



HAL
open science

Twin-width and generalized coloring numbers

Jan Dreier, Jakub Gajarsky, Yiting Jiang, Patrice Ossona de Mendez,
Jean-Florent Raymond

► **To cite this version:**

Jan Dreier, Jakub Gajarsky, Yiting Jiang, Patrice Ossona de Mendez, Jean-Florent Raymond.
Twin-width and generalized coloring numbers. *Discrete Mathematics*, 2022, 345 (3), pp.112746.
10.1016/j.disc.2021.112746 . hal-03450765

HAL Id: hal-03450765

<https://uca.hal.science/hal-03450765v1>

Submitted on 29 Nov 2021

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution 4.0 International License

Twin-width and generalized coloring numbers

Jan Dreier

Vienna University of Technology

Jakub Gajarský

University of Warsaw

Yiting Jiang

Université de Paris, CNRS, IRIF, F-75006, Paris, France and Department of Mathematics, Zhejiang Normal University, China

Patrice Ossona de Mendez

Centre d'Analyse et de Mathématiques Sociales (CNRS, UMR 8557), Paris, France and Computer Science Institute of Charles University, Praha, Czech Republic

Jean-Florent Raymond

CNRS, LIMOS, Université Clermont Auvergne, France

Abstract

In this paper, we prove that a graph G with no $K_{s,s}$ -subgraph and twin-width d has r -admissibility and r -coloring numbers bounded from above by an exponential function of r and that we can construct graphs achieving such a dependency in r .

Keywords: Twin-width, generalized coloring numbers

1. Introduction

In this paper we consider the twin-width graph parameter, defined by Bonnet, Kim, Thomassé and Watrigant [5] as a generalization of a width invariant for classes of permutations defined by Guillemot and Marx [10]. This parameter was intensively studied recently in the context of many structural and algorithmic questions such as FPT model checking [5], graph enumeration [1], graph coloring [3], matrices and ordered graphs [4], and transductions of permutations [2]. (We postpone the formal definition of twin-width to Section 2.1.)

It is known that a graph class with bounded twin-width excludes some biclique as a subgraph if and only if it has bounded expansion [1]. Recall that a class \mathcal{C} has *bounded expansion* if, for

Email addresses: dreier@ac.tuwien.ac.at (Jan Dreier), gajarsky@mimuw.edu.pl (Jakub Gajarský), yjiang@irif.fr (Yiting Jiang), pom@ehess.fr (Patrice Ossona de Mendez), j-florent.raymond@uca.fr (Jean-Florent Raymond)

each integer r the class of all the minors of graphs of \mathcal{C} obtained by contracting vertex disjoint connected subgraphs with radius at most r and deleting some edges and vertices have bounded average degree, which may depend on r . (We refer the interested reader to [12] for an in-depth study of classes with bounded expansion.) Among the numerous characterizations of classes with bounded expansion, three relate to the *generalized colouring numbers* $wcol_r$ and $scol_r$ introduced by Kierstead and Yang [11] and to the *r-admissibility* adm_r introduced by Dvořák [7]. Indeed, as proved by Zhu [13], the following are equivalent for a class \mathcal{C} :

1. \mathcal{C} has bounded expansion;
2. $\sup\{wcol_r(G) : G \in \mathcal{C}\} < \infty$ for every integer r ;
3. $\sup\{scol_r(G) : G \in \mathcal{C}\} < \infty$ for every integer r .

Moreover, using the inequality $adm_r(G) \leq scol_r(G) \leq wcol_r(G) \leq \frac{adm_r(G)^{r+1}-1}{adm_r(G)-1}$ (see [7]), we get yet another equivalent property.

4. $\sup\{adm_r(G) : G \in \mathcal{C}\} < \infty$ for every integer r .

One can show [1] that for every integer r there exists a function $f_r : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that if G is a graph with twin-width t and no $K_{s,s}$ -subgraph, then we have $wcol_r(G) \leq f_r(t, s)$. Similar bounds also exist for $scol_r$ and adm_r . However, the proof given in [1] that biclique-free graphs with bounded twin-width have bounded expansion does not indicate how to compute such bounding functions.

In this paper, we prove that a graph G with no $K_{s,s}$ -subgraph and twin-width d has adm_r , $scol_r$ and $wcol_r$ bounded from above by an exponential function of r , and that we can construct graphs achieving such a dependency in r . In particular, $scol_r(G) \leq (d^r + 3)s$ (Theorem 1). On the other hand, one can choose G such that $scol_r(G) \geq (\frac{d-4}{8})^r s$ (Corollary 4).

2. Definitions and Notations

2.1. Twin-width

We define twin-width with the help of *trigraphs*. The notion of trigraphs used in this work is slightly different from the notion used in [5]. Both notions are nevertheless equivalent up to isomorphism. A *trigraph* \mathbf{G} on a graph $G = (V, E)$ is a binary structure with two binary relations, the *black adjacency* E and the *red adjacency* R , whose domain is a partition of V , and whose black and red adjacencies are exclusive (that is: no two elements of \mathbf{G} can be adjacent in both relations). Thus, the elements of \mathbf{G} are subsets of the vertices of G . To distinguish the elements of \mathbf{G} from the vertices of G , we will call them *nodes* and denote them by capital letters, like X, Y, Z . The set of nodes of \mathbf{G} is denoted by $V(\mathbf{G})$. The elements of $E(\mathbf{G})$ and $R(\mathbf{G})$ are respectively called *black edges* and *red edges*. The *set of neighbours* $N_{\mathbf{G}}(X)$ of a node X in a trigraph \mathbf{G} consists of all the nodes adjacent to X by a black or red edge; the *set of E-neighbours* $N_{\mathbf{G}}^E(X)$ consists of all nodes adjacent to X by a black edge and the *set of R-neighbours* $N_{\mathbf{G}}^R(X)$ consists of all nodes adjacent to X by a red edge. A *d-trigraph* is a trigraph \mathbf{G} with maximum red degree at most d , i.e., $|N_{\mathbf{G}}^R(X)| \leq d$ for all $X \in V(\mathbf{G})$.

