Exact controllability of semilinear heat equations through a constructive approach

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Abstract

The exact distributed controllability of the semilinear heat equation $\partial_t y - \Delta y + f(y) = v\,1_\omega$ posed over multi-dimensional and bounded domains, assuming that f is locally Lipschitz continuous and satisfies the growth condition $\limsup_{|r|\to\infty} |f(r)|/(|r|\ln^{3/2}|r|) \leqslant \beta$ for some β small enough has been obtained by Fernández-Cara and Zuazua in 2000. The proof based on a non constructive fixed point arguments makes use of precise estimates of the observability constant for a linearized heat equation. Under the same assumption, by introducing a different fixed point application, we present a simpler proof of the exact controllability, which is not based on the cost of observability of the heat equation with respect to potentials. Then, assuming that f is locally Lipschitz continuous and satisfies the growth condition $\limsup_{|r|\to\infty} |f'(r)|/\ln^{3/2}|r| \leqslant \beta$ for some β small enough, we show that the above fixed point application is contracting yielding a constructive method to compute the controls for the semilinear equation. Numerical experiments illustrate the results.

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1 Introduction and main results

Let Ω be a bounded connected open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^\star$) with $\mathcal{C}^2$ boundary, $\omega \subset\subset \Omega$ be any non-empty open set and let $T > 0$. We set $Q_T = \Omega \times (0,T)$, $q_T = \omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. We are concerned with the null controllability problem for the following semilinear heat equation

$$\begin{cases} \partial_t y - \Delta y + f(y) = v 1_\omega \quad \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = u_0 \text{ in } \Omega, \end{cases} \tag{1}$$

where $u_0 \in L^2(\Omega)$ is the initial state of y, $v \in L^2(q_T)$ is a *control* function and $f : \mathbb{R} \mapsto \mathbb{R}$ is a nonlinear function. Recall that if f is locally Lipschitz-continuous and satisfies the condition $|f'(r)| \leqslant C(1+|r|^{4+d})$ for all $r \in \mathbb{R}$, then (1) possesses exactly one local in time solution

Moreover, in accordance with the results in [8, Section 5], under the growth condition $|f(r)| \leqslant C(1+|r|\ln(1+|r|))$ for all $r \in \mathbb{R}$ and some $C > 0$, the solutions to (1) are globally defined in $[0,T]$ and one has

$$y \in \mathcal{C}^0([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^1(\Omega)). \tag{2}$$

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Without the above growth condition, the solutions to (1) can blow up before $t = T$; in general, the blow-up time depends on f and the size of $\|u_0\|_{L^2(\Omega)}$. We refer to [25] and to [20, Section 2 and Section 5] for a survey on this issue.

The system (1) is said to be *exactly controllable to trajectories* at time T if, for any globally defined trajectory $y^\star \in \mathcal{C}^0([0,T];L^2(\Omega))$ (corresponding to data $u_0^\star \in L^2(\Omega)$ and $v^\star \in L^2(q_T)$), for any $u_0 \in L^2(\Omega)$, there exist controls v such that $v1_\omega \in L^2(q_T)$ and associated states y satisfying (2) and

$$y(x,T) = y^\star(x,T), \quad x \in \Omega. \tag{3}$$

When such property is true for a specific trajectory $y^\star$ and for any choice of initial datum u_0, we say that system (1) is globally exactly controllable to $y^\star$ at time T. The uniform controllability strongly depends on the nonlinearity f. Assuming a growth condition on f at infinity, Fernández-Cara and Zuazua in [18] showed the following result.

Theorem 1. *[18] Let $T > 0$ be given. Assume that (1) admits at least one solution $y^\star$, globally defined in $[0,T]$ and bounded in Q_T associated with $v^\star \in L^\infty(q_T)$. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz continuous and satisfies*

$(\mathbf{H_0})$ $|f'(r)| \leqslant C(1+|r|^{4+d})$ *a.e. in* $\mathbb{R}$.

