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Sticky nonlinear SDEs and convergence of McKean-Vlasov equations without confinement

Alain Durmus, Andreas Eberle, Arnaud Guillin, Katharina Schuh

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Abstract

We develop a new approach to study the long time behaviour of solutions to nonlinear stochastic differential equations in the sense of McKean, as well as propagation of chaos for the corresponding mean-field particle system approximations. Our approach is based on a sticky coupling between two solutions to the equation. We show that the distance process between the two copies is dominated by a solution to a one-dimensional nonlinear stochastic differential equation with a sticky boundary at zero. This new class of equations is then analyzed carefully. In particular, we show that the dominating equation has a phase transition. In the regime where the Dirac measure at zero is the only invariant probability measure, we prove exponential convergence to equilibrium both for the one-dimensional equation, and for the original nonlinear SDE. Similarly, propagation of chaos is shown by a componentwise sticky coupling and comparison with a system of one dimensional nonlinear SDEs with sticky boundaries at zero. The approach applies to equations without confinement potential and to interaction terms that are not of gradient type.

1 Introduction

The main objective of this paper is to study and quantify convergence to equilibrium for McKean-Vlasov type nonlinear stochastic differential equations of the form

$$d\bar{X}_t = \left[\int_{\mathbb{R}^d} b(\bar{X}_t - x) d\bar{\mu}_t(x) \right] dt + dB_t, \quad \bar{\mu}_t = \text{Law}(\bar{X}_t), \quad (1)$$

where $(B_t)_{t \geq 0}$ is a d -dimensional standard Brownian motion and $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a Lipschitz continuous function. This nonlinear SDE is the probabilistic counterpart of the *Fokker-Planck equation*

$$\frac{\partial}{\partial t} u_t = \nabla \cdot \left[(1/2) \nabla u_t - (b * u_t) u_t \right], \quad (2)$$

which describes the time evolution of the density u_t of $\bar{\mu}_t$ with respect to the Lebesgue measure on \mathbb{R}^d . Moreover, we also study uniform in time propagation of chaos for the approximating mean-field interacting particle systems

$$dX_t^{i,N} = \frac{1}{N} \sum_{j=1}^N b(X_t^{i,N} - X_t^{j,N})dt + dB_t^i, \quad i \in \{1, \dots, N\}, \quad (3)$$

with i.i.d. initial values $X_0^{1,N}, \dots, X_0^{N,N}$, and driven by independent d -dimensional Brownian motions $\{(B_t^i)_{t \geq 0}\}_{i=1}^N$. Our results are based on a new probabilistic approach relying on sticky couplings and comparison with solutions to a class of nonlinear stochastic differential equations on the real interval $[0, \infty)$ with a sticky boundary at 0. The study of this type of equations carried out below might also be of independent interest.

The equations (1) and (2) have been studied in many works. Often a slightly different setup is considered, where the interaction b is assumed to be of gradient type, i.e., $b = -\nabla W$ for an *interaction potential* function $W : \mathbb{R}^d \rightarrow \mathbb{R}$, and an additional *confinement potential* function $V : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfying $\lim_{|x| \rightarrow \infty} V(x) = \infty$ is included in the equations. The corresponding Fokker-Planck equation

$$\frac{\partial}{\partial t} u_t = \nabla \cdot \left[(1/2) \nabla u_t + (\nabla V + \nabla W * u_t) u_t \right], \quad (4)$$

occurs for example in the modelling of granular media, see [58, 6] and the references therein. Existence and uniqueness of solutions to (1), (2) and (4) have been studied intensively. Introductions to this topic can be found for example in [27, 44, 45, 56], while recent results have been established in [48, 30]. Under appropriate conditions, it can be shown that the solutions converge to a unique stationary distribution at some given rate, see e.g. [13, 14, 10, 22, 19, 28]. In the case without confinement considered here, convergence to equilibrium of $(\bar{\mu}_t)_{t \geq 0}$ defined by (1) can only be expected for centered solutions, or after recentering around the center of mass of $\bar{\mu}_t$. It has first been analyzed in [13, 14] by an analytic approach and under the assumption that $b = -\nabla W$ for a convex function W . In particular, exponential convergence to equilibrium has been established under the strong convexity assumption $\text{Hess}(W) \geq \rho \text{Id}$ for some $\rho > 0$, and polynomial convergence in the case where W is only degenerately strictly convex. Similar results and some extensions have been derived in [42, 15] using a probabilistic approach.

Our first contribution aims at complementing these results, and extending them to non-convex interaction potentials and interaction functions that are not of gradient type. More precisely, suppose that

$$b(x) = -Lx + \gamma(x), \quad x \in \mathbb{R}^d, \quad (5)$$

where $L \in (0, \infty)$ is a positive real constant, and $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a bounded function. Then we give conditions on γ ensuring exponential convergence of centered solutions to (1) to a

unique stationary distribution in the standard L^1 Wasserstein metric. More generally, we show in Theorem 1 that under these conditions there exist constants $M, c \in (0, \infty)$ that depend only on L and γ such that if $(\bar{\mu}_t)_{t \geq 0}$ and $(\bar{\nu}_t)_{t \geq 0}$ are the marginal distributions of two solutions of (1), then for all $t \geq 0$,

$$\mathcal{W}_1(\bar{\mu}_t, \bar{\nu}_t) \leq M e^{-ct} \mathcal{W}_1(\bar{\mu}_0, \bar{\nu}_0) .$$

Using a coupling approach, related results have been derived in the previous works [22, 19] for the case where an additional confinement term is included in the equations. However, the arguments in these works rely on treating the equation with confinement and interaction term as a perturbation of the corresponding equation without interaction term, which has good ergodic properties. In the unconfined case this approach does not work, since the equation without interaction is transient and hence does not admit an invariant probability measure. Therefore, we have to develop a new approach for analyzing the equation without confinement.

Our approach is based on sticky couplings, an idea first developed in [23] to control the total variation distance between the marginal distributions of two non degenerate diffusion processes with identical noise but different drift coefficients. Since two solutions of (1) differ only in their drifts, we can indeed couple them using a sticky coupling in the sense of [23]. It can then be shown that the coupling distance process can be controlled by the solution $(r_t)_{t \geq 0}$ of a nonlinear SDE on $[0, \infty)$ with a sticky boundary at 0 of the form

$$dr_t = [\tilde{b}(r_t) + a\mathbb{P}(r_t > 0)]dt + 2\mathbb{1}_{(0, \infty)}(r_t)dW_t , \quad (6)$$

Here \tilde{b} is a real-valued function on $[0, \infty)$ satisfying $\tilde{b}(0) = 0$, a is a positive constant, and $(W_t)_{t \geq 0}$ is a one-dimensional standard Brownian motion. Solutions to SDEs with diffusion coefficient $r \mapsto \mathbb{1}_{(0, \infty)}(r)$, as in (6), have a sticky boundary at 0, i.e., if the drift at 0 is strictly positive, then the set of all time points $t \in [0, \infty)$ such that $r_t = 0$ is a fractal set with strictly positive Lebesgue measure that does not contain any open interval. Sticky SDEs have attracted wide interest, starting from [25, 26] in the one-dimensional case. Multivariate extensions have been considered in [34, 60, 61] building upon results obtained in [43, 52, 53], while corresponding martingale problems have been investigated in [54]. Note that in general no strong solution for this class of SDEs exists as illustrated in [16]. We refer to [24, 3] and the references therein for recent contributions on this topic. Note, however, that in contrast to standard sticky SDEs, the equation (6) is nonlinear in the sense of McKean. We are not aware of previous studies of such nonlinear sticky equations, which seems to be a very interesting topic on its own.

Intuitively, one would expect that as time evolves, more mass gets stuck at 0, i.e., $\mathbb{P}(r_t > 0)$ decreases. As a consequence, the drift at 0 in Equation (6) decreases, which again forces even more mass to get stuck at 0. Therefore, one could hope that $\mathbb{P}(r_t = 0)$ converges to 1 as $t \rightarrow \infty$. On the other hand, if a is too large then the drift at 0 might be too strong so that not all of the mass gets stuck at 0 eventually. This indicates that there might be a *phase*

transition for the nonlinear sticky SDE depending on the size of the constant a compared to \tilde{b} . In Section 3, we prove rigorously that this intuition is correct. Under appropriate conditions on \tilde{b} , we show at first that existence and uniqueness in law holds for solutions of (6). Then we prove that for a sufficiently small, the Dirac measure at 0 is the unique invariant probability measure, and geometric ergodicity holds. As a consequence, under corresponding assumptions, the sticky coupling approach yields exponential convergence to equilibrium for the original nonlinear SDE (1). On the other hand, we prove the existence of multiple invariant probability measures for (6) if the smallness condition on a is not satisfied. Our results for (1) can also be adapted to deal with nonlinear SDEs over the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, as considered in [18]. As an example, we discuss the application to the Kuramoto model for which a more explicit analysis is available [1, 7, 8, 12].

Finally, in addition to studying the long-time behaviour of the nonlinear SDE (1), we are also interested in establishing *propagation of chaos* for the mean-field particle system approximation (3). The propagation of chaos phenomenon first introduced by Kac [37] describes the convergence of the empirical measure of the mean-field particle system (3) to the solution (1). More precisely, in [56, 45] it has been shown under weak assumptions on W that for i.i.d. initial laws, the random variables $X_t^{i,N}$, $i \in \{1, \dots, N\}$, become asymptotically independent as $N \rightarrow \infty$, and the common law μ_t^N of each of these random variables converges to $\bar{\mu}_t$. However, the original results are only valid uniformly over a finite time horizon. Quantifying the convergence uniformly for all times $t \in \mathbb{R}_+$ is an important issue. The case with a confinement potential has been studied for example in [19], see also the references therein. Again, the case when there is only interaction is more difficult. Malrieu [42] seems the first to consider the case without confinement. By applying a synchronous coupling, he proved uniform in time propagation of chaos for strongly convex interaction potentials. Later on, assuming that the interaction potential is losing strict convexity only in a finite number of points (e.g., $W(x) = |x|^3$), Cattiaux, Guillin and Malrieu [15] have shown uniform in time propagation of chaos with a rate getting worse with the degeneracy in convexity. In a very recent work, Delarue and Tse [17] prove uniform in time weak propagation of chaos (i.e., observable by observable) on the torus via Lions derivative methods. Remarkably, their results are not limited to the unique invariant measure case.

Our contribution is in the same vein using probabilistic tools in place of analytic ones. We endow the space \mathbb{R}^{Nd} consisting of N particle configurations $x = (x^i)_{i=1}^N$ with the semi-metric $l^1 \circ \pi$, where

$$l^1(x, y) = \frac{1}{N} \sum_{i=1}^N |x^i - y^i| \quad (7)$$

is a normalized l^1 -distance between configurations $x, y \in \mathbb{R}^{Nd}$, and

$$\pi(x, y) = \left(\left(x^i - \frac{1}{N} \sum_{j=1}^N x^j \right)_{i=1}^N, \left(y^i - \frac{1}{N} \sum_{j=1}^N y^j \right)_{i=1}^N \right), \quad (8)$$

is a projection from $\mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ to the subspace $\mathbf{H}_N \times \mathbf{H}_N$, where

$$\mathbf{H}_N = \{x \in \mathbb{R}^{Nd} : \sum_{i=1}^N x^i = 0\}. \quad (9)$$

Let $\mathcal{W}_{l^1 \circ \pi}$ denote the L^1 Wasserstein semimetric on probability measures on \mathbb{R}^{Nd} corresponding to the cost function $l^1 \circ \pi$. Then under assumptions stated below, we prove uniform in time propagation of chaos for the mean-field particle system in the following sense: Suppose that $(X_t^{1,N}, \dots, X_t^{N,N})_{t \geq 0}$ is a solution of (3) such that $X_0^{1,N}, \dots, X_0^{N,N}$ are i.i.d. with distribution $\bar{\mu}_0$ having finite second moment. Let ν_t^N denote the joint law of the random variables $X_t^{i,N}$, $i \in \{1, \dots, N\}$, and let $\bar{\mu}_t$ denote the law of the solution of (1) with initial law $\bar{\mu}_0$. Then there exists a constant $C \in [0, \infty)$ such that for any $N \in \mathbb{N}$,

$$\sup_{t \geq 0} \mathcal{W}_{l^1 \circ \pi}(\bar{\mu}_t^{\otimes N}, \nu_t^N) \leq CN^{-1/2}. \quad (10)$$

The proof is based on a componentwise sticky coupling, and a comparison of the coupling difference process with a system of one-dimensional sticky nonlinear SDEs.

The paper is organised as follows. In Section 2, we state our main results regarding the long-time behaviour of (1). The main results on one-dimensional nonlinear SDEs with a sticky boundary at zero are stated in Section 3. Sections 4 and 5 contain the corresponding results on uniform (in time) propagation of chaos and mean-field systems of sticky SDEs. All the proofs are given in Section 6. In Appendix A, we carry the results over to nonlinear sticky SDEs over \mathbb{T} and consider the application to the Kuramoto model.

Notation The Euclidean norm on \mathbb{R}^d is denoted by $|\cdot|$. For $x \in \mathbb{R}$, we write $x_+ = \max(0, x)$. For some space \mathbb{X} , which here is either \mathbb{R}^d , \mathbb{R}^{Nd} or \mathbb{R}_+ , we denote its Borel σ -algebra by $\mathcal{B}(\mathbb{X})$. The space of all probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is denoted by $\mathcal{P}(\mathbb{X})$. Let $\mu, \nu \in \mathcal{P}(\mathbb{X})$. A coupling ξ of μ and ν is a probability measure on $(\mathbb{X} \times \mathbb{X}, \mathcal{B}(\mathbb{X}) \otimes \mathcal{B}(\mathbb{X}))$ with marginals μ and ν . $\Gamma(\mu, \nu)$ denotes the set of all couplings of μ and ν . The L^1 Wasserstein distance with respect to a distance function $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ is defined by

$$\mathcal{W}_d(\mu, \nu) = \inf_{\xi \in \Gamma(\mu, \nu)} \int_{\mathbb{X} \times \mathbb{X}} d(x, y) \xi(dx dy).$$

We write \mathcal{W}_1 if the underlying distance function is the Euclidean distance.

We denote by $\mathcal{C}(\mathbb{R}_+, \mathbb{X})$ the set of continuous functions from \mathbb{R}_+ to \mathbb{X} , and by $\mathcal{C}^2(\mathbb{R}_+, \mathbb{X})$ the set of twice continuously differentiable functions.

Consider a probability space (Ω, \mathcal{A}, P) and a measurable function $r : \Omega \rightarrow \mathcal{C}(\mathbb{R}_+, \mathbb{X})$. Then $\mathbb{P} = P \circ r^{-1}$ denotes the law on $\mathcal{C}(\mathbb{R}_+, \mathbb{X})$, and $P_t = P \circ r_t^{-1}$ the marginal law on \mathbb{X} at time t .

2 Long-time behaviour of McKean-Vlasov diffusions

We establish our results regarding (1) and (3) under the following assumption on b .

B1. *The function $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is Lipschitz continuous and anti-symmetric, i.e., $b(z) = -b(-z)$, and there exist $L \in (0, \infty)$, a function $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$ and a Lipschitz continuous function $\kappa : [0, \infty) \rightarrow \mathbb{R}$ such that*

$$b(z) = -Lz + \gamma(z) \quad \text{for all } z \in \mathbb{R}^d, \quad (11)$$

and the following conditions are satisfied for all $x, \tilde{x}, y \in \mathbb{R}^d$:

$$\langle x - y, \gamma(x - \tilde{x}) - \gamma(y - \tilde{x}) \rangle \leq \kappa(|x - y|)|x - y|^2, \quad (12)$$

and

$$\limsup_{r \rightarrow \infty} (\kappa(r) - L) < 0. \quad (13)$$

Let $\bar{b}(r) = (\kappa(r) - L)r$. If (13) holds, then there exist $R_0, R_1 \geq 0$ such that for

$$\bar{b}(r) < 0, \quad \text{for any } r \geq R_0, \quad (14)$$

$$\bar{b}(r)/r \leq -4/[R_1(R_1 - R_0)], \quad \text{for any } r \geq R_1. \quad (15)$$

In addition, we assume

B2.

$$\|\gamma\|_\infty \leq \left(4 \int_0^{R_1} \exp\left(\frac{1}{2} \int_0^s \bar{b}(r)_+ dr\right) ds\right)^{-1}.$$

We consider the following condition on the initial distribution.

B3. *The initial distribution μ_0 satisfies $\int_{\mathbb{R}^d} \|x\|^4 \mu_0(dx) < +\infty$ and $\int_{\mathbb{R}^d} x \mu_0(dx) = 0$.*

Note that under conditions **B1** and **B3**, unique strong solutions $(\bar{X}_t)_{t \geq 0}$ and $(\{X_t^{i,N}\}_{i=1}^N)_{t \geq 0}$ exist for (1) and (3), see e.g. [15, Theorem 2.6]. In addition, note that since b is assumed to be anti-symmetric, by an easy localisation argument, we get that $d\mathbb{E}[\bar{X}_t]/dt = \mathbb{E}[b * \mu_t(\bar{X}_t)] = 0$ and $d\mathbb{E}[N^{-1} \sum_{i=1}^N X_t^{i,N}]/dt = 0$. Thus, if \bar{X}_0 and $\{X_0^{i,N}\}_{i=1}^N$ have distribution μ_0 and $\mu_0^{\otimes N}$, respectively, with μ_0 satisfying **B3**, then it holds $\mathbb{E}[\bar{X}_t] = 0$ and $\mathbb{E}[N^{-1} \sum_{i=1}^N X_t^{i,N}] = 0$ for all $t \geq 0$.

Suppose $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is an increasing, concave function vanishing at zero. Then $d(x, y) = f(|x - y|)$ defines a distance. The corresponding L^1 Wasserstein distance is denoted by \mathcal{W}_f . Note that in the case $f(t) = t$ for any $t \geq 0$, \mathcal{W}_f is simply \mathcal{W}_1 .

Theorem 1 (Contraction for nonlinear SDE). *Assume **B1**. Let $\bar{\mu}_0, \bar{\nu}_0$ be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying **B3**. For any $t \geq 0$, let $\bar{\mu}_t$ and $\bar{\nu}_t$ denote the laws of \bar{X}_t and \bar{Y}_t where $(\bar{X}_s)_{s \geq 0}$ and $(\bar{Y}_s)_{s \geq 0}$ are solutions of (1) with initial distribution $\bar{\mu}_0$ and $\bar{\nu}_0$, respectively. Then, for all $t \geq 0$,*

$$\mathcal{W}_f(\bar{\mu}_t, \bar{\nu}_t) \leq e^{-\tilde{c}t} \mathcal{W}_f(\bar{\mu}_0, \bar{\nu}_0) \quad \text{and} \quad \mathcal{W}_1(\bar{\mu}_t, \bar{\nu}_t) \leq M_1 e^{-\tilde{c}t} \mathcal{W}_1(\bar{\mu}_0, \bar{\nu}_0), \quad (16)$$

where the function f is defined by (37) and the constants \tilde{c} and M_1 are given by

$$\tilde{c}^{-1} = 2 \int_0^{R_1} \int_0^s \exp\left(\frac{1}{2} \int_r^s \bar{b}(u)_+ du\right) dr ds, \quad (17)$$

$$M_1 = 2 \exp\left(\frac{1}{2} \int_0^{R_0} \bar{b}(s)_+ ds\right). \quad (18)$$

Proof. The proof is postponed to Section 6.2.2. □

The construction and definition of the underlying distance function $f(|x-y|)$ mentioned in Theorem 1 is based on the one introduced by [20].

To prove Theorem 1 we use a coupling $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$ of two copies of solutions to the nonlinear stochastic differential equation (1) with different initial conditions. The coupling $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$ will be defined as the weak limit of a family of couplings $(\bar{X}_t^\delta, \bar{Y}_t^\delta)_{t \geq 0}$, parametrized by $\delta > 0$. Roughly, this family is mixture of synchronous and reflection couplings and can be described as follows. For $\delta > 0$, $(\bar{X}_t^\delta, \bar{Y}_t^\delta)_{t \geq 0}$ behaves like a reflection coupling if $|\bar{X}_t^\delta - \bar{Y}_t^\delta| \geq \delta$, and like a synchronous coupling if $|\bar{X}_t^\delta - \bar{Y}_t^\delta| = 0$. For $|\bar{X}_t^\delta - \bar{Y}_t^\delta| \in (0, \delta)$ we take an interpolation of synchronous and reflection coupling. We argue that the family of couplings $\{(\bar{X}_t^\delta, \bar{Y}_t^\delta)_{t \geq 0} : \delta > 0\}$ is tight and that a subsequence $\{(\bar{X}_t^{\delta_n}, \bar{Y}_t^{\delta_n})_{t \geq 0} : n \in \mathbb{N}\}$ converges to a limit $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$. This limit is a coupling which we call the sticky coupling associated to (1).

To carry out the construction rigorously, we take two Lipschitz continuous functions $\text{rc}^\delta, \text{sc}^\delta : \mathbb{R}_+ \rightarrow [0, 1]$ for $\delta > 0$ such that

$$\text{rc}^\delta(0) = 0, \quad \text{rc}^\delta(r) = 1 \text{ for } r \geq \delta, \quad \text{rc}^\delta(r) > 0 \text{ for } r > 0 \text{ and } \text{rc}^\delta(r)^2 + \text{sc}^\delta(r)^2 = 1 \text{ for } r \geq 0. \quad (19)$$

Further, we assume that there exists $\epsilon_0 > 0$ such that for any $\delta \leq \epsilon_0$, rc^δ satisfies

$$\text{rc}^\delta(r) \geq \frac{\|\gamma\|_{\text{Lip}}}{2\|\gamma\|_\infty} r \quad \text{for any } r \in (0, \delta), \quad (20)$$

where $\|\gamma\|_{\text{Lip}} < \infty$ denotes the Lipschitz norm of γ . This assumption is satisfied for example if $\text{rc}^\delta(r) = \sin((\pi/2\delta)r) \mathbb{1}_{r < \delta} + \mathbb{1}_{r \geq \delta}$ and $\text{sc}^\delta(r) = \cos((\pi/2\delta)r) \mathbb{1}_{r < \delta}$ with $\delta \leq \epsilon_0 = 2\|\gamma\|_\infty / \|\gamma\|_{\text{Lip}}$.

Let $(B_t^1)_{t \geq 0}$ and $(B_t^2)_{t \geq 0}$ be two d -dimensional Brownian motions. We define the coupling $(\bar{X}_t^\delta, \bar{Y}_t^\delta)_{t \geq 0}$ as a process in \mathbb{R}^{2d} satisfying the following nonlinear stochastic differential

equation

$$\begin{aligned} d\bar{X}_t^\delta &= b * \bar{\mu}_t^\delta(\bar{X}_t^\delta)dt + \text{rc}^\delta(\bar{r}_t^\delta)dB_t^1 + \text{sc}^\delta(\bar{r}_t^\delta)dB_t^2, & \bar{\mu}_t^\delta &= \text{Law}(\bar{X}_t^\delta), \\ d\bar{Y}_t^\delta &= b * \bar{\nu}_t^\delta(\bar{Y}_t^\delta)dt + \text{rc}^\delta(\bar{r}_t^\delta)(\text{Id} - 2\bar{e}_t^\delta(\bar{e}_t^\delta)^T)dB_t^1 + \text{sc}^\delta(\bar{r}_t^\delta)dB_t^2, & \bar{\nu}_t^\delta &= \text{Law}(\bar{Y}_t^\delta) \end{aligned} \quad (21)$$

with initial condition $(\bar{X}_0^\delta, \bar{Y}_0^\delta) = (x_0, y_0)$. Here we set $\bar{Z}_t^\delta = \bar{X}_t^\delta - \bar{Y}_t^\delta$, $\bar{r}_t^\delta = |\bar{Z}_t^\delta|$ and $\bar{e}_t^\delta = \bar{Z}_t^\delta / \bar{r}_t^\delta$ if $\bar{r}_t^\delta \neq 0$. For $\bar{r}_t^\delta = 0$, \bar{e}_t^δ is some arbitrary unit vector, whose exact choice is irrelevant since $\text{rc}^\delta(0) = 0$.

Theorem 2. *Assume **B1**. Let $\bar{\mu}_0$ and $\bar{\nu}_0$ be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying **B3**. Then, $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$ is the subsequential limit as $\delta \rightarrow 0$ of $\{(\bar{X}_t^\delta, \bar{Y}_t^\delta)_{t \geq 0} : \delta > 0\}$ where $(\bar{X}_t)_{t \geq 0}$ and $(\bar{Y}_t)_{t \geq 0}$ are solutions of (1) with initial distribution $\bar{\mu}_0$ and $\bar{\nu}_0$. Further, there exists a process $(r_t)_{t \geq 0}$ defined on the same probability space as $(\bar{X}_t, \bar{Y}_t)_{t \geq 0}$ satisfying for any $t \geq 0$, $|\bar{X}_t - \bar{Y}_t| \leq r_t$ almost surely and which is a weak solution of*

$$dr_t = (\bar{b}(r_t) + 2\|\gamma\|_\infty \mathbb{P}(r_t > 0))dt + 2\mathbb{1}_{(0, \infty)}(r_t)d\tilde{W}_t, \quad (22)$$

where $(\tilde{W}_t)_{t \geq 0}$ is a one-dimensional Brownian motion.

