

Counting functions for Dirichlet series and compactness of composition operators

Frédéric Bayart

▶ **To cite this version:**

Frédéric Bayart. Counting functions for Dirichlet series and compactness of composition operators. 2021. hal-03471965

HAL Id: hal-03471965
https://hal.science/hal-03471965

Preprint submitted on 9 Dec 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COUNTING FUNCTIONS FOR DIRICHLET SERIES AND COMPACTNESS OF COMPOSITION OPERATORS

FRÉDÉRIC BAYART

Abstract. We give a sufficient condition for a composition operator with positive characteristic to be compact on the Hardy space of Dirichlet series.

1. Introduction

1.1. **Description of the results.** Let $\mathcal{H}^2$ be the Hilbert space of Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ with square summable coefficients endowed with $\|f\|^2 = \sum_{n\geq 1} |a_n|^2$. By the Cauchy-Schwarz inequality, Dirichlet series in $\mathcal{H}^2$ generate holomorphic functions in $\mathbb{C}_{1/2}$, where $\mathbb{C}_\theta = \{s \in \mathbb{C} : \ \Re e(s) > \theta\}$. Let $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ be analytic. The composition operator with symbol φ is defined on $\mathcal{H}^2$ by $C_\varphi(f) = f\circ\varphi$. In [9], Gordon and Hedenmalm determined which symbols φ generate a bounded composition operator on $\mathcal{H}^2$: this happens if and only if φ belongs to the Gordon-Hedenmalm class $\mathcal{G}$ of the analytic functions $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ which may be written $\varphi(s) = c_0 + \psi(s)$ where c_0 is a non-negative integer, ψ is a Dirichlet series that converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and satisfies the following properties:

(a) if $c_0 = 0$, then $\psi_0(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$;
(b) if $c_0 \geq 1$, then either $\psi(\mathbb{C}_0) \subset \mathbb{C}_0$ or $\psi_0 \equiv 0$.

The non-negative integer c_0 is called the characteristic of φ and we will use the notation $\mathcal{G}_0$ and $\mathcal{G}_{\geq 1}$, respectively, for the subclasses (a) and (b).

Once you know your operator is continuous the next step is to study whether it is compact. In our context we try to characterize compactness of C_φ from properties of its symbol φ. This has been investigated in many papers (like [1], [2], [3], [6], [7], [8], [11]). Following the seminal paper of Shapiro [12] for composition operators on $H^2(\mathbb{D})$, a natural way for doing so is to characterize compactness of C_φ by mean of some counting function related to φ. This was recently achieved in [7] for the subclass $\mathcal{G}_0$ of composition operators with zero-characteristic.

In the present paper, we mostly concentrate on the subclass $\mathcal{G}_{\geq 1}$. In [2], the Nevanlinna counting function of $\varphi = c_0 s + \psi \in \mathcal{G}_{\geq 1}$ was defined on $\mathbb{C}_0$ by

$$\mathcal{N}_\varphi(w) = \sum_{\varphi(s)=w} \Re e(s), \ w \in \mathbb{C}_0.$$

In that paper, it was shown that provided $|\Im m(\psi)|$ is bounded, the condition $\mathcal{N}_\varphi(w) = o(\Re e(w))$ as $\Re e(w)$ tends to 0 implies the compactness of C_φ. Conversely, in [1], Bailleul established that if ψ is supported on a finite set of prime numbers

Date: December 9, 2021.
2010 *Mathematics Subject Classification.* Primary 47B33, Secondary 30B50, 46E15.
Key words and phrases. Composition operator, Dirichlet series, compactness.

and is finitely valent, the compactness of C_φ implies the above condition on $\mathcal{N}_\varphi$. Moreover these two properties are equivalent if ψ is supported on a single prime number (see [6]). We recall that if $\mathcal{Q}$ is a subset of the set of prime numbers, a Dirichlet series $\sum_n a_n n^{-s}$ is supported on $\mathcal{Q}$ provided $a_n = 0$ as soon as there exists a prime number not in $\mathcal{Q}$ such that $p|n$.