There exists a $\beta^\star = \beta(y^\star) > 0$ such that if

$(\mathbf{H_1})$ $\limsup_{|r|\to\infty} \dfrac{|f(r)|}{|r|\ln_+^{3/2}|r|} \leqslant \beta^\star$

then system (1) is globally exactly controllable to $y^\star$ at time T with controls in $L^\infty(q_T)$.

Here and in the sequel, we note

$$\ln_+|r| = \begin{cases} 0 \text{ if } |r| \leqslant 1 \\ \ln|r| \text{ else.} \end{cases}$$

Therefore, if $|f(r)|$ does not grow at infinity faster than $|r|\ln_+^p(|r|)$ for any $p < 3/2$, then (1) is exactly controllable to trajectories. We also mention [2] which gives a similar result assuming the additional sign condition $f(r)r \geqslant -C(1+r^2)$ for all $r \in \mathbb{R}$ and some $C > 0$. On the contrary, if f is too "super-linear" at infinity, precisely, if $p > 2$, then for some initial data, the control cannot compensate the blow-up phenomenon occurring in $\Omega\backslash\overline{\omega}$ (see [18, Theorem 1.1]). The problem remains open when f behaves at infinity like $|r|\ln^p|r|$ with $3/2 \leqslant p \leqslant 2$. We mention however the recent work of Le Balc'h [26] where uniform controllability results in large time are obtained for $p \leqslant 2$ assuming additional sign conditions on f, notably that $f(r) > 0$ for $r > 0$ or $f(r) < 0$ for $r < 0$, a condition not satisfied for $f(r) = -r\ln_+^p|r|$. Eventually, we also mention [11] where a positive boundary controllability result is proved for a specific class of initial and final data and T large enough.

Theorem 1 is deduced in [18] from a null controllability result corresponding to the null trajectory, i.e. $y^\star \equiv 0$ corresponding to $v^\star \equiv 0, u_0^\star \equiv 0$ and assuming $f(0) = 0$. The proof of the null controllability given in [18] is based on a fixed point method, initially introduced in [33] for a one dimensional semilinear wave equation. Precisely, it is shown that the operator $\Lambda : L^\infty(Q_T) \to L^\infty(Q_T)$, where $y := \Lambda(z)$ is a null controlled solution corresponding to the control of minimal L^∞ norm of the linear boundary value problem

$$\begin{cases} \partial_t y - \Delta y + y\,\widetilde{f}(z) = v1_\omega \quad \text{in} \quad Q_T \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = u_0 \quad \text{in} \quad \Omega \end{cases}, \qquad \widetilde{f}(r) := \begin{cases} f(r)/r & r \neq 0 \\ f'(0) & r = 0 \end{cases} \tag{4}$$

maps a closed ball $B(0,M) \subset L^\infty(Q_T)$ into itself, for some $M > 0$. The Kakutani's theorem then provides the existence of a fixed point for the operator Λ, which is also a controlled solution for (1).

This allows to obtain controlled solutions in $L^\infty(Q_T)$ but requires refined L^1 Carleman type estimates (see [18, Proposition 3.2]).

The general goal considered in this work is the approximation of the controllability problem associated with (1), that is to construct an explicit sequence $(v_k)_{k\in\mathbb{N}}$ converging strongly toward a control for (1). The controllability of nonlinear partial differential equations has attracted a large number of works in the last decades (see the monography [10] and references therein). However, as far as we know, few are concerned with the approximation of exact controls for nonlinear partial differential equations, and the construction of convergent control approximations for nonlinear equations remains a challenge. A natural strategy is to take advantage of the method used in [18, 26] and consider, for any element $y_0 \in L^\infty(Q_T)$, the Picard iterations defined by $y_{k+1} = \Lambda(y_k)$, $k \geqslant 0$ associated with the operator Λ. Numerical experiments reported in [14] exhibit the non convergence of the sequences $(y_k)_{k\in\mathbb{N}}$ and $(v_k)_{k\in\mathbb{N}}$ for some initial conditions large enough. This phenomenon is related to the fact that the operator Λ is in general not contracting, including the cases for which $\widetilde{f}$ is globally Lipschitz. We also refer to [4, 5] where this strategy is implemented.

