\mathbb{R}^+}\left(\|\rho^\varepsilon(t)-\bar\rho(t)\|_{k-2}^2+\varepsilon^2\|v^\varepsilon(t)-\bar v(t)\|_{k-2}^2+\|\nabla\phi^\varepsilon(t)-\nabla\bar\phi(t)\|_{k-1}^2\right)\\ +\ &\int_0^{+\infty}\left(\|\rho^\varepsilon(t)-\bar\rho(t)\|_{k-1}^2+\|\nabla\phi^\varepsilon(t)-\nabla\bar\phi(t)\|_{k-1}^2+\|v^\varepsilon(t)-\bar v(t)\|_{k-2}^2\right)dt\le C_4\varepsilon^{2q_1},\end{aligned}$$

where $q_1=\min(q,1)$.

3.2. **Proof of Theorem 3.1.** For simplicity, we adopt the similar notations

$$(N^\varepsilon,w^\varepsilon,F^\varepsilon)=(\rho^\varepsilon-\bar\rho,v^\varepsilon-\bar v,\nabla\phi^\varepsilon-\nabla\bar\phi).$$

From (3.1) and (1.8), we have

$$\mathrm{div}F^\varepsilon=\Delta\phi^\varepsilon-\Delta\bar\phi=-N^\varepsilon.$$

Then

$$\mathrm{div}(\partial_tF^\varepsilon)=-\partial_tN^\varepsilon=\mathrm{div}(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v).$$

Consequently, there exists a function M^ε, such that

$$\partial_tF^\varepsilon=(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v)+\nabla\times M^\varepsilon.$$

We conclude that

$$\begin{cases}F^\varepsilon=\nabla(\phi^\varepsilon-\bar\phi),\\ \mathrm{div}F^\varepsilon=-N^\varepsilon,\\ \partial_tF^\varepsilon=(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v)+\nabla\times M^\varepsilon,\end{cases}\tag{3.8}$$

which will be frequently used in the later proof. According to the definition, F^ε is a stream function of (1.15) with corresponding $K=\nabla\times M^\varepsilon$.

For a multi-index $\alpha\in\mathbb{N}^3$, we denote

$$\begin{cases}(\rho_\alpha,v_\alpha,\phi_\alpha)=(\partial^\alpha\rho^\varepsilon,\partial^\alpha v^\varepsilon,\partial^\alpha\phi^\varepsilon),\\ (\bar\rho_\alpha,\bar v_\alpha,\bar\phi_\alpha)=(\partial^\alpha\bar\rho,\partial^\alpha\bar v,\partial^\alpha\bar\phi),\end{cases}$$

and

$$(N_\alpha,w_\alpha,F_\alpha)=(\partial^\alpha N^\varepsilon,\partial^\alpha w^\varepsilon,\partial^\alpha F^\varepsilon).$$

We start with the following result.

Lemma 3.1. *It holds*

$$\sup_{t\in\mathbb{R}^+}\left(\|N^\varepsilon(t)\|_{k-2}^2+\|F^\varepsilon(t)\|_{k-1}^2\right)+\int_0^{+\infty}\left(\|N^\varepsilon(\tau)\|_{k-1}^2+\|F^\varepsilon(\tau)\|_{k-1}^2\right)d\tau\le C\varepsilon^{2q_1},\tag{3.9}$$

where $q_1>0$ is defined in Theorem 3.1.

Proof. Subtracting (3.3) from the second equation in (3.1) leads to

$$\varepsilon^2\left(\partial_t(\rho^\varepsilon v^\varepsilon)+\mathrm{div}(\rho^\varepsilon v^\varepsilon\otimes v^\varepsilon)\right)+\nabla\big(P(\rho^\varepsilon)-P(\bar\rho)\big)=-(\rho^\varepsilon\nabla\phi^\varepsilon-\bar\rho\nabla\bar\phi)-(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v). \quad (3.10)$$