Let \mathbf{G} be a trigraph on a graph G and let X and Y be (non-necessarily adjacent) nodes of \mathbf{G} . We say a trigraph \mathbf{G}' on G is obtained from \mathbf{G} by *contracting* X and Y if $V(\mathbf{G}') = V(\mathbf{G}) \setminus \{X, Y\} \cup \{X \cup Y\}$, $N_{\mathbf{G}'}(X \cup Y) = N_{\mathbf{G}}(X) \cup N_{\mathbf{G}}(Y)$, $N_{\mathbf{G}'}^E(X \cup Y) = N_{\mathbf{G}}^E(X) \cap N_{\mathbf{G}}^E(Y)$ (and $N_{\mathbf{G}'}^R(X \cup Y) = N_{\mathbf{G}'}(X \cup Y) \setminus N_{\mathbf{G}'}^E(X \cup Y)$), and the red and black adjacencies between all other nodes of \mathbf{G}' are as in \mathbf{G} .

A d -contraction sequence of a graph $G = (V, E)$ with n vertices is a sequence $\mathbf{G}_n, \dots, \mathbf{G}_1$ of d -trigraphs on G , where \mathbf{G}_n is the trigraph isomorphic to G defined by $V(\mathbf{G}_n) = \{\{v\} : v \in V\}$, $E(\mathbf{G}_n) = \{(\{u\}, \{v\}) : (u, v) \in E(G)\}$, and $R(\mathbf{G}_n) = \emptyset$, \mathbf{G}_1 is the trigraph with single node V , and \mathbf{G}_i is obtained from \mathbf{G}_{i+1} by performing a single contraction. The minimum d such that there exists a d -contraction sequence of a graph G is the *twin-width* of G , denoted by $\text{tww}(G)$. For a contraction sequence $\mathbf{G}_n, \dots, \mathbf{G}_1$, we define the *universe* $\mathcal{U} = \bigcup_{i=1}^n V(\mathbf{G}_i)$ to be the union of all node sets.

A given contraction sequence $\mathbf{G}_n, \dots, \mathbf{G}_1$ on a graph $G = (V, E)$ (with universe \mathcal{U}) can also be reversed to $\mathbf{G}_1, \dots, \mathbf{G}_n$ and seen as an *uncontraction sequence* where we start with a single node (the trigraph \mathbf{G}_1) and a node Z of \mathbf{G}_i is split into two nodes X and Y with no edge, black edge or red edge between them in \mathbf{G}_{i+1} . With this picture in mind, we define for every $X \in \mathcal{U}$, the *birth time* $\text{bt}(X)$ as the minimum integer i with $X \in V(\mathbf{G}_i)$ and the *split time* $\text{st}(X)$ as the maximum integer i with $X \in V(\mathbf{G}_i)$. Observe that for every $i \in \{1, \dots, n-1\}$, there is a unique $X \in \mathcal{U}$ with $\text{st}(X) = i$; the subsets $X \in \mathcal{U}$ with $\text{st}(X) = n$ are the nodes of \mathbf{G}_n , that is the singletons $\{v\}$ with $v \in V(G)$. If $X \in \mathcal{U} \setminus \{V\}$, the *parent* of X is the minimal set $Y \in \mathcal{U}$ with $Y \supseteq X$. Conversely, if $|X| > 1$, the *children* of X are the two maximal sets Y and Z in \mathcal{U} with $Y \subsetneq X$ and $Z \subsetneq X$. Note that $\{Y, Z\}$ is a partition of X and that $\text{bt}(Y) = \text{bt}(Z) = \text{st}(X) + 1$.

2.2. Generalized Colouring Numbers and Admissibility

Let $\Pi(G)$ be the set of all linear orders of the vertices of the graph G , and let $L \in \Pi(G)$. (We denote by \leq_L the corresponding binary relation for better readability.) Let $u, v \in V(G)$, and let r be a positive integer.

We say that u is *weakly r -reachable* from v with respect to L , if there exists a path P of length at most r between u and v such that $u \leq_L w$ for all vertices w of P . Let $\text{WReach}_r[L, v]$ be the set of vertices that are weakly r -reachable from v with respect to L . Note that $v \in \text{WReach}_r[L, v]$.

We say that u is *strongly r -reachable* from v with respect to L , if there is a path P of length at most r connecting u and v such that $u \leq_L v$ and all inner vertices w of P satisfy $v <_L w$. Let $\text{Sreach}_r[L, v]$ be the set of vertices that are strongly r -reachable from v with respect to L . Note that again we have $v \in \text{Sreach}_r[L, v]$.

The *r -backconnectivity* $b_r(L, v)$ of a vertex v is the maximum number of paths of length at most r in G that start in v , share no other vertices except v , and end at vertices that lie before v in the ordering L .