Proof. The proof is postponed to Section 6.2.1. □

Therefore, next we study sticky nonlinear SDEs given by (6).

3 Nonlinear SDEs with sticky boundaries

Consider nonlinear SDEs with a sticky boundary at 0 of the form

$$dr_t = (\tilde{b}(r_t) + P_t(g))dt + 2\mathbb{1}_{(0, \infty)}(r_t)dW_t, \quad P_t = \text{Law}(r_t), \quad (23)$$

where $\tilde{b} : [0, \infty) \rightarrow \mathbb{R}$ is some continuous function and $P_t(g) = \int_{\mathbb{R}_+} g(r)P_t(dr)$ for some measurable function $g : [0, \infty) \rightarrow \mathbb{R}$.

In this section we establish existence, uniqueness in law and comparison results for solutions of (6). Consider a filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t), P)$ and a probability measure μ on \mathbb{R}_+ . We call an (\mathcal{F}_t) adapted process $(r_t, W_t)_{t \geq 0}$ a *weak solution* of (23) with initial distribution μ if the following holds: $\mu = P \circ r_0^{-1}$, the process $(W_t)_{t \geq 0}$ is a one-dimensional (\mathcal{F}_t) Brownian motion w.r.t. P , the process $(r_t)_{t \geq 0}$ is non-negative and continuous, and satisfies almost-surely

$$r_t - r_0 = \int_0^t (\tilde{b}(r_s) + P_s(g))ds + \int_0^t 2 \cdot \mathbb{1}_{(0, \infty)}(r_s)dW_s, \quad \text{for } t \in \mathbb{R}_+.$$

Note that the sticky nonlinear SDE given in (6) is a special case of (23) with $g(r) = a\mathbb{1}_{(0, \infty)}(r)$ since $\mathbb{P}(r_t > 0) = \int_{\mathbb{R}_+} \mathbb{1}_{(0, \infty)}(y)dP_t(y)$ with $P_t = P \circ r_t^{-1}$.

3.1 Existence, uniqueness in law, and a comparison result

Let $\mathbb{W} = \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ be the space of continuous functions endowed with the topology of uniform convergence on compact sets, and let $\mathcal{B}(\mathbb{W})$ be the corresponding Borel σ -algebra. Suppose $(r_t, W_t)_{t \geq 0}$ is a solution of (23) on (Ω, \mathcal{A}, P) , then we denote by $\mathbb{P} = P \circ r^{-1}$ its law on $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$. We say that *uniqueness in law* holds for (23) if for any two solutions $(r_t^1)_{t \geq 0}$ and $(r_t^2)_{t \geq 0}$ of (23) with the same initial law, the distributions of $(r_t^1)_{t \geq 0}$ and $(r_t^2)_{t \geq 0}$ on $(\mathbb{W}, \mathcal{B}(\mathbb{W}))$ are equal.

We impose the following assumptions on \tilde{b} , g and the initial condition μ :

H1. \tilde{b} is a Lipschitz continuous function with Lipschitz constant \tilde{L} and $\tilde{b}(0) = 0$.

H2. g is a left-continuous, non-negative, non-decreasing and bounded function.

H3. There exists $p > 2$ such that the p -th order moment of the law μ is finite.

Note that for (6), the condition **H2** is satisfied if a is a positive constant. It follows from **H1** and **H2** that there is a constant $C < \infty$ such that for all $r \in \mathbb{R}_+$, the following linear growth condition holds,

$$\tilde{b}(r) + \sup_{p \in \mathcal{P}(\mathbb{R}_+)} p(g) \leq C(1 + |r|). \quad (24)$$

In order to get a solution to (23) on \mathbb{R}_+ we extend the function \tilde{b} to \mathbb{R} by setting $\tilde{b}(r) = 0$ for $r < 0$. Note that any solution $(r_t)_{t \geq 0}$ with initial distribution supported on \mathbb{R}_+ satisfies almost surely $r_t \geq 0$ for all $t \geq 0$. This follows from Ito-Tanaka formula applied to $F(r) = \mathbb{1}_{(-\infty, 0)}(r)r$, cf. [51, Chapter 6, Theorem 1.1]. Indeed

$$\begin{aligned} \mathbb{1}_{(-\infty, 0)}(r_t)r_t &= \int_0^t \mathbb{1}_{(-\infty, 0)}(r_s)dr_s - \frac{1}{2}\ell_t^0(r) \\ &= \int_0^t \mathbb{1}_{(-\infty, 0)}(r_s)(\tilde{b}(r_s) + P_s(g))ds + \int_0^t \mathbb{1}_{(-\infty, 0)}2\mathbb{1}_{(0, \infty)}(r_s)dW_s - \frac{1}{2}\ell_t^0(r) \\ &= \int_0^t \mathbb{1}_{(-\infty, 0)}P_s(g)ds > 0, \end{aligned}$$

where $\ell_t^0(r)$ is the local time at 0, which vanishes, since $d[r]_s = \mathbb{1}_{(0, \infty)}(r_s)ds$.

Existence and uniqueness in law of (23) is a direct consequence of a stronger result that we now introduce. To study existence and uniqueness and to compare two solutions of (23) with different drifts, we establish existence of a synchronous coupling of two copies of (23),

$$\begin{aligned} dr_t &= (\tilde{b}(r_t) + P_t(g))dt + 2\mathbb{1}_{(0, \infty)}(r_t)dW_t, \\ ds_t &= (\hat{b}(s_t) + \hat{P}_t(h))dt + 2\mathbb{1}_{(0, \infty)}(s_t)dW_t, \quad \text{Law}(r_0, s_0) = \eta, \end{aligned} \quad (25)$$

where $P_t = P \circ r_t^{-1}$, $\hat{P}_t = P \circ s_t^{-1}$, $(W_t)_{t \geq 0}$ is a Brownian motion and where $\eta \in \Gamma(\mu, \nu)$ for $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$.

Theorem 3. Suppose that (\tilde{b}, g) and (\hat{b}, h) satisfy **H1** and **H2**. Let $\eta \in \Gamma(\mu, \nu)$ where the probability measures μ and ν on \mathbb{R}_+ satisfy **H3**. Then there exists a weak solution $(r_t, s_t)_{t \geq 0}$ of the sticky stochastic differential equation (25) with initial distribution η defined on a probability space (Ω, \mathcal{A}, P) with values in $(\mathbb{W} \times \mathbb{W}, \mathcal{B}(\mathbb{W}) \otimes \mathcal{B}(\mathbb{W}))$. If additionally,

$$\begin{aligned} \tilde{b}(r) \leq \hat{b}(r) \quad \text{and} \quad g(r) \leq h(r) & \quad \text{for any } r \in \mathbb{R}_+, \text{ and} \\ P[r_0 \leq s_0] = 1, \end{aligned}$$

then $P[r_t \leq s_t \text{ for all } t \geq 0] = 1$.

Proof. The proof is postponed to Section 6.3.1. □

Remark 4. We note that by the comparison result we can deduce uniqueness in law for the solution of (23).

3.2 Invariant measures and phase transition

Under the following conditions on the drift function \tilde{b} we exhibit a phase transition phenomenon for the model (6).

Theorem 5. Suppose **H1** holds and $\limsup_{r \rightarrow \infty} (r^{-1} \tilde{b}(r)) < 0$. Then, the Dirac measure at 0, δ_0 , is an invariant probability measure for (6). If there exists $p \in (0, 1)$ solving

$$(2/a) = (1 - p)I(a, p) \tag{26}$$

with

$$I(a, p) = \int_0^\infty \exp\left(\frac{1}{2}apx + \frac{1}{2} \int_0^x \tilde{b}(r) dr\right) dx, \tag{27}$$

then the probability measure π on $[0, \infty)$ given by

$$\pi(dx) \propto \frac{1}{Z} \left(\frac{2}{ap} \delta_0(dx) + \exp\left(\frac{1}{2}apx + \frac{1}{2} \int_0^x \tilde{b}(r) dr\right) \lambda_{(0, \infty)}(dx) \right) \tag{28}$$

is another invariant probability measure for (6).

Proof. The proof is postponed to Section 6.3.2. □

In our next result we specify a necessary and sufficient condition for the existence of a solution of (26).

Proposition 6. Suppose that $\tilde{b}(r)$ in (6) is of the form $\tilde{b}(r) = -\tilde{L}r$ with constant a $\tilde{L} > 0$. If $a/\sqrt{\tilde{L}} > 2/\sqrt{\pi}$, then there exists a unique \hat{p} solving (27). In particular, the Dirac measure δ_0 and the measure π given in (28) with \hat{p} are invariant measures for (6). On the other hand, if $a/\sqrt{\tilde{L}} \leq 2/\sqrt{\pi}$, then there exists no \hat{p} solving (27).

Proof. The proof is postponed to Section 6.3.2. □

3.3 Convergence for sticky nonlinear SDEs

Under **H1** and the following additional assumption we establish geometric convergence in Wasserstein distance for the marginal law of the solution r_t of (6) to the Dirac measure at 0:

H4. It holds $\limsup_{r \rightarrow \infty} (r^{-1} \tilde{b}(r)) < 0$ and $a \leq (2 \int_0^{\tilde{R}_1} \exp(\frac{1}{2} \int_0^s \tilde{b}(u)_+ du) ds)^{-1}$ with \tilde{R}_0, \tilde{R}_1 defined by

$$\tilde{R}_0 = \inf\{s \in \mathbb{R}_+ : \tilde{b}(r) \leq 0 \forall r \geq s\} \quad \text{and} \quad (29)$$

$$\tilde{R}_1 = \inf\{s \geq \tilde{R}_0 : -\frac{s}{r}(s - \tilde{R}_0)\tilde{b}(r) \geq 4 \forall r \geq s\}. \quad (30)$$

Theorem 7. Suppose **H1** and **H4** holds. Then, the Dirac measure at 0, δ_0 , is the unique invariant probability measure of (6). Moreover if $(r_s)_{s \geq 0}$ is a solution of (6) with r_0 distributed with respect to an arbitrary probability measure μ on $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$, it holds for all $t \geq 0$,

$$\mathbb{E}[f(r_t)] \leq e^{-ct} \mathbb{E}[f(r_0)], \quad (31)$$

where f and c are given by (37) and (36) with a and \tilde{b} given in (6) and \tilde{R}_0 and \tilde{R}_1 given in (29) and (30).

Proof. The proof is postponed to Section 6.3.3. □

4 Uniform in time propagation of chaos

To prove uniform in time propagation of chaos, we consider the L^1 Wasserstein distance with respect to the cost function $\bar{f}_N \circ \pi : \mathbb{R}^{Nd} \times \mathbb{R}^{Nd} \rightarrow \mathbb{R}_+$ with π given in (8), and \bar{f}_N given by

$$\bar{f}_N((x^{i,N})_{i=1}^N, (y^{i,N})_{i=1}^N) = \frac{1}{N} \sum_{i=1}^N f(|x^i - y^i|), \quad (32)$$

with $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined in (37). This distance is denoted by $\mathcal{W}_{f,N}$. Note that \bar{f}_N is equivalent to l^1 defined in (7).

We note that since π defines a projection from \mathbb{R}^{Nd} to the hyperplane $\mathbb{H}_N \subset \mathbb{R}^{Nd}$ given in (9), for $\hat{\mu}$ and $\hat{\nu}$ on \mathbb{H}_N , $\mathcal{W}_{f,N}(\hat{\mu}, \hat{\nu})$ coincides with the Wasserstein distance given by

$$\hat{\mathcal{W}}_{f,N}(\hat{\mu}, \hat{\nu}) = \inf_{\xi \in \Gamma(\hat{\mu}, \hat{\nu})} \int_{\mathbb{H}_N \times \mathbb{H}_N} \bar{f}_N(x, y) \xi(dx dy) \quad (33)$$

and $\mathcal{W}_{l^1 \circ \pi}(\hat{\mu}, \hat{\nu}) = \hat{\mathcal{W}}_{l^1}(\hat{\mu}, \hat{\nu})$, where \bar{f}_N and l^1 are given in (32) and (7), respectively, and where $\hat{\mathcal{W}}_{l^1}(\hat{\mu}, \hat{\nu})$ is defined as in (33) with respect to the distance l^1 .

Theorem 8 (Uniform in time propagation of chaos). *Let $N \in \mathbb{N}$ and assume **B1**. Let $\bar{\mu}_0$ and ν_0 be probability measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ satisfying **B3**. For $t \geq 0$, denote by $\bar{\mu}_t$ and ν_t^N the law of \bar{X}_t and $\{X_t^{i,N}\}_{i=1}^N$ where $(\bar{X}_s)_{s \geq 0}$ and $(\{X_s^{i,N}\}_{i=1}^N)_{s \geq 0}$ are solutions of (1) and (3), respectively, with initial distributions $\bar{\mu}_0$ and $\nu_0^{\otimes N}$. Then for all $t \geq 0$,*

$$\begin{aligned} \mathcal{W}_{f,N}(\bar{\mu}_t^{\otimes N}, \nu_t^N) &\leq e^{-\tilde{c}t} \mathcal{W}_{f,N}(\bar{\mu}_0^{\otimes N}, \nu_0^{\otimes N}) + \tilde{C} \tilde{c}^{-1} N^{-1/2}, \\ \mathcal{W}_{l^1 \circ \pi}(\bar{\mu}_t^{\otimes N}, \nu_t^N) &\leq M_1 e^{-\tilde{c}t} \mathcal{W}_{l^1 \circ \pi}(\bar{\mu}_0^{\otimes N}, \nu_0^{\otimes N}) + M_1 \tilde{C} \tilde{c}^{-1} N^{-1/2}, \end{aligned}$$

where f is defined by (37), M_1 by (18), \tilde{c} by (17) and \tilde{C} is a finite constant depending on $\|\gamma\|_\infty$, L and the second moment of $\bar{\mu}_0$ and given in (58).

Proof. The proof is postponed to Section 6.2.3. \square

Remark 9. Denote by μ_t^N and ν_t^N the distribution of $\{X_t^{i,N}\}_{i=1}^N$ and $\{Y_t^{i,N}\}_{i=1}^N$ where the two processes $(\{X_s^{i,N}\}_{i=1}^N)_{s \geq 0}$ and $(\{Y_s^{i,N}\}_{i=1}^N)_{s \geq 0}$ are solutions of (3) with initial probability distributions $\mu_0^N, \nu_0^N \in \mathcal{P}(\mathbb{R}^{Nd})$, respectively, with finite fourth moment. An easy inspection and adaptation of the proof of Theorem 8 show that if **B1** holds, then

$$\mathcal{W}_{f,N}(\mu_t^N, \nu_t^N) \leq e^{-\tilde{c}t} \mathcal{W}_{f,N}(\mu_0^{\otimes N}, \nu_0^{\otimes N}), \quad \mathcal{W}_{l^1 \circ \pi}(\mu_t^N, \nu_t^N) \leq 2M_1 e^{-\tilde{c}t} \mathcal{W}_{l^1 \circ \pi}(\mu_0^{\otimes N}, \nu_0^{\otimes N}),$$

where f , \tilde{c} and M_1 are defined as in Theorem 8.

5 System of N sticky SDEs

Consider a system of N one-dimensional SDEs with sticky boundaries at 0 given by

$$dr_t^i = \left(\tilde{b}(r_t^i) + \frac{1}{N} \sum_{j=1}^N g(r_t^j) \right) dt + 2\mathbb{1}_{(0,\infty)}(r_t^i) dW_t^i, \quad i = 1, \dots, N. \quad (34)$$

The results on existence, uniqueness and the comparison theorem for solutions of sticky nonlinear SDEs mostly carry directly over to a solution of (34) and are applied to prove propagation of chaos in Theorem 8.

Let μ be a probability distribution on \mathbb{R}_+ . For $N \in \mathbb{N}$, $(\{r_t^i, W_t^i\}_{i=1}^N)_{t \geq 0}$ is a weak solution on the filtered probability space $(\Omega, \mathcal{A}, (\mathcal{F}_t), P)$ of (34) with initial distribution $\mu^{\otimes N}$ if the following hold: $\mu^{\otimes N} = P \circ (\{r_0^i\}_{i=1}^N)^{-1}$, $(\{W_t^i\}_{i=1}^N)_{t \geq 0}$ is a N -dimensional (\mathcal{F}_t) Brownian motion w.r.t. P , the process $(r_t^i)_{t \geq 0}$ is non-negative, continuous and satisfies almost surely for any $i \in \{1, \dots, N\}$ and $t \in \mathbb{R}_+$,

$$r_t^i - r_0^i = \int_0^t \left(\tilde{b}(r_s^i) + \frac{1}{N} \sum_{j=1}^N g(r_s^j) \right) ds + \int_0^t 2\mathbb{1}_{(0,\infty)}(r_s^i) dW_s^i.$$

To show existence and uniqueness in law of a weak solution $(\{r_t^i, W_t^i\}_{i=1}^N)_{t \geq 0}$, we suppose **H1** and **H2** for \tilde{b} and g .

It follows that there exists a constant $C < \infty$ such that for all $\{r^i\}_{i=1}^N \in \mathbb{R}_+^N$, it holds $\sum_{i=1}^N |\tilde{b}(r^i)| + |g(r^i)| \leq C(1 + \sum_{i=1}^N |r^i|)$, and a possible solution $(\{r_t^i\}_{i=1}^N)_{t \geq 0}$ is non-explosive. If the initial distribution is supported on \mathbb{R}_+^N , then in the same line as for the nonlinear SDE in Section 3.1, the solution $(\{r_t^i\}_{i=1}^N)_{t \geq 0}$ satisfies $r_t^i > 0$ almost surely for any $i = 1, \dots, N$ and $t \geq 0$ by **H1** and **H2**.

Existence and uniqueness in law of (34) is a direct consequence of a stronger result that we now introduce. To study existence and uniqueness and to compare two solutions of (34) with different drifts, we establish existence of a synchronous coupling of two copies of (34),

$$\begin{aligned} dr_t^i &= \left(\tilde{b}(r_t^i) + \frac{1}{N} \sum_{j=1}^N g(r_t^j) \right) dt + 2\mathbb{1}_{(0,\infty)}(r_t^i) dW_t^i, \\ ds_t^i &= \left(\hat{b}(s_t^i) + \frac{1}{N} \sum_{j=1}^N h(s_t^j) \right) dt + 2\mathbb{1}_{(0,\infty)}(s_t^i) dW_t^i, \quad \text{for } i \in \{1, \dots, N\} \\ \text{Law}(r_0^i, s_0^i) &= \eta, \end{aligned} \tag{35}$$

where $(\{W_t^i\}_{i=1}^N)_{t \geq 0}$ are N i.i.d. 1-dimensional Brownian motions and where $\eta \in \Gamma(\mu, \nu)$ for $\mu, \nu \in \mathcal{P}(\mathbb{R}_+)$.

Let $\mathbb{W}^N = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^N)$ be the space of continuous functions from \mathbb{R}_+ to \mathbb{R}^N endowed with the topology of uniform convergence on compact sets, and let $\mathcal{B}(\mathbb{W}^N)$ denote its Borel σ -Algebra.

Theorem 10. *Assume that (\tilde{b}, g) and (\hat{b}, h) satisfy **H1** and **H2**. Let $\eta \in \Gamma(\mu, \nu)$ where μ and ν are the probability measure on \mathbb{R}_+ satisfying **H3**. Then there exists a weak solution $(\{r_t^i, s_t^i\}_{i=1}^N)_{t \geq 0}$ of the sticky stochastic differential equation (35) with initial distribution $\eta^{\otimes N}$ defined on a probability space (Ω, \mathcal{A}, P) with values in $\mathbb{W}^N \times \mathbb{W}^N$. If additionally,*

$$\begin{aligned} \tilde{b}(r) &\leq \hat{b}(r) \quad \text{and} \quad g(r) \leq h(r), & \text{for any } r \in \mathbb{R}_+, \\ P[r_0^i \leq s_0^i \text{ for all } i = 1, \dots, N] &= 1, \end{aligned}$$

then $P[r_t^i \leq s_t^i \text{ for all } t \geq 0 \text{ and } i = 1, \dots, N] = 1$.

Proof. The proof is postponed to Section 6.4. □

Remark 11. *We note that by the comparison result we can deduce uniqueness in law for the solution of (34).*

6 Proofs

6.1 Definition of the metrics

In Theorem 1, Theorem 7 and Theorem 8 we consider Wasserstein distances based on carefully designed concave function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ that we now define. In addition we

derive useful properties of this function that will be used in our proofs of Theorem 1, Theorem 8 and Theorem 7. Let $a \in \mathbb{R}_+$ and $\tilde{b} : \mathbb{R}_+ \rightarrow \mathbb{R}$ be such that **H4** is satisfied with \tilde{R}_0 and \tilde{R}_1 defined in (29). We define

$$\begin{aligned} \varphi(r) &= \exp\left(-\int_0^r \{\tilde{b}(s)_+/2\} ds\right), & \Phi(r) &= \int_0^r \varphi(s) ds, & \text{and} \\ g(r) &= 1 - \frac{c}{2} \int_0^{r \wedge \tilde{R}_1} \{\Phi(s)/\varphi(s)\} ds - \frac{a}{2} \int_0^{r \wedge \tilde{R}_1} \{1/\varphi(s)\} ds \end{aligned}$$

where

$$c = \left(2 \int_0^{\tilde{R}_1} \{\Phi(s)/\varphi(s)\} ds\right)^{-1}, \quad (36)$$

and \tilde{R}_1 is given in (30). It holds $\varphi(r) = \varphi(\tilde{R}_0)$ for $r \geq \tilde{R}_0$ with \tilde{R}_0 given in (29), $g(r) = g(\tilde{R}_1) \in [1/2, 3/4]$ for $r \geq \tilde{R}_1$ and $g(r) \in [1/2, 1]$ for all $r \in \mathbb{R}_+$ by (36) and **H4**. We define the increasing function $f : [0, \infty) \rightarrow [0, \infty)$ by

$$f(t) = \int_0^t \varphi(r)g(r)dr. \quad (37)$$

Note that f is concave, since φ and g are decreasing. Since for all $r \in \mathbb{R}_+$

$$\varphi(\tilde{R}_0)r/2 \leq \Phi(r)/2 \leq f(r) \leq \Phi(r) \leq r, \quad (38)$$

$(x, y) \mapsto f(|x - y|)$ defines a distance on \mathbb{R}^d equivalent to the Euclidean distance on \mathbb{R}^d .