Let $T>0$. For a multi-index $\alpha\in\mathbb{N}^3$ with $|\alpha|\le k-1$, applying ∂^α to (3.10), taking the inner product with F_α in L^2 and integrating over $[0,T]$, we have

$$\begin{aligned}0 &= \int_0^T\varepsilon^2\left\langle F_\alpha,\partial^\alpha\partial_t(\rho^\varepsilon v^\varepsilon)\right\rangle dt+\int_0^T\left\langle F_\alpha,\partial^\alpha(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v)\right\rangle dt\\ &\quad+\int_0^T\left\langle F_\alpha,\partial^\alpha\nabla(P(\rho^\varepsilon)-P(\bar\rho))\right\rangle dt+\int_0^T\left\langle F_\alpha,\partial^\alpha(\rho^\varepsilon F^\varepsilon)\right\rangle dt\\ &\quad+\int_0^T\left\langle F_\alpha,\partial^\alpha(N^\varepsilon\nabla\bar\phi)\right\rangle dt+\int_0^T\varepsilon^2\left\langle F_\alpha,\partial^\alpha\mathrm{div}(\rho^\varepsilon v^\varepsilon\otimes v^\varepsilon)\right\rangle dt\\ &\overset{def}{=} J_1+J_2+J_3+J_4+J_5+J_6, \end{aligned}\quad (3.11)$$

with the natural correspondence for J_1, J_2, J_3 ,J_4, J_5 and J_6, which are treated term by term as follows. First, taking the integration by parts with respect to time in J_1, we have

$$J_1=-\int_0^T\varepsilon^2\left\langle\partial_tF_\alpha,\partial^\alpha(\rho^\varepsilon v^\varepsilon)\right\rangle dt+\int_0^T\frac{d}{dt}\left\langle F_\alpha,\partial^\alpha(\rho^\varepsilon v^\varepsilon)\right\rangle dt.$$

Noticing (3.8),

$$\partial_t\Delta(\phi^\varepsilon-\bar\phi)=-\partial_tN^\varepsilon=\mathrm{div}(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v),$$

which implies

$$\|\partial_tF_\alpha\|=\|\partial_t\partial^\alpha\nabla(\phi^\varepsilon-\bar\phi)\|\le C\|\partial^\alpha(\rho^\varepsilon v^\varepsilon-\bar\rho\bar v)\|.$$

Thus, using (3.4), the Cauchy-Schwarz inequality and the Moser-type inequalities, we have

$$\left|\int_0^T\varepsilon^2\left\langle\partial_tF_\alpha,\partial^\alpha(\rho^\varepsilon v^\varepsilon)\right\rangle dt\right|\le C\varepsilon^2\int_0^T(\|v^\varepsilon(\tau)\|_{k-1}^2+\|\bar v(\tau)\|_{k-1}^2)d\tau\le C\varepsilon^2,$$

and similar to (2.26),

$$\left|\int_0^T\frac{d}{dt}\left\langle F_\alpha,\partial^\alpha(\rho^\varepsilon v^\varepsilon)\right\rangle dt\right|\le\frac14\|F_\alpha(T)\|^2+C\varepsilon^{2q_1}.$$

Hence, we have

$$|J_1|\le\frac14\|F_\alpha(T)\|^2+C\varepsilon^{2q_1}.$$

For J_2, noticing (3.8), we find that F_α is rotation-free. Hence,

$$J_2=\int_0^T\left\langle F_\alpha,\partial_tF_\alpha-\nabla\times M^\varepsilon\right\rangle dt=\frac12\|F_\alpha(T)\|^2-\frac12\|F_\alpha(0)\|^2\ge\frac12\|F_\alpha(T)\|^2-C\varepsilon^{2q_1}.$$

Similarly, using (2.4), (2.5), (3.4), (3.5), (3.6), (3.7), (3.8), the Young's inequality, the Cauchy-Schwarz inequality and the Moser-type inequalities, J_3, J_4, J_5 and J_6 can be treated

in a similar way as in (2.23). Then we have

$$\begin{aligned} J_3 &\geq c_1 \int_0^T \|N_\alpha(\tau)\|^2 d\tau - C\delta \int_0^T \|N^\varepsilon(\tau)\|_{k-1}^2 d\tau, \\ J_4 &\geq 3c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau - C\delta \int_0^T \|F^\varepsilon(\tau)\|_{k-1}^2 d\tau, \\ |J_5| &\leq c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau + C\delta \int_0^T \|N^\varepsilon(\tau)\|_{k-1}^2 d\tau, \\ |J_6| &\leq c_2 \int_0^T \|F_\alpha(\tau)\|^2 d\tau + C\varepsilon^2. \end{aligned}$$