The *weak r -colouring number* $\text{wcol}_r(G)$ of G is defined as

$$\text{wcol}_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} |\text{WReach}_r[L, v]|,$$

and the *strong r -colouring number* $\text{scol}_r(G)$ of G is defined as

$$\text{scol}_r(G) := \min_{L \in \Pi(G)} \max_{v \in V(G)} |\text{Sreach}_r[L, v]|,$$

The *r -admissibility* of G is defined as

$$\text{adm}_r(G) = \min_{L \in \Pi(G)} \max_{v \in V(G)} b_r(L, v).$$

3. From Strong Colouring to Weak Colouring

It is known that the weak and strong colouring numbers are related by $\text{scol}_r(G) \leq \text{wcol}_r(G) \leq \text{scol}_r(G)^r$ [11]. However it is possible to improve the upper bound in the case where the strong coloring numbers increase at least at an exponential rate.

Lemma 1. *For every graph G and every positive integer r we have*

$$\text{wcol}_r(G) \leq 2^{r-1} \max_{1 \leq k \leq r} \text{scol}_k(G)^{r/k}.$$

Proof. Let r be a positive integer and let L be a linear order on $V(G)$ that minimizes

$$\max_{v \in V(G)} |\text{WReach}_r[L, v]|.$$

Let u be a vertex of G , $v \in \text{WReach}_r[L, u]$, and consider a path P certifying that v is weakly r -reachable from u ; in particular P has length at most r . Let $C(r)$ be the set of all *compositions* of r , that is of all tuples (r_1, \dots, r_k) with $r_i > 0$ (for $1 \leq i \leq k$) and $\sum_{1 \leq i \leq k} r_i = r$. A *milestone* of P is a vertex v of P such that all the vertices of P from u (included) to v (excluded) are greater than v . Let $v_1, \dots, v_k = v$ be the milestones of P other than u , and let r_1, \dots, r_{k-1} be the lengths of the paths from $v_0 = u$ to v_1, \dots, v_{k-2} to v_{k-1} , and let $r_k = r - \sum_{i=1}^{k-1} r_i$, so that $(r_1, \dots, r_k) \in C(r)$. The subpath of P from v_{i-1} to v_i witnesses that v_i is strongly r_i -reachable from v_{i-1} . Note that strong r_k -reachability requires the existence of a witness path of length at most r_k , hence it is safe to consider r_k instead of the length of the subpath of P linking v_{k-1} and v_k . We deduce that

$$\text{WReach}_r[L, u] \subseteq \bigcup_{(r_1, \dots, r_k) \in C(r)} \bigcup_{v_1 \in \text{Sreach}_{r_1}[L, u]} \dots \bigcup_{v_{k-1} \in \text{Sreach}_{r_{k-1}}[L, v_{k-2}]} \text{Sreach}_{r_k}[L, v_{k-1}].$$

(Note that we actually have equality, the reverse inclusion following from the concatenation of paths witnessing $v_1 \in \text{Sreach}_{r_1}[L, v]$, \dots , $u \in \text{Sreach}_{r_k}[L, v_{k-1}]$.) Thus we have

$$\text{wcol}_r(G) \leq \sum_{(r_1, \dots, r_k) \in C(r)} \prod_{i=1}^k \text{scol}_{r_i}(G).$$

Let $z = \max_{1 \leq k \leq r} \text{scol}_k(G)^{1/k}$. Then $\text{scol}_{r_i}(G) \leq z^{r_i}$. Thus

$$\text{wcol}_r(G) \leq \sum_{(r_1, \dots, r_k) \in C(r)} \prod_{i=1}^k z^{r_i} = |C(r)| z^r = 2^{r-1} z^r.$$

□

4. Upper bounds

Let $\text{b}\omega(G)$ denote the maximum integer s such that $K_{s,s}$ is a subgraph of G .

Theorem 1. *For every graph G and every positive integer r we have*

$$\text{scol}_r(G) \leq \left(3 + \text{tw}\omega(G) \sum_{i=0}^{r-1} (\text{tw}\omega(G) - 1)^i\right) \text{b}\omega(G) \leq (\text{tw}\omega(G)^r + 3) \text{b}\omega(G). \quad (1)$$

Proof. Let $d = \text{tw}(G)$ and $s = \text{b}\omega(G)$. Without loss of generality, we can assume that G is connected and contains more than s vertices. We consider a d -uncontraction sequence $\mathbf{G}_1, \dots, \mathbf{G}_n$ of G with universe \mathcal{U} . For every $i \in \{1, \dots, n\}$ we say a node of \mathbf{G}_i is *small* if it contains at most s vertices and it is *big*, otherwise. A set $X \in \mathcal{U}$ is *nice at step i* with $\text{bt}(X) \leq i \leq \text{st}(X)$ if X is small and some black edge is incident to it in \mathbf{G}_i . Note that if X is nice at step i , it is nice at step j for all $i \leq j \leq \text{st}(X)$. The set X is *nice* if it is nice at some step (equivalently, at step $\text{st}(X)$). For every nice set X we define $\rho(X)$ as the minimum i such that X is nice at step i . (Note that $\rho(X) > 1$ as \mathbf{G}_1 is edgeless.) As G is connected, it is clear that every $X \in \mathcal{U}$ has a subset $Y \in \mathcal{U}$ that is nice. Also, if $X, Y \in \mathcal{U}$, $X \subseteq Y$ and Y is nice, then X is also nice. It follows that the family \mathcal{N} of all the maximal nice sets in \mathcal{U} form a partition of V . We order the elements of \mathcal{N} as N_1, \dots, N_k in such a way that for all $i < j$, either $\rho(N_i) < \rho(N_j)$ holds or $\rho(N_i) = \rho(N_j)$ and $\text{bt}(N_i) \geq \text{bt}(N_j)$. We now fix any linear ordering L of V such that for all $v \in N_i$, $w \in N_j$ with $i < j$ holds $v <_L w$. See also Remark 1 for an equivalent algorithmic way to define the order L . We will use this ordering to bound the strong coloring numbers of G .