Our aim here is to show that the result of [2] is still true without any assumption on $|\Im m(\psi)|$.

Theorem 1.1. *Let $\varphi \in \mathcal{G}_{\geq 1}$ and let us assume that $\mathcal{N}_\varphi(w) = o(\Re e(w))$ as $\Re e(w)$ tends to 0. Then C_φ is compact on $\mathcal{H}^2$.*

1.2. **Background material.** We briefly review some basic facts on Dirichlet series. Let $\mathbb{T}^\infty$ be the infinite polycircle endowed with its Haar measure m. It can be identified to the group of characters of $(\mathbb{Q}_+, +)$ via the prime number factorization: to any $z \in \mathbb{T}^\infty$ we associate the character χ defined by

$$\chi(n) = z_1^{\alpha_1} \cdots z_d^{\alpha_d} \text{ for } n = \prod_{j=1}^{d} p_j^{\alpha_j}.$$

For $f(s) = \sum_n a_n n^{-s}$ a Dirichlet series and χ a character, we denote by f_χ the Dirichlet series $f_\chi(s) = \sum_n a_n \chi(n) n^{-s}$. If f converges uniformly in $\overline{\mathbb{C}_\theta}$ for some $\theta \in \mathbb{R}$, then for any $\chi \in \mathbb{T}^\infty$, there exists a sequence of real numbers (τ_n) such that f_χ is the uniform limit in $\overline{\mathbb{C}_\theta}$ of the vertical translates $(f(\cdot + i\tau_n))$. Conversely all uniform limits in $\overline{\mathbb{C}_\theta}$ of vertical translates are equal to some f_χ, which justifies that the functions f_χ are called vertical limit functions.

If we now assume that f belongs to $\mathcal{H}^2$, then for almost every $\chi \in \mathbb{T}^\infty$, f_χ converges in $\mathbb{C}_0$ and one can compute the norm of f via the following Littlewood-Paley formula:

$$\mu(\mathbb{R})\|f\|^2 = \mu(\mathbb{R})|f(+\infty)|^2 + 4\int_{\mathbb{T}^\infty}\int_0^{+\infty}\int_{\mathbb{R}} \sigma |f'_\chi(\sigma + it)|^2 d\mu(t) d\sigma dm(\chi),$$

where μ is any finite positive measure on $\mathbb{R}$.

Regarding composition operators, the notion of vertical limits is extended to symbols $\varphi \in \mathcal{G}_{\geq 1}$ by defining $\varphi_\chi = c_0 s + \psi_\chi$. The composition operators C_φ and C_{φ_χ} are related by the formula

$$(f \circ \varphi)_\chi = f_{\chi^{c_0}} \circ \varphi_\chi.$$

2. On counting functions for Dirichlet series

Let φ belong to $\mathcal{G}_{\geq 1}$. The classical Nevanlinna counting function associated to φ is defined on $\mathbb{C}_0$ by

$$\mathcal{N}_\varphi(w) = \sum_{\varphi(s)=w} \Re e(s),\ w \in \mathbb{C}_0.$$

It was introduced in [2] where the two following important properties were proved:

(NC1) $\mathcal{N}_\varphi(w) \leq \frac{1}{c_0}\Re e(w)$ for all $w \in \mathbb{C}_0$.

(NC2) If $\mathcal{N}_\varphi(w) = o(\Re e(w))$ as $\Re e(w) \to 0$, then for all $\varepsilon > 0$, there exists $\theta > 0$ such that, for all $w \in \mathbb{C}_0$ with $\Re e(w) < \theta$, for all $\chi \in \mathbb{T}^\infty$, $\mathcal{N}_{\varphi_\chi}(w) \leq \varepsilon \Re e(w)$ (namely, the o bound is uniform with respect to $\chi \in \mathbb{T}^\infty$).