In the one-dimensional case, a least-squares type approach, based on the minimization over $Z := L^2((T-t)^{-1}, Q_T)$ of the functional $\mathcal{R} : Z \to \mathbb{R}^+$ defined by $\mathcal{R}(z) := \|z - \Lambda(z)\|_Z^2$ has been introduced and analyzed in [14]. Assuming $u_0 \in L^\infty(\Omega)$, $\widetilde{f} \in \mathcal{C}^1(\mathbb{R})$ and $(\widetilde{f})' \in L^\infty(\mathbb{R})$, it is proved that $\mathcal{R} \in \mathcal{C}^1(Z;\mathbb{R}^+)$ and that, for some constant $C > 0$

$$(1 - C\|(\widetilde{f})'\|_{L^\infty(\mathbb{R})}\|u_0\|_\infty)\sqrt{2\mathcal{R}(z)} \leqslant \|\mathcal{R}'(z)\|_{L^2(Q_T)} \quad \forall z \in L^2(Q_T)$$

implying that if $\|(\widetilde{f})'\|_\infty\|u_0\|_\infty$ is small enough, then any critical point for $\mathcal{R}$ is a fixed point for Λ (see [14, Proposition 3.2]). Under such smallness assumption on the data, numerical experiments reported in [14] display the convergence of gradient based minimizing sequences for $\mathcal{R}$ and a better behavior than the ones associated with the Picard iterates for Λ. The analysis of convergence is however not performed.

More recently, a constructive method has been developed for the one-dimensional case in [28] by introducing the following (non convex) least-squares problem

$$\inf_{(y,v)\in\mathcal{A}} E_s(y,v), \quad E_s(y,v) := \|\partial_t y - \Delta y + f(y) - v\,1_\omega\|^2_{L^2(\rho_0(s),Q_T)}$$

where $\mathcal{A}$ is a convex space which incorporates the initial and controllability requirement and where ρ_0 denotes a Carleman type weight parametrized by s and blowing up as $t \to T^-$. Assuming slightly stronger assumption on f than in Theorem 1, a strong convergent approximation of a controlled pair is obtained:

Theorem 2. *[28] Let $T > 0$ be given. Let $d = 1$. Assume that (1) admits at least one solution $y^\star$, globally defined in $[0,T]$ and bounded in Q_T associated with $v^\star \in L^2(q_T)$ and $u_0^\star \in L^2(\Omega)$. Assume that $f \in \mathcal{C}^1(\mathbb{R})$ satisfies the growth condition*

$$(\mathbf{H_1'}) \quad \exists \alpha > 0, \ s.t. \ |f'(r)| \leqslant (\alpha + \beta^\star \ln_+ |r|)^{3/2}, \quad \forall r \in \mathbb{R}$$

for some $\beta^\star = \beta^\star(y^\star) > 0$ small enough and

$$(\overline{\mathbf{H}}_\mathbf{p}) \quad \exists p \in [0,1] \text{ such that } \sup_{\substack{a,b\in\mathbb{R}\\ a\neq b}} \frac{|f'(a) - f'(b)|}{|a-b|^p} < +\infty.$$

Then, for any $u_0 \in L^2(\Omega)$, one can construct a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly to a controlled pair for (1) satisfying (3). Moreover, after a finite number of iterations, the convergence is of order at least $1 + p$.