For $1 \leq i \leq n$, we define \mathcal{B}_i to be the set of all nodes of \mathbf{G}_i that are not nice at step i . We first establish some easy properties of \mathcal{B}_i .

▷ *Claim 1.* No small node in \mathcal{B}_i is incident to a black edge in \mathbf{G}_i .

Proof of the claim. Assume $X \in \mathcal{B}_i$ is small. Then it is not adjacent to a black edge as it is not nice at step i . ◁

▷ *Claim 2.* No two nodes in \mathcal{B}_i are adjacent in \mathbf{G}_i by a black edge.

Proof of the claim. Assume for contradiction that X and Y are nodes in \mathcal{B}_i that are adjacent by a black edge in \mathbf{G}_i . Hence, $G[X \cup Y]$ includes $K_{|X|, |Y|}$ as a subgraph. According to Claim 1, both X and Y are big, contradicting the assumption $\text{b}\omega(G) = s$. ◁

▷ *Claim 3.* Every node $X \in \mathcal{B}_i$ is such that $|\bigcup N_{\mathbf{G}_i}^E(X)| \leq s$.

Proof of the claim. Let $X \in \mathcal{B}_i$ and let $Y = \bigcup N_{\mathbf{G}_i}^E(X)$. Then X and Y induce a biclique in G thus $\min(|X|, |Y|) \leq s$. As only big nodes in \mathcal{B}_i are adjacent to black edges (by Claim 1), we deduce $|X| > s$ and therefore $|Y| \leq s$. ◁

Let us consider a vertex $v \in V$. In the remainder of the proof, we will bound the number of vertices in G that are strongly r -reachable from v with respect to L . Let $a \in \{1, \dots, k\}$ be such that $v \in N_a$ and let $t = \rho(N_a) - 1$. Let S be the unique node of \mathbf{G}_t with $\text{st}(S) = t$, and let Y, Z be the two children of S .

Let $\mathcal{L} = \{X \in V(\mathbf{G}_t) : (\exists i < a), N_i \supseteq X\}$. By definition of L , all the vertices of G that belong to the nodes in \mathcal{L} appear before v in L . If we set $\mathcal{R} = V(\mathbf{G}_t) \setminus \mathcal{L}$, then $\mathcal{R} \subseteq \mathcal{B}_t$.

Case 1: $v \in S$. Note that if a vertex $u \in V(G)$ is strongly r -reachable from v , then u belongs either to S or to a set in \mathcal{L} . We consider a BFS-tree T in \mathbf{G}_t , starting at S , following only red edges, with depth r , and stopping each time it reaches a node in \mathcal{L} . Note that, by construction, T has no internal node in \mathcal{L} , but it may have leaves that are not in \mathcal{L} (e.g. nodes at depth r). We further remove from T any node with no descendant (in T) belonging to \mathcal{L} . This way we get a tree T rooted at S , with internal nodes in \mathcal{R} , with depth at most r , and with leaves in \mathcal{L} . Let \mathcal{I} denote the sets of all internal nodes of T and let \mathcal{E} be the sets of all leaves of T . Then $|\mathcal{I}| \leq 1 + \sum_{\ell=0}^{r-2} d(d-1)^\ell$ and $|\mathcal{E}| \leq d(d-1)^{r-1}$. Consider any vertex u that is strongly r -reachable

from v , and let P be a path from v to u in G witnessing this. We can project P onto \mathbf{G}_t by mapping every vertex to the node of \mathbf{G}_t containing it. The projection is a walk from S to a node X_u containing u . From this walk we extract a path Q of length at most r from S to X_u . All the internal nodes of Q as well as S belong to \mathcal{R} , hence all the edges of Q (but maybe the last one) are red (according to Claim 2). Moreover, X_u is either S or it belongs to \mathcal{L} . So, either $u \in S$, or X_u has been reached by a black edge from some internal node of T , or X_u is a leaf of T . It is easily checked that at most $|T|s$ vertices of G can be of the second type (according to Claim 3), and at most $|\mathcal{E}|s$ are of the last type (as leaves belong to \mathcal{L} , so they are small). Regarding the first type, we assume without loss of generality that $v \in Z$ and observe that either $u \in Z$, so there are at most s choices for u (as $Z = N_a$ is nice hence small at time $t + 1$), or $u \in Y$ but then, as $u \leq_L v$, Y is nice as well at time $t + 1$ so $|Y| \leq s$. Thus at most $2s$ vertices of G can be of the first type. Altogether, we get

$$\begin{aligned} |\text{Sreach}[G, L, v]| &\leq \left(2 + 1 + d + \dots + d(d-1)^{r-2} + d(d-1)^{r-1}\right) s \\ &\leq \left(3 + d \sum_{\ell=0}^{r-1} (d-1)^\ell\right) s. \end{aligned}$$