In this paper, we will be also interested in a restricted version of the counting function, which is defined by

$$N_\varphi(w) = \sum_{\substack{\varphi(s)=w \\ |\Im m(s)|<1}} \Re e(s),\ w \in \mathbb{C}_0.$$

A similar restricted counting function has been introduced in [6], when φ is supported on a single prime number. The main interest of working with N_φ instead of $\mathcal{N}_\varphi$ comes from the following enhancement of (NC1), which is inspired by [6, Lemma 6.3].

Proposition 2.1. *Let $\varphi = c_0 s + \psi \in \mathcal{G}_{\geq 1}$. There exists $C > 0$ such that, for all $\chi \in \mathbb{T}^\infty$, for all $w \in \mathbb{C}_0$ with $\Re e(w) < c_0$,*

$$N_{\varphi_\chi}(w) \leq C \frac{\Re e(w)}{1 + (\Im m(w))^2}.$$

Proof. Let Θ be the conformal map from $\mathbb{D}$ onto the half-strip

$$S = \{s = \sigma + it :\ \sigma > 0,\ |t| < 2\}$$

normalized by $\Theta(0) = 2$ and $\Theta'(0) > 0$. By standard regularity results on conformal maps there exists $C_1 > 0$ such that, for all s with $0 < \Re e(s) < 1$ and $|\Im m(s)| < 1$,

$$\Re e(s) \leq C_1 \log \frac{1}{|\Theta^{-1}(s)|}. \tag{2.1}$$

Fix $w \in \mathbb{C}_0$ and $\chi \in \mathbb{T}^\infty$ with $0 < \Re e(w) < c_0$. Define G on $\mathbb{D}$ by

$$G(z) = G_{w,\chi}(z) := \frac{\varphi_\chi(\Theta(z)) - w}{\varphi_\chi(\Theta(z)) + \bar{w}} \tag{2.2}$$

which is a self-map of $\mathbb{D}$. The Littlewood inequality for the standard Nevanlinna counting function of G says that

$$\sum_{z \in G^{-1}(\{0\})} \log \frac{1}{|z|} \leq \log \frac{1}{|G(0)|}. \tag{2.3}$$

Now $G(z) = 0$ if and only if $\varphi_\chi(\Theta(z)) = w$ so that (2.3) becomes

$$\sum_{\Theta(z) \in \varphi_\chi^{-1}(\{w\})} \log \frac{1}{|z|} \leq \log \left| \frac{\varphi_\chi(\Theta(0) + \bar{w}}{\varphi_\chi(\Theta(0)) - w} \right|$$

which itself can be rewritten

$$\sum_{s \in \varphi_\chi^{-1}(\{w\})} \log \frac{1}{|\Theta^{-1}(s)|} \leq \log \left| \frac{\varphi_\chi(2) + \bar{w}}{\varphi_\chi(2) - w} \right|.$$

Observe now that, when $\varphi_\chi(s) = w$, then $0 < \Re e(s) < 1$ so that, using (2.1),

$$\begin{aligned} N_{\varphi_\chi}(w) &= \sum_{\substack{\varphi_\chi(s)=w \\ |\Im m(s)|<1,\ \Re e(s)<1}} \Re e(s) \\ &\leq C_1 \sum_{s \in \varphi_\chi^{-1}(\{w\})} \log \frac{1}{|\Theta^{-1}(s)|} \end{aligned}$$

$$\leq C_1 \log \left| \frac{\varphi_\chi(2)+\bar{w}}{\varphi_\chi(2)-w} \right|.$$

Now we apply [7, Lemma 2.3] to get

$$N_{\varphi_\chi}(w) \leq 2C_1 \frac{\Re e(\varphi_\chi(2))\Re e(w)}{|w-\varphi_\chi(2)|^2}$$
$$\leq 2C_1 \frac{\Re e(\varphi_\chi(2))\Re e(w)}{(\Re e(\varphi_\chi(2))-\Re e(w))^2+(\Im m(\varphi_\chi(2))-\Im m(w))^2}.$$