Moreover, f satisfies

$$2f''(0) = -\tilde{b}(0)_+ - a = -a, \quad (39)$$

and

$$2f''(r) \leq 2f''(0) - f'(r)\tilde{b}(r) - cf(r), \quad \text{for all } r \in \mathbb{R}_+ \setminus \{\tilde{R}_1\}. \quad (40)$$

Indeed by construction of f , $f''(r) = -\tilde{b}(r)_+f'(r)/2 - c\Phi(r)/2 - a/2$ for $0 \leq r < \tilde{R}_1$ and so (40) holds for $0 \leq r < \tilde{R}_1$ by (38). To show (40) for $r > \tilde{R}_1$ note that $f''(r) = 0$ and $f'(r) \geq \varphi(\tilde{R}_0)/2$ hold for $r > \tilde{R}_1$. Hence, by the definition (30) of \tilde{R}_1 , for $r > \tilde{R}_1$,

$$f''(r) + f'(r)\tilde{b}(r)/2 \leq \varphi(\tilde{R}_0)\tilde{b}(r)/4 \leq -(\tilde{R}_1(\tilde{R}_1 - \tilde{R}_0))^{-1}\varphi(\tilde{R}_0)r. \quad (41)$$

Since $\varphi(r) = \varphi(\tilde{R}_0)$ for $r \geq \tilde{R}_0$, it holds $\Phi(r) = \Phi(\tilde{R}_0) + (r - \tilde{R}_0)\varphi(\tilde{R}_0)$ for $r \geq \tilde{R}_0$. Further, it holds $\Phi(\tilde{R}_0) \geq \tilde{R}_0\varphi(\tilde{R}_0)$ since φ is decreasing for $r \leq \tilde{R}_0$. Hence,

$$\frac{r}{\tilde{R}_1} = \frac{(r - \tilde{R}_1)(\Phi(\tilde{R}_0) + (\tilde{R}_1 - \tilde{R}_0)\varphi(\tilde{R}_0))}{\tilde{R}_1\Phi(\tilde{R}_1)} + 1 \geq \frac{(r - \tilde{R}_1)\tilde{R}_1\varphi(\tilde{R}_0)}{\tilde{R}_1\Phi(\tilde{R}_1)} + 1 = \frac{\Phi(r)}{\Phi(\tilde{R}_1)}. \quad (42)$$

Furthermore, we have

$$\begin{aligned} \int_{\tilde{R}_0}^{\tilde{R}_1} \{\Phi(s)/\varphi(s)\} ds &= \int_{\tilde{R}_0}^{\tilde{R}_1} \frac{\Phi(\tilde{R}_0) + (s - \tilde{R}_0)\varphi(\tilde{R}_0)}{\varphi(\tilde{R}_0)} ds \\ &= (\tilde{R}_1 - \tilde{R}_0) \frac{\Phi(\tilde{R}_0)}{\varphi(\tilde{R}_0)} + \frac{1}{2}(\tilde{R}_1 - \tilde{R}_0)^2 \geq \frac{1}{2}(\tilde{R}_1 - \tilde{R}_0) \frac{\Phi(\tilde{R}_1)}{\varphi(\tilde{R}_0)}. \end{aligned} \quad (43)$$

We insert (42) and (43) in (41) and use (36) to obtain

$$f''(r) + f'(r)\tilde{b}(r)/2 \leq -\Phi(r)\Phi(\tilde{R}_1)^{-1}(\tilde{R}_1 - \tilde{R}_0)^{-1}\varphi(\tilde{R}_0) \quad (44)$$

$$\leq -\frac{\Phi(r)}{2 \int_{\tilde{R}_0}^{\tilde{R}_1} \{\Phi(s)/\varphi(s)\} ds} \leq -\frac{cf(r)}{2} - \frac{c\Phi(r)}{2}. \quad (45)$$

By **H4** and (36), we get

$$-\frac{c\Phi(r)}{2} \leq -\frac{\Phi(\tilde{R}_1)}{4 \int_0^{\tilde{R}_1} \{\Phi(s)/\varphi(s)\} ds} \leq -\frac{1}{4 \int_0^{\tilde{R}_1} \{1/\varphi(s)\} ds} \leq -\frac{a}{2} = f''(0).$$

Combining this estimate with (44) gives (40) for $r > \tilde{R}_1$.

6.2 Proof of Section 2

6.2.1 Proof of Theorem 2

Note that the nonlinear SDE (21) has Lipschitz continuous coefficients. The existence and the uniqueness of the coupling $(\bar{X}_t^\delta, \bar{Y}_t^\delta)_{t \geq 0}$ follows from [45, Theorem 2.2]. By Levy's characterization, $(\bar{X}_t^\delta, \bar{Y}_t^\delta)_{t \geq 0}$ is indeed a coupling of two copies of solutions of (1). Further, we remark that $W_t^\delta = \int_0^t (\bar{e}_s^\delta)^T \cdot dB_s^1$ is a one-dimensional Brownian motion. In the next step, we analyse $|\bar{X}_t^\delta - \bar{Y}_t^\delta|$.

Lemma 12. *Suppose that the conditions **B1** and **B3** are satisfied. Then, it holds for any $\epsilon < \epsilon_0$, where ϵ_0 is given by (20), setting $\bar{r}_t^\delta = |\bar{X}_t^\delta - \bar{Y}_t^\delta|$*

$$d\bar{r}_t^\delta = \left(-L\bar{r}_t^\delta + \langle \bar{e}_t^\delta, \mathbb{E}_{x \sim \bar{\mu}_t^\delta, y \sim \bar{\nu}_t^\delta} [\gamma(\bar{X}_t^\delta - x) - \gamma(\bar{Y}_t^\delta - y)] \rangle \right) dt + 2rc^\delta(\bar{r}_t^\delta) dW_t^\delta \quad (46)$$

$$\leq \left(\bar{b}(\bar{r}_t^\delta) + 2\|\gamma\|_\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} rc^\epsilon(|x - y|) \bar{\mu}_t^\delta(dx) \bar{\nu}_t^\delta(dy) \right) dt + 2rc^\delta(\bar{r}_t^\delta) dW_t^\delta, \quad (47)$$

almost surely for all $t \geq 0$, where $\bar{\mu}_t^\delta$ and $\bar{\nu}_t^\delta$ are the laws of \bar{X}_t^δ and \bar{Y}_t^δ , respectively.

Proof. Using (21), **B1** and **B3**, the stochastic differential equation of the process $((\bar{r}_t^\delta)^2)_{t \geq 0}$ is given by

$$\begin{aligned} d((\bar{r}_t^\delta)^2) &= 2 \left\langle Z_t^\delta, -LZ_t^\delta + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \gamma(\bar{X}_t^\delta - x) - \gamma(\bar{Y}_t^\delta - y) \bar{\mu}_t^\delta(dx) \bar{\nu}_t^\delta(dy) \right\rangle dt \\ &\quad + 4rc^\delta(\bar{r}_t^\delta)^2 dt + 4rc^\delta(\bar{r}_t^\delta) \langle Z_t^\delta, e_t^\delta \rangle dW_t^\delta. \end{aligned}$$

For $\varepsilon > 0$ we define as in [23, Lemma 8] a \mathcal{C}^2 approximation of the square root by

$$S_\varepsilon(r) = \begin{cases} (-1/8)\varepsilon^{-3/2}r^2 + (3/4)\varepsilon^{-1/2}r + (3/8)\varepsilon^{1/2} & \text{for } r < \varepsilon \\ \sqrt{r} & \text{otherwise.} \end{cases}$$

Then, by Ito's formula,

$$\begin{aligned} dS_\varepsilon((\bar{r}_t^\delta)^2) &= S'_\varepsilon((\bar{r}_t^\delta)^2)d(\bar{r}_t^\delta)^2 + \frac{1}{2}S''_\varepsilon((\bar{r}_t^\delta)^2)d[(\bar{r}_t^\delta)^2]_t \\ &= 2S'_\varepsilon((\bar{r}_t^\delta)^2)\langle Z_t^\delta, -LZ_t^\delta + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \gamma(\bar{X}_t^\delta - x) - \gamma(\bar{Y}_t^\delta - y)\bar{\mu}_t^\delta(dx)\bar{\nu}_t^\delta(dy)\rangle dt \\ &\quad + S'_\varepsilon((\bar{r}_t^\delta)^2)4\text{rc}^\delta(\bar{r}_t^\delta)^2 dt + S'_\varepsilon((\bar{r}_t^\delta)^2)4\text{rc}^\delta(\bar{r}_t^\delta)\langle Z_t^\delta, e_t^\delta \rangle dW_t^\delta + 8S''_\varepsilon((\bar{r}_t^\delta)^2)(\text{rc}^\delta(\bar{r}_t^\delta))^2(\bar{r}_t^\delta)^2 dt. \end{aligned}$$

We take the limit $\varepsilon \rightarrow 0$. Then for $r > 0$, $\lim_{\varepsilon \rightarrow 0} S'_\varepsilon(r) = (1/2)r^{-1/2}$ and $\lim_{\varepsilon \rightarrow 0} S''_\varepsilon(r) = -(1/4)r^{-3/2}$. Since $\sup_{0 \leq r \leq \varepsilon} |S'_\varepsilon(r)| \lesssim \varepsilon^{-1/2}$, $\sup_{0 \leq r \leq \varepsilon} |S''_\varepsilon(r)| \lesssim \varepsilon^{-3/2}$ and rc^δ is Lipschitz continuous with $\text{rc}^\delta(0) = 0$, we apply Lebesgue's dominated convergence theorem to show convergence for the integrals with respect to time t . More precisely, we note that the integrand $(4S'_\varepsilon((\bar{r}_t^\delta)^2) + 8S''_\varepsilon((\bar{r}_t^\delta)^2))\text{rc}^\delta(\bar{r}_t^\delta)^2(\bar{r}_t^\delta)^2$ is dominated by $3\varepsilon^{1/2}\|\text{rc}^\delta\|_{\text{Lip}}$. For any $\varepsilon < \varepsilon_0$ for fixed $\varepsilon_0 > 0$, the integrand $2S'_\varepsilon((\bar{r}_t^\delta)^2)\langle Z_t^\delta, -LZ_t^\delta + \int \int \gamma(\bar{X}_t^\delta - x) - \gamma(\bar{Y}_t^\delta - y)d\bar{\mu}_t^\delta(x)d\bar{\nu}_t^\delta(y)\rangle$ is dominated by $(3/2)(L \max(\varepsilon_0^{(1/2)}, \bar{r}_t^\delta) + 2\|\gamma\|_\infty)$.

For the stochastic integral it holds $|S'_\varepsilon((\bar{r}_t^\delta)^2)4\text{rc}^\delta(\bar{r}_t^\delta)\bar{r}_t^\delta| \leq 3$. Hence, the stochastic integral converges along a subsequence almost surely, to $\int_0^t 2\text{rc}^\delta(\bar{r}_s^\delta)dW_s^\delta$, see [51, Chapter 4, Theorem 2.12]. Hence, we obtain (46). By **B1** and (20), we obtain for $\varepsilon < \varepsilon_0$

$$\begin{aligned} &\langle \bar{e}_t^\delta, \mathbb{E}[\gamma(\bar{X}_t^\delta - \hat{X}_t^\delta) - \gamma(\bar{Y}_t^\delta - \hat{Y}_t^\delta)] \rangle \\ &\leq \langle \bar{e}_t^\delta, \mathbb{E}_{x \sim \bar{\mu}_t^\delta, y \sim \bar{\nu}_t^\delta}[\gamma(\bar{X}_t^\delta - x) - \gamma(\bar{Y}_t^\delta - x) + \gamma(\bar{Y}_t^\delta - x) - \gamma(\bar{Y}_t^\delta - y)] \rangle \\ &\leq \kappa(\bar{r}_t^\delta)\bar{r}_t^\delta + \mathbb{E}_{x \sim \bar{\mu}_t^\delta, y \sim \bar{\nu}_t^\delta}[2\|\gamma\|_\infty \text{rc}^\varepsilon(|x - y|)], \end{aligned}$$

and hence (47) holds. \square

We define a one-dimensional process $(r_t^{\delta, \varepsilon})_{t \geq 0}$ by

$$dr_t^{\delta, \varepsilon} = \left(\bar{b}(r_t^{\delta, \varepsilon}) + 2\|\gamma\|_\infty \int \text{rc}^\varepsilon(u)dP_t^{\delta, \varepsilon}(u)\right)dt + 2 \cdot \text{rc}^\delta(r_t^{\delta, \varepsilon})dW_t^\delta \quad (48)$$

with initial condition $r_0^{\delta, \varepsilon} = \bar{r}_0^\delta$, $P_t^{\delta, \varepsilon} = \text{Law}(r_t^{\delta, \varepsilon})$ and $W_t^\delta = \int_0^t (\bar{e}_s^\delta)^T dB_s^1$. This process will allow us to control the distance of \bar{X}_t^δ and \bar{Y}_t^δ .

By [45, Theorem 2.2], under **B1** and **B3**, $(U_t^{\delta, \varepsilon})_{t \geq 0} = (\bar{X}_t^\delta, \bar{Y}_t^\delta, r_t^{\delta, \varepsilon})_{t \geq 0}$ exists and is unique, where $(\bar{X}_t^\delta, \bar{Y}_t^\delta)_{t \geq 0}$ solves uniquely (21), $(\bar{r}_t^\delta)_{t \geq 0}$ and $(r_t^{\delta, \varepsilon})_{t \geq 0}$ solve uniquely (46) and (48), respectively, with $W_t^\delta = \int_0^t (\bar{e}_s^\delta)^T \cdot dB_s^1$.

Lemma 13. *Assume **B1** and **B3**. Then, $|\bar{X}_t^\delta - \bar{Y}_t^\delta| = \bar{r}_t^\delta \leq r_t^{\delta, \varepsilon}$, almost surely for all t and $\varepsilon < \varepsilon_0$.*

Proof. Note that $(\bar{r}_t^\delta)_{t \geq 0}$ and $(r_t^{\delta, \epsilon})_{t \geq 0}$ have the same initial distribution and are driven by the same noise. Since the drift of $(\bar{r}_t^\delta)_{t \geq 0}$ is smaller than the drift of $(r_t^{\delta, \epsilon})_{t \geq 0}$ for $\epsilon < \epsilon_0$, the result follows by Lemma 18. \square

Proof of Theorem 2. We consider the nonlinear process $(U_t^{\delta, \epsilon})_{t \geq 0} = (\bar{X}_t^\delta, \bar{Y}_t^\delta, r_t^{\delta, \epsilon})_{t \geq 0}$ on \mathbb{R}^{2d+1} for each $\epsilon, \delta > 0$. We denote by $\mathbb{P}^{\delta, \epsilon}$ the law of $U_t^{\delta, \epsilon}$ on the space $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^{2d+1})$. We define by $\mathbf{X}, \mathbf{Y} : \mathcal{C}(\mathbb{R}_+, \mathbb{R}^{2d+1}) \rightarrow \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and $\mathbf{r} : \mathcal{C}(\mathbb{R}_+, \mathbb{R}^{2d+1}) \rightarrow \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ the canonical projections onto the first d components, onto the second d components and onto the last component, respectively. By **B1** and **B3** following the same line as the proof of Lemma 19, see (65), it holds for each $T > 0$

$$E[|U_{t_2}^{\delta, \epsilon} - U_{t_1}^{\delta, \epsilon}|^4] \leq C|t_2 - t_1|^2 \quad \text{for } t_1, t_2 \in [0, T], \quad (49)$$

for some constant C depending on $T, L, \|\gamma\|_{\text{Lip}}, \|\gamma\|_\infty$ and on the fourth moment of μ_0 and ν_0 . As in Lemma 19 the law $\mathbb{P}_T^{\delta, \epsilon}$ of $(U_t^{\delta, \epsilon})_{0 \leq t \leq T}$ on $C([0, T], \mathbb{R}^{2d+1})$ is tight for each $T > 0$ by [38, Corollary 14.9] and for each $\epsilon > 0$ there exists a subsequence $\delta_n \rightarrow 0$ such that $(\mathbb{P}_T^{\delta_n, \epsilon})_{n \in \mathbb{N}}$ on $C([0, T], \mathbb{R}^{2d+1})$ converge to a measure \mathbb{P}_T^ϵ on $C([0, T], \mathbb{R}^{2d+1})$. By a diagonalization argument and since $\{\mathbb{P}_T^\epsilon : T \geq 0\}$ is a consistent family, cf. [38, Theorem 5.16], there exists a probability measure \mathbb{P}^ϵ on $C(\mathbb{R}_+, \mathbb{R}^{2d+1})$ such that for all ϵ there exists a subsequence δ_n such that $(\mathbb{P}^{\delta_n, \epsilon})_{n \in \mathbb{N}}$ converges along this subsequence to \mathbb{P}^ϵ , i.e., the family $(\mathbb{P}^{\delta, \epsilon})_{\delta > 0}$ is tight. As in the proof of Lemma 20 we repeat this argument for the family of measures $(\mathbb{P}^\epsilon)_{\epsilon > 0}$. Hence, there exists a subsequence $\epsilon_n \rightarrow 0$ such that $(\mathbb{P}^{\epsilon_n})_{n \in \mathbb{N}}$ converges to a measure \mathbb{P} . Let $(\bar{X}_t, \bar{Y}_t, r_t)_{t \geq 0}$ be some process on \mathbb{R}^{2d+1} with distribution \mathbb{P} on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$.

Since $(\bar{X}_t^\delta)_{t \geq 0}$ and $(\bar{Y}_t^\delta)_{t \geq 0}$ are solutions of (1) which are unique in law, we have that for any $\epsilon, \delta > 0$, $\mathbb{P}^{\delta, \epsilon} \circ (\mathbf{X})^{-1} = \mathbb{P} \circ (\mathbf{X})^{-1}$ and $\mathbb{P}^{\delta, \epsilon} \circ (\mathbf{Y})^{-1} = \mathbb{P} \circ (\mathbf{Y})^{-1}$. And therefore $(\bar{X}_t)_{t \geq 0}$ and $(\bar{Y}_t)_{t \geq 0}$ are solutions of (1) as well with the same initial condition. Hence $\mathbb{P} \circ (\mathbf{X}, \mathbf{Y})^{-1}$ is a coupling of two copies of (1).

In addition, the statement of Lemma 13 carries over to the limiting process $(r_t)_{t \geq 0}$, i.e., $|\bar{X}_t - \bar{Y}_t| \leq r_t$ for all $t \geq 0$, since by the weak convergence and the Portmanteau theorem, $\bar{P}(|\bar{X}_t - \bar{Y}_t| \leq r_t) \geq \limsup_{\epsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} P(|\bar{X}_t^\delta - \bar{Y}_t^\delta| \leq r_t^{\delta, \epsilon}) = 1$.

Finally, similarly to the proof of Lemma 19 and Lemma 20 there exist an extended probability space and a one-dimensional Brownian motion $(W_t)_{t \geq 0}$ such that $(r_t, W_t)_{t \geq 0}$ is a solution to

$$dr_t = (\bar{b}(r_t) + 2\|\gamma\|_\infty \mathbb{P}(r_t > 0))dt + 2\mathbb{1}_{(0, \infty)}(r_t)dW_t.$$

\square

6.2.2 Proof of Theorem 1

Proof of Theorem 1. We consider the process $(\bar{X}_t, \bar{Y}_t, r_t)_{t \geq 0}$ defined in Theorem 2 and satisfying $|\bar{X}_t - \bar{Y}_t| \leq r_t$ for any $t \geq 0$, and $(r_t)_{t \geq 0}$ is a weak solution of (22). Set $a = 2\|\gamma\|_\infty$ and $\tilde{b}(r) = \bar{b}(r)$. With this notation, **B1** and **B2** imply **H4** and $\tilde{R}_0 = R_0$ and $\tilde{R}_1 = R_1$ by (14), (15), (29) and (30). By Ito-Tanaka formula, cf. [51, Chapter 6, Theorem 1.1], using that f' is absolutely continuous, we have,

$$\begin{aligned} df(r_t) &\leq f'(r_t)(\bar{b}(r_t) + 2\|\gamma\|_\infty \mathbb{P}(r_t > 0))dt + 2f''(r_t)\mathbb{1}_{(0,\infty)}(r_t)dt \\ &\quad + f'(r_t)2\mathbb{1}_{(0,\infty)}(r_t)dW_t. \end{aligned}$$

Taking expectation we obtain by (39) and (40)

$$\frac{d}{dt}\mathbb{E}[f(r_t)] \leq \mathbb{E}[f'(r_t)\tilde{b}(r_t)_+ + 2(f''(r_t) - f''(0))] + \mathbb{E}[(a + 2f''(0))\mathbb{1}_{r_t > 0}] \leq -\tilde{c}\mathbb{E}[f(r_t)],$$

where \tilde{c} is given by (17). Therefore by Gronwall's lemma,

$$\mathbb{E}[f(|\bar{X}_t - \bar{Y}_t|)] \leq \mathbb{E}[f(r_t)] \leq e^{-\tilde{c}t}\mathbb{E}[f(r_0)] = e^{-\tilde{c}t}\mathbb{E}[f(|\bar{X}_0 - \bar{Y}_0|)].$$

Hence, it holds

$$\mathcal{W}_f(\bar{\mu}_t, \bar{\nu}_t) \leq \mathbb{E}[f(|\bar{X}_t - \bar{Y}_t|)] \leq e^{-\tilde{c}t} \int f(|x - y|)\xi(dx dy)$$

for an arbitrary coupling $\xi \in \Gamma(\mu_0, \nu_0)$. Taking the infimum over all couplings $\xi \in \Gamma(\mu_0, \nu_0)$, we obtain the first inequality of (16). By (38), we get the second inequality of (16). \square

6.2.3 Proof of Theorem 8

To show Theorem 8, we first give a uniform in time bound for the second moment of the process $(\bar{X}_t)_{t \geq 0}$ solving (1).

Lemma 14. *Let $(\bar{X}_t)_{t \geq 0}$ be a solution of (1) with $\mathbb{E}[|\bar{X}_0|^2] < \infty$. Assume **B1**. Then there exists $C \in (0, \infty)$ depending on d, W and the second moment of \bar{X}_0 such that*

$$C = \sup_{t \geq 0} \mathbb{E}[|\bar{X}_t|^2] < \infty. \quad (50)$$

Proof of Lemma 14. By Ito's formula, it holds

$$\frac{1}{2}d|\bar{X}_t|^2 = \langle \bar{X}_t, b * \bar{\mu}_t(\bar{X}_t) \rangle dt + \bar{X}_t^T dB_t + \frac{1}{2}d \, dt.$$

Taking expectation and using symmetry, we get

$$\begin{aligned}
\frac{d}{dt}\mathbb{E}[|\bar{X}_t|^2] &= \mathbb{E}[\langle \bar{X}_t - \tilde{X}_t, b(\bar{X}_t - \tilde{X}_t) \rangle] + d \\
&= -\mathbb{E}[\langle \bar{X}_t - \tilde{X}_t, L(\bar{X}_t - \tilde{X}_t) - \gamma(\bar{X}_t - \tilde{X}_t) \rangle \mathbb{1}_{|\bar{X}_t - \tilde{X}_t| > R_0}] \\
&\quad - \mathbb{E}[\langle \bar{X}_t - \tilde{X}_t, L(\bar{X}_t - \tilde{X}_t) - \gamma(\bar{X}_t - \tilde{X}_t) \rangle \mathbb{1}_{|\bar{X}_t - \tilde{X}_t| \leq R_0}] + d \\
&\leq \mathbb{E}[|\bar{X}_t|^2(-2L + \kappa(|\bar{X}_t - \tilde{X}_t|) \mathbb{1}_{|\bar{X}_t - \tilde{X}_t| > R_0})] + \|\gamma\|_\infty R_0 + d.
\end{aligned}$$

Hence by definition (14) of R_0 and by Gronwall's lemma we obtain the result (50). \square

Let $N \in \mathbb{N}$. We construct a sticky coupling of N i.i.d. realizations of solutions $(\{\bar{X}_t^i\}_{i=1}^N)_{t \geq 0}$ to (1) and of the solution $(\{Y_t^i\}_{i=1}^N)_{t \geq 0}$ to the mean field particle system (3). Then, we consider a weak limit for $\delta \rightarrow 0$ of Markovian couplings which are constructed similar as in Section 2. Let $\text{rc}^\delta, \text{sc}^\delta$ satisfy (19) and (20). The coupling $(\{\bar{X}_t^{i,\delta}, Y_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ is defined as process in \mathbb{R}^{2Nd} satisfying a system of SDEs given by

$$\begin{aligned}
d\bar{X}_t^{i,\delta} &= b * \bar{\mu}_t^\delta(\bar{X}_t^{i,\delta}) dt + \text{rc}^\delta(\tilde{r}_t^{i,\delta}) dB_t^{i,1} + \text{sc}^\delta(\tilde{r}_t^{i,\delta}) dB_t^{i,2} \\
dY_t^{i,\delta} &= \frac{1}{N} \sum_{j=1}^N b(Y_t^{i,\delta} - Y_t^{j,\delta}) dt + \text{rc}^\delta(\tilde{r}_t^{i,\delta}) (\text{Id} - 2\tilde{e}_t^{i,\delta} (\tilde{e}_t^{i,\delta})^T) dB_t^{i,1} + \text{sc}^\delta(\tilde{r}_t^{i,\delta}) dB_t^{i,2}, \tag{51}
\end{aligned}$$

where $\text{Law}(\{\bar{X}_0^{i,\delta}, Y_0^{i,0}\}_{i=1}^N) = \bar{\mu}_0^{\otimes N} \otimes \nu_0^{\otimes N}$, and where $(\{B_t^{i,1}\}_{i=1}^N)_{t \geq 0}, (\{B_t^{i,2}\}_{i=1}^N)_{t \geq 0}$ are i.i.d. d -dimensional standard Brownian motions. We set $\tilde{X}_t^{i,\delta} = \bar{X}_t^{i,\delta} - \frac{1}{N} \sum_{j=1}^N \bar{X}_t^{j,\delta}$, $\tilde{Y}_t^{i,\delta} = Y_t^{i,\delta} - \frac{1}{N} \sum_{j=1}^N Y_t^{j,\delta}$, $\tilde{Z}_t^{i,\delta} = \tilde{X}_t^{i,\delta} - \tilde{Y}_t^{i,\delta}$, $\tilde{r}_t^{i,\delta} = |\tilde{Z}_t^{i,\delta}|$ and $\tilde{e}_t^{i,\delta} = \tilde{Z}_t^{i,\delta} / \tilde{r}_t^{i,\delta}$ for $\tilde{r}_t^{i,\delta} \neq 0$. The value $\tilde{e}_t^{i,\delta}$ for $\tilde{r}_t^{i,\delta} = 0$ is irrelevant as $\text{rc}^{i,\delta}(0) = 0$. By Levy's characterization $(\{\bar{X}_t^{i,\delta}, Y_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ is indeed a coupling of (1) and (3). Existence and uniqueness of the coupling given in (51) hold by [45, Theorem 2.2]. In the next step we analyse $\tilde{r}_t^{i,\delta}$.