Since $J_2 + J_3 + J_4 = J_1 - J_5 - J_6$, combining (3.11) and the estimates for $J_i\,(1 \leq i \leq 6)$ and summing up all $|\alpha| \leq k-1$, it yields

$$\|F^\varepsilon(T)\|_{k-1}^2 + \int_0^T \left(\|N^\varepsilon(\tau)\|_{k-1}^2 + \|F^\varepsilon(\tau)\|_{k-1}^2\right) d\tau \leq C\varepsilon^{2q_1},$$

provided that δ is sufficiently small. Besides, from (3.8), we have

$$\|N^\varepsilon\|_{k-2} \leq \|\nabla F^\varepsilon\|_{k-2},$$

which implies

$$\|N^\varepsilon(T)\|_{k-2}^2 + \|F^\varepsilon(T)\|_{k-1}^2 + \int_0^T \left(\|N^\varepsilon(\tau)\|_{k-1}^2 + \|F^\varepsilon(\tau)\|_{k-1}^2\right) d\tau \leq C\varepsilon^{2q_1}.$$

This proves (3.9) since $T > 0$ is arbitrary. □

The proof of Theorem 3.1 follows from the following estimate.

Lemma 3.2. *It holds*

$$\sup_{t\in\mathbb{R}^+} \|\varepsilon w^\varepsilon(t)\|_{k-2}^2 + \int_0^{+\infty} \|w^\varepsilon(\tau)\|_{k-2}^2 d\tau \leq C\varepsilon^{2q_1}. \tag{3.12}$$

Proof. Noticing that the second equation in (3.1) is equivalent to

$$\varepsilon^2(\partial_t v^\varepsilon + (v^\varepsilon \cdot \nabla)v^\varepsilon) + \nabla h(\rho^\varepsilon) = -\nabla\phi^\varepsilon - v^\varepsilon.$$

Subtracting (3.3) from the above, we obtain

$$\varepsilon^2 \partial_t w^\varepsilon + w^\varepsilon = R_2^\varepsilon, \tag{3.13}$$

where

$$R_2^\varepsilon = -\varepsilon^2(\partial_t \bar{v} + (v^\varepsilon \cdot \nabla)v^\varepsilon) - (\nabla h(\rho^\varepsilon) - \nabla h(\bar{\rho})) - F^\varepsilon.$$

Let $T > 0$. For a multi-index $\alpha \in \mathbb{N}^3$ with $|\alpha| \leq k-2$, applying ∂^α to (3.13), taking the inner product with w_α in L^2 and integrating over $[0, T]$, it yields

$$\|\varepsilon w_\alpha(T)\|^2 + \int_0^T \|w_\alpha(\tau)\| d\tau \leq C \int_0^T \|\partial^\alpha R_2^\varepsilon(\tau)\| d\tau + \|\varepsilon w_\alpha(0)\|^2. \tag{3.14}$$

The two terms on right-hand side of the above inequality are treated similarly as in (2.38), which gives the following

$$\begin{aligned}\int_0^T \|\partial^\alpha R_2^\varepsilon(\tau)\|d\tau &\leq C\varepsilon^{2q_1},\\ \|\varepsilon w_\alpha(0)\|^2 &\leq C\varepsilon^{2q_1}.\end{aligned}$$

Substituting these estimates into (3.14), summing up all $|\alpha| \leq k-2$, it follows

$$\|\varepsilon w^\varepsilon(T)\|_{k-2}^2 + \int_0^T \|w^\varepsilon(\tau)\|_{k-2}^2 d\tau \leq C\varepsilon^{2q_1},$$

which implies (3.12) since $T > 0$ is arbitrary. □

Declarations of interest. None

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References

[1] C. Besse, P. Degond, F. Deluzet, J. Claudel, G. Gallice and C. Tessieras, A model hierarchy for ionospheric plasma modeling, *Math. Models Methods Appl. Sci.*, 14 (2004), no. 3, 393–415.

[2] F. Chen, *Introduction to Plasma Physics and Controlled Fusion*, vol. 1, Plenum Press, New York, 1984.