Our restriction on $\Re e(w)$ shows that $\Re e(\varphi_\chi(2))-\Re e(w) \geq c_0$. On the other hand, since ψ is a Dirichlet series uniformly convergent in each $\mathbb{C}_\varepsilon$, $\varepsilon>0$, $\psi(2+i\mathbb{R})$ is bounded. ψ_χ being a vertical limit function of ψ, there exists $C_2>0$ such that $|\varphi_\chi(2)| \leq C_2$ for all $\chi \in \mathbb{T}^\infty$. If $|\Im m(w)| \leq 2C_2$, then we write

$$N_{\varphi_\chi}(w) \leq \frac{2C_1C_2\Re e(w)}{c_0^2} \leq \frac{2C_1C_2}{c_0^2}(1+4C_2^2)\frac{\Re e(w)}{1+(\Im m(w))^2}$$

whereas, if $|\Im m(w)|>2C_2$, then $\Im m(w)-\Im m(\varphi_\chi(2)) \geq \frac{1}{2}\Im m(w)$ which yields

$$N_{\varphi_\chi}(w) \leq 2C_1C_2 \frac{\Re e(w)}{c_0+\frac{1}{4}(\Im m(w))^2}.$$

Therefore, in all cases, Proposition 2.1 is proved. □

Proposition 2.1 will be used to give a uniform bound on N_{φ_χ}, in the spirit of (NC2), upon the assumption $\mathcal{N}_\varphi(w)=o(\Re e(w))$.

Corollary 2.2. *Let $\varphi \in \mathcal{G}_{\geq 1}$ and $\delta \in (0,1)$. Let us assume that $\mathcal{N}_\varphi(w)=o(\Re e(w))$ as $\Re e(w)$ goes to 0. Then for all $\varepsilon>0$ there exists $\theta>0$ such that, for all $\chi \in \mathbb{T}^\infty$, for all $w \in \mathbb{C}_0$ with $\Re e(w)<\theta$,*

$$N_{\varphi_\chi}(w) \leq \varepsilon \frac{\Re e(w)}{\left(1+(\Im m(w))^2\right)^{(1+\delta)/2}}. \tag{2.4}$$

Proof. Assume that the result is false. Then we can find $\varepsilon>0$, a sequence (w_k) in $\mathbb{C}_0$ with $\Re e(w_k) \to 0$ and a sequence (χ_k) in $\mathbb{T}^\infty$ such that, for all k,

$$\mathcal{N}_{\varphi_{\chi_k}}(w_k) \geq N_{\varphi_{\chi_k}}(w_k) > \varepsilon \frac{\Re e(w_k)}{\left(1+(\Im m(w_k))^2\right)^{(1+\delta)/2}}.$$

If $\Im m(w_k)$ is unbounded, this contradicts Proposition 2.1 whereas if $\Im m(w_k)$ is bounded, this contradicts that $\mathcal{N}_\varphi(w_k)=o(\Re e(w_k))$ and in particular (NC2). □

3. Compact composition operators

3.1. **Compactness on Hardy spaces.**

Proof of Theorem 1.1. Let (f_n) be a sequence in $\mathcal{H}^2$ that converges weakly to zero. We will let $f_{n,\chi}$ the vertical limit function of f_n with respect to the character χ. By the Littlewood-Paley formula applied with $d\mu(t)=\frac{1}{2}\mathbf{1}_{[-1,1]}dt$, setting $s=\sigma+it$,

$$\|C_\varphi(f_n)\|^2 = |f_n(+\infty)|^2 + 2\int_{\mathbb{T}^\infty}\int_{\mathbb{R}_+}\int_{-1}^{1} |f'_{n,\chi^{c_0}}(\varphi_\chi(s))|^2 |\varphi'_\chi(s)|^2 \sigma dt d\sigma dm(\chi).$$

Our assumption on (f_n) implies that $(f_n(+\infty))$ tends to zero. In the inner-most integrals we do the non-univalent change of variables $w = u + iv = \varphi_\chi(\sigma + it)$. Observe that this change of variables involves the restricted Nevanlinna counting function N_φ whereas in [2] we used $\mathcal{N}_\varphi$ (but we only obtained an inequality). Hence

$$\|C_\varphi(f_n)\|^2 = |f_n(+\infty)|^2 + 2\int_{\mathbb{T}^\infty}\int_{\mathbb{R}_+}\int_{\mathbb{R}} |f'_{n,\chi^{c_0}}(w)|^2 N_{\varphi_\chi}(w)dvdudm(\chi).$$