Lemma 15. *Assume B1 holds. Then, for $\epsilon < \epsilon_0$, where ϵ_0 is given in (20), and for any $i \in \{1, \dots, N\}$, it holds almost surely,*

$$\begin{aligned}
d\tilde{r}_t^{i,\delta} &= -L\tilde{r}_t^{i,\delta} dt + \langle \tilde{e}_t^{i,\delta}, \frac{1}{N} \sum_{j=1}^N \gamma(\tilde{X}_t^{i,\delta} - \tilde{X}_t^{j,\delta}) - \gamma(\tilde{Y}_t^{i,\delta} - \tilde{Y}_t^{j,\delta}) \rangle dt \\
&\quad + 2\sqrt{1 + \frac{1}{N}} \text{rc}^\delta(\tilde{r}_t^{i,\delta}) dW_t^{i,\delta} + \langle \tilde{e}_t^{i,\delta}, \Theta_t^{i,\delta} + \frac{1}{N} \sum_{k=1}^N \Theta_t^{k,\delta} \rangle dt \tag{52} \\
&\leq (\bar{b}(\tilde{r}_t^{i,\delta}) + 2\|\gamma\|_\infty \frac{1}{N} \sum_{j=1}^N \text{rc}^\epsilon(\tilde{r}_t^{j,\delta})) dt + 2\sqrt{1 + \frac{1}{N}} \text{rc}^\delta(\tilde{r}_t^{i,\delta}) dW_t^{i,\delta} \\
&\quad + (A_t^{i,\delta} + \frac{1}{N} \sum_{k=1}^N A_t^{k,\delta}) dt.
\end{aligned}$$

with $\Theta_t^{i,\delta} = b * \bar{\mu}_t^\delta(\bar{X}_t^{i,\delta}) - \frac{1}{N} \sum_{j=1}^N b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta})$ and

$$A_t^{i,\delta} = \left| \Theta_t^{i,\delta} \right| = \left| b * \bar{\mu}_t^\delta(\bar{X}_t^{i,\delta}) - \frac{1}{N} \sum_{j=1}^N b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta}) \right| \quad (53)$$

and where $(\{W_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ are N one-dimensional Brownian motions given by

$$W_t^{i,\delta} = \sqrt{\frac{N}{N+1}} \left(\int_0^t (\tilde{e}_s^{i,\delta})^T dB_s^{i,1} + \frac{1}{N} \sum_{j=1}^N \int_0^t (\tilde{e}_s^{j,\delta})^T dB_s^{j,1} \right), \quad i = 1, \dots, N. \quad (54)$$

Proof. By (51) and since γ is anti-symmetric, it holds by Ito's formula for any $i \in \{1, \dots, N\}$,

$$\begin{aligned} d(\tilde{r}_t^{i,\delta})^2 &= -2L(\tilde{r}_t^{i,\delta})^2 dt + 2\langle \tilde{Z}_t^{i,\delta}, \frac{1}{N} \sum_{j=1}^N \gamma(\tilde{X}_t^{i,\delta} - \tilde{X}_t^{j,\delta}) - \gamma(\tilde{Y}_t^{i,\delta} - \tilde{Y}_t^{j,\delta}) \rangle dt \\ &\quad + 4\left(1 + \frac{1}{N}\right) \text{rc}^\delta(\tilde{r}_t^{i,\delta})^2 dt + 4\sqrt{1 + \frac{1}{N}} \text{rc}^\delta(\tilde{r}_t^{i,\delta}) \langle \tilde{Z}_t^{i,\delta}, \tilde{e}_t^{i,\delta} \rangle dW_t^{i,\delta} \\ &\quad + 2\langle \tilde{Z}_t^{i,\delta}, b * \bar{\mu}_t^\delta(\bar{X}_t^{i,\delta}) - \frac{1}{N} \sum_{j=1}^N b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta}) \rangle dt \\ &\quad + 2\langle \tilde{Z}_t^{i,\delta}, -\frac{1}{N} \sum_{k=1}^N (b * \bar{\mu}_t^\delta(\bar{X}_t^{k,\delta}) - \frac{1}{N} \sum_{j=1}^N b(\bar{X}_t^{k,\delta} - \bar{X}_t^{j,\delta})) \rangle dt. \end{aligned}$$

where $(\{W_t^i\}_{i=1}^N)_{t \geq 0}$ are N i.i.d. one-dimensional Brownian motions given by (54). Note that the prefactor $(N/(N+1))^{1/2}$ ensures that the quadratic variation satisfies $[W^i]_t = t$ for $t \geq 0$, and hence $(\{W_t^i\}_{i=1}^N)_{t \geq 0}$ are Brownian motions. This definition of $(\{W_t^i\}_{i=1}^N)_{t \geq 0}$ leads to $(1 + 1/N)^{1/2}$ in the diffusion term of the SDE. Applying the \mathcal{C}^2 approximation of the square root used in the proof of Lemma 12 and taking $\varepsilon \rightarrow 0$ in the approximation yields the stochastic differential equations of $(\{\tilde{r}_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$. We obtain its upper bound for $\varepsilon < \varepsilon_0$ by B1 and (20) similarly to the proof of Lemma 12. \square

Next, we state a bound for (53).

Lemma 16. *Under the same assumption as in Lemma 17, it holds for any $i = 1, \dots, N$*

$$E\left[|A_t^{i,\delta}|^2\right] \leq C_1 N^{-1} \quad \text{and} \quad E\left[A_t^{i,\delta}\right] \leq C_2 N^{-1/2},$$

where $A_t^{i,\delta}$ is given in (53) and C_1 and C_2 are constants depending on $\|\gamma\|_\infty$, L and C given in Lemma 14.

Proof. By **B3**, it holds $\mathbb{E}[|\bar{X}_0^{i,\delta}|^2] < \infty$ for $i = 1, \dots, N$. Note that given $\bar{X}_t^{i,\delta}$, $\bar{X}_t^{j,\delta}$ are i.i.d. with law $\bar{\mu}_t^\delta$ for all $j \neq i$. Hence,

$$\mathbb{E}[b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta}) | \bar{X}_t^{i,\delta}] = b * \bar{\mu}_t^\delta(\bar{X}_t^{i,\delta}).$$

Since γ is anti-symmetric, $b(0) = 0$, and we have

$$\begin{aligned} & \mathbb{E}\left[|b * \bar{\mu}_t^\delta(\bar{X}_t^{i,\delta}) - \frac{1}{N-1} \sum_{j=1}^N b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta})|^2 | \bar{X}_t^{i,\delta}\right] \\ &= \mathbb{E}\left[\left|\frac{1}{N-1} \sum_{j=1}^N \mathbb{E}[b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta}) | \bar{X}_t^{i,\delta}] - \frac{1}{N-1} \sum_{j=1}^N b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta})\right|^2 | \bar{X}_t^{i,\delta}\right] \\ &= \frac{1}{N-1} \text{Var}_{\bar{\mu}_t^\delta}(b(\bar{X}_t^{i,\delta} - \cdot)). \end{aligned}$$

By (11), **B1**, **B3** and Lemma 14, we obtain

$$\begin{aligned} \text{Var}_{\bar{\mu}_t^\delta}(b(\bar{X}_t^{i,\delta} - \cdot)) &= \int_{\mathbb{R}^d} \left| \left(-L(\bar{X}_t^{i,\delta} - x) + \int_{\mathbb{R}^d} L(\bar{X}_t^{i,\delta} - \tilde{x}) \bar{\mu}_t^\delta(d\tilde{x}) \right) \right. \\ &\quad \left. + \left(\gamma(\bar{X}_t^{i,\delta} - x) - \int_{\mathbb{R}^d} \gamma(\bar{X}_t^{i,\delta} - \tilde{x}) \bar{\mu}_t^\delta(d\tilde{x}) \right) \right|^2 \bar{\mu}_t^\delta(dx) \\ &= \int_{\mathbb{R}^d} \left| Lx + \left(\gamma(\bar{X}_t^{i,\delta} - x) - \int_{\mathbb{R}^d} \gamma(\bar{X}_t^{i,\delta} - \tilde{x}) \bar{\mu}_t^\delta(d\tilde{x}) \right) \right|^2 \bar{\mu}_t^\delta(dx) \\ &\leq 2L^2 \int_{\mathbb{R}^d} |x|^2 \bar{\mu}_t^\delta(dx) + 8\|\gamma\|_\infty^2 \leq 2L^2 C^2 + 8\|\gamma\|_\infty^2. \end{aligned}$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}[(A_t^{i,\delta})^2] &\leq 2\mathbb{E}\left[|b * \bar{\mu}_t^\delta(\bar{X}_t^{i,\delta}) - \frac{1}{N-1} \sum_{j=1}^N b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta})|^2\right] \\ &\quad + 2\left(\frac{1}{N-1} - \frac{1}{N}\right)^2 \mathbb{E}\left[\left|\sum_{j=1}^N b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta})\right|^2\right] \\ &\leq 2\frac{1}{N-1} \mathbb{E}[\text{Var}_{\bar{\mu}_t^\delta}(b(\bar{X}_t^{i,\delta} - \cdot))] + \frac{1}{N^2(N-1)} \mathbb{E}\left[\sum_{j=1}^N |b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta})|^2\right] \\ &\leq \frac{4L^2}{N-1} C + \frac{16\|\gamma\|_\infty^2}{N-1} + \frac{1}{N^2} (8CL^2 + 4\|\gamma\|_\infty^2) \\ &\leq N^{-1} C_1 < \infty, \end{aligned}$$

where C_1 depends on $\|\gamma\|_\infty$, L and the second moment bound C . Similarly, it holds

$$\begin{aligned} \mathbb{E}[A_t^{i,\delta}] &\leq \mathbb{E}\left[\left|b * \bar{\mu}_t^\delta(\bar{X}_t^{i,\delta}) - \frac{1}{N-1} \sum_{j=1}^N b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta})\right|\right] \\ &\quad + \left(\frac{1}{N-1} - \frac{1}{N}\right) \sum_{j=1}^N \mathbb{E}\left[\left|b(\bar{X}_t^{i,\delta} - \bar{X}_t^{j,\delta})\right|\right] \\ &\leq \frac{\sqrt{2}L}{\sqrt{N-1}} C^{1/2} + \frac{\sqrt{8}\|\gamma\|_\infty}{\sqrt{N-1}} + \frac{1}{N} (\sqrt{2}C^{1/2}L + \|\gamma\|_\infty) \\ &\leq N^{-1/2} C_2 < \infty, \end{aligned}$$

where $C_2 = 2LC^{1/2} + 4\|\gamma\|_\infty + (\sqrt{2}C^{1/2} + \|\gamma\|_\infty)$. \square

To control $(\{\tilde{r}_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$, we consider $(\{r_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0}$ given as solution of

$$\begin{aligned} dr_t^{i,\delta,\epsilon} &= \bar{b}(r_t^{i,\delta,\epsilon})dt + \frac{1}{N} \sum_{j=1}^N 2\|\gamma\|_\infty rc^\epsilon(r_t^{j,\delta,\epsilon})dt + \left(A_t^{i,\delta} + \frac{1}{N} \sum_{k=1}^N A_t^{k,\delta}\right)dt \\ &\quad + 2\sqrt{1 + \frac{1}{N}} rc^\delta(r_t^{i,\delta,\epsilon})dW_t^{i,\delta} \end{aligned} \quad (55)$$

with initial condition $r_0^{i,\delta,\epsilon} = \tilde{r}_0^{i,\delta}$ for all $i = 1, \dots, N$, $A_t^{i,\delta}$ given in (53) and $W_t^{i,\delta}$ given in (54).

By [45, Theorem 2.2], under **B1** and **B3**, $(\{U_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0} = (\{\bar{X}_t^{i,\delta}, Y_t^{i,\delta}, r_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0}$ exists and is unique, where $(\{\bar{X}_t^{i,\delta}, \bar{Y}_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ solves uniquely (51), $(\{\tilde{r}_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ and $(\{r_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0}$ solve uniquely (52) and (55), respectively, with $(\{W_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ given by (54).

Lemma 17. *Assume **B1** and **B3**. Then for any $i = 1, \dots, N$, $|\bar{X}_t^{i,\delta} - Y_t^{i,\delta} - \frac{1}{N} \sum_j (\bar{X}_t^{j,\delta} - Y_t^{j,\delta})| = \tilde{r}_t^{i,\delta} \leq r_t^{i,\delta,\epsilon}$, almost surely for all $t \geq 0$ and $\epsilon < \epsilon_0$.*

Proof. Note, that both processes $(\{\tilde{r}_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ and $(\{r_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0}$ have the same initial condition and are driven by the same noise. Since the drift for $(\{r_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0}$ is larger than the drift for $(\{\tilde{r}_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ for $\epsilon < \epsilon_0$ by (20), we can conclude $\tilde{r}_t^{i,\delta} \leq r_t^{i,\delta,\epsilon}$ almost surely for all $t \geq 0$, $\epsilon < \epsilon_0$ and $i = 1, \dots, N$ by Lemma 21. \square

Proof of Theorem 8. Consider the process $(\{U_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0} = (\{\bar{X}_t^{i,\delta}, Y_t^{i,\delta}, r_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0}$ on $\mathbb{R}^{N(2d+1)}$ for each $\epsilon, \delta > 0$. We denote by $\mathbb{P}^{\delta,\epsilon}$ the law of $\{U_t^{\delta,\epsilon}\}_{i=1}^N$ on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^{N(2d+1)})$. We define the canonical projections $\mathbf{X}, \mathbf{Y}, \mathbf{r}$ onto the first Nd , second Nd and last N components.

By **B1** and **B3** it holds in the same line as in the proof of Lemma 22 for each $T > 0$

$$E[|\{U_{t_2}^{i,\delta,\epsilon} - U_{t_1}^{i,\delta,\epsilon}\}_{i=1}^N|^4] \leq C|t_2 - t_1|^2 \quad \text{for } t_1, t_2 \in [0, T], \quad (56)$$

for some constant C depending on $T, L, \|\gamma\|_{\text{Lip}}, \|\gamma\|_{\infty}, N$ and on the fourth moment of μ_0 and ν_0 . Note that we used here that the additional drift terms $(A_t^{i,\delta})_{t \geq 0}$ occurring in the SDE of $(\{r_t^{i,\delta,\epsilon}\}_{i=1}^N)_{t \geq 0}$ are Lipschitz continuous in $(\{\bar{X}_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$. Then as in the proofs of Lemma 22 and Lemma 23, $\mathbb{P}^{\delta,\epsilon}$ is tight and converges weakly along a subsequence to a measure \mathbb{P} by Kolmogorov's continuity criterion, cf. [38, Corollary 14.9].

As in Lemma 22 the law $\mathbb{P}_T^{\delta,\epsilon}$ of $(\{U_t^{i,\delta,\epsilon}\}_{i=1}^N)_{0 \leq t \leq T}$ on $C([0, T], \mathbb{R}^{N(2d+1)})$ is tight for each $T > 0$ by [38, Corollary 14.9] and for each $\epsilon > 0$ there exists a subsequence $\delta_n \rightarrow 0$ such that $(\mathbb{P}_T^{\delta_n,\epsilon})_{n \in \mathbb{N}}$ on $C([0, T], \mathbb{R}^{N(2d+1)})$ converge to a measure \mathbb{P}_T^ϵ on $C([0, T], \mathbb{R}^{N(2d+1)})$. By a diagonalization argument and since $\{\mathbb{P}_T^\epsilon : T \geq 0\}$ is a consistent family, cf. [38, Theorem 5.16], there exists a probability measure \mathbb{P}^ϵ on $C(\mathbb{R}_+, \mathbb{R}^{N(2d+1)})$ such that for all ϵ there exists a subsequence δ_n such that $(\mathbb{P}^{\delta_n,\epsilon})_{n \in \mathbb{N}}$ converges along this subsequence to \mathbb{P}^ϵ , i.e., the family $(\mathbb{P}^{\delta,\epsilon})_{\delta > 0}$ is tight. As in the proof of Lemma 23 we repeat this argument for the family of measures $(\mathbb{P}^\epsilon)_{\epsilon > 0}$. Hence, there exists a subsequence $\epsilon_n \rightarrow 0$ such that $(\mathbb{P}^{\epsilon_n})_{n \in \mathbb{N}}$ converges to a measure \mathbb{P} . Let $(\{\bar{X}_t^i, Y_t^i, r_t^i\}_{i=1}^N)_{t \geq 0}$ be some process on $\mathbb{R}^{N(2d+1)}$ with distribution \mathbb{P} on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$.

Since $(\{\bar{X}_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ and $(\{Y_t^{i,\delta}\}_{i=1}^N)_{t \geq 0}$ are solutions that are unique in law, we have that for any $\delta, \epsilon > 0$, $\mathbb{P}^{\delta,\epsilon} \circ (\mathbf{X})^{-1} = \mathbb{P} \circ (\mathbf{X})^{-1}$ and $\mathbb{P}^{\delta,\epsilon} \circ (\mathbf{Y})^{-1} = \mathbb{P} \circ (\mathbf{Y})^{-1}$. Hence, $\mathbb{P} \circ (\mathbf{X}, \mathbf{Y})^{-1}$ is a coupling of (1) and (3). In addition, the statement of Lemma 17 carries over to the limiting process $(\{r_t^i\}_{i=1}^N)_{t \geq 0}$, since by the weak convergence and the Portmanteau, it is preserved under weak convergence by Portmanteau theorem, $\bar{P}(|\bar{X}_t^i - \tilde{Y}_t^i| \leq r_t^i \text{ for } i = 1, \dots, N) \geq \limsup_{\epsilon \rightarrow 0} \limsup_{\delta \rightarrow 0} P(|\tilde{X}_t^{i,\delta} - \tilde{Y}_t^{i,\delta}| \leq r_t^{i,\delta,\epsilon} \text{ for } i = 1, \dots, N) = 1$, where $\tilde{X}_t^i = \bar{X}_t^i - (1/N) \sum_{j=1}^N \bar{X}_t^j$ and $\tilde{Y}_t^i = \bar{Y}_t^i - (1/N) \sum_{j=1}^N \bar{Y}_t^j$ for all $t \geq 0$ and $i = 1, \dots, N$.

Similarly to the proof of Lemma 22 and Lemma 23 there exist an extended underlying probability space and N i.i.d one-dimensional Brownian motion $(\{W_t^i\}_{i=1}^N)_{t \geq 0}$ such that $(\{r_t^i, W_t^i\}_{i=1}^N)_{t \geq 0}$ is a solution of

$$\begin{aligned} dr_t^i &= \bar{b}(r_t^i)dt + \frac{1}{N} \sum_{j=1}^N 2\|\gamma\|_{\infty} \mathbb{1}_{(0,\infty)}(r_t^j)dt + \left(A_t^i + \frac{1}{N} \sum_{k=1}^N A_t^k \right) dt \\ &\quad + 2\sqrt{1 + \frac{1}{N}} \mathbb{1}_{(0,\infty)}(r_t^i) dW_t^i, \end{aligned}$$

where $A_t^i = |b * \bar{\mu}_t(\bar{X}_t^i) - \frac{1}{N} \sum_{j=1}^N b(\bar{X}_t^i - \bar{X}_t^j)|$.

Using Ito-Tanaka formula, c.f. [51, Chapter 6, Theorem 1.1], and f' is absolutely

continuous, we obtain for f defined in (37) with $\tilde{b}(r) = (\kappa(r) - L)r$ and $a = 2\|\gamma\|_\infty$,

$$\begin{aligned} d\left(\frac{1}{N} \sum_{i=1}^N f(r_t^i)\right) &= \frac{1}{N} \sum_{i=1}^N \left(\tilde{b}(r_t^i) f'(r_t^i) + f''(r_t^i) 2\frac{N+1}{N} \mathbb{1}_{(0,\infty)}(r_t^i)\right) dt \\ &\quad + \frac{1}{N^2} \sum_{i=1}^N \sum_{j=1}^N 2f'(r_t^i) \|\gamma\|_\infty \mathbb{1}_{(0,\infty)}(r_t^j) dt \\ &\quad + \frac{1}{N} \sum_{i=1}^N f'(r_t^i) 2\sqrt{1 + \frac{1}{N}} \mathbb{1}_{(0,\infty)}(r_t^i) dW_t^i + \frac{1}{N} \sum_{i=1}^N f'(r_t^i) \left(A_t^i + \frac{1}{N} \sum_{k=1}^N A_t^k\right) dt. \end{aligned}$$

Taking expectation, we get using $f'(r) \leq 1$ for all $r \geq 0$,

$$\begin{aligned} \frac{d}{dt} \mathbb{E}\left[\frac{1}{N} \sum_{i=1}^N f(r_t^i)\right] &\leq \frac{1}{N} \sum_{i=1}^N \left\{ \mathbb{E}\left[\tilde{b}(r_t^i) f'(r_t^i) + 2\frac{N+1}{N} (f''(r_t^i) - f''(0))\right] \right. \\ &\quad \left. + \mathbb{E}\left[2\left(\|\gamma\|_\infty + \frac{N+1}{N} f''(0)\right) \mathbb{1}_{(0,\infty)}(r_t^i)\right] + \mathbb{E}\left[2A_t^i\right] \right\}. \end{aligned} \quad (57)$$

By (39) and (40), the first two terms are bounded by $-\tilde{c} \frac{1}{N} \sum_i f(r_t^i)$ with \tilde{c} given in (17).

By Lemma 16 the last term in (57) is bounded by

$$2E[A_t^i] \leq \tilde{C} N^{-1/2},$$

where

$$\tilde{C} = 2C_2 = 4LC^{1/2} + 8\|\gamma\|_\infty + 2(\sqrt{2}C^{1/2}L + \|\gamma\|_\infty). \quad (58)$$

Hence, we obtain

$$\frac{d}{dt} \mathbb{E}\left[\frac{1}{N} \sum_i f(r_t^i)\right] \leq -\tilde{c} \frac{1}{N} \sum_i \mathbb{E}[f(r_t^i)] + \tilde{C} N^{-1/2}$$

for $t \geq 0$ which leads by Gronwall's lemma to

$$\mathbb{E}\left[\frac{1}{N} \sum_i f(r_t^i)\right] \leq e^{-\tilde{c}t} \mathbb{E}\left[\frac{1}{N} \sum_i f(r_0^i)\right] + \frac{1}{\tilde{c}} \tilde{C} N^{-1/2}.$$

For an arbitrary coupling $\xi \in \Gamma(\bar{\mu}_0^{\otimes N}, \nu_0^{\otimes N})$, we have

$$\mathcal{W}_{f,N}((\bar{\mu}_t)^{\otimes N}, \nu_t^{\otimes N}) \leq e^{-\tilde{c}t} \int_{\mathbb{R}^{2Nd}} \frac{1}{N} \sum_{i=1}^N f\left(\left|x^i - y^i - \frac{1}{N} \sum_{j=1}^N (x^j - y^j)\right|\right) \xi(dx dy) + \frac{\tilde{C}}{\tilde{c}N^{1/2}},$$

as $\mathbb{E}[f(r_0^i)] \leq \int_{\mathbb{R}^{2Nd}} \frac{1}{N} \sum_{i=1}^N f(|x^i - y^i - \frac{1}{N} \sum_{j=1}^N (x^j - y^j)|) \xi(dx dy)$. Taking the infimum over all couplings $\xi \in \Gamma(\bar{\mu}_0^{\otimes N}, \nu_0^{\otimes N})$ gives the first bound. By (38), the second bound follows. \square

6.3 Proof of Section 3

6.3.1 Proof of Theorem 3

We show Theorem 3 via a family of stochastic differential equations, indexed by $n, m \in \mathbb{N}$, with Lipschitz continuous coefficients,

$$\begin{aligned} dr_t^{n,m} &= (\tilde{b}(r_t^{n,m}) + P_t^{n,m}(g^m))dt + 2\theta^n(r_t^{n,m})dW_t \\ ds_t^{n,m} &= (\hat{b}(s_t^{n,m}) + \hat{P}_t^{n,m}(h^m))dt + 2\theta^n(s_t^{n,m})dW_t, \quad \text{Law}(r_0^{n,m}, s_0^{n,m}) = \eta_{n,m}, \end{aligned} \quad (59)$$

where $P_t^{n,m} = \text{Law}(r_t^{n,m})$, $\hat{P}_t^{n,m} = \text{Law}(s_t^{n,m})$ and $P_t^{n,m}(g^m) = \int_{\mathbb{R}_+} g^m(r)P_t^{n,m}(dx)$ and $\hat{P}_t^{n,m}(h^m) = \int_{\mathbb{R}_+} h^m(r)\hat{P}_t^{n,m}(dx)$ for some measurable functions $(g^m)_{m \in \mathbb{N}}$ and $(h^m)_{m \in \mathbb{N}}$, and where $\eta_{n,m} \in \Gamma(\mu_{n,m}, \nu_{n,m})$ for $\mu_{n,m}, \nu_{n,m} \in \mathcal{P}(\mathbb{R}_+)$. We identify the weak limit for $n \rightarrow \infty$ as solution of a family of stochastic differential equations, indexed by $m \in \mathbb{N}$, given by

$$\begin{aligned} dr_t^m &= (\tilde{b}(r_t^m) + P_t^m(g^m))dt + 2\mathbb{1}_{(0,\infty)}(r_t^m)dW_t \\ ds_t^m &= (\hat{b}(s_t^m) + \hat{P}_t^m(h^m))dt + 2\mathbb{1}_{(0,\infty)}(s_t^m)dW_t, \quad \text{Law}(r_0^m, s_0^m) = \eta_m. \end{aligned} \quad (60)$$

with $P_t^m = \text{Law}(r_t^m)$ and $\hat{P}_t^m = \text{Law}(s_t^m)$, and where $\eta_m \in \Gamma(\mu_m, \nu_m)$ for $\mu_m, \nu_m \in \mathcal{P}(\mathbb{R}_+)$. Taking the limit $m \rightarrow \infty$, we show in the next step that the solution of (60) converges to a solution of (25).