[3] G.-Q. Chen, J.W. Jerome and D.H. Wang, Compressible Euler-Maxwell equations. *Trans. Theory Stat. Phys.*, 29 (2000), no. 3-5, 311-331.

[4] R. Duan, Global smooth flows for the compressible Euler-Maxwell system : the relaxation case, *J. Hyperbolic Differ. Equ.*, 8 (2011), no. 2, 375-413.

[5] T. Goudon and C. Lin, Analysis of the M1 model: well-posedness and diffusion asymptotics, *J. Math. Anal. Appl.*, 402 (2013), no. 2,579-593.

[6] M.-L. Hajjej and Y.-J. Peng, Initial layers and zero-relaxation limits of Euler-Maxwell equations, *J. Differential Equations*, 252 (2012), no. 2, 1441-1465.

[7] L. Hsiao, P.A. Markowich and S. Wang, The asymptotic behavior of globally smooth solutions of the multidimensional isentropic hydrodynamic model for semiconductors, *J. Differential Equations*, 192 (2003), no. 1, 111-133.

[8] S. Junca and M. Rascle, Strong relaxation of the isothermal Euler system to the heat equation, *Z. Angew. Math. Phys.*, 53 (2002), no. 2, 239-264.

[9] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Ration. Mech. Anal.*, 58 (1975), no. 3, 181-205.

[10] C. Lattanzio, On the 3-D bipolar isentropic Euler-Poisson model for semiconductors and the drift-diffusion limit, *Math. Models Meth. Appl. Sci.*, 10 (2000), no. 3, 351-360.

[11] C. Lattanzio and P. Marcati, The relaxation to the drift-diffusion system for the 3-D isentropic Euler-Poisson model for semiconductors, *Discrete Contin. Dyn. Syst.*, 5 (1999), 449-455.

[12] P.D. Lax, Hyperbolic systems of conservation laws and the mathematical theory of shock waves, *SIAM Regional Conf. Lecture*, No. 11, Philadelphia, 1973.

[13] Y. Li, Y.J. Peng and L. Zhao, Convergence rate from hyperbolic systems of balance laws to parabolic systems, *Appl. Anal.*, 100 (2021), no. 5, 1079–1095.

[14] T.P. Liu, Hyperbolic conservation laws with relaxation, *Comm. Math. Phys.*, 108 (1987), no. 1, 153-175.

[15] A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Springer-Verlag, New York, 1984.

[16] P. Marcati and B. Rubino, Hyperbolic to parabolic relaxation theory for quasilinear first order systems, *J. Diff. Equations*, 162 (2000), 359-399.

[17] P. A. Markowich, C. A. Ringhofer and C. Schmeiser, *Semiconductor Equations*, Springer-Verlag, New York, 1990.

[18] R. Natalini, Recent results on hyperbolic relaxation problems. *Analysis of systems of conservation laws (Aachen, 1997)*, 128-198, Chapman & Hall/CRC Monogr. Surv. Pure Appl. Math., 99, Boca Raton, FL, 1999.

[19] Y.J. Peng, Uniformly global smooth solutions and convergence of Euler-Poisson systems with small parameters, *SIAM J. Math. Anal.*, 47 (2015), no. 2, 1355-1376.

[20] Y.J. Peng, S. Wang and Q.L. Gu, Relaxation limit and global existence of smooth solutions of compressible Euler-Maxwell equations, *SIAM J. Math. Anal.*, 43 (2011), no. 2, 944-970.

[21] V. Wasiolek, Uniform global existence and convergence of Euler-Maxwell systems with small parameters, *Comm. Pure Appl. Anal.*, 15 (2016), no. 6, 2007-2021.

[22] G.B. Whitham, *Linear and Nonlinear Waves*, Wiley, New York, 1974.

[23] Z. Wu and W. Wang, Decay of the solution to the bipolar Euler-Poisson system with damping in dimension three, *Commun. Math. Sci.*, 12 (2014), no. 7, 1257-1276.

[24] J. Xu, Global classical solutions to the compressible Euler-Maxwell equations, *SIAM J. Math. Anal.*, 43 (2011), no. 6, 2688-2718.

[25] W.-A. Yong, Diffusive relaxation limit of multidimensional isentropic hydrodynamical models for semiconductors, *SIAM. J. Appl. Math.*, 64 (2004), no. 5, 1737-1748.