The hypothesis on f are stronger here than in Theorem 1: it should be noted however that the function $f(r) = a + br + \beta r \ln(1 + |r|)^{3/2}$, $a, b \in \mathbb{R}$ which is somehow the limit case in $(\mathbf{H_1})$ satisfies

($\mathbf{H_1'}$) and ($\overline{\mathbf{H}}_\mathbf{1}$). On the other hand, Theorem 2 devoted to the one dimensional case is constructive, contrary to Theorem 1. A similar construction is performed in a multi-dimensional case with $d \leqslant 3$ in [27] assuming that f is globally Lipschitz. The extension of Theorem 2 to the case $d \leqslant 3$ can be obtained as well. The minimizing sequence for E_s constructed in [27, 28] are related to the operator $\Lambda_N : \mathcal{A} \to \mathcal{A}$ defined by $y = \Lambda_N(z)$ controlled solution of

$$\begin{cases} \partial_t y - \Delta y + f'(z)y = v1_\omega + f'(z)z - f(z) \quad \text{in} \quad Q_T \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot,0) = u_0 \quad \text{in} \quad \Omega \end{cases}, \tag{5}$$

through the control v of minimal $L^2(\rho_0(s), q_T)$ norm. The analysis in [28] makes use of global L^2 Carleman estimates as initially introduced in this context in [19].

In this paper, we prove the following result.

Theorem 3. *Let $T > 0$ be given. Let $d \leqslant 5$. Assume that (1) admits at least one solution $y^\star$, globally defined in $[0,T]$ and bounded in Q_T associated with a control function $v^\star \in L^2(q_T)$ and $u_0^\star \in L^\infty(\Omega)$. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz continuous.*

- *Assume that there exists $\beta^\star > 0$ small enough such that f satisfies ($\mathbf{H_1}$). Then system (1) is globally exactly controllable from any initial datum in $L^\infty(\Omega)$ to $y^\star$ at time T with controls in $\{v^*\} + L^{p_d}(q_T)$, where p_d is defined in (19).*
- *If $f \in \mathcal{C}^1(\mathbb{R})$ satisfies ($\mathbf{H_1'}$) for $\beta^* > 0$ small enough, then for any $u_0 \in L^\infty(\Omega)$, one may construct a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly in $L^2(Q_T) \times L^2(q_T)$ to a controlled pair (y_∞, v_∞) for (1). Besides, the convergence of $(y_k - y_\infty, v_k - v_\infty)_{k\in\mathbb{N}}$ holds at least with a linear rate for the norm $L^2(\rho_0(s), Q_T) \times L^2(\rho_0(s), q_T)$, where ρ_0 is defined in (15) and s is chosen suitably large depending on $\|u_0\|_{L^\infty(\Omega)}$.*

The first part of Theorem 3 differs from Theorem 1 on the functional spaces as it is obtained from a null controllability result based on a different fixed point application leading to a simpler proof. In particular, it is not based on the analysis of the cost of observability of the heat equation with potential. We believe that this different approach could possibly help analyzing non-linearities f which behave like $|r| \log_+(|r|)^p$ for $p \in [3/2, 2]$.

On the other hand, the second part relaxed the Hölder type assumption ($\overline{\mathbf{H}}_\mathbf{p}$) on f' but still leads in multi-dimensional cases to a constructive method. As we shall see, this is related to an appropriate choice of the parameter s related to the norm of the initial condition.

The null controllability result we shall prove and leading to Theorem 3 (by simply considering $y - y^*$ instead of y and $v - v^*$ instead of v, see [18, p.603]; the proof of Theorem 3 is then left to the reader) reads as follows.