We assume for $(g^m)_{m \in \mathbb{N}}$, $(h^m)_{m \in \mathbb{N}}$, $(\theta^n)_{n \in \mathbb{N}}$ and the initial distributions:

H5. $(g^m)_{m \in \mathbb{N}}$ and $(h^m)_{m \in \mathbb{N}}$ are sequences of non-decreasing non-negative uniformly bounded Lipschitz continuous functions such that for all $r \geq 0$, $g^m(r) \leq g^{m+1}(r)$ and $h^m(r) \leq h^{m+1}(r)$ and $\lim_{m \rightarrow +\infty} g^m(r) = g(r)$ and $\lim_{m \rightarrow +\infty} h^m(r) = h(r)$ where g, h are left-continuous non-negative non-decreasing bounded functions. In addition, there exists $K_m < \infty$ for any m such that for all $r, s \in \mathbb{R}$

$$|g^m(r) - g^m(s)| \leq K_m|r - s| \quad \text{and} \quad |h^m(r) - h^m(s)| \leq K_m|r - s|.$$

H6. $(\theta^n)_{n \in \mathbb{N}}$ is a sequence of Lipschitz continuous functions from \mathbb{R}_+ to $[0, 1]$ with $\theta^n(0) = 0$, $\theta^n(r) = 1$ for all $r \geq 1/n$ and $\theta^n(r) > 0$ for all $r > 0$.

H7. $(\mu_{n,m})_{m,n \in \mathbb{N}}$, $(\nu_{n,m})_{m,n \in \mathbb{N}}$, $(\mu_m)_{m \in \mathbb{N}}$, $(\nu_m)_{m \in \mathbb{N}}$ are families of probability distributions on \mathbb{R}_+ and $(\eta_{n,m})_{n,m \in \mathbb{N}}$, $(\eta_m)_{m \in \mathbb{N}}$ families of probability distributions on \mathbb{R}_+^2 such that for any $n, m \in \mathbb{N}$ $\eta_{n,m} \in \Gamma(\mu_{n,m}, \nu_{n,m})$ and $\eta_m \in \Gamma(\mu_m, \nu_m)$ and for any $m \in \mathbb{N}$, $(\eta_{n,m})_{n \in \mathbb{N}}$ converges weakly to η_m and $(\eta_m)_{m \in \mathbb{N}}$ converges weakly to η . Further, the p -th order moments of $(\mu_{n,m})_{n,m \in \mathbb{N}}$, $(\nu_{n,m})_{n,m \in \mathbb{N}}$, $(\mu_m)_{m \in \mathbb{N}}$ and $(\nu_m)_{m \in \mathbb{N}}$ are uniformly bounded for $p > 2$ given in **H3**.

Note that by **H5** for any non-decreasing sequence $(u_m)_{m \in \mathbb{N}}$, which converges to $u \in \mathbb{R}_+$, $g^m(u_m)$ and $h^m(u_m)$ converge to $g(u)$ and $h(u)$, respectively. More precisely, it holds for

for all $m \in \mathbb{N}$, $g^m(u_m) - g(u) \leq 0$ and for $m \geq n$, $g^m(u_m) \geq g^m(u_n)$ and therefore, $\lim_{m \rightarrow \infty} g^m(u_n) - g(u) \geq \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} = \lim_{n \rightarrow \infty} g(u_n) - g(u) = 0$ by left-continuity of g . Hence, $\lim_{m \rightarrow \infty} g^m(u_m) - g(u) = 0$ and analogously $\lim_{m \rightarrow \infty} h^m(u_m) - h(u) = 0$. By **H5**, $\Gamma = \max(\|h\|_\infty, \|g\|_\infty)$ is a uniform upper bound of $(g^m)_{m \in \mathbb{N}}$ and $(h^m)_{m \in \mathbb{N}}$.

Consider a probability space $(\Omega_0, \mathcal{A}_0, Q)$ and a one-dimensional Brownian motion $(W_t)_{t \geq 0}$. Under **H5**, **H6** and **H7**, for all $m, n \in \mathbb{N}$, there exists random variables $r^{n,m}, s^{n,m} : \Omega_0 \rightarrow \mathbb{W}$ for each n, m such that $(r_t^{n,m}, s_t^{n,m})_{t \geq 0}$ is a unique strong solution to (59) associated to $(W_t)_{t \geq 0}$ by [45, Theorem 2.2]. We denote by $\mathbb{P}^{n,m} = Q \circ (r^{n,m}, s^{n,m})^{-1}$ the corresponding distribution on $\mathbb{W} \times \mathbb{W}$.

Before studying the two limits $n, m \rightarrow \infty$ and proving Theorem 3, we state a modification of the comparison theorem by Ikeda and Watanabe to compare two solutions of (59), cf. [35, Section VI, Theorem 1.1].

Lemma 18. *Let $(r_t^{n,m}, s_t^{n,m})_{t \geq 0}$ be a solution of (59) for fixed $n, m \in \mathbb{N}$. Assume **H1**, **H5** and **H6**. If $Q[r_0^{n,m} \leq s_0^{n,m}] = 1$, $\tilde{b}(r) \leq \hat{b}(r)$ and $g^m(r) \leq h^m(r)$ for any $r \in \mathbb{R}_+$, then*

$$Q[r_t^{n,m} \leq s_t^{n,m} \text{ for all } t \geq 0] = 1. \quad (61)$$

Proof. For simplicity, we drop the dependence on n, m in $(r_t^{n,m})$ and $(s_t^{n,m})$. Denote by ρ the Lipschitz constant of θ^n . Let $(a_k)_{k \in \mathbb{N}}$ be a decreasing sequence, $1 > a_1 > a_2 > \dots > a_k > \dots > 0$, such that $\int_{a_1}^1 \rho^{-2} x^{-1} dx = 1$, $\int_{a_2}^{a_1} \rho^{-2} x^{-1} dx = 2, \dots, \int_{a_k}^{a_{k-1}} \rho^{-2} x^{-1} dx = k$. We choose a sequence $\Psi_k(u)$, $k = 1, 2, \dots$, of continuous functions such that its support is contained in (a_k, a_{k-1}) , $\int_{a_k}^{a_{k-1}} \Psi_k(u) du = 1$ and $0 \leq \Psi_k(u) \leq 2/k \cdot \rho^{-2} u^{-2}$. Such a function exists. We set

$$\varphi_k(x) = \begin{cases} \int_0^x dy \int_0^y \Psi_k(u) du & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Note that for any $k \in \mathbb{N}$, $\varphi_k \in \mathcal{C}^2(\mathbb{R}_+)$, $|\varphi_k'(x)| \leq 1$, $\varphi_k(x) \rightarrow x_+$ as $k \uparrow \infty$ and $\varphi_k'(x) \uparrow \mathbb{1}_{(0, \infty)}(x)$. Applying Ito's formula to $\varphi_k(r_t - s_t)$, we obtain

$$\varphi_k(r_t - s_t) = \varphi_k(r_0 - s_0) + I_1(k) + I_2(k) + I_3(k),$$

where

$$\begin{aligned} I_1(k) &= \int_0^t \varphi_k'(r_u - s_u) [\theta^n(r_u) - \theta^n(s_u)] dB_u, \\ I_2(k) &= \int_0^t \varphi_k'(r_u - s_u) [b(r_u) - \hat{b}(s_u) + P_u(g^m) - \hat{P}_u(h^m)] du, \\ I_3(k) &= \frac{1}{2} \int_0^t \varphi_k''(r_u - s_u) [\theta^n(r_u) - \theta^n(s_u)]^2 du, \end{aligned}$$

with $P_u = Q \circ r_u^{-1}$ and $\hat{P}_u = Q \circ s_u^{-1}$. It holds by boundedness and Lipschitz continuity of θ^n

$$\mathbb{E}[I_1(k)] = 0, \quad \text{and} \quad \mathbb{E}[I_3(k)] \leq \frac{1}{2} \mathbb{E} \left[\int_0^t \varphi_k''(r_u - s_u) \rho^2 |r_u - s_u|^2 du \right] \leq \frac{t}{k}.$$

We note that by **H5** $\mathbb{E}[(g^m(r_u) - h^m(s_u)) \mathbb{1}_{r_u - s_u < 0}] \leq 0$ and

$$\begin{aligned} \mathbb{E}[(g^m(r_u) - h^m(s_u)) \mathbb{1}_{r_u - s_u \geq 0}] &\leq \mathbb{E}[(g^m(r_u) - g^m(s_u) + g^m(s_u) - h^m(s_u)) \mathbb{1}_{r_u - s_u \geq 0}] \\ &\leq \mathbb{E}[(g^m(r_u) - g^m(s_u)) \mathbb{1}_{r_u - s_u \geq 0}] \\ &\leq K_m \mathbb{E}[|r_u - s_u| \mathbb{1}_{r_u - s_u \geq 0}] \end{aligned} \quad (62)$$

by Lipschitz continuity of g^m , by $g^m(r) \leq h^m(r)$ and since g^m and h^m are non-decreasing. Hence for I_2 , we obtain

$$\begin{aligned} I_2(k) &= \int_0^t \varphi_k'(r_u - s_u) [\tilde{b}(r_u) - \hat{b}(r_u) + \hat{b}(r_u) - \hat{b}(s_u)] du \\ &\quad + \int_0^t \varphi_k'(r_u - s_u) \left(\mathbb{E}[(g^m(r_u) - h^m(s_u)) \mathbb{1}_{r_u - s_u \geq 0}] + \mathbb{E}[(g^m(r_u) - h^m(s_u)) \mathbb{1}_{r_u - s_u < 0}] \right) du \\ &\leq \int_0^t \varphi_k'(r_u - s_u) \tilde{L} |r_u - s_u| du + \int_0^t \varphi_k'(r_u - s_u) K_m \mathbb{E}[|r_u - s_u| \mathbb{1}_{r_u - s_u \geq 0}] du. \end{aligned}$$

Taking the limit $k \rightarrow \infty$ and using that $\mathbb{E}[r_0 - s_0] = 0$, we obtain

$$\mathbb{E}[(r_t - s_t)_+] \leq \tilde{L} \mathbb{E} \left[\int_0^t (r_u - s_u)_+ du \right] + K_m \mathbb{E} \left[\int_0^t \mathbb{1}_{(0, \infty)}(r_u - s_u) \mathbb{E}[(r_u - s_u)_+] du \right], \quad (63)$$

by the monotone convergence theorem and since $(\varphi_k')_{k \in \mathbb{N}}$ is a monotone increasing sequence which converges pointwise to $\mathbb{1}_{(0, \infty)}(x)$. Assume there exists $t^* = \inf \{t \geq 0 : \mathbb{E}[(r_t - s_t)_+] > 0\} < \infty$. Then, $\int_0^{t^*} \mathbb{E}[(r_u - s_u)_+] du > 0$ or $\int_0^{t^*} \mathbb{E}[\mathbb{1}_{(0, \infty)}(r_u - s_u)] \mathbb{E}[(r_u - s_u)_+] du > 0$. By definition of t^* , $\mathbb{E}[(r_u - s_u)_+] = 0$ for all $u < t^*$ and hence both terms are zero. This contradicts the definition of t^* . Hence, (61) holds. \square

Next, we show that the distribution of the solution of (59) converges as $n \rightarrow \infty$.

Lemma 19. *Assume that \tilde{b} , \hat{b} , g and h satisfy **H1** and **H2**. Let $\eta \in \Gamma(\mu, \nu)$ where the probability measures μ and ν on \mathbb{R}_+ satisfy **H3**. Assume that $(g^m)_{m \in \mathbb{N}}$, $(h^m)_{m \in \mathbb{N}}$, $(\theta^m)_{m \in \mathbb{N}}$, $(\mu_{n,m})_{m,n \in \mathbb{N}}$, $(\nu_{n,m})_{m,n \in \mathbb{N}}$ and $(\eta_{n,m})_{m,n \in \mathbb{N}}$ satisfy condition **H5**, **H7** and **H6**. Then for any $m \in \mathbb{N}$, there exists a random variable (r^m, s^m) defined on some probability space $(\Omega^m, \mathcal{A}^m, P^m)$ with values in $\mathbb{W} \times \mathbb{W}$, such that $(r_t^m, s_t^m)_{t \geq 0}$ is a weak solution of the stochastic differential equation (60). More precisely, for all $m \in \mathbb{N}$ the sequence of laws $Q \circ (r^{n,m}, s^{n,m})^{-1}$ converges weakly to the distribution $P^m \circ (r^m, s^m)^{-1}$. If additionally,*

$$\begin{aligned} \tilde{b}(r) &\leq \hat{b}(r) \quad \text{and} \quad g^m(r) \leq h^m(r), & \text{for any } r \in \mathbb{R}_+ \text{ and} \\ Q[r_0^{n,m} \leq s_0^{n,m}] &= 1 & \text{for any } n, m \in \mathbb{N}, \end{aligned}$$

then $P^m[r_t^m \leq s_t^m \text{ for all } t \geq 0] = 1$.

Proof. Fix $m \in \mathbb{N}$. The proof is divided in three parts. First we show tightness of the sequences of probability measures. Then we identify the limit of the sequence of stochastic processes. Finally, we compare the two limiting processes.

Tightness: We show that the sequence of probability measures $(\mathbb{P}^{n,m})_{n \in \mathbb{N}}$ on $(\mathbb{W} \times \mathbb{W}, \mathcal{B}(\mathbb{W}) \otimes \mathcal{B}(\mathbb{W}))$ is tight by applying Kolmogorov's continuity theorem. Consider $p > 2$ such that the p -th moment in **H3** and **H7** are uniformly bounded. Fix $T > 0$. Then the p -th moment of $r_t^{n,m}$ for $t < T$ can be bounded using Ito's formula,

$$\begin{aligned} d|r_t^{n,m}|^p &\leq p|r_t^{n,m}|^{p-2} \langle r_t^{n,m}, (\tilde{b}(r_t^{n,m}) + P_t^{n,m}(g^m)) \rangle dt + 2\theta^n(r_t^{n,m})p|r_t^{n,m}|^{p-2}r_t^{n,m}dW_t \\ &\quad + p(p-1)|r_t^{n,m}|^{p-2}2\theta^n(r_t^n)^2dt \\ &\leq p\left(|r_t^{n,m}|^p\tilde{L} + \Gamma|r_t^{n,m}|^{p-1} + 2(p-1)|r_t^{n,m}|^{p-2}\right)dt + 2\theta^n(r_t^{n,m})p(r_t^{n,m})^{p-1}dW_t \\ &\leq p(\tilde{L} + \Gamma + 2(p-1))|r_t^{n,m}|^pdt + p(\Gamma + 2(p-1))dt + 2\theta^n(r_t^{n,m})p(r_t^{n,m})^{p-1}dW_t, \end{aligned}$$

where $\Gamma = \max(\|g\|_\infty, \|h\|_\infty)$. Taking expectation yields

$$\frac{d}{dt}\mathbb{E}[|r_t^{n,m}|^p] \leq p(\tilde{L} + \Gamma + 2(p-1))\mathbb{E}|r_t^{n,m}|^p + p(\Gamma + 2(p-1)).$$

Then by Gronwall's lemma

$$\sup_{t \in [0, T]} \mathbb{E}[|r_t^{n,m}|^p] \leq e^{p(\tilde{L} + \Gamma + 2(p-1))T}(\mathbb{E}[|r_0^{n,m}|^p] + Tp(\Gamma + 2(p-1))) < C_p < \infty, \quad (64)$$

where C_p depends on T and the p -th moment of the initial distribution, which is finite by **H6**. Similarly, it holds $\sup_{t \in [0, T]} \mathbb{E}[|s_t^{n,m}|^p] < C_p$ for $t \leq T$. Using this moment bound, it holds for all $t_1, t_2 \in [0, T]$ by **H1**, **H5** and **H6**,

$$\begin{aligned} \mathbb{E}[|r_{t_2}^{n,m} - r_{t_1}^{n,m}|^p] &\leq C_1(p)\left(\mathbb{E}\left[\left|\int_{t_1}^{t_2} \tilde{b}(r_u^{n,m}) + P_u^{n,m}(g^m)du\right|^p\right] + \mathbb{E}\left[\left|\int_{t_1}^{t_2} 2\theta^n(r_u^{n,m})dW_u\right|^p\right]\right) \\ &\leq C_2(p)\left(\left(\mathbb{E}\left[\frac{\tilde{L}^p}{|t_2 - t_1|} \int_{t_1}^{t_2} |r_u^{n,m}|^p du\right] + \Gamma^p\right)|t_2 - t_1|^p + \mathbb{E}\left[\int_{t_1}^{t_2} 2\theta^n(r_u^{n,m})du\right]^{p/2}\right) \\ &\leq C_2(p)\left(\left(\frac{\tilde{L}^p}{|t_2 - t_1|} \int_{t_1}^{t_2} \mathbb{E}[|r_u^{n,m}|^p]du + \Gamma^p\right)|t_2 - t_1|^p + 2^{p/2}|t_2 - t_1|^{p/2}\right) \\ &\leq C_3(p, T, \tilde{L}, \Gamma, C_p)|t_2 - t_1|^{p/2}, \end{aligned}$$

where $C_i(\cdot)$ are constants depending on the stated argument and which are independent of n, m . Note that in the second step, we used Burkholder-Davis-Gundy inequality, see [50, Chapter IV, Theorem 48]. It holds similarly, $\mathbb{E}[|s_{t_2}^{n,m} - s_{t_1}^{n,m}|^p] \leq C_3(p, T, \tilde{L}, \Gamma, C_p)|t_2 - t_1|^{p/2}$. Hence,

$$\mathbb{E}[|(r_{t_2}^{n,m}, s_{t_2}^{n,m}) - (r_{t_1}^{n,m}, s_{t_1}^{n,m})|^p] \leq C_4(p, T, \tilde{L}, \Gamma, C_p)|t_2 - t_1|^{p/2} \quad (65)$$

for all $t_1, t_2 \in [0, T]$. Hence, by Kolmogorov's continuity criterion, cf. [38, Corollary 14.9], there exists a constant \tilde{C} depending on p and γ such that

$$\mathbb{E} \left[[(r^{n,m}, s^{n,m})]_\gamma^p \right] \leq \tilde{C} \cdot C_4(p, T, \tilde{L}, \Gamma, C_p), \quad (66)$$

where $[\cdot]_\gamma^p$ is defined by $[x]_\gamma = \sup_{t_1, t_2 \in [0, T]} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\gamma}$ and $(r_t^{n,m}, s_t^{n,m})_{n \in \mathbb{N}, t \geq 0}$ is tight in $\mathcal{C}([0, T], \mathbb{R}^2)$. Hence, for each $T > 0$ there exists a subsequence $n_k \rightarrow \infty$ and a probability measure \mathbb{P}_T^m on $\mathcal{C}([0, T], \mathbb{R}^2)$. Since $\{\mathbb{P}_T^m\}_T$ is a consistent family, there exists by [38, Theorem 5.16] a probability measure \mathbb{P}^m on $(\mathbb{W} \times \mathbb{W}, \mathcal{B}(\mathbb{W}) \otimes \mathcal{B}(\mathbb{W}))$ such that there is a subsequence $(n_k)_{k \in \mathbb{N}}$ such that $\mathbb{P}^{n_k, m}$ converges along this subsequence to \mathbb{P}^m . Note that here we can take by a diagonalization argument the same subsequence $(n_k)_{k \in \mathbb{N}}$ for all m .

Characterization of the limit measure: In the following we drop for simplicity the index k in the subsequence. Denote by $(\mathbf{r}_t, \mathbf{s}_t)(\omega) = \omega(t)$ the canonical process on $\mathbb{W} \times \mathbb{W}$. Since $\mathbb{P}^{n,m} \circ (\mathbf{r}_0, \mathbf{s}_0)^{-1} = \eta_{n,m}$ converges weakly to η_m by **H7**, it holds $\mathbb{P}^m \circ (\mathbf{r}_0, \mathbf{s}_0)^{-1} = \eta_m$. We define the maps $M^{n,m}, N^{n,m} : \mathbb{W} \times \mathbb{W} \rightarrow \mathbb{W}$ by

$$M_t^{n,m} = \mathbf{r}_t - \mathbf{r}_0 - \int_0^t (\tilde{b}(\mathbf{r}_u) + P_u^n(g^m)) du \quad \text{and} \quad N_t^{n,m} = \mathbf{s}_t - \mathbf{s}_0 - \int_0^t (\hat{b}(\mathbf{s}_u) + \hat{P}_u^n(h^m)) du,$$

where $P_u^n = \mathbb{P}^{n,m} \circ (\mathbf{r}_u)^{-1}$ and $\hat{P}_u^n = \mathbb{P}^{n,m} \circ (\mathbf{s}_u)^{-1}$. For each $m, n \in \mathbb{N}$, $(M_t^{n,m}, \mathcal{F}_t, \mathbb{P}^{n,m})$ and $(N_t^{n,m}, \mathcal{F}_t, \mathbb{P}^{n,m})$ are martingales with respect to the canonical filtration $\mathcal{F}_t = \sigma((\mathbf{r}_u, \mathbf{s}_u)_{0 \leq u \leq t})$ by Ito's formula and the moment estimate (64). Further the family $(M_t^{n,m}, \mathbb{P}^{n,m})_{n \in \mathbb{N}, t \geq 0}$ and $(N_t^{n,m}, \mathbb{P}^{n,m})_{n \in \mathbb{N}, t \geq 0}$ are uniformly integrable by Lipschitz continuity of \tilde{b} and \hat{b} and by boundedness of g^m and h^m . Further, the mappings $M^{n,m}$ and $N^{n,m}$ are continuous in \mathbb{W} . We show that $\mathbb{P}^{n,m} \circ (\mathbf{r}, \mathbf{s}, M^{n,m}, N^{n,m})^{-1}$ converges weakly to $\mathbb{P}^m \circ (\mathbf{r}, \mathbf{s}, M^m, N^m)^{-1}$ as $n \rightarrow \infty$, where

$$M_t^m = \mathbf{r}_t - \mathbf{r}_0 - \int_0^t (\tilde{b}(\mathbf{r}_u) + P_u^m(g^m)) du \quad \text{and} \quad N_t^m = \mathbf{s}_t - \mathbf{s}_0 - \int_0^t (\hat{b}(\mathbf{s}_u) + \hat{P}_u^m(h^m)) du, \quad (67)$$

with $P_u^m = \mathbb{P}^m \circ \mathbf{r}_u^{-1}$ and $\hat{P}_u^m = \mathbb{P}^m \circ \mathbf{s}_u^{-1}$. To show weak convergence to $\mathbb{P}^m \circ (\mathbf{r}, \mathbf{s}, M^m, N^m)^{-1}$, we note that (M^m, N^m) is continuous in \mathbb{W} and we consider for a Lipschitz continuous and bounded function $G : \mathbb{W} \rightarrow \mathbb{R}$,

$$\begin{aligned} & \left| \int_{\mathbb{W}} G(\omega) d\mathbb{P}^{n,m} \circ (M^{n,m})^{-1}(\omega) - \int_{\mathbb{W}} G(\omega) d\mathbb{P}^m \circ (M^m)^{-1}(\omega) \right| \\ & \leq \left| \int_{\mathbb{W}} G(\omega) d\mathbb{P}^{n,m} \circ (M^{n,m})^{-1}(\omega) - \int_{\mathbb{W}} G(\omega) d\mathbb{P}^{n,m} \circ (M^m)^{-1}(\omega) \right| \\ & \quad + \left| \int_{\mathbb{W}} G(\omega) d\mathbb{P}^{n,m} \circ (M^m)^{-1}(\omega) - \int_{\mathbb{W}} G(\omega) d\mathbb{P}^m \circ (M^m)^{-1}(\omega) \right|. \end{aligned}$$

The second term converges to 0 as $n \rightarrow \infty$, since (M^m) is continuous. For the first term it holds

$$\begin{aligned} & \left| \int_{\mathbb{W}} G(\omega) d\mathbb{P}^{n,m} \circ (M^{n,m})^{-1}(\omega) - \int_{\mathbb{W}} G(\omega) d\mathbb{P}^{n,m} \circ (M^m)^{-1}(\omega) \right| \\ &= \left| \int_{\mathbb{W}} (G \circ M^{n,m})(\omega) d\mathbb{P}^{n,m}(\omega) - \int_{\mathbb{W}} (G \circ M^m)(\omega) d\mathbb{P}^{n,m}(\omega) \right| \\ &\leq \|G\|_{\text{Lip}} \sup_{\omega \in \mathbb{W}} d_{\mathbb{W}}(M^{n,m}(\omega), M^m(\omega)), \end{aligned}$$

where $d_{\mathbb{W}}(f, g) = \sum_{k=1}^{\infty} \sup_{t \in [0, k]} 2^{-k} |f(t) - g(t)|$. This term converges to 0 for $n \rightarrow \infty$, since for all $T > 0$ and $\omega \in \mathbb{W}$, for $n \rightarrow \infty$

$$\sup_{t \in [0, T]} |M_t^{n,m}(\omega) - M_t^m(\omega)| \leq \int_0^T |(\mathbb{P}^{n,m} \circ \mathbf{r}_s^{-1})(g^m) - (\mathbb{P}^m \circ \mathbf{r}_s^{-1})(g^m)| ds \rightarrow 0,$$

by Lebesgue dominated convergence theorem, since g is bounded. Hence,

$$\left| \int_{\mathbb{W}} G(\omega) d\mathbb{P}^{n,m} \circ (M^{n,m})^{-1}(\omega) - \int_{\mathbb{W}} G(\omega) d\mathbb{P}^m \circ (M^m)^{-1}(\omega) \right| \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

and similarly for $(N^{n,m})$, and therefore by the Portmanteau theorem [39, Theorem 13.16], weak convergence of $\mathbb{P}^{n,m} \circ (\mathbf{r}, \mathbf{s}, M^{n,m}, N^{n,m})^{-1}$ to $\mathbb{P}^m \circ (\mathbf{r}, \mathbf{s}, M^m, N^m)^{-1}$ holds.