Now let $\varepsilon > 0$ and let $\theta > 0$ be given by Corollary 2.2 for $\delta = 1/2$. We split the integral over $\mathbb{R}_+$ into $\int_0^\theta + \int_\theta^{+\infty}$. For the first integral, say I_0, we use (2.4) to get

$$\begin{aligned} I_0 &\le \varepsilon \int_{\mathbb{T}^\infty}\int_{\mathbb{R}_+}\int_{\mathbb{R}} |f'_{n,\chi^{c_0}}(w)|^2 u \frac{dv}{(1+v^2)^{3/4}} dudm(\chi) \\ &\le C_1\varepsilon\|f_n\|^2 \end{aligned}$$

where we have used the Littlewood-Paley equality with the finite measure $d\mu(v) = \frac{dv}{(1+v^2)^{3/4}}$. To hande the integral over $[\theta, +\infty[$, say I_2, we now use Proposition 2.1 to write

$$\begin{aligned} I_1 &\le C\int_{\mathbb{T}^\infty}\int_\theta^{+\infty}\int_{\mathbb{R}} |f'_{n,\chi^{c_0}}(w)|^2 \frac{u}{1+v^2} dvdudm(\chi) \\ &\le C\int_\theta^{+\infty}\int_{\mathbb{R}} \sum_n |a_{n,k}|^2 (\log k)^2 k^{-2u} \frac{u}{1+v^2} dvdudm(\chi) \\ &\le C_2 \sum_{k\ge 1} |a_{n,k}|^2 (\log^2 k) \int_\theta^{+\infty} k^{-2u} du \end{aligned}$$

where we have written $f_n(s) = \sum_{k\ge1} a_{n,k}k^{-s}$. Now the argument of [2] shows that $\limsup_{n\to+\infty} \sum_{k\ge1} |a_{n,k}|^2(\log^2 k)\int_\theta^{+\infty} k^{-2u}du = 0$, which finishes the proof of Theorem 1.1. □

Remark 3.1. Using the Littlewood-Paley formula for $\mathcal{H}^p$ and the argument of [4], a variant of the above proof shows that the condition $\mathcal{N}_\varphi(w) = o(\Re e(w))$ as $\Re e(w)$ tends to zero is also sufficient to prove the compactness of C_φ on $\mathcal{H}^p$, $p \ge 1$.

Remark 3.2. Proposition 2.1 provides also an alternative approach to the continuity of C_φ when $\varphi \in \mathcal{G}_{\ge1}$. Nevertheless, as it is written, we loose the fact that C_φ is a contraction.

3.2. **Compactness on Bergman spaces.** In this section, we turn to the Bergman spaces of Dirichlet series introduced by McCarthy in [10]. For $\alpha > -1$ define

$$\mathcal{A}_\alpha = \left\{ f(s) = \sum_{n=1}^{+\infty} a_n n^{-s} : \ \|f\|_\alpha^2 = \sum_{n=1}^{+\infty} \frac{|a_n|^2}{(1+\log n)^{1+\alpha}} < +\infty \right\}$$

(as in the case of the disc, the Hardy space $\mathcal{H}^2$ corresponds to the limiting case $\alpha = -1$). The norm of an element f of $\mathcal{A}_\alpha$ can be evaluated thanks to the following Littlewood-Paley formula:

$$\|f\|_\alpha^2 \asymp |f(+\infty)|^2 + \int_{\mathbb{T}^\infty}\int_0^{+\infty}\int_{\mathbb{R}} |f'_\chi(\sigma+it)|^2 \sigma^{2+\alpha} d\mu(t)d\sigma dm(\chi)$$

where μ is any finite positive measure on $\mathbb{R}$. Moreover, if f is a Dirichlet series which converges uniformly in each half-plane $\mathbb{C}_\varepsilon$, $\varepsilon > 0$, an easy adaptation of [7, Lemma 2.2] shows that

$$\|f\|_\alpha^2 \asymp |f(+\infty)|^2 + \lim_{\sigma_0 \to 0^+} \lim_{T \to +\infty} \frac{1}{T} \int_{\sigma_0}^{+\infty} \int_{-T}^{T} |f'(\sigma + it)|^2 \sigma^{2+\alpha} dt d\sigma. \tag{3.1}$$