Theorem 4. *Let $T > 0$ be given. Let $d \leqslant 5$ and $s > 0$ large enough.*

- *There exists $\beta^\star > 0$ such that if $f \in \mathcal{C}^0(\mathbb{R})$ satisfies*

  ($\mathbf{H_2}$) $\quad \exists \alpha > 0, \ s.t. \ |f(r)| \leqslant |r|\big(\alpha + \beta^\star \ln_+ |r|\big)^{3/2}, \quad \forall r \in \mathbb{R}$

  then system (1) is globally null-controllable at time T for initial data in $L^\infty(\Omega)$ with controls in $L^2(q_T) \cap L^{p_d}(q_T)$, where p_d is defined in (19) and corresponding controlled solution in $L^\infty(Q_T)$.
- *If $f \in \mathcal{C}^1(\mathbb{R})$ satisfies ($\mathbf{H_1'}$) for $\beta^\star > 0$ small enough and $f(0) = 0$ then for any $u_0 \in L^\infty(\Omega)$, one can construct a sequence $(y_k, v_k)_{k\in\mathbb{N}}$ converging strongly in $L^2(Q_T) \times L^2(Q_T)$ to a controlled pair for (1). Besides, the convergence of $(y_k, v_k)_{k\in\mathbb{N}}$ holds at least with a linear rate for the norm $L^2(\rho_0(s), Q_T) \times L^2(\rho_0(s), q_T)$, where ρ_0 is defined in (15) and s is chosen suitably large depending on $\|u_0\|_{L^\infty(\Omega)}$.*

Remark that the first part of Theorem 4 relaxes the regularity assumption to $f \in \mathcal{C}^0(\mathbb{R})$ instead of f locally Lipschitz continuous. Therefore, with such non-linearities, we do not know the local existence in time for the uncontrolled system. It should be noted that $(\mathbf{H_2})$ implies, in addition to $f(0) = 0$ that $|f(r)|/|r|$ is bounded in any neighborhood of 0.

Theorem 4 is obtained by introducing the following linearized controllability problem: for $\widehat{y}$ in a suitable class $\mathcal{C}(s)$ depending on a free parameter $s \geqslant 1$, find the control v such that the solution y of

$$\begin{cases} \partial_t y - \Delta y = v 1_\omega - f(\widehat{y}) \quad \text{in} \quad Q_T \\ y = 0 \text{ on } \Sigma_T, \quad y(\cdot, 0) = u_0 \quad \text{in} \quad \Omega \end{cases}, \tag{6}$$

satisfies $y(\cdot, T) = 0$ in Ω, and (y, v) corresponds to the minimizer of a functional J_s depending on s and involving Carleman weight functions (see Remark 1), which can also be computed as the solution of an affine problem involving Carleman weight functions (see (20)).

This will define an operator $\Lambda_s : \widehat{y} \mapsto y$ from some suitable class $\mathcal{C}(s)$ into itself, on which we can use fixed point theorems for s sufficiently large depending on $\|u_0\|_{L^\infty(\Omega)}$, namely Schauder fixed point theorem for the first item of Theorem 4, and Banach-Picard fixed point theorem for the second item, allowing to exhibit a simple sequence of convergent approximations of the control and controlled trajectory.

The main trick here is thus to keep the parameter s free in the whole construction of the fixed point operator and to get suitable estimates on the control and controlled trajectories.

In order to do that, we will use the Carleman estimate introduced in [1], which presents the advantage of not degenerating as $t \to 0$, allowing to handle initial conditions in a somewhat more natural way than with the classical parabolic Carleman estimate of [19].

Let us also point out that this idea of introducing Carleman estimates within the control process and choose the Carleman parameter large to limit the influence of lower order terms is also very natural in the context of inverse problems, and in some extent, it should be compared to the techniques developed in [3] or in [23] for instance, where Carleman estimates have been used to design numerical methods to recover the unknown coefficient.

The paper is organized as follows. In Section 2, we derive a controllability result for the linear heat equation with precise estimates. Then, in Section 3, we prove, for any time $T > 0$ and control domain ω the uniform null controllability of (1) assuming that f is continuous and satisfies the condition $(\mathbf{H_2})$. Then in Section 4, assuming the hypothesis $(\mathbf{H_1'})$, we show that the operator Λ_s is contracting, yielding the convergence of the Picard iterates $y_{k+1} = \Lambda_s(y_k)$. Section 5 illustrates the result with some numerical experiments in the one dimensional case while Section 6 concludes with some perspectives.