Let $G : \mathbb{W} \rightarrow \mathbb{R}_+$ be a \mathcal{F}_s -measurable, bounded, non-negative function. By uniformly integrability of $(M_t^{n,m}, \mathbb{P}^{n,m})_{n \in \mathbb{N}, t \geq 0}$, for any $s \leq t$,

$$\begin{aligned} \mathbb{E}^m[G(M_t^m - M_s^m)] &= \mathbb{E}^m[G(\int_s^t (\tilde{b}(\mathbf{r}_u) + P_u(g^m)) du)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^{n,m}[G(\int_s^t (\tilde{b}(\mathbf{r}_u) + P_u^n(g^m)) du)] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}^{n,m}[G(M_t^{n,m} - M_s^{n,m})] = 0, \end{aligned} \tag{68}$$

and analogously for $(N_t^{n,m})_{t \geq 0}$ and hence, $(M_t^m, \mathcal{F}_t, \mathbb{P}^m)$ and $(N_t^m, \mathcal{F}_t, \mathbb{P}^m)$ are continuous martingales. The quadratic variation $([M^m, N^m]_t)$ exists \mathbb{P}^m -almost surely. To complete the identification of the limit, it suffices to note that the quadratic variation is given by

$$\begin{aligned} [M^m] &= 4 \int_0^\cdot \mathbb{1}_{(0, \infty)}(\mathbf{r}_u) du && \mathbb{P}^m\text{-almost surely,} \\ [N^m] &= 4 \int_0^\cdot \mathbb{1}_{(0, \infty)}(\mathbf{s}_u) du && \mathbb{P}^m\text{-almost surely, and} \\ [M^m, N^m] &= 4 \int_0^\cdot \mathbb{1}_{(0, \infty)}(\mathbf{r}_u) \mathbb{1}_{(0, \infty)}(\mathbf{s}_u) du && \mathbb{P}^m\text{-almost surely,} \end{aligned} \tag{69}$$

which holds following the computations in the proof of [23, Theorem 22]. We show that $((M_t^m)^2 - 4 \int_0^t \mathbb{1}_{(0, \infty)} \mathbf{r}_u du)$ is a sub- and a supermartingale and hence a martingale using a

monotone class argument by noting first that for any bounded continuous and non-negative function $G : \mathbb{W} \rightarrow \mathbb{R}_+$,

$$\mathbb{E}^m[G(M_t^m)^2] = \lim_{n \rightarrow \infty} \mathbb{E}^{n,m}[G(M_t^{n,m})^2] \quad (70)$$

holds using uniform integrability of $((M_t^{n,m})^2, \mathbb{P}^{n,m})_{n \in \mathbb{N}, t \geq 0}$ which holds similarly as above. Note that

$$\mathbb{E}^m \left[G \int_s^t \mathbb{1}_{(0,\infty)}(\mathbf{r}_u) du \right] \leq \lim_{\epsilon \downarrow 0} \liminf_{n \rightarrow \infty} \mathbb{E}^{n,m} \left[G \int_s^t \mathbb{1}_{(\epsilon,\infty)}(\mathbf{r}_u) du \right] \quad (71)$$

holds by lower semicontinuity of $\omega \rightarrow \int_0^t \mathbb{1}_{(\epsilon,\infty)}(\omega_s) ds$ for each $\epsilon > 0$, Fatou's lemma and the Portmanteau theorem. For any fixed $\epsilon > 0$,

$$\liminf_{n \rightarrow \infty} \mathbb{E}^{n,m} \left[G \left(\int_s^t \theta^n(\mathbf{r}_u)^2 du - \int_s^t \mathbb{1}_{(\epsilon,\infty)}(\mathbf{r}_u) du \right) \right]. \quad (72)$$

Then by (70), (71) and (72)

$$\begin{aligned} & \mathbb{E}^m \left[G \left((M_t^m)^2 - (M_s^m)^2 - 4 \int_s^t \mathbb{1}_{(0,\infty)}(\mathbf{r}_u) du \right) \right] \\ & \geq \lim_{\epsilon \downarrow 0} \liminf_{n \rightarrow \infty} \mathbb{E}^{n,m} \left[G \left((M_t^{n,m})^2 - (M_s^{n,m})^2 - 4 \int_s^t \theta^n(\mathbf{r}_u)^2 du \right) \right] = 0 \end{aligned}$$

and hence by a monotone class argument, cf. [50, Chapter 1, Theorem 8], $((M_t^m)^2 - 4 \int_0^t \mathbb{1}_{(0,\infty)}(\mathbf{r}_u) du, \mathbb{P}^m)$ is a submartingale. To show that it is also a supermartingale we note that $((M_t^m)^2 - 4t, \mathbb{P}^m)$ is a supermartingale by (70). By the uniqueness of the Doob-Meyer decomposition, cf. [50, Chapter 3, Theorem 8], $t \rightarrow [M^m]_t - 4t$ is \mathbb{P}^m -almost surely decreasing. Note further, that $(\mathbf{r}_t, \mathcal{F}_t, \mathbb{P}^m)$ is a continuous semimartingale with $[\mathbf{r}] = [M^m]$. Then by Ito-Tanaka formula, cf. [51, Chapter 6, Theorem 1.1],

$$\int_0^t \mathbb{1}_{\{0\}}(\mathbf{r}_u) d[M^m]_u = \int_0^t \mathbb{1}_{\{0\}}(\mathbf{r}_u) d[\mathbf{r}]_u = \int_0^t \mathbb{1}_{\{0\}}(y) \ell_t^y(\mathbf{r}) dy = 0,$$

where $\ell_t^y(\mathbf{r})$ is the local time of \mathbf{r} in y . Therefore, for any $0 \leq s < t$,

$$[M^m]_t - [M^m]_s = \int_0^t \mathbb{1}_{(0,\infty)}(\mathbf{r}_u) d[M^m]_u \leq 4 \int_0^t \mathbb{1}_{(0,\infty)}(\mathbf{r}_u) du$$

and hence, for any \mathcal{F}_s -measurable, bounded, non-negative function $G : \mathbb{W} \rightarrow \mathbb{R}_+$,

$$\mathbb{E}^m \left[G \left((M_t^m)^2 - (M_s^m)^2 - 4 \int_s^t \mathbb{1}_{(0,\infty)}(\mathbf{r}_u) du \right) \right] \leq 0.$$

As before, by a monotone class argument, $((M_t^m)^2 - 4 \int_0^t \mathbb{1}_{(0,\infty)}(\mathbf{r}_u) du, \mathbb{P}^m)$ is a supermartingale, and hence a martingale.

Hence, we obtain the quadratic variation $[M^m]_t$ given in (69). The other characterizations in (69) follow by analogous arguments. Then by a martingale representation theorem, see [35, Chapter II, Theorem 7.1], we conclude, that there are a probability space $(\Omega^m, \mathcal{A}^m, P^m)$ and a Brownian motion W and random variables (r^m, s^m) on this space such that $P^m \circ (r^m, s^m)^{-1} = \mathbb{P}^m \circ (\mathbf{r}^m, \mathbf{s}^m)^{-1}$ and such that (r^m, s^m, W) is a weak solution of (60). Finally, note that we have weak convergence of $Q \circ (r^{n,m}, s^{n,m})^{-1}$ to $P^m \circ (r^m, s^m)^{-1}$ not only along a subsequence since the characterization of the limit holds for any subsequence $(n_k)_{k \in \mathbb{N}}$.

Comparison of two solutions: To show $P^m[r_t^m \leq s_t^m \text{ for all } t \geq 0] = 1$ we note that by Lemma 18, $Q[r_t^n \leq s_t^n \text{ for all } t \geq 0] = 1$. The monotonicity carries over to the limit by the Portmanteau theorem for closed sets, since we have weak convergence of $\mathbb{P}^{n,m} \circ (\mathbf{r}, \mathbf{s})^{-1}$ to $\mathbb{P}^m \circ (\mathbf{r}, \mathbf{s})^{-1}$. \square

We show in the next step that the distribution of the solution of (60) converges as $m \rightarrow \infty$. For each $m \in \mathbb{N}$ let $(\Omega^m, \mathcal{A}^m, P^m)$ be a probability space and random variables $r^m, s^m : \Omega^m \rightarrow \mathbb{W}$ such that $(r_t^m, s_t^m)_{t \geq 0}$ is a solution of (60). Let $\mathbb{P}^m = P^m \circ (r^m, s^m)^{-1}$ denote the law on $\mathbb{W} \times \mathbb{W}$.

Lemma 20. *Assume that (\tilde{b}, g) and (\hat{b}, h) satisfy **H1** and **H2**. Let $\eta \in \Gamma(\mu, \nu)$ where the probability measures μ and ν on \mathbb{R}_+ satisfy **H3**. Assume that $(g^m)_{m \in \mathbb{N}}$, $(h^m)_{m \in \mathbb{N}}$, $(\mu_m)_{m \in \mathbb{N}}$, $(\nu_m)_{m \in \mathbb{N}}$ and $(\eta_m)_{m \in \mathbb{N}}$ satisfy conditions **H5** and **H7**. Then there exists a random variable (r, s) defined on some probability space (Ω, \mathcal{A}, P) with values in $\mathbb{W} \times \mathbb{W}$, such that $(r_t, s_t)_{t \geq 0}$ is a weak solution of the sticky stochastic differential equation (25). Furthermore, the sequence of laws $P^m \circ (r^m, s^m)^{-1}$ converges weakly to the law $P \circ (r, s)^{-1}$. If additionally,*

$$\begin{aligned} \tilde{b}(r) \leq \hat{b}(r), \quad g(r) \leq h(r) \quad \text{and} \quad g^m(r) \leq h^m(r) & \quad \text{for any } r \in \mathbb{R}_+, \text{ and} \\ P^m[r_0^m \leq s_0^m] = 1 & \quad \text{for any } m \in \mathbb{N} \end{aligned}$$

then $P[r_t \leq s_t \text{ for all } t \geq 0] = 1$.

Proof. The proof is structured as the proof of Lemma 19. First analogously to the proof of (64) we show under **H1**, **H5** and **H7**,

$$\sup_{t \in [0, T]} \mathbb{E}[|r_t^m|^p] < \infty. \tag{73}$$

Tightness of the sequence of probability measures $(\mathbb{P}^m)_{m \in \mathbb{N}}$ on $(\mathbb{W} \times \mathbb{W}, \mathcal{B}(\mathbb{W}) \otimes \mathcal{B}(\mathbb{W}))$ holds adapting the steps of the proof of Lemma 19 to (60). Note that (64) and (65) hold analogously for $(r_t^m, s_t^m)_{m \in \mathbb{N}}$ by **H1**, **H5** and **H7**. Hence by Kolmogorov's continuity criterion, cf. [38, Corollary 14.9], we can deduce that there exists a probability measure \mathbb{P} on $(\mathbb{W} \times \mathbb{W}, \mathcal{B}(\mathbb{W}) \otimes \mathcal{B}(\mathbb{W}))$ such that there is a subsequence $(m_k)_{k \in \mathbb{N}}$ along which \mathbb{P}^{m_k} converge towards \mathbb{P} . To characterize the limit, we first note that by Skorokhod representation

theorem, cf. [9, Chapter 1, Theorem 6.7], without loss of generality we can assume that (r^m, s^m) are defined on a common probability space (Ω, \mathcal{A}, P) with expectation E and converge almost surely to (r, s) with distribution \mathbb{P} . By **H5**, $P_t^m(g^m) = E[g^m(r_t^m)]$ and the monotone convergence theorem, $P_t^m(g^m)$ converges to $P_t(g)$ for any $t \geq 0$. Then, by Lebesgue convergence theorem it holds almost surely for all $t \geq 0$

$$\lim_{m \rightarrow \infty} \int_0^t (\tilde{b}(r_t^m) + P_u^m(g^m)) du = \int_0^t (\tilde{b}(r_t) + P_u(g)) du, \quad (74)$$

where $P_u^m = P \circ (r_u^m)^{-1}$ and $P_u = P \circ (r_u)^{-1}$. A similar statement holds for $(s_t)_{t \geq 0}$.

Consider the mappings $M^m, N^m : \mathbb{W} \times \mathbb{W} \rightarrow \mathbb{W}$ defined by (67). Then for all $m \in \mathbb{N}$, $(M_t^m, \mathcal{F}_t, \mathbb{P}^m)$ and $(N_t^m, \mathcal{F}_t, \mathbb{P}^m)$ are martingales with respect to the canonical filtration $\mathcal{F}_t = \sigma((\mathbf{r}_u, \mathbf{s}_u)_{0 \leq u \leq t})$. Further the family $(M_t^m, \mathbb{P}^m)_{m \in \mathbb{N}, t \geq 0}$ and $(N_t^m, \mathbb{P}^m)_{m \in \mathbb{N}, t \geq 0}$ are uniformly integrable by (73). In the same line as in the proof of Lemma 19 and by (74), $\mathbb{P}^m \circ (\mathbf{r}, \mathbf{s}, M^m, N^m)$ converges weakly to $\mathbb{P} \circ (\mathbf{r}, \mathbf{s}, M, N)$ where

$$M_t = \mathbf{r}_t - \mathbf{r}_0 - \int_0^t (\tilde{b}(\mathbf{r}_u) + P_u(g)) du \quad \text{and} \quad N_t = \mathbf{s}_t - \mathbf{s}_0 - \int_0^t (\hat{b}(\mathbf{s}_u) + \hat{P}_u(h)) du.$$

Let $G : \mathbb{W} \rightarrow \mathbb{R}_+$ be a \mathcal{F}_s -measurable bounded, non-negative function. By uniform integrability, for any $s \leq t$,

$$\begin{aligned} \mathbb{E}[G(M_t - M_s)] &= \mathbb{E}[G(\int_s^t (\tilde{b}(\mathbf{r}_u) + P_u(g)) du)] = \lim_{m \rightarrow \infty} \mathbb{E}^m[G(\int_s^t (\tilde{b}(\mathbf{r}_u) + P_u(g^m)) du)] \\ &= \lim_{m \rightarrow \infty} \mathbb{E}^m[G(M_t^m - M_s^m)] = 0, \end{aligned}$$

and analogously for $(N_t)_{t \geq 0}$. Hence, $(M_t, \mathcal{F}_t, \mathbb{P})$ and $(N_t, \mathcal{F}_t, \mathbb{P})$ are martingales. Further, the quadratic variation $([(M, N)]_t)$ exists \mathbb{P} -almost surely and is given by (69) \mathbb{P} -almost surely, which holds following the computations in the proof of Lemma 19. As in Lemma 19, we conclude by a martingale representation theorem that there are a probability space (Ω, \mathcal{A}, P) and a Brownian motion W and random variables (r, s) on this space such that $P \circ (r, s)^{-1} = \mathbb{P} \circ (\mathbf{r}, \mathbf{s})^{-1}$ and such that (r, s, W) is a weak solution of (25). Note that the limit identification holds for all subsequences $(m_k)_{k \in \mathbb{N}}$ and hence $P^m \circ (r^m, s^m)^{-1}$ converges weakly to $P \circ (r, s)^{-1}$ for $m \rightarrow \infty$. The monotonicity $P^m[r_t^m \leq s_t^m \text{ for all } t \geq 0] = 1$ carries over to the limit by Portmanteau theorem, since $\mathbb{P}^m \circ (\mathbf{r}, \mathbf{s})^{-1}$ converges weakly to $\mathbb{P} \circ (\mathbf{r}, \mathbf{s})^{-1}$. \square

Proof of Theorem 3. The proof is a direct consequence of Lemma 19 and Lemma 20. \square

6.3.2 Proof of Theorem 5

Proof of Theorem 5. Note that the Dirac at 0, δ_0 , is by definition an invariant measure of $(r_t)_{t \geq 0}$ solving (6). Assume that the process starts from an invariant probability measure

π , hence $\mathbb{P}(r_t > 0) = p = \pi((0, \infty))$ for any $t \geq 0$. Note that for $p = 0$ the drift vanishes. If the initial measure is the Dirac measure in 0, δ_0 , then the diffusion coefficient disappears. Hence, $\text{Law}(r_t) = \delta_0$ for any $t \geq 0$. It remains to investigate the case $p \neq 0$. Here, we are in the regime of [23, Lemma 24] where an invariant measure is of the form (28). Since $p = \mathbb{P}(r_t > 0)$, the invariant measure π satisfies additionally the necessary condition

$$p = \pi((0, \infty)) = \frac{I(a, p)}{2/(ap) + I(a, p)} \quad (75)$$

with $I(a, p)$ given in (27). For $p \neq 0$, this expression is equivalent to (26). \square

Proof of Proposition 6. By Theorem 5, it suffices to study the solutions of (26). By (27) and since $\hat{b}(r) = -\tilde{L}r$, it holds for $\hat{I}(a, p) = (1 - p)I(a, p)$,

$$\hat{I}(a, p) = \left(\sqrt{\frac{\pi}{2}} + \int_0^{\frac{ap}{\sqrt{2\tilde{L}}}} \exp(-x^2/2) dx \right) \sqrt{\frac{2}{\tilde{L}}} \exp\left(\frac{a^2 p^2}{4\tilde{L}}\right) (1 - p). \quad (76)$$

In the case $a/\sqrt{\tilde{L}} \leq 2/\sqrt{\pi}$, $\hat{I}(a, 0) = \sqrt{\pi/\tilde{L}}$ by (76). Further, by $1 + x \leq e^x$ and $a/\sqrt{\tilde{L}} \leq 2/\sqrt{\pi}$,

$$\begin{aligned} \left(\sqrt{\frac{\pi}{2}} + \int_0^{\frac{ap}{\sqrt{2\tilde{L}}}} e^{-\frac{x^2}{2}} dx \right) (1 - p) e^{\frac{a^2 p^2}{4\tilde{L}}} &\leq \sqrt{\frac{\pi}{2}} \left(1 + \sqrt{\frac{2}{\pi}} \int_0^{\frac{ap}{\sqrt{2\tilde{L}}}} e^{-\frac{x^2}{2}} dx \right) e^{-p} e^{\frac{p^2}{\pi}} \\ &\leq \sqrt{\frac{\pi}{2}} \left(1 + \frac{2p}{\pi} \right) e^{-p} e^{\frac{p^2}{\pi}} \leq \sqrt{\frac{\pi}{2}} e^{p(\frac{3}{\pi} - 1)} < \sqrt{\frac{\pi}{2}} \end{aligned}$$

for $p \in (0, 1]$. Hence, $\hat{I}(a, p) < \hat{I}(a, 0)$ by (76). Therefore, $\hat{I}(a, p) < \hat{I}(a, 0) \leq \frac{2}{a}$ for all $p \in (0, 1]$ and so δ_0 is the unique invariant probability measure for $a/\sqrt{\tilde{L}} \leq 2/\sqrt{\pi}$.

To show that for $a/\sqrt{\tilde{L}} > 2/\sqrt{\pi}$, there exists a unique p solving (26), we note that $\hat{I}(a, p)$ is continuous with $\hat{I}(a, 0) > 2/a$ and $\hat{I}(a, 1) = 0$. By the mean value theorem, there exists at least one $p \in (0, 1)$ satisfying (26). In the following we drop the dependence on a in $I(a, p)$ and $\hat{I}(a, p)$. We show uniqueness of the solution p by contradiction. Assume that $p_1 < p_2$ are the two smallest solutions of (26). Hence, it holds either $\hat{I}'(p_1) < 0$ or $\hat{I}'(p) = 0$ for p_1 . Note that the derivative is given by

$$\begin{aligned} \hat{I}'(p_i) &= -I(p_i) + (1 - p_i)I'(p_i) = -I(p_i) + (1 - p_i) \left(p_i \frac{a^2}{2\tilde{L}} I(p_i) + \frac{a}{\tilde{L}} \right) \\ &= -\frac{2}{a(1 - p_i)} + (1 - p_i) \frac{a}{\tilde{L}} \left(\frac{p_i}{1 - p_i} + 1 \right) = -\frac{2}{a(1 - p_i)} + \frac{a}{\tilde{L}}. \end{aligned} \quad (77)$$

Then, for $p_2 > p_1$, it holds

$$\hat{I}'(p_2) = -\frac{2}{a(1 - p_2)} + \frac{a}{\tilde{L}} < -\frac{2}{a(1 - p_1)} + \frac{a}{\tilde{L}} = \hat{I}'(p_1) \leq 0.$$

If $\hat{I}'(p_1) < 0$, it holds $\hat{I}'(p_2) < 0$ which contradicts that p_1 and p_2 are the two smallest solutions. In the second case, when $\hat{I}'(p_1) = 0$, we note that the second derivative of $\hat{I}(p)$ at p_1 is given by

$$\begin{aligned}\hat{I}''(p_1) &= -2I'(p_1) + (1 - p_1)I''(p_1) \\ &= \left(-2 + (1 - p_1)\frac{a^2 p_1}{2\tilde{L}}\right)\left(I(p_1)\frac{a^2 p_1}{2\tilde{L}} + \frac{a}{\tilde{L}}\right) + (1 - p_1)I(p_1)\frac{a^2}{2\tilde{L}} \\ &= \left(-2 + (1 - p_1)\frac{a^2 p_1}{2\tilde{L}}\right)\frac{a}{\tilde{L}(1 - p_1)} + \frac{a}{\tilde{L}} = -\frac{a}{\tilde{L}(1 - p_1)} < 0.\end{aligned}$$

Hence, in this case there is a maximum at p_1 , which contradicts that p_1 is the smallest solution. Thus, there exists a unique solution p_1 of (26) for $a/\sqrt{\tilde{L}} > 2/\sqrt{\pi}$. \square

6.3.3 Proof of Theorem 7

Proof of Theorem 7. To show (31) we extend the function f to a concave function on \mathbb{R} by setting $f(x) = x$ for $x < 0$. Note that f is continuously differentiable and f' is absolutely continuous and bounded. Using Ito-Tanaka formula, c.f. [51, Chapter 6, Theorem 1.1] we obtain

$$df(r_t) = f'(r_t)(\tilde{b}(r_t) + a\mathbb{P}(r_t > 0))dt + 2f''(r_t)\mathbb{1}_{(0,\infty)}(r_t)dt + dM_t,$$

where $M_t = 2\int_0^t f'(r_s)\mathbb{1}_{(0,\infty)}(r_s)dB_s$ is a martingale. Taking expectation, we get

$$\begin{aligned}\frac{d}{dt}\mathbb{E}[f(r_t)] &= \mathbb{E}[f'(r_t)(\tilde{b}(r_t) + a\mathbb{P}(r_t > 0))] + 2\mathbb{E}[f''(r_t)\mathbb{1}_{(0,\infty)}(r_t)] \\ &= \mathbb{E}[f'(r_t)\tilde{b}(r_t) + 2(f''(r_t) - f''(0))] + \mathbb{E}[af'(r_t) + 2f''(0)]\mathbb{P}(r_t > 0) \\ &\leq -c\mathbb{E}[f(r_t)],\end{aligned}$$

where the last step holds by (39) and (40). By applying Gronwall's lemma, we obtain (31). \square

6.4 Proof of Section 5

As for the nonlinear case we show Theorem 10 via a family of stochastic differential equations, with Lipschitz continuous coefficients,

$$\begin{aligned}dr_t^{i,n,m} &= \left(\tilde{b}(r_t^{i,n,m}) + \frac{1}{N}\sum_{j=1}^N g^m(r_t^{j,n,m})\right)dt + 2\theta^n(r_t^{i,n,m})dW_t^i \\ ds_t^{i,n,m} &= \left(\hat{b}(s_t^{i,n,m}) + \frac{1}{N}\sum_{j=1}^N h^m(s_t^{j,n,m})\right)dt + 2\theta^n(s_t^{i,n,m})dW_t^i \\ \text{Law}(r_0^{i,n,m}, s_0^{i,n,m}) &= \eta_{n,m}, \quad i \in \{1, \dots, N\},\end{aligned}\tag{78}$$

where $\eta_{n,m} \in \Gamma(\mu_{n,m}, \nu_{n,m})$. Under **H1**, **H2**, **H5**, **H6** and **H7** we identify the weak limit of $(\{r_t^{i,n,m}, s_t^{i,n,m}\}_{i=1, n, m \in \mathbb{N}}^N)_{t \geq 0}$ solving (78) for $n \rightarrow \infty$ by $(\{r_t^{i,m}, s_t^{i,m}\}_{i=1, m \in \mathbb{N}}^N)_{t \geq 0}$ solving the family of SDEs given by

$$\begin{aligned} dr_t^{i,m} &= \left(\tilde{b}(r_t^{i,m}) + \frac{1}{N} \sum_{j=1}^N g^m(r_t^{j,m}) \right) dt + 2\mathbb{1}_{(0,\infty)}(r_t^{i,m}) dW_t^i, \\ ds_t^{i,m} &= \left(\hat{b}(s_t^{i,m}) + \frac{1}{N} \sum_{j=1}^N h^m(s_t^{j,m}) \right) dt + 2\mathbb{1}_{(0,\infty)}(s_t^{i,m}) dW_t^i, \\ \text{Law}(r_0^{i,m}, s_0^{i,m}) &= \eta_m, \quad i \in \{1, \dots, N\}, \end{aligned} \tag{79}$$

where $\eta_m \in \Gamma(\mu_m, \nu_m)$.