Case 2: $v \notin S$. Note that if u is strongly r -reachable from v , then u belongs either to S or to N_a , or to a set in \mathcal{L} . We consider a BFS-tree T in \mathbf{G}_t , starting at N_a , following only red edges, with depth r , and stopping each time it reaches a node in \mathcal{L} . We further remove from T any node with no descendant (in T) belonging to $\mathcal{L} \cup \{S\}$. This way we get a tree T rooted at N_a , with internal nodes in \mathcal{R} , with depth at most r , and with leaves in \mathcal{L} . Let \mathcal{I} denote the sets of all internal nodes of T and let \mathcal{E} be the sets of all leaves of T . Then $|\mathcal{I}| \leq 1 + d + \dots + d(d-1)^{r-2}$ and $|\mathcal{E}| \leq d(d-1)^{r-1}$. Consider any vertex u that is strongly r -reachable from v , and let P be a path from v to u witnessing this. The path P projects on \mathbf{G}_t as a walk with length at most r from N_a to the vertex X_u containing u . From this walk we extract a path Q with length at most r from N_a to X_u . All the internal nodes of Q belong to \mathcal{R} hence all the edges of Q (but maybe the last one) are red (according to Claim 2). Moreover, X_u is either N_a , or S , or it belongs to \mathcal{L} . So, either $u \in N_a$, or $u \in S$, or X_u has been reached by a black edge from some internal node of T , or X_u is a leaf of T . The first type correspond to at most s vertices. The second type correspond to at most $2s$ vertices because in this case, Y, Z , or both, have been ordered by L before N_a which mean they are nice at step $t + 1$, hence small. The third type correspond to at most $(|\mathcal{I}| - 1)s$, as the root N_a is small hence adjacent to no black edge. The last type correspond to at most $|\mathcal{E}|s$ vertices. Altogether, we get

$$\begin{aligned} |\text{Sreach}[G, L, v]| &\leq \left(1 + 2 + (1 + d + \dots + d(d-1)^{r-2} - 1) + d(d-1)^{r-1}\right) s \\ &\leq \left(3 + d \sum_{\ell=0}^{r-1} (d-1)^\ell\right) s. \end{aligned}$$

Thus in both cases we have that every graph G with $\text{tw}(G) = d$ and $\text{b}\omega(G) = s$ satisfies (1). \square

Remark 1. We describe an algorithmic procedure that also yields the order L defined in Theorem 1. We are given an uncontraction sequence $\mathbf{G}_n, \dots, \mathbf{G}_1$. For each $i \in \{1, \dots, n-1\}$ we define a function $\text{origin}_i: V(\mathbf{G}_{i+1}) \rightarrow V(\mathbf{G}_i)$ that, informally, assigns each node in \mathbf{G}_{i+1} to the node in \mathbf{G}_i that it originates from. Let us be more precise: assume \mathbf{G}_{i+1} is constructed from \mathbf{G}_i by splitting a

node Z into two nodes X and Y , then $\text{origin}_i(X) = \text{origin}_i(Y) = Z$ and $\text{origin}_i(W) = W$ for every other node W of \mathbf{G}_{i+1} .

Remember that a node of \mathbf{G}_i is *small* if it contains at most s vertices and is *big*, otherwise. A node $X \in V(\mathbf{G}_i)$ is a *nice* node of \mathbf{G}_i if X is small and some black edge is incident to X in \mathbf{G}_i . We incrementally construct for all i an ordering $<_i$ on the nice nodes of G_i . Since all nodes of \mathbf{G}_n are nice and correspond to singletons, the ordering $<_n$ then corresponds to an ordering of the vertices of G . This order will be equivalent to the ordering L defined in Theorem 1 (up to non-determinism).

Since G_1 has no nice nodes, $<_1$ is the ordering on the empty set. Assuming that $<_i$ is already constructed, we construct $<_{i+1}$ such that it satisfies for all nice $X, Y \in V(\mathbf{G}_{i+1})$ the following conditions.

1. If $\text{origin}_i(X)$ and $\text{origin}_i(Y)$ are nice in \mathbf{G}_i and $\text{origin}_i(X) <_i \text{origin}_i(Y)$ then $X <_{i+1} Y$.
2. If $\text{origin}_i(X)$ is nice in \mathbf{G}_i and $\text{origin}_i(Y)$ is not nice in \mathbf{G}_i then $X <_{i+1} Y$.
3. If both $\text{origin}_i(X)$ and $\text{origin}_i(Y)$ are not nice in \mathbf{G}_i and $\text{bt}(X) > \text{bt}(Y)$ then $X \leq_{i+1} Y$.

Each order $<_i$ represents a partial order on V that is refined over time as i increases, until we reach a total order on V . Rule 1. states that the old order is preserved when possible, rule 2. states that new nice sets are appended at the end and rule 3. makes sure that we append new nice sets in order of their birth.

In the proof of Theorem 1, we fix a vertex $v \in V$ and pick t maximal such that in \mathbf{G}_t the node N containing v is not nice. We then partition the nodes of \mathbf{G}_t into sets \mathcal{L} and \mathcal{R} . One can show that \mathcal{L} contains precisely those nodes of \mathbf{G}_t that are strictly smaller than N with respect to $<_t$.