Notations. In this article, C denotes generic constants depending on Ω, ω, and T, which may change from line to line, but is independent of the Carleman parameter s.

2 Controllability result for the linear heat equation

This section is devoted to a controllability result for a linear heat equation with a right hand side $B = B(x, t)$ in some class defined in the sequel. More precisely we are interested by the existence of a control function v such that the solution z of

$$\begin{cases} \partial_t z - \Delta z = v 1_\omega + B \quad \text{in} \quad Q_T, \\ z = 0 \text{ on } \Sigma_T, \quad z(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{7}$$

satisfies

$$z(\cdot, T) = 0 \text{ in } \Omega. \tag{8}$$

This control problem has already been analyzed many times in the literature since the pioneering work [19], and it is known that null-controllability can be achieved only if the source term B vanishes exponentially fast close to T.

In this section, we follow the usual strategy of [19] to construct a solution to the null-controllability problem (7)–(8), using Carleman estimates as a fundamental tool, and getting suitable estimates on this linear control problem.

In order to do that, instead of using the classical Carleman estimates of [19], we use the one in [1], for which it is easier to deal with initial data as the weight function does not blow up as $t \to 0$.

2.1 Carleman estimates

Before introducing the Carleman estimate, we define the several weight functions which will be involved in it.

We start by choosing a function $\widetilde{\psi} \in \mathcal{C}^2(\overline{\Omega}; [0,1])$ satisfying $\widetilde{\psi} = 0$ on $\partial\Omega$ and $\inf_{\Omega\setminus\omega} |\nabla\widetilde{\psi}| > 0$. We then set $\widehat{\psi} = \widetilde{\psi} + 6$.

We then introduce two free parameters $s \geqslant 1$ and $\lambda \geqslant 1$.

Setting $\mu = s\lambda^2 e^{2\lambda}$, we then choose a function $\theta \in \mathcal{C}^2([0,T))$, depending on the free parameters s and λ through the choice of μ, such that

$$\theta(t) = \begin{cases} 1 + \left(1 - \dfrac{4t}{T}\right)^{\mu} & \forall t \in [0, T/4], \\ 1 & \forall t \in [T/4, T - T/2], \\ \theta \text{ is increasing} & \text{on } [T - T/2, T - 3T/4], \\ 1 + \dfrac{1}{T-t} & \forall t \in [T - 3T/4, T). \end{cases} \tag{9}$$

(In [1, Theorem 2.5], the function θ was chosen equal to $1/(T-t)$ on $[T - 3T/4, T]$, which is possible only if $T \leqslant 4$, but one can easily adapt the proof of [1, Theorem 2.5] to handle a function θ as in (9), which also allows to consider any time horizon T.)

We then set φ, ξ and ρ as follows

$$\varphi(x,t) = \theta(t)\big(\lambda \exp(12\lambda) - \exp(\lambda\widehat{\psi}(x))\big), \qquad (x,t) \in \overline{\Omega} \times [0,T), \tag{10}$$

$$\xi(x,t) = \theta(t) \exp(\lambda\widehat{\psi}(x)), \qquad (x,t) \in \overline{\Omega} \times [0,T), \tag{11}$$

$$\rho(x,t) = \exp\big(s\varphi(x,t)\big), \qquad (x,t) \in \overline{\Omega} \times [0,T). \tag{12}$$

Finally, we introduce the weight functions $\tilde{\rho}_0$, $\tilde{\rho}_1$, and $\tilde{\rho}_2$ as follows:

$$\tilde{\rho}_0 = \xi^{-3/2}\rho, \quad \tilde{\rho}_1 = \xi^{-1}\rho, \quad \tilde{\rho}_2 = \xi^{-1/2}\rho. \tag{13}$$

We emphasize that these weights ρ, $\tilde{\rho}_0$, $\tilde{\rho}_1$, and $\tilde{\rho}_2$ all blow up at $t \to T^-$ and not at $t = 0$.