Compactness of composition operators on $\mathcal{A}_\alpha$ has been studied in [2] and in [1]. The interest of working in Bergman spaces is that one often can replace a sufficient condition on some counting function by a sufficient condition on the symbol itself. This is what happens again here.

Theorem 3.3. *Let $\varphi \in \mathcal{G}$ and $\alpha > -1$.*

a) If $\varphi \in \mathcal{G}_{\geq 1}$ and $\frac{\Re e(\varphi(s))}{\Re e(s)} \to +\infty$ as $\Re e(s) \to 0$, then C_φ is compact on $\mathcal{A}_\alpha$.

b) If $\varphi \in \mathcal{G}_0$ and $\frac{\Re e(\varphi(s))-1/2}{\Re e(s)} \to +\infty$ as $\Re e(s) \to 0$, then C_φ is compact on $\mathcal{A}_\alpha$.

Proof. Let us start with a). Let $N_{\alpha,\varphi} = \sum_{\substack{\varphi(s)=w \\ |\Im m(s)|<1}} \big(\Re e(s)\big)^{2+\alpha}$ be the appropriated Nevanlinna counting function for $\mathcal{A}_\alpha$. We first observe that, for all $\chi \in \mathbb{T}^\infty$ and for all $w \in \mathbb{C}_0$,

$$\begin{aligned} N_{\alpha,\varphi_\chi}(w) &\leq \left(\sum_{\substack{\varphi_\chi(s)=w \\ |\Im m(s)|<1}} \Re e(s) \right)^{2+\alpha} \\ &\leq C \frac{(\Re e(w))^{2+\alpha}}{1 + (\Im m(w))^{2+\alpha}} \end{aligned}$$

for some $C > 0$. Moreover, let $\varepsilon > 0$ and $\theta > 0$ be such that $\Re e(s) < \theta \implies \Re e(s) < \varepsilon \Re e(\varphi(s))$. For all $\chi \in \mathbb{T}^\infty$, one gets by vertical limit $\Re e(s) \leq \varepsilon \Re e(\varphi_\chi(s))$. Let $w \in \mathbb{C}_0$ with $\Re e(w) < \theta$ and let $\chi \in \mathbb{T}^\infty$. If $\varphi_\chi^{-1}(\{w\}) = \varnothing$ then $N_{\alpha,\varphi_\chi}(w) = 0$. Otherwise, any $s \in \varphi_\chi^{-1}(\{w\})$ satisfy $\Re e(s) < \theta$ so that

$$\begin{aligned} N_{\alpha,\varphi_\chi}(w) &= \sum_{\substack{\varphi_\chi(s)=1 \\ |\Im m(s)|<1}} \big(\Re e(s)\big)^{2+\alpha} \\ &\leq \varepsilon^{\alpha+1} \big(\Re e(w)\big)^{\alpha+1} \sum_{\substack{\varphi_\chi(s)=1 \\ |\Im m(s)|<1}} \Re e(s) \\ &\leq C\varepsilon^{\alpha+1} \frac{\big(\Re e(w)\big)^{2+\alpha}}{1 + (\Im m(w))^2} \end{aligned}$$

where we have used Proposition 2.1. We then conclude exactly as in the proof of Theorem 1.1. Details are left to the reader.