Taking the limit $m \rightarrow \infty$, we obtain (35) as the weak limit of (79). In the case $g(r) = \mathbb{1}_{(0,\infty)}(r)$, we can choose $g^m = \theta^m$.

Consider a probability space $(\Omega_0, \mathcal{A}_0, Q)$ and N i.i.d.1-dimensional Brownian motions $(\{W_t^i\}_{i=1}^N)_{t \geq 0}$. Note that under **H1–H7**, there are random variables $\{r^{i,n,m}\}_{i=1}^N, \{s^{i,n,m}\}_{i=1}^N : \Omega_0 \rightarrow \mathbb{W}^N$ for each n, m such that $(\{r^{i,n,m}, s^{i,n,m}\}_{i=1}^N)$ is a unique solution to (78) by [45, Theorem 2.2]. We denote by $\mathbb{P}^{n,m} = Q \circ (\{r^{i,n,m}, s^{i,n,m}\}_{i=1}^N)^{-1}$ the law on $\mathbb{W}^N \times \mathbb{W}^N$.

Before taking the two limits and proving Theorem 10, we introduce a modification of Ikeda and Watanabe's comparison theorem, to compare two solutions of (78), cf. [35, Section VI, Theorem 1.1].

Lemma 21. *Suppose a solution $(\{r_t^{i,n,m}, s_t^{i,n,m}\}_{i=1}^N)_{t \geq 0}$ of (78) is given for fixed $n, m \in \mathbb{N}$. Assume **H5** for g^m and h^m , **H1** for \tilde{b} and \hat{b} , **H6** for θ^n . If $Q[r_0^{i,n,m} \leq s_0^{i,n,m}$ for all $i = 1, \dots, N] = 1$, $\tilde{b}(r) \leq \hat{b}(r)$ and $g^m(r) \leq h^m(r)$ for any $r \in \mathbb{R}_+$, then*

$$Q[r_t^{i,n,m} \leq s_t^{i,n,m} \text{ for all } t \geq 0 \text{ and } i = 1, \dots, N] = 1$$

Proof. The proof is similar for each component $i = 1, \dots, N$ to the proof of Lemma 18. It holds for the interaction part similarly to (62) using the properties of g^m and h^m ,

$$\frac{1}{N} \sum_{j=1}^N (g^m(r_t^{j,n,m}) - h^m(s_t^{j,n,m})) \leq K_m \frac{1}{N} \sum_{j=1}^N |r_t^{j,n,m} - s_t^{j,n,m}| \mathbb{1}_{(0,\infty)}(r_t^{j,n,m} - s_t^{j,n,m}).$$

Hence, we obtain analogously to (63),

$$\mathbb{E}[(r_t^{i,n,m} - s_t^{i,n,m})_+] \leq \tilde{L} \mathbb{E} \left[\int_0^t (r_u^{i,n,m} - s_u^{i,n,m})_+ du \right] + K_m \mathbb{E} \left[\int_0^t \frac{1}{N} \sum_{j=1}^N (r_u^{j,n,m} - s_u^{j,n,m})_+ du \right]$$

for all $i = 1, \dots, N$. Assume analogously to the nonlinear case $t^* = \inf\{t \geq 0 : \mathbb{E}[(r_t^{i,n,m} - s_t^{i,n,m})_+] > 0 \text{ for some } i\} < \infty$. Then, there exists $i \in \{1, \dots, N\}$ such that $\int_0^{t^*} \mathbb{E}[(r_u^{i,n,m} - s_u^{i,n,m})_+] du > 0$. But, by definition of t^* , for all $i, u < t^*$, $\mathbb{E}[(r_u^{i,n,m} - s_u^{i,n,m})_+] = 0$. This contradicts the definition of t^* . Hence, $Q[r_t^{i,n,m} \leq s_t^{i,n,m} \text{ for all } i, t \geq 0] = 1$. \square

In the next step, we prove that the distribution of the solution of (78) converges as $n \rightarrow \infty$.

Lemma 22. *Assume that **H1** and **H2** is satisfied for (\tilde{b}, g) and (\hat{b}, h) . Further, let $(\theta^n)_{n \in \mathbb{N}}$, $(g^m)_{m \in \mathbb{N}}$, $(h^m)_{m \in \mathbb{N}}$, $(\mu_{n,m})_{n,m \in \mathbb{N}}$, $(\nu_{n,m})_{n,m \in \mathbb{N}}$ and $(\eta_{n,m})_{n,m \in \mathbb{N}}$ be such that **H5**, **H6** and **H7** hold. Let $m \in \mathbb{N}$. Then there exists a random variable $(\{r_t^{i,m}, s_t^{i,m}\}_{i=1}^N)$ defined on some probability space $(\Omega^m, \mathcal{A}^m, P^m)$ with values in $\mathbb{W}^N \times \mathbb{W}^N$ such that $(\{r_t^{i,m}, s_t^{i,m}\}_{i=1}^N)_{t \geq 0}$ is a weak solution of (79). Moreover, the laws $Q \circ (\{r^{i,n,m}, s^{i,n,m}\}_{i=1}^N)^{-1}$ converge weakly to $P^m \circ (\{r^{i,m}, s^{i,m}\}_{i=1}^N)^{-1}$. If in addition,*

$$\begin{aligned} \tilde{b}(r) &\leq \hat{b}(r) \quad \text{and} \quad g^m(r) \leq h^m(r) && \text{for any } r \in \mathbb{R}_+, \\ Q[r_0^{i,n,m} \leq s_0^{i,n,m}] &= 1 && \text{for any } n \in \mathbb{N}, i = 1, \dots, N, \end{aligned}$$

then $P^m[r_t^{i,m} \leq s_t^{i,m}] = 1$ for all $t \geq 0$ and $i \in \{1, \dots, N\}$.

Proof. Fix $m \in \mathbb{N}$. The proof is divided in three parts and is similar to the proof of Lemma 19. First we show tightness of the sequences of probability measures. Then we identify the limit of the sequence of stochastic processes. Finally, we compare the two limiting processes.

Tightness: We show analogously as in the proof of Lemma 19 that the sequence of probability measures $(\mathbb{P}^{n,m})_{n \in \mathbb{N}}$ on $(\mathbb{W}^N \times \mathbb{W}^N, \mathcal{B}(\mathbb{W}^N) \otimes \mathcal{B}(\mathbb{W}^N))$ is tight by applying Kolmogorov's continuity theorem. We consider $p > 2$ such that the p -th moment in **H7** are uniformly bounded. Fix $T > 0$. Then the p -th moment of $r_t^{i,n,m}$ and $s_t^{i,n,m}$ for $t < T$ is bounded using Ito's formula,

$$\begin{aligned} d|r_t^{i,n,m}|^p &\leq p|r_t^{i,n,m}|^{p-2} \langle r_t^{i,n,m}, (\tilde{b}(r_t^{i,n,m}) + \frac{1}{N} \sum_{j=1}^N g^m(r_t^{j,n,m})) \rangle dt \\ &\quad + 2\theta^n (r_t^{i,n,m}) p |r_t^{i,n,m}|^{p-2} r_t^{i,n,m} dW_t^i + p(p-1) |r_t^{i,n,m}|^{p-2} 2\theta^n (r_t^{i,n,m})^2 dt \\ &\leq p \left(|r_t^{i,n,m}|^p \tilde{L} + \Gamma |r_t^{i,n,m}|^{p-1} + 2(p-1) |r_t^{i,n,m}|^{p-2} \right) dt + 2\theta^n (r_t^{i,n,m}) p (r_t^{i,n,m})^{p-1} dW_t^i \\ &\leq p \left(\tilde{L} + \Gamma + 2(p-1) \right) |r_t^{i,n,m}|^p dt + p(\Gamma + 2(p-1)) dt + 2\theta^n (r_t^{i,n,m}) p (r_t^{i,n,m})^{p-1} dW_t^i. \end{aligned}$$

Taking expectation yields

$$\frac{d}{dt} \mathbb{E}[|r_t^{i,n,m}|^p] \leq p(L + \Gamma + 2(p-1)) \mathbb{E}[|r_t^{i,n,m}|^p] + p(\Gamma + 2(p-1)).$$

Then by Gronwall's lemma

$$\sup_{t \in [0, T]} \mathbb{E}[|r_t^{i,n,m}|^p] \leq e^{p(L + \Gamma + 2(p-1))T} (\mathbb{E}[|r_0^{i,n,m}|^p] + Tp(\Gamma + 2(p-1))) < C_p < \infty, \quad (80)$$

where C_p depends on T and the p -th moment of the initial distribution, which is by assumption finite. Similarly, it holds $\sup_{t \in [0, T]} \mathbb{E}[|s_t^{i, n, m}|^p] < C_p$ for $t \leq T$. Using these moment bounds, it holds for all $t_1, t_2 \in [0, T]$ by **H1**, **H5** and **H6**,

$$\begin{aligned}
& \mathbb{E}[|r_{t_2}^{i, n, m} - r_{t_1}^{i, n, m}|^p] \\
& \leq C_1(p) \left(\mathbb{E}\left[\left| \int_{t_1}^{t_2} \tilde{b}(r_u^{i, n, m}) du \right|^p \right] + \frac{1}{N} \sum_{j=1}^N \mathbb{E}\left[\left| \int_{t_1}^{t_2} g^m(r_t^{j, n, m}) du \right|^p \right] + \mathbb{E}\left[\left| \int_{t_1}^{t_2} 2\theta^n(r_u^{i, n, m}) dW_u^i \right|^p \right] \right) \\
& \leq C_2(p) \left(\left(\mathbb{E}\left[\frac{\tilde{L}^p}{|t_2 - t_1|} \int_{t_1}^{t_2} |r_u^{i, n, m}|^p du \right] + \Gamma^p \right) |t_2 - t_1|^p + \mathbb{E}\left[\left| \int_{t_1}^{t_2} 2\theta^n(r_u^{i, n, m}) du \right|^{p/2} \right] \right) \\
& \leq C_2(p) \left(\left(\frac{\tilde{L}^p}{|t_2 - t_1|} \int_{t_1}^{t_2} \mathbb{E}[|r_u^{i, n, m}|^p] du + \Gamma^p \right) |t_2 - t_1|^p + 2^{p/2} |t_2 - t_1|^{p/2} \right) \\
& \leq C_3(p, T, \tilde{L}, \Gamma, C_p) |t_2 - t_1|^{p/2},
\end{aligned}$$

where $C_k(\cdot)$ are constants depending on the stated arguments, but independent of n, m . Note that in the second step, we use Burkholder-Davis-Gundy inequality, see [50, Chapter IV, Theorem 48]. It holds similarly, $\mathbb{E}[|s_{t_2}^{i, n, m} - s_{t_1}^{i, n, m}|^p] \leq C_3(p, T, \tilde{L}, \Gamma, C_p) |t_2 - t_1|^{p/2}$. Hence,

$$\begin{aligned}
& \mathbb{E}\left[\left| \left(\{r_{t_2}^{i, n, m}, s_{t_2}^{i, n, m}\}_{i=1}^N \right) - \left(\{r_{t_1}^{i, n, m}, s_{t_1}^{i, n, m}\}_{i=1}^N \right) \right|^p \right] \\
& \leq C_4(p, N) \left(\sum_{i=1}^N \left(\mathbb{E}[|r_{t_2}^{i, n, m} - r_{t_1}^{i, n, m}|^p] + \mathbb{E}[|s_{t_2}^{i, n, m} - s_{t_1}^{i, n, m}|^p] \right) \right) \\
& \leq C_5(p, N, T, \tilde{L}, \Gamma, C_p) |t_2 - t_1|^{p/2}
\end{aligned}$$

for all $t_1, t_2 \in [0, T]$. Hence, by Kolmogorov's continuity criterion, cf. [38, Corollary 14.9], there exists a constant \tilde{C} depending on p and γ such that

$$\mathbb{E}\left[\left| \left(\{r_{t_2}^{i, n, m}, s_{t_2}^{i, n, m}\}_{i=1}^N \right) \right|_\gamma^p \right] \leq \tilde{C} \cdot C_5(p, N, T, \tilde{L}, \Gamma, C_p). \quad (81)$$

where $[\cdot]_\gamma^p$ is defined by $[x]_\gamma = \sup_{t_1, t_2 \in [0, T]} \frac{|x(t_1) - x(t_2)|}{|t_1 - t_2|^\gamma}$ and $(\{r_t^{i, n, m}, s_t^{i, n, m}\}_{i=1}^N)_{n \in \mathbb{N}, t \geq 0}$ is tight in $\mathcal{C}([0, T], \mathbb{R}^{2N})$. Hence, for each $T > 0$ there exists a subsequence $n_k \rightarrow \infty$ and a probability measure \mathbb{P}_T on $\mathcal{C}([0, T], \mathbb{R}^{2N})$. Since $\{\mathbb{P}_T^m\}_T$ is a consistent family, there exists by [38, Theorem 5.16] a probability measure \mathbb{P}^m on $(\mathbb{W}^N \times \mathbb{W}^N, \mathcal{B}(\mathbb{W}^N) \otimes \mathcal{B}(\mathbb{W}^N))$ such that $\mathbb{P}^{n_k, m}$ converges weakly to \mathbb{P}^m . Note that we can take here the same subsequence (n_k) for all m using a diagonalization argument.

Characterization of the limit measure: Denote by $(\{\mathbf{r}_t^i, \mathbf{s}_t^i\}_{i=1}^N) = \omega(t)$ the canonical process on $\mathbb{W}^N \times \mathbb{W}^N$. To characterize the measure \mathbb{P}^m we first note that $\mathbb{P}^m \circ (\mathbf{r}_0^i, \mathbf{s}_0^i)^{-1} = \eta_m$ for all $i \in \{1, \dots, N\}$, since $\mathbb{P}^{n, m}(\mathbf{r}_0^i, \mathbf{s}_0^i)^{-1} = \eta_{n, m}$ converges weakly to η_m by assump-

tion. We define maps $M^{i,m}, N^{i,m} : \mathbb{W}^N \times \mathbb{W}^N \rightarrow \mathbb{W}$ by

$$\begin{aligned} M_t^{i,m} &= \mathbf{r}_t^i - \mathbf{r}_0^i - \int_0^t \left(\tilde{b}(\mathbf{r}_u^i) + \frac{1}{N} \sum_{j=1}^N g^m(\mathbf{r}_u^j) \right) du, \quad \text{and} \\ N_t^{i,m} &= \mathbf{s}_t^i - \mathbf{s}_0^i - \int_0^t \left(\hat{b}(\mathbf{s}_u^i) + \frac{1}{N} \sum_{j=1}^N h^m(\mathbf{s}_u^j) \right) du. \end{aligned} \tag{82}$$

For each $n, m \in \mathbb{N}$ and $i = 1, \dots, N$, $(M_t^{i,m}, \mathcal{F}_t, \mathbb{P}^{n,m})$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(\{\mathbf{r}_u^j, \mathbf{s}_u^j\} : j = 1, \dots, N, 0 \leq u \leq t)$. The family $(\{M_t^{i,m}\}_{i=1}^N, \mathbb{P}^{n,m})_{n \in \mathbb{N}, t \geq 0}$ and $(\{N_t^{i,m}\}_{i=1}^N, \mathbb{P}^{n,m})_{n \in \mathbb{N}, t \geq 0}$ are uniformly integrable. Since the mappings $M^{i,m}$ and $N^{i,m}$ are continuous in \mathbb{W} , $\mathbb{P}^{n,m} \circ (\{\mathbf{r}^i, \mathbf{s}^i, M^{i,m}, N^{i,m}\}_{i=1}^N)^{-1}$ converges weakly to $\mathbb{P}^m \circ (\{\mathbf{r}^i, \mathbf{s}^i, M^{i,m}, N^{i,m}\}_{i=1}^N)^{-1}$ by the continuous mapping theorem. Then applying the same argument as in (68), $(M_t^{m,i}, \mathcal{F}_t, \mathbb{P}^m)$ and $(N_t^{m,i}, \mathcal{F}_t, \mathbb{P}^m)$ are continuous martingales for all $i = 1, \dots, N$ and the quadratic variation $([\{M^{i,m}, N^{i,m}\}_{i=1}^N]_t)_{t \geq 0}$ exists \mathbb{P}^m -almost surely. To complete the identification of the limit, it suffices to identify the quadratic variation. Similar to the computations in the proof of Lemma 19, it holds

$$\begin{aligned} [M^{i,m}] &= 4 \int_0^\cdot \mathbb{1}_{(0,\infty)}(\mathbf{r}_u^i) du && \mathbb{P}^m\text{-almost surely,} \\ [N^{i,m}] &= 4 \int_0^\cdot \mathbb{1}_{(0,\infty)}(\mathbf{s}_u^i) du && \mathbb{P}^m\text{-almost surely, and} \\ [M^{i,m}, N^{i,m}] &= 4 \int_0^\cdot \mathbb{1}_{(0,\infty)}(\mathbf{r}_u^i) \mathbb{1}_{(0,\infty)}(\mathbf{s}_u^i) du && \mathbb{P}^m\text{-almost surely,} \end{aligned} \tag{83}$$

Further, $[M^{i,m}, M^{j,m}]_t = [N^{i,m}, N^{j,m}]_t = [M^{i,m}, N^{j,m}]_t = 0$ $\mathbb{P}^{n,m}$ -almost surely for $i \neq j$ and $(M_t^{i,m} M_t^{j,m}, \mathbb{P}^{n,m})$, $(N_t^{i,m} N_t^{j,m}, \mathbb{P}^{n,m})$ and $(M_t^{i,m} N_t^{j,m}, \mathbb{P}^{n,m})$ are martingales. For any bounded, continuous non-negative function $G : \mathbb{W} \rightarrow \mathbb{R}$, it holds

$$E^m[G(M_t^{i,m} M_t^{j,m} - M_s^{i,m} M_s^{j,m})] = \lim_{n \rightarrow \infty} E^{n,m}[G(M_t^{i,m} M_t^{j,m} - M_s^{i,m} M_s^{j,m})] = 0,$$

respectively, $E^m[G(N_t^{i,m} N_t^{j,m} - N_s^{i,m} N_s^{j,m})] = 0$ and $E^m[G(M_t^{i,m} N_t^{j,m} - M_s^{i,m} N_s^{j,m})] = 0$. Then

$$[M^{i,m}, M^{j,m}] = [N^{i,m}, N^{j,m}] = [M^{i,m}, N^{j,m}] = 0 \quad \mathbb{P}^m\text{-almost surely, for all } i \neq j. \tag{84}$$

Then by a martingale representation theorem, cf. [35, Chapter II, Theorem 7.1], there is a probability space $(\Omega^m, \mathcal{A}^m, P^m)$ and a Brownian motion $\{W^i\}_{i=1}^N$ and random variables $(\{r^{i,m}, s^{i,m}\}_{i=1}^N)$ on this space, such that $P^m \circ (\{r^{i,m}, s^{i,m}\}_{i=1}^N)^{-1} = \mathbb{P}^m \circ (\{\mathbf{r}^i, \mathbf{s}^i\}_{i=1}^N)^{-1}$ and such that $(\{r^{i,m}, s^{i,m}, W^i\}_{i=1}^N)$ is a weak solution of (79).

Comparison of two solutions: To show $P^m[r_t^{i,m} \leq s_t^{i,m}$ for all $t \geq 0$ and $i = 1, \dots, N] = 1$ it suffices to note that $P^{n,m}[r_t^{i,n,m} \leq s_t^{i,n,m}$ for all $t \geq 0$ and $i = 1, \dots, N] = 1$, which holds by Lemma 21, carries over to the limit by the Portmanteau theorem, since we have weak convergence of $\mathbb{P}^{n,m} \circ (\{\mathbf{r}^i, \mathbf{s}^i\}_{i=1}^N)^{-1}$ to $\mathbb{P}^m \circ (\{\mathbf{r}^i, \mathbf{s}^i\}_{i=1}^N)^{-1}$. \square

We show in the next step that the distribution of the solution of (79) converges as $m \rightarrow \infty$. Consider a probability space $(\Omega^m, \mathcal{A}^m, P^m)$ for each $m \in \mathbb{N}$ and random variables $\{r_t^{i,m}\}_{i=1}^N, \{s_t^{i,m}\}_{i=1}^N : \Omega^m \rightarrow \mathbb{W}^N$ such that $(\{r_t^{i,m}, s_t^{i,m}\}_{i=1}^N)_{t \geq 0}$ is a solution to (79). Denote by $\mathbb{P}^m = P^m \circ (\{r_t^{i,m}, s_t^{i,m}\}_{i=1}^N)^{-1}$ the law on $\mathbb{W}^N \times \mathbb{W}^N$.

Lemma 23. *Assume that **H1** and **H2** is satisfied for (\tilde{b}, g) and (\hat{b}, h) . Let $\eta \in \Gamma(\mu, \nu)$ where the probability measures μ and ν on \mathbb{R}_+ satisfy **H3**. Further, let $(g^m)_{m \in \mathbb{N}}, (h^m)_{m \in \mathbb{N}}, (\mu_m)_{m \in \mathbb{N}}, (\nu_m)_{m \in \mathbb{N}}$ and $(\eta_m)_{m \in \mathbb{N}}$ be such that **H5** and **H7** hold. Then there exists a random variable $(\{r^i, s^i\}_{i=1}^N)$ defined on some probability space (Ω, \mathcal{A}, P) with values in $\mathbb{W}^N \times \mathbb{W}^N$ such that $(\{r_t^i, s_t^i\}_{i=1}^N)$ is a weak solution of (35). Moreover, the laws $P^m \circ (\{r_t^{i,m}, s_t^{i,m}\}_{i=1}^N)^{-1}$ converge weakly to $P \circ (\{r^i, s^i\}_{i=1}^N)^{-1}$. If in addition,*

$$\begin{aligned} \tilde{b}(r) &\leq \hat{b}(r), \quad g(r) \leq h(r), \quad \text{and} \quad g^m(r) \leq h^m(r) && \text{for any } r \in \mathbb{R}_+, \text{ and} \\ P^m[r_0^{i,m} \leq s_0^{i,m} \text{ for all } t \geq 0 \text{ and } i \in \{1, \dots, N\}] &= 1 && \text{for any } m \in \mathbb{N}, \end{aligned}$$

then $P[r_t^i \leq s_t^i \text{ for all } t \geq 0 \text{ and } i \in \{1, \dots, N\}] = 1$.