Let us also point out that, although it does not explicitly appear in the above notations (as it is usually done), all the above weight functions depend on the parameters $s \geqslant 1$ and $\lambda \geqslant 1$. In fact, to make the dependence in the parameter s explicit, we will write the weights ρ, $\tilde{\rho}_0$, $\tilde{\rho}_1$, and $\tilde{\rho}_2$ as functions of $(x,t) \in \Omega \times [0,T)$ and $s \geqslant 1$.

The controllability property for the linear system (7) is based on the following Carleman estimate.

Lemma 1. *[1, Theorem 2.5] There exists $\lambda_0 \geqslant 1$ and $s_0 \geqslant 1$ such for all $\lambda \geqslant \lambda_0$ and for all $s \geqslant s_0$, for all $p \in L^2(0,T; H^2 \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$,*

$$\int_\Omega \rho^{-2}(\cdot,0,s)|\nabla p(0)|^2 + s^2\lambda^3 e^{14\lambda} \int_\Omega \rho^{-2}(\cdot,0,s)|p(0)|^2 + s\lambda^2 \int_{Q_T} \tilde{\rho}_2^{-2}(s)|\nabla p|^2 + s^3\lambda^4 \int_{Q_T} \tilde{\rho}_0^{-2}(s)|p|^2$$
$$\leqslant C\bigg(\int_{Q_T} \rho^{-2}(s)|-\partial_t p - \Delta p|^2 + s^3\lambda^4 \int_{q_T} \tilde{\rho}_0^{-2}(s)|p|^2\bigg). \tag{14}$$

In the following, we fixed $\lambda = \lambda_0$ and we only keep the parameter s free. This allows to replace in (14) all the weights $\tilde{\rho}_0$, $\tilde{\rho}_1$, and $\tilde{\rho}_2$ in (13) by the following ones:

$$\rho_0 = \theta^{-3/2}\rho, \quad \rho_1 = \theta^{-1}\rho, \quad \rho_2 = \theta^{-1/2}\rho. \tag{15}$$

Modifying the constant if needed, it is easy to check that Lemma 1 implies that there exists a constant $C > 0$ such that for all $s \geqslant s_0$, for all $p \in L^2(0,T;H^2 \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$,

$$\begin{aligned}\int_\Omega \rho^{-2}(\cdot,0,s)|\nabla p(0)|^2 + s^2\int_\Omega \rho^{-2}(\cdot,0,s)|p(0)|^2 + s\int_{Q_T}\rho_2^{-2}(s)|\nabla p|^2 + s^3\int_{Q_T}\rho_0^{-2}(s)|p|^2 \\ \leqslant C\bigg(\int_{Q_T}\rho^{-2}(s)|-\partial_t p - \Delta p|^2 + s^3\int_{q_T}\rho_0^{-2}(s)|p|^2\bigg).\end{aligned} \tag{16}$$

For further purposes, we will also need some additional $L^2(0,T;H^2(\Omega))\cap H^1(0,T;L^2(\Omega))$ estimates on p from the right hand side. This is the goal of the main Lemma.

In order to state it precisely, we define two additional weight functions $\varphi_* = \varphi_*(t,s)$ and $\rho_3 = \rho_3(t,s)$ depending only on the time variable and on the free parameter $s \geqslant s_0$:

$$\begin{aligned}&\forall t \in [0,T), \quad \varphi_*(t) = \sup_{x\in\overline{\Omega}}\varphi(x,t),\\ &\forall t \in [0,T), \quad \rho_3(t,s) = e^{\frac{3}{2}s\varphi_*(t)}.\end{aligned}$$

We then get the following result.