Corollary 1. *For every graph G and every positive integer r we have*

$$\text{scol}_r(G) \leq \begin{cases} 2 \text{b}\omega(G) & \text{if } \text{tww}(G) = 0, \\ 3 \text{b}\omega(G) & \text{if } \text{tww}(G) = 1, \\ 5 \text{b}\omega(G) & \text{if } \text{tww}(G) = 2, \\ 3(\text{tww}(G) - 1)^r \text{b}\omega(G) & \text{if } \text{tww}(G) \geq 3. \end{cases}$$

Proof. If $\text{tww}(G) \geq 3$ we have

$$\begin{aligned} \text{scol}_r(G) &\leq \left(3 + \text{tww}(G) \frac{(\text{tww}(G) - 1)^r - 1}{\text{tww}(G) - 2} \right) \text{b}\omega(G) \\ &\leq (3 + 3((\text{tww}(G) - 1)^r - 1)) \text{b}\omega(G) \\ &\leq 3(\text{tww}(G) - 1)^r \text{b}\omega(G). \end{aligned}$$

The cases where $\text{tww}(G) = 1$ or 2 follow from the theorem. If $\text{tww}(G) = 0$ then G is a cograph. Let us then show that for every cograph G it holds that $\text{scol}_r(G) \leq 2 \text{b}\omega(G)$.

The proof is by induction on the number of vertices. The base case $|V(G)| = 1$ is trivial so we consider a cograph with at least two vertices and assume that the desired bound holds for all cographs on less vertices. Being a cograph, G can be obtained from two cographs G_1 and G_2 by disjoint union or complete join [6]. Without loss of generality we assume $|V(G_1)| \leq |V(G_2)|$. By induction, for every $i \in \{1, 2\}$ there is an ordering L_i of $V(G_i)$ such that $\text{scol}_r(G_i) \leq 2 \text{b}\omega(G_i)$.

Then the order L is obtained by putting first L_1 , then L_2 . If G is the disjoint union of G_1 and G_2 then $\text{scol}_r(G, L) = \max(\text{scol}_r(G, L_1), \text{scol}_r(G, L_2))$ and the result follows; if G is the complete join of G_1 and G_2 then $\text{scol}_r(G, L) \leq \text{scol}_r(G_2, L_2) + |V(G_1)|$ and $\text{b}\omega(G) \geq \text{b}\omega(G_2) + |V(G_1)|/2$.

As $\text{scol}_r(G_2, L_2) \leq 2 \text{b}\omega(G_2)$, we get $\text{scol}_r(G_2, L_2) \leq 2 \text{b}\omega(G) - |V(G_1)|$ hence $\text{scol}_r(G, L) \leq 2 \text{b}\omega(G)$. \square

Combining Lemma 1 with Theorem 1 we get the following.

Corollary 2. *For every graph G and every positive integer r we have*

$$\text{wcol}_r(G) \leq \frac{1}{2}((2 \text{tww}(G) + 6) \text{b}\omega(G))^r.$$

Note that the base of the exponential comes from the degeneracy of G . In order to improve this upper bound it is thus natural to try to improve the degeneracy bound. Hence the following problem:

Problem 1. *What is the maximum degeneracy of a $K_{s+1, s+1}$ -free graph with twin-width at most d ?*

Recall that a *depth r minor* of a graph G is a graph H obtained from G by taking a subgraph and contracting vertex disjoint subgraphs of radius at most r . The *greatest reduced average density* (grad) of G with rank r is the maximum ratio $|E(H)|/|V(H)|$ over all (non-empty) depth r minors of a graph G ; it is denoted by $\nabla_r(G)$. Hence, by definition, a class \mathcal{C} has bounded expansion if, for each positive integer r , we have $\sup\{\nabla_r(G) : G \in \mathcal{C}\} < \infty$. It is known that $\nabla_r(G) \leq \text{wcol}_{2r+1}(G)$ [13]. Hence the next corollary directly follows from Corollary 2.

Corollary 3. *For every graph G and every positive integer r we have*

$$\nabla_r(G) \leq \frac{1}{2}((2 \text{tww}(G) + 6) \text{b}\omega(G))^{2r+1}. \quad (2)$$

In particular, every class of graphs of bounded clique-width that exclude a biclique as a subgraph has (at most) exponential expansion.

5. Lower bounds

It is known that high-girth graphs have large strong coloring numbers [9]. On the other hand, there exist expander graphs with high girth and small twinwidth [1]. We combine both results to construct graphs with small twinwidth whose strong r -coloring numbers grow exponentially in r .

Proposition 1 ([9, Theorem 5.1]). *Let G be a d -regular graph of girth at least $4g + 1$, where $d \geq 7$. Then for every $r \leq g$,*

$$\text{scol}_r(G) \geq \frac{d}{2} \left(\frac{d-2}{4} \right)^{2^{\log_2 r} - 1}.$$

Lemma 2. *For every integer $\Delta \geq 7$ and every integers r and $g \geq 4r + 1$ there exists a Δ -regular graph G with girth at least g , $2\Delta - 1 \leq \text{tww}(G) \leq 2\Delta$, and*

$$\text{scol}_r(G) \geq \frac{\Delta}{2} \left(\frac{\Delta-2}{4} \right)^{2^{\log_2 r} - 1} \geq \frac{\text{tww}(G)}{4} \left(\frac{\text{tww}(G) - 4}{8} \right)^{2^{\log_2 r} - 1}.$$

Proof. We will construct a sequence G_0, G_1, \dots of Δ -regular graphs of twin-width at most 2Δ and increasing girth. Once we reach a graph with girth at least $g \geq 4r + 1$, the result of this lemma follows from Proposition 1. Note that the twin-width of a Δ -regular graph with girth at least 5 is at least $2\Delta - 1$ (because of the first contraction).