Let us turn to b). We are inspired by [7] but working in a Bergman space and using this very strong assumption we will avoid most of the technical difficulties which appear here. First we observe that for any $f \in \mathcal{A}_\alpha$, $\sigma_u(f \circ \varphi) \leq 0$, which

implies that we can estimate the norm of $f \circ \varphi$ using (3.1). We then consider, for $\sigma_0, T > 0$ and $w \in \mathbb{C}_0$, the counting function

$$\mathcal{M}_{\alpha,\varphi}(\sigma_0, T; w) = \frac{1}{T} \sum_{\substack{\varphi(s)=w \\ |\Im m(s)|<T,\ \Re e(s)>\sigma_0}} \big(\Re e(s)\big)^{2+\alpha}.$$

The nonunivalent change of variables $w = \varphi_\chi(s)$ leads to

$$\|C_\varphi f\|_\alpha^2 \asymp |f(\varphi(+\infty))|^2 + \lim_{\sigma_0 \to 0^+} \lim_{T \to +\infty} \int_{\mathbb{C}_{1/2}} |f'(w)|^2 \mathcal{M}_{\alpha,\varphi}(\sigma_0, T; w) d\sigma dt.$$

Let $\varepsilon > 0$ and $0 < \theta < \Re e(\varphi(+\infty))/2$ such that $\Re e(\varphi(s)) < \frac{1}{2} + \theta$ implies $\Re e(s) < \varepsilon \left(\Re e(\varphi(s)) - \frac{1}{2}\right)$. Then

$$\begin{aligned}
\mathcal{M}_{\alpha,\varphi}(\sigma_0, T; w) &\le \frac{1}{T} \sum_{\substack{\varphi(s)=w \\ |\Im m(s)|<T}} \big(\Re e(s)\big)^{2+\alpha} \\
&\le \varepsilon^{1+\alpha} \left(\Re e(w) - \frac{1}{2}\right)^{1+\alpha} \frac{1}{T} \sum_{\substack{\varphi(s)=w \\ |\Im m(s)|<T}} \Re e(s) \\
&\le C \frac{\varepsilon^{1+\alpha} \left(\Re e(w) - \frac{1}{2}\right)^{2+\alpha}}{|w - \varphi(+\infty)|^2}
\end{aligned}$$

by [7, Lemma 2.3 and 2.4] where the constant C is uniform in T for all $T \ge \sigma_1$, for some $\sigma_1 > 0$. The compactness of C_φ now follows from an argument similar to that of [7, Theorem 1.4], using that the estimate on $\mathcal{M}_{\alpha,\varphi}(\sigma_0, T; w)$ for $\Re e(w) < \theta$ is uniform with respect to σ_0 and T. □

Corollary 3.4. *Let $\varphi \in \mathcal{G}$ and $\alpha > -1$.*

(1) If $\varphi \in \mathcal{G}_{\ge 1}$ and φ is supported on a finite set of prime numbers, then

$$C_\varphi \text{ is compact on } \mathcal{A}_\alpha \iff \frac{\Re e(\varphi(s))}{\Re e(s)} \xrightarrow{\Re e(s)\to 0} +\infty.$$

(2) If $\varphi \in \mathcal{G}_0$ and φ is supported on a single prime number, then

$$C_\varphi \text{ is compact on } \mathcal{A}_\alpha \iff \frac{\Re e(\varphi(s)) - 1/2}{\Re e(s)} \xrightarrow{\Re e(s)\to 0} +\infty.$$

Remark 3.5. For the case of positive characteristic, we do not know whether the condition $\Re e(\varphi(s))/\Re e(s) \to +\infty$ is always necessary for compactness.

3.3. **Concluding remarks.**

Question 3.6. On $\mathcal{H}^2$, can we get a necessary condition using some counting function without any extra assumption on $\varphi \in \mathcal{G}_{\ge 1}$? Or at least, if $\varphi \in \mathcal{G}_{\ge 1}$ is supported on a finite set of prime numbers, do we have

$$C_\varphi \text{ is compact on } \mathcal{H}^2 \iff \mathcal{N}_\varphi(w) = o\big(\Re e(w)\big) \text{ as } \Re e(w) \to 0?$$

Remark 3.7. The condition $\frac{\Re e(\varphi(s))-1/2}{\Re e(s)} \xrightarrow{\Re e(s)\to 0} +\infty$ is not necessary for compactness on $\mathcal{A}_\alpha$ when the symbol is not supported on a single prime number. For instance, $\varphi(s) = \frac{5}{2} - 2^{-s} - 3^{-s}$ generates a compact operator on $\mathcal{A}_\alpha$ (the proof of [3, Theorem 2], done in $\mathcal{H}^2$, can be adapted to $\mathcal{A}_\alpha$) which does not satisfy the above condition.