Proof. The proof is structured as the proof of Lemma 22. Tightness of the sequence of probability measures $(\mathbb{P}^m)_{m \in \mathbb{N}}$ on $(\mathbb{W}^N \times \mathbb{W}^N, \mathcal{B}(\mathbb{W}^N) \otimes \mathcal{B}(\mathbb{W}^N))$ holds adapting the steps of the proof of Lemma 22 to (79). Note that (80) and (81) hold analogously for $(\{r_t^{i,m}, s_t^{i,m}\}_{i=1}^N)$ by **H1**, **H5** and **H7**. Hence by Kolmogorov's continuity criterion, cf. [38, Corollary 14.9], we can deduce that there exists a probability measure \mathbb{P} on $(\mathbb{W}^N \times \mathbb{W}^N, \mathcal{B}(\mathbb{W}^N) \otimes \mathcal{B}(\mathbb{W}^N))$ such that there is a subsequence $(m_k)_{k \in \mathbb{N}}$ along which \mathbb{P}^{m_k} converge towards \mathbb{P} .

To characterize the limit, we first note that by Skorokhod representation theorem, cf. [9, Chapter 1, Theorem 6.7], without loss of generality we can assume that $(\{r_t^{i,m}, s_t^{i,m}\}_{i=1}^N)$ are defined on a common probability space (Ω, \mathcal{A}, P) with expectation E and converge almost surely to $(\{r^i, s^i\}_{i=1}^N)$ with distribution \mathbb{P} . Then, by **H5** and Lebesgue convergence theorem it holds almost surely for all $t \geq 0$,

$$\lim_{m \rightarrow \infty} \int_0^t \tilde{b}(r_t^{i,m}) + \frac{1}{N} \sum_{j=1}^N g^m(r_u^{j,m}) du = \int_0^t \tilde{b}(r_t^i) + \frac{1}{N} \sum_{j=1}^N g^m(r_u^j) du. \quad (85)$$

Consider the mappings $M^{i,m}, N^{i,m} : \mathbb{W}^N \times \mathbb{W}^N \times \mathcal{P}(\mathbb{W}^N \times \mathbb{W}^N) \rightarrow \mathbb{W}$ defined by (82) Then for all $m \in \mathbb{N}$ and $i = 1, \dots, N$, $(M_t^{i,m}, \mathcal{F}_t, \mathbb{P}^m)$ and $(N_t^{i,m}, \mathcal{F}_t, \mathbb{P}^m)$ are martingales with respect to the canonical filtration $\mathcal{F}_t = \sigma(\{\mathbf{r}_u^i, \mathbf{s}_u^i\}_{i=1}^N)_{0 \leq u \leq t}$. Further the family $(\{M_t^{i,m}\}_{i=1}^N, \mathbb{P}^m)_{m \in \mathbb{N}, t \geq 0}$ and $(\{N_t^{i,m}\}_{i=1}^N, \mathbb{P}^m)_{m \in \mathbb{N}, t \geq 0}$ are uniformly integrable. In the same line as weak convergence is shown in the proof of Lemma 19 and by (85),

$\mathbb{P}^m \circ (\{\mathbf{r}^i, \mathbf{s}^i, M^{i,m}, N^{i,m}\}_{i=1}^N)^{-1}$ converges weakly to $\mathbb{P} \circ (\{\mathbf{r}^i, \mathbf{s}^i, M^i, N^i\}_{i=1}^N)^{-1}$ where

$$M_t^i = \mathbf{r}_t^i - \mathbf{r}_0^i - \int_0^t \left(\tilde{b}(\mathbf{r}_u^i) + \frac{1}{N} \sum_{j=1}^N g(\mathbf{r}_u^j) \right) du, \quad \text{and}$$

$$N_t^i = \mathbf{s}_t^i - \mathbf{s}_0^i - \int_0^t \left(\hat{b}(\mathbf{s}_u^i) + \frac{1}{N} \sum_{j=1}^N h(\mathbf{s}_u^j) \right) du.$$

Then $(\{M_t^i\}_{i=1}^N, \mathcal{F}_t, \mathbb{P})$ and $(\{N_t^i\}_{i=1}^N, \mathcal{F}_t, \mathbb{P})$ are continuous martingales using the same argument as in (68). Further, the quadratic variation $([\{M_t^i, N_t^i\}_{i=1}^N]_t)_{t \geq 0}$ exists \mathbb{P} -almost surely and is given by (83) and (84) \mathbb{P} -almost surely, which holds following the computations in the proof of Lemma 19 and Lemma 22. As in Lemma 22, we conclude by a martingale representation theorem that there are a probability space (Ω, \mathcal{A}, P) and a Brownian motion $\{W^i\}_{i=1}^N$ and random variables $(\{r^i\}_{i=1}^N, \{s^i\}_{i=1}^N)$ on this space such that $P \circ (\{r^i, s^i\}_{i=1}^N)^{-1} = \mathbb{P} \circ (\{\mathbf{r}^i, \mathbf{s}^i\}_{i=1}^N)^{-1}$ and such that $(\{r^i, s^i, W^i\}_{i=1}^N)$ is a weak solution of (25).

By the Portmanteau theorem the monotonicity carries over to the limit, since $\mathbb{P}^m \circ (\{\mathbf{r}^i, \mathbf{s}^i\}_{i=1}^N)^{-1}$ converges weakly to $\mathbb{P} \circ (\{\mathbf{r}^i, \mathbf{s}^i\}_{i=1}^N)^{-1}$. \square

Proof of Theorem 10. The proof is a direct consequence of Lemma 22 and Lemma 23. \square

A Appendix

A.1 Kuramoto model

Lower bounds on the contraction rate can also be shown for nonlinear SDEs on the one-dimensional torus using the same approach. Here, we consider the Kuramoto model given by

$$dX_t = -k \left[\int_{\mathbb{T}} \sin(X_t - x) d\mu_t(x) \right] dt + dB_t \quad (86)$$

on the torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.

Theorem 24. *Let μ_t and ν_t be laws of X_t and Y_t where $(X_s)_{s \geq 0}$ and $(Y_s)_{s \geq 0}$ are two solutions of (86) with initial distributions μ_0 and ν_0 on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$, respectively. If*

$$4k \int_0^\pi \exp(2k - 2k \cos(r/2)) dr \leq 1 \quad (87)$$

holds, then for all $t \geq 0$,

$$\mathcal{W}_{\tilde{f}}(\mu_t, \nu_t) \leq e^{-c_{\mathbb{T}} t} \mathcal{W}_{\tilde{f}}(\mu_0, \nu_0) \quad \text{and} \quad \mathcal{W}_1(\mu_t, \nu_t) \leq 2 \exp(2k) e^{-c_{\mathbb{T}} t} \mathcal{W}_1(\mu_0, \nu_0),$$

where

$$c_{\mathbb{T}} = 1/(2 \int_0^\pi \int_0^r \exp[2k(\cos(r/2) - \cos(s/2))] ds dr) \quad (88)$$

and \tilde{f} is a concave, increasing function given in (92).

In [18, Appendix A], a contraction result is stated for a general drift using a similar approach.

We prove Theorem 24 via a sticky coupling approach. In the same line as in Section 2 the coupling $(X_t, Y_t)_{t \geq 0}$ is defined as the weak limit of Markovian couplings $\{(X_t^\delta, Y_t^\delta)_{t \geq 0} : \delta > 0\}$ on $\mathbb{T} \times \mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}/(2\pi\mathbb{Z})$ given by

$$\begin{aligned} dX_t^\delta &= -k \left[\int_{\mathbb{T}} \sin(X_t^\delta - x) d\mu_t^\delta(x) \right] dt + rc^\delta(\bar{r}_t^\delta) dB_t^1 + sc^\delta(\bar{r}_t^\delta) dB_t^2 \\ dY_t^\delta &= -k \left[\int_{\mathbb{T}} \sin(Y_t^\delta - x) d\nu_t^\delta(x) \right] dt - rc^\delta(\bar{r}_t^\delta) dB_t^1 + sc^\delta(\bar{r}_t^\delta) dB_t^2, \end{aligned} \quad (89)$$

where $\bar{r}_t^\delta = d_{\mathbb{T}}(X_t^\delta, Y_t^\delta)$ with $d_{\mathbb{T}}(\cdot, \cdot)$ defined by

$$d_{\mathbb{T}}(x, y) = \begin{cases} (|x - y| \bmod 2\pi) & \text{if } (|x - y| \bmod 2\pi) \leq \pi, \\ (2\pi - |x - y| \bmod 2\pi) & \text{otherwise.} \end{cases} \quad (90)$$

The functions rc^δ, sc^δ are given by (19) and satisfy that there exists $\epsilon_0 > 0$ such that $rc^\delta(r) \geq r/2$ for any $0 \leq r \leq \delta \leq \epsilon_0$.

Theorem 25. *Assume (87). Let μ_0 and ν_0 be probability measures on $(\mathbb{T}, \mathcal{B}(\mathbb{T}))$ having finite forth moment. Then, $(X_t, Y_t)_{t \geq 0}$ is the subsequential limit as $\delta \rightarrow 0$ of $\{(X_t^\delta, Y_t^\delta)_{t \geq 0} : \delta > 0\}$, where $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are solutions of (86) with initial distributions μ_0 and ν_0 , respectively. Further, there exists a process $(r_t)_{t \geq 0}$ satisfying for any $t \geq 0$, $d_{\mathbb{T}}(X_t, Y_t) \leq r_t$ almost surely, and which is a weak solution of*

$$dr_t = (2k \sin(r_t/2) + 2k\mathbb{P}(r_t))dt + 2\mathbb{1}_{(0, \pi]}(r_t)dW_t - 2d\ell_t^\pi, \quad (91)$$

where $(W_t)_{t \geq 0}$ is a one-dimensional Brownian motion on \mathbb{T} and ℓ^π is the local time at π .

Proof. The proof works analogously to the proof of Theorem 2 stated in Section 6.2.2. It holds similarly to Lemma 12 by Meyer-Tanaka's formula, cf. [51, Chapter 6, Theorem 1.1], and using (90),

$$\begin{aligned} \bar{r}_t^\delta - \bar{r}_0^\delta &= \int_0^t \text{sgn}(X_t^\delta - Y_t^\delta)(-k)e_t \left[\int_{\mathbb{T}} \sin(X_t^\delta - x) d\mu_t(x) - \int_{\mathbb{T}} \sin(Y_t^\delta - x) d\nu_t(x) \right] dt \\ &\quad + \int_0^t \text{sgn}(X_t^\delta - Y_t^\delta) 2rc^\delta(\bar{r}_t^\delta) e_t dB_t^1 + \int_{\mathbb{R}} 2rc^\delta(\bar{r}_t^\delta)^2 \ell_t^a(\delta_0 - \delta_\pi)(da), \end{aligned}$$

where $\text{sgn}(x) = \mathbb{1}_{(0,\pi]}(x) - \mathbb{1}_{(\pi,2\pi]}(x)$, $(\ell_t^a)_{t \geq 0}$ is the local time at a associated with $(X_t^\delta - Y_t^\delta)_{t \geq 0}$ and $e_t = (X_t^\delta - Y_t^\delta)/d_{\mathbb{T}}(X_t^\delta, Y_t^\delta)$ for $\bar{r}_t^\delta \neq 0$. For $\bar{r}_t^\delta = 0$, e_t is some arbitrary unit vector. For any a the support of ℓ_t^a as a function of t is a subset of the set of t such that $r_t = a$ [38, Theorem 19.1], hence $\mathbb{1}_{(0,\pi]}(r_t)\ell_t^0 = 0$ almost surely. so the term involving the local time reduces to $-2\ell_t^\pi$. Further, we note that $W_t = \int_0^t \text{sgn}(X_t^\delta - Y_t^\delta)e_t dB_t^1$ is a Brownian motion. As in Lemma 12, it holds for the process $(\bar{r}_t^\delta)_{t \geq 0}$ for $\epsilon < \epsilon_0$ with ϵ_0 given by (20),

$$d\bar{r}_t^\delta \leq (2k \sin(\bar{r}_t^\delta/2) + 2k\mathbb{E}_{x \sim \mu_t^\delta, y \sim \nu_t^\delta}(\text{rc}^\epsilon(d_{\mathbb{T}}(x, y))))dt + 2\text{rc}^\delta(\bar{r}_t^\delta)dW_t - 2d\ell_t^\pi,$$

where we used the properties of rc^δ and

$$(x - y) \cdot (\sin(x - \tilde{x}) - \sin(y - \tilde{x})) \leq 2 \sin(|x - y|/2)|x - y|$$

for any $x, y, \tilde{x} \in \mathbb{T}$. Consider $(r_t^{\delta, \epsilon})_{t \geq 0}$ given by

$$dr_t^{\delta, \epsilon} = (2k \sin(r_t^{\delta, \epsilon}/2) + 2k \int_0^\pi \text{rc}^\epsilon(u) dP_t^{\delta, \epsilon}(u))dt + 2\text{rc}^\delta(r_t^{\delta, \epsilon})dW_t - 2d\ell_t^\pi,$$

where $P_t^{\delta, \epsilon}$ is the law of $r_t^{\delta, \epsilon}$. Then as in Lemma 13, for the processes $(\bar{r}_t^\delta)_{t \geq 0}$ and $(r_t^{\delta, \epsilon})_{t \geq 0}$ with the same initial condition and driven by the same noise it holds $\bar{r}_t^\delta \leq r_t^{\delta, \epsilon}$ almost surely for every t and $\epsilon < \epsilon_0$.

Consider the process $(U_t^{\delta, \epsilon})_{t \geq 0} = (X_t^\delta, Y_t^\delta, r_t^{\delta, \epsilon})_{t \geq 0}$ on $\mathbb{T}^2 \times [0, \pi]$ for each $\epsilon, \delta > 0$. We define by $\mathbf{X}, \mathbf{Y} : \mathcal{C}(\mathbb{R}_+, \mathbb{T}^2 \times [0, \pi]) \rightarrow \mathcal{C}(\mathbb{R}_+, \mathbb{T})$ and $\mathbf{r} : \mathcal{C}(\mathbb{R}_+, \mathbb{T}^2 \times [0, \pi]) \rightarrow \mathcal{C}(\mathbb{R}_+, [0, \pi])$ the canonical projections onto the first component, onto the second component and onto the last component, respectively. Analogously to the proof of Theorem 2, the law $\mathbb{P}^{\delta, \epsilon}$ of the process $(U_t^{\delta, \epsilon})_{t \geq 0}$ converges along a subsequence $(\delta_k, \epsilon_k)_{k \in \mathbb{N}}$ to a probability measure \mathbb{P} . Let $(X_t, Y_t, r_t)_{t \geq 0}$ be some process on $\mathbb{T}^2 \times [0, \pi]$ with distribution \mathbb{P} on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$. Since $(X_t^\delta)_{t \geq 0}$ and $(Y_t^\delta)_{t \geq 0}$ are solutions of (86) which are unique in law, we have that for any $\epsilon, \delta > 0$, $\mathbb{P}^{\delta, \epsilon} \circ (\mathbf{X})^{-1} = \mathbb{P} \circ (\mathbf{X})^{-1}$ and $\mathbb{P}^{\delta, \epsilon} \circ (\mathbf{Y})^{-1} = \mathbb{P} \circ (\mathbf{Y})^{-1}$. And therefore $(X_t)_{t \geq 0}$ and $(Y_t)_{t \geq 0}$ are solutions of (86) as well with the same initial condition. Hence $\mathbb{P} \circ (\mathbf{X}, \mathbf{Y})^{-1}$ is a coupling of two copies of (86).

Further, the monotonicity $\bar{r}_t^\delta \leq r_t^{\delta, \epsilon}$ carries over to the limit by the Portmanteau theorem. Finally, similarly to the proof of Lemma 19 and Lemma 20 there exist an extended probability space and a one-dimensional Brownian motion $(W_t)_{t \geq 0}$ such that $(r_t, W_t)_{t \geq 0}$ is a solution to (97). □

Proof of Theorem 24. Similarly to (37) we consider a function \tilde{f} on $[0, \pi]$ defined by

$$\tilde{f}(t) = \int_0^t \tilde{\varphi}(r)\tilde{g}(r)dr, \tag{92}$$

where

$$\begin{aligned}\tilde{\varphi}(r) &= \exp\{2k(\cos(r/2) - 1)\}, & \tilde{\Phi}(r) &= \int_0^r \tilde{\varphi}(s)ds, \\ \tilde{g}(r) &= 1 - \frac{c_{\mathbb{T}}}{2} \int_0^r \{\tilde{\Phi}(s)/\tilde{\varphi}(s)\}ds - k \int_0^r \{1/\tilde{\varphi}(s)\}ds, \\ c_{\mathbb{T}} &= \left(2 \int_0^\pi \{\tilde{\Phi}(s)/\tilde{\varphi}(s)\}ds\right)^{-1}.\end{aligned}$$

Then for k satisfying (87), $\tilde{g}(r) \in [1/2, 1]$ and \tilde{f} is a concave function satisfying similarly to (38)

$$\exp(-2k)/2r \leq \tilde{f} \leq \tilde{\Phi}(r) \leq r \quad (93)$$

and

$$\begin{aligned}\tilde{f}''(0) &= -k \\ 2(\tilde{f}''(r) - \tilde{f}''(0)) &\leq -2k \sin(r/2)\tilde{f}'(r) - c_{\mathbb{T}}\tilde{f}(r) \quad \text{for all } r \in [0, \pi].\end{aligned} \quad (94)$$

By Ito's formula it holds

$$\begin{aligned}d\tilde{f}(r_t) &= \tilde{f}'(r_t)(2k \sin(r/2) + 2k\mathbb{P}(r_t > 0))dt + 2\tilde{f}'(r_t)\mathbb{1}_{(0, \pi]}(r_t)dW_t - 2\tilde{f}'(r_t)d\ell_t^\pi \\ &\quad + 2\tilde{f}''(r_t)\mathbb{1}_{(0, \pi]}(r_t)dt.\end{aligned}$$

Taking expectation and using that the term involving the local time is negative, we obtain

$$\begin{aligned}\frac{d}{dt}\mathbb{E}[\tilde{f}(r_t)] &\leq \mathbb{E}[2(\tilde{f}''(r_t) - \tilde{f}''(0)) + \tilde{f}'(r_t)2k \sin(r_t/2)] + (2\tilde{f}''(0) + 2k)\mathbb{P}(r_t > 0) \\ &\leq -c_{\mathbb{T}}\mathbb{E}[\tilde{f}(r_t)],\end{aligned}$$

where the last step holds by (94). Then

$$\mathbb{E}[\tilde{f}(d_{\mathbb{T}}(\bar{X}_t, \bar{Y}_t))] \leq \mathbb{E}[\tilde{f}(r_t)] \leq e^{-c_{\mathbb{T}}t}\mathbb{E}[\tilde{f}(r_0)] = e^{-c_{\mathbb{T}}t}\mathbb{E}[\tilde{f}(d_{\mathbb{T}}(\bar{X}_0, \bar{Y}_0))], \quad (95)$$

provided (87) holds. Thus

$$\mathcal{W}_{\tilde{f}}(\mu_t, \nu_t) \leq e^{-c_{\mathbb{T}}t}\mathcal{W}_{\tilde{f}}(\mu_0, \nu_0),$$

and by (93)

$$\mathcal{W}_1(\mu_t, \nu_t) \leq 2 \exp(2k)e^{-c_{\mathbb{T}}t}\mathcal{W}_1(\mu_0, \nu_0).$$

□

Remark 26. Let us finally remark that we can relax the condition (87) and we can obtain contraction with a modified contraction rate $c_{\mathbb{T}}$ for all $k < k_0$, where k_0 is given by

$$k_0 \int_0^\pi \exp(2k_0 - 2k_0 \cos(r/2)) dr = 1. \quad (96)$$

More precisely, set $\zeta = 1 - k \int_0^\pi \exp(2k - 2k \cos(r/2)) dr$ and $c_{\mathbb{T}} = \zeta \left(\int_0^\pi \{\tilde{\Phi}(s)/\tilde{\varphi}(s)\} ds \right)^{-1}$. Then, $\tilde{g}(r) \in [\zeta/2, 1]$ and $\zeta \exp(-2k)/2r \leq \tilde{f}(r) \leq r$. Following the previous computations, we obtain

$$\mathcal{W}_1(\mu_t, \nu_t) \leq 2 \exp(2k)/\zeta e^{-c_{\mathbb{T}} t} \mathcal{W}_1(\mu_0, \nu_0),$$

where for k close to k_0 , the contraction rate becomes small and the prefactor $2 \exp(2k)/\zeta$ explodes.

A.2 Sticky nonlinear SDEs on bounded state space

In the same line as in Theorem 3, existence, uniqueness in law and comparison results hold for solutions to the sticky SDE on $[0, \pi]$ given by

$$dr_t = (\tilde{b}(r_t) + 2k\mathbb{P}(r_t > 0))dt + 2\mathbb{1}_{(0, \pi)}(r_t)dW_t - 2d\ell_t^\pi, \quad (97)$$

where $k \in \mathbb{R}_+$ and ℓ^π is the local time at π .

The analysis of invariant measures and phase transitions can be easily adapted to the case of the sticky SDE on $[0, \pi]$ given by (97).

Theorem 27. Let $(r_t)_{t \geq 0}$ be a solution of (97) with drift \tilde{b} satisfying **H1**. Then, the Dirac measure at zero, δ_0 , is an invariant probability measure on $[0, \pi]$ for (97). If there exists $p \in (0, 1)$ solving $(1/k) = (1-p)I(k, p)$ where

$$I(k, p) = \int_0^\pi \exp\left(kpx + \frac{1}{2} \int_0^x \tilde{b}(r) dr\right) dx,$$

then the probability measure π on $[0, \pi]$ given by

$$\pi(dx) \propto \frac{1}{kp} \delta_0(dx) + \exp\left(kpx + \frac{1}{2} \int_0^x \tilde{b}(r) dr\right) \lambda_{(0, \pi)}(dx) \quad (98)$$

is another invariant probability measure for (97).

Proof of Theorem 27. The proof works analogously to the proof of Theorem 5 for sticky SDEs on \mathbb{R}_+ . Note that here the condition (75) transforms for $p \in (0, 1]$ to

$$p = \pi((0, \pi)) = \frac{I(k, p)}{1/(kp) + I(k, p)} \Leftrightarrow (1-p)I(k, p) = 1/k.$$

□

Example 28. Consider a solution $(r_t)_{t \geq 0}$ of (97) with drift $\tilde{b}(r) = 2k \sin(r/2)$. Consider a solution $p \in (0, 1]$ solving $1/k = (1-p)I(k, p)$ with

$$I(k, p) = \int_0^\pi \exp\left(kpx + \int_0^x k \sin(r/2) dr\right) dx = \int_0^\pi \exp\left(kpx + 2k - 2k \cos(x/2)\right) dx .$$

Then by Theorem 27, the Dirac measure at zero, δ_0 and the probability measure

$$\pi(dx) \propto \frac{1}{kp} \delta_0(dx) + \exp(kpx + 2k - 2k \cos(x/2)) \lambda_{(0, \pi)}(dx) \quad (99)$$

are invariant probability measures for (97). We specify a necessary and sufficient condition for the existence of a solution p satisfying $1/k = (1-p)I(k, p)$. We define $\hat{I}(k, p) = (1-p)I(k, p)$. We first consider the case $1/k < \hat{I}(k, 0) = \int_0^\pi \exp(2k - 2k \cos(x/2)) dx$. Then since $1/k > \hat{I}(k, 1) = 0$ and by the mean value theorem there exists a p solving $1/k = \hat{I}(k, p)$ and therefore there exist multiple invariant distributions for (99). On the other hand, if $1/k > \hat{I}(k, 0) = \int_0^\pi \exp(2k - 2k \cos(x/2)) dx$, since $\pi \leq \int_0^\pi \exp(2k - 2k \cos(x/2)) dx$ and for $k < 1/\pi$, it holds

$$\begin{aligned} \frac{d}{dp} \hat{I}(k, p) &= -I(k, p) + (1-p) \int_0^\pi kx \exp(kpx + 2k - 2k \cos(x/2)) dx \\ &= \int_0^\pi ((1-p)kx - 1) \exp(kpx + 2k - 2k \cos(x/2)) dx \leq 0 , \end{aligned}$$

there is no p satisfying (99).

Remark 29. The contraction result given in Theorem 7 carries over to the sticky diffusion (r_t) given by (97) on $[0, \pi]$ with $\tilde{b}(r) = 2k \sin(r/2)$. If (87) holds, then for $t \geq 0$, (31) holds with \tilde{f} defined in (92) and $c_{\mathbb{T}}$ defined in (88) using (95). Moreover by Remark 26, we can deduce that if (96) holds, the Dirac measure at zero, δ_0 , is the unique invariant measure and contraction towards δ_0 holds.

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