We define G_0 to be the complete graph with $\Delta + 1$ vertices. We fix a graph G_{k-1} with edges e_1, \dots, e_m and describe how to construct G_k . For every edge e_i of G_{k-1} , we define a mapping $\theta_{e_i}: \{0, 1\}^m \rightarrow \{0, 1\}^m$ that flips the i th coordinate and preserves all other coordinates, i.e., $\theta_{e_i}((x_1, \dots, x_m)) = (y_1, \dots, y_m)$ with

$$y_j = \begin{cases} 1 - x_j & \text{if } j = i \\ x_j & \text{otherwise.} \end{cases}$$

We define G_k to be the graph with $V(G_k) = V(G_{k-1}) \times \{0, 1\}^m$ and $E(G_k) = \{(u, \mathbf{x}), (v, \theta_{uv}(\mathbf{x})) : uv \in E(G_{k-1}) \text{ and } \mathbf{x} \in \{0, 1\}^m\}$.

A *2-lift* of a graph G is a graph obtained by adding for every vertex v of G two vertices v_1 and v_2 , and adding for every edge uv of G either the edges u_1v_1 and u_2v_2 (parallel edges) or the edges u_1v_2 and u_2v_1 (crossing edges). The graph G_{k+1} can be obtained by a sequence of 2-lifts from G_k and therefore also by a sequence of 2-lifts from $G_0 = K_{\Delta+1}$. We can construct a contraction sequence that “undoes” these 2-lifts by repeatedly contracting all pairs of duplicates. Once we reach $K_{\Delta+1}$, we simply contract the remaining vertices one by one. While doing so, the red degree never exceeds 2Δ (see also [1, Lemma 26]). Hence $\text{tw}(G_k) \leq 2\Delta$.

It remains to show that the girth of G_k is higher than the girth of G_{k-1} . Let γ be a shortest cycle of G_k . Let $p_V: V(G_k) \rightarrow V(G_{k-1})$ be the standard projection, and let $p_E: E(G_k) \rightarrow E(G_{k-1})$ be the associated projection. It is easily checked that applying p_E to a cyclic graph yields a cyclic graph, and thus $p_E(\gamma)$ includes a cycle. If we apply the composition of all the mappings $\theta_{p_E(e)}$ for $e \in \gamma$ then the starting vertex of γ is fixed. It follows that each θ_e is applied an even number of times. Thus the length of γ is at least twice the length of $p_E(\gamma)$. Hence, the girth of G_k is at least twice the girth of G_{k-1} . \square

Corollary 4. *For every integer $d \geq 14$, every positive integer s , and every integer r of the form 2^k , there exists a graph G with $\text{tw}(G) \leq d$, $\text{b}\omega(G) = s$, and*

$$\text{scol}_r(G) \geq \frac{ds}{4} \left(\frac{d-4}{8} \right)^{r-1} \geq 2 \left(\frac{\text{tw}(G) - 4}{8} \right)^r \text{b}\omega(G).$$

Proof. Take the lexicographic product of a graph obtained by Lemma 2 and K_s . This way we get a graph with twin-width at most $d \geq 14$ and no $K_{s+1, s+1}$. \square

Remark 2. *The 2-lift construction used in the proof of Lemma 2 was used in [1] to prove that there exist cubic expander graphs with twin-width at most 6. It follows from this result and the characterization of classes with polynomial expansion [8] that for $d \geq 6$, the value $\nabla_r(G)$ is not bounded on the $K_{2,2}$ -free graphs with twin-width at most d by a polynomial function of r . We leave as a question whether $\sup\{\nabla_r(G) : \text{tw}(G) \leq d \text{ and } \text{b}\omega(G) \leq s\}$ increases exponentially with r for sufficiently large d .*

Admissibility being a lower bound for strong coloring numbers, the above results do not provide any lower bound for admissibility. We show below how to construct classes of graphs that have no $K_{2,2}$ -subgraph with low twin-width and high admissibility.

Lemma 3 ([1, Proposition 28]). *For $d \geq 0$ and $k > 0$, if the clique K_n subdivided k times has twin-width less than d , then $k \geq \log_d(n-1) - 1$.*

Lemma 4 ([1, Proposition 31]). *For any $c > 0$, the class of cliques K_n subdivided at least $\frac{\log n}{c}$ times has twin-width at most $f(c)$ for some triple exponential function f .*

Lemma 5. *For every integers d and $r \geq 4$ there is a graph G such that*

- G has no $K_{2,2}$ subgraph;
- $d \leq \text{tw}(G) \leq f(2 \log d)$;
- $\text{adm}_r(G) \geq d^{2(r-1)}$,

where f is the function of Lemma 4.

In particular, the above lemma implies that for every d , there is a graph class of bounded twin-width, whose members contain no $K_{2,2}$, but which has r -admissibility (and thus r -weak and strong coloring numbers) at least $d^{2(r-1)}$.