Remark 3.8. We can also use the restricted Nevanlinna counting function introduced here to simplify the results of [5], deleting an unnecessary assumption of boundedness. For instance, we can replace Theorem 5.5 of [5] by: let φ_0 and $\varphi_1 \in \mathcal{G}_{\geq 1}$ and write them $\varphi_0 = c_0 s + \psi_0$, $\varphi_1 = c_0 s + \psi_1$. Assume moreover that there exists $C > 0$ such that

- $|\varphi_0 - \varphi_1| \leq C \min(\operatorname{Re}\varphi_0, \operatorname{Re}\varphi_1)$;
- $|\varphi_0' - \varphi_1'| \leq C$.

Then C_{φ_0} and C_{φ_1} belong to the same component of $\mathcal{C}(\mathcal{H})$, the set of composition operators on $\mathcal{H}$. If moreover we assume that

$$|\varphi_0 - \varphi_1| = o\big(\min(\operatorname{Re}\varphi_0, \operatorname{Re}\varphi_1)\big) \text{ and } \varphi_0' - \varphi_1' \to 0 \text{ as } \min(\operatorname{Re}\varphi_0, \operatorname{Re}\varphi_1) \to 0$$

then $C_{\varphi_0} - C_{\varphi_1}$ is compact.

References

[1] M. Bailleul, *Composition operators on weighted Bergman spaces of Dirichlet series*, J. Math. Anal. Appl. **426** (2015), 340–363.

[2] F. Bayart, *Compact composition operators on a Hilbert space of Dirichlet series*, Illinois J. Math. **47** (2003), 725–743.

[3] F. Bayart and O. F. Brevig, *Compact composition operators with non-linear symbols on the H^2 space of Dirichlet series*, Pacific Journal of Math **291** (2017), 81–120.

[4] F. Bayart, H. Queffélec, and K. Seip, *Approximation numbers of composition operators on H^p spaces of Dirichlet series*, Annales de l'Institut Fourier **66** (2016), 551–588.

[5] F. Bayart, M. Wang, and X. Yao, *Topological structure of the space of composition operators on the Hardy space of Dirichlet series*, preprint (2021), arXiv:2109.09581.

[6] O. Brevig and K-M. Perfekt, *Orthogonal decomposition of composition operators on the H^2 space of Dirichlet series*, preprint, arXiv:2108.12787.

[7] ———, *A mean counting function for Dirichlet series and compact composition operators*, Adv. Math. **385** (2021), 107775.

[8] C. Finet, H. Queffélec, and A. Volberg, *Compactness of composition operators on a Hilbert space of Dirichlet series*, J. Funct. Anal. **211** (2004), 271–287.

[9] J. Gordon and H. Hedenmalm, *The composition operators on the space of Dirichlet series with square summable coefficients*, Michigan Math. J. **46** (1999), 313–329.

[10] J. McCarthy, *Hilbert spaces of Dirichlet series and their multipliers*, Trans. Amer. Math. Soc. **356** (2004), 881–893.

[11] H. Queffélec and K. Seip, *Approximation numbers of composition operators on the H^2 space of Dirichlet series*, J. Funct. Anal. **268** (2015), 1612–1648.

[12] J.H. Shapiro, *The essential norm of a composition operator*, Ann. of Math. **125** (1987), 375–404.

Laboratoire de Mathématiques Blaise Pascal UMR 6620 CNRS, Université Clermont Auvergne, Campus universitaire des Cézeaux, 3 place Vasarely, 63178 Aubière Cedex, France.

Email address: frederic.bayart@uca.fr