Proof. Let $d, r \in \mathbb{N}$ and $n = d^{2(r-1)}$. We define G_r^d as the graph obtained by subdividing $r-1$ times each edge of K_n . By construction, G_r^d has no $K_{2,2}$ subgraph. Let $c = 2 \log d$ and notice that $r-1 = \frac{\log n}{c}$. According to Lemma 4, G_r^d has twin-width at most $f(c)$. On the other hand, as $r > 3$ we have

$$\begin{aligned} r-1 &< 2(r-1) - 2 \\ &\leq \log_d(d^{2(r-1)}) - 2 \\ &< \log_d(n-1) - 1, \end{aligned}$$

and therefore Lemma 3 implies that $d \leq \text{tw}(G)$.

In order to prove the bound on admissibility, let us now consider an arbitrary ordering σ of $V(G_r^d)$. Notice that G_r^d has two types of vertices: n vertices of degree $n-1$, which correspond to the vertices of the n -clique that was used to construct G_r^d , and vertices of degree 2, which have been introduced by subdivisions. Let x denote the vertex of degree $n-1$ that appears the latest in σ . Notice that there are $n-1$ paths of length r that start in x , are otherwise disjoint and end at the $n-1$ other vertices of degree $n-1$ of G . By definition of x , these $n-1$ vertices appear before x in the ordering. This implies $\text{adm}_r(G_r^d, \sigma) \geq n = d^{2(r-1)}$. As σ was chosen arbitrarily, the same bound holds for the r -admissibility of G_r^d . \square

Corollary 5. *For all integers $d, r \geq 4$, there is a constant $\varepsilon > 0$ such that for all positive integers r and n there is a $K_{s+1, s+1}$ -free graph G with $|V(G)| \geq n$, $\text{b}\omega(G) = s$, and*

$$\text{adm}_r(G) \geq (\log \log \text{tw}(G))^\varepsilon \text{b}\omega(G).$$

Proof. We first consider the case where $s = 1$. Let G_0 be the graph given by Lemma 5. If $|V(G_0)| \geq n$ then G_0 is the desired graph. Otherwise, we denote by G the disjoint union of n copies of G_0 . Clearly this does not create any $K_{2,2}$ thus $\text{b}\omega(G_0) = 1$. We have $\text{adm}_r(G) \geq d^{2(r-1)}$, as otherwise any ordering of G with smaller r -admissibility would give an ordering with smaller r -admissibility for G_0 . Finally, as the twin-width of the disjoint union of two graphs is the maximum of the twin-width of each of them, we have $\text{tw}(G) = \text{tw}(G_0)$, so $\text{tw}(G) \leq$

$f(2 \log d)$. The existence of the constant $\varepsilon > 0$ then follows from the fact that f is a triple exponential function.

The case where $s > 1$ follows by considering the lexicographic product of the graphs obtained above by K_s . \square

Acknowledgments

These results have been obtained at the occasion of the workshop “Generalized coloring numbers and friends” of the Sparse Graphs Coalition, which was co-organized by Michał Pilipczuk and Piotr Micek.

References

- [1] E. Bonnet, C. Geniet, E.J. Kim, S. Thomassé, and R. Watrigant, *Twin-width II: small classes*, Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA), 2021, pp. 1977–1996.
- [2] E. Bonnet, J. Nešetřil, P. Ossona de Mendez, and S. Siebertz, *Twin-width and permutations*, arXiv:2102.06880 [cs.LO], 2021.
- [3] Édouard Bonnet, Colin Geniet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant, *Twin-width III: Max Independent Set and Coloring*, arXiv:2007.14161 [cs.DS], 2020.
- [4] Édouard Bonnet, Ugo Giocanti, Patrice Ossona de Mendez, and Stéphan Thomassé, *Twin-width IV: low complexity matrices*, arXiv:2102.03117 [math.CO], 2021.
- [5] Édouard Bonnet, Eun Jung Kim, Stéphan Thomassé, and Rémi Watrigant, *Twin-width I: tractable FO model checking*, 61st IEEE Annual Symposium on Foundations of Computer Science, FOCS 2020, Durham, NC, USA, November 16-19, 2020, IEEE, 2020, pp. 601–612.
- [6] Derek G Corneil, Helmut Lerchs, and L Stewart Burlingham, *Complement reducible graphs*, Discrete Applied Mathematics **3** (1981), no. 3, 163–174.
- [7] Z. Dvořák, *Constant-factor approximation of domination number in sparse graphs*, European J. Combin. **34** (2013), no. 5, 833–840.
- [8] Zdenek Dvorák and Sergey Norin, *Strongly sublinear separators and polynomial expansion*, SIAM Journal on Discrete Mathematics **30** (2016), no. 2, 1095–1101.
- [9] M. Grohe, S. Kreutzer, R. Rabinovich, S. Siebertz, and K. Stavropoulos, *Colouring and covering nowhere dense graphs*, SIAM J. Discrete Math. **32** (2018), no. 4, 2467–2481.
- [10] Sylvain Guillemot and Dániel Marx, *Finding small patterns in permutations in linear time*, Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2014, Portland, Oregon, USA, January 5-7, 2014, 2014, pp. 82–101.
- [11] H.A. Kierstead and D. Yang, *Orderings on graphs and game coloring number*, Order **20** (2003), 255–264.
- [12] J. Nešetřil and P. Ossona de Mendez, *Sparsity (graphs, structures, and algorithms)*, Algorithms and Combinatorics, vol. 28, Springer, 2012, 465 pages.
- [13] X. Zhu, *Colouring graphs with bounded generalized colouring number*, Discrete Math. **309** (2009), no. 18, 5562–5568.