Lemma 2. *There exists a constant $C > 0$ such that for all $s \geqslant s_0$, for all $p \in L^2(0,T;H^2 \cap H_0^1(\Omega)) \cap H^1(0,T;L^2(\Omega))$,*

$$\|\partial_t(\rho_3^{-1}(s)p)\|^2_{L^2(Q_T)} + \|\rho_3^{-1}(s)p\|^2_{L^2(0,T;H^2(\Omega))} \leqslant C\bigg(\int_{Q_T}\rho^{-2}(s)|-\partial_t p - \Delta p|^2 + s^3\int_{q_T}\rho_0^{-2}(s)|p|^2\bigg). \tag{17}$$

Proof. Denoting $p_1 = \rho_3^{-1}(s)p$ we have

$$\partial_t p_1 + \Delta p_1 = \rho_3^{-1}(s)(\partial_t p + \Delta p) + \frac{3}{2}s\partial_t\varphi_*(t)\rho_3^{-1}(s)p,$$

and from maximal regularity results for the heat equation

$$\begin{aligned}\|\partial_t p_1\|^2_{L^2(Q_T)} + \|p_1\|^2_{L^2(0,T;H^2(\Omega))} \\ \leqslant C\big(\|\rho_3^{-1}(s)(\partial_t p + \Delta p)\|^2_{L^2(Q_T)} + \|s\partial_t\varphi_*(t)\rho_3^{-1}(s)p\|^2_{L^2(Q_T)} + \|\nabla p_1(0)\|^2_{L^2(\Omega)}\big).\end{aligned} \tag{18}$$

Let us estimate each term of the right hand side of this inequality.

Since $\rho_3^{-1}(s) \leqslant \rho^{-1}(s)$, the first term of the right hand-side (18) is easily bounded as follows:

$$\|\rho_3^{-1}(s)(\partial_t p + \Delta p)\|^2_{L^2(Q_T)} \leqslant \|\rho^{-1}(s)(\partial_t p + \Delta p)\|^2_{L^2(Q_T)}.$$

For the second term of the right hand-side (18), from the definition of φ_* we have $\partial_t\varphi_* = \frac{\partial_t\theta}{\theta}\varphi_*$. Besides, from the definition of θ we have, for some constant C independent of s:

$$\Big|\frac{\partial_t\theta}{\theta}(t)\Big| \leqslant \begin{cases} Cs & \forall t \in [0,T/4],\\ 0 & \forall t \in [T/4, T-T/2],\\ C\theta & \forall t \in [T-T/2,T).\end{cases}$$

Since $\theta \geqslant 1$, we deduce that $|\partial_t\theta/\theta| \leqslant Cs\theta$ for some C independent of s, and thus we get that $|s\varphi_*'| \leqslant Cs\theta\varphi_* \leqslant Cs\theta^2$.

Accordingly, for $s \geqslant s_0$,

$$|s\partial_t\varphi_*(t)\rho_3^{-1}(s)| \leqslant Cs\theta^2\rho_3^{-1}(s) \leqslant Cs\theta^2\exp(-s\varphi_*/2)\rho^{-1}(s) \leqslant Cs^{3/2}\theta^{3/2}\rho^{-1}(s) = Cs^{3/2}\rho_0^{-1}(s),$$

where we used that $s\theta^2\exp(-s\varphi_*/2) \leqslant Cs^{3/2}\theta^{3/2}$ for $s \geqslant s_0$: this is obvious since $\varphi_* \geqslant c\theta_*$ for some strictly positive constant $c > 0$, $s^{1/2}\theta^{1/2}\exp(-cs\theta) \leqslant \|\tau^{1/2}\exp(-c\tau)\|_{L^\infty(\mathbb{R}_+)}$ and $s \geqslant 1$.

These estimates entail that

$$\|s\partial_t\varphi_*(t)\rho_3^{-1}(s)p\|^2_{L^2(Q_T)} \leqslant Cs^3\|\rho_0^{-1}(s)p\|^2_{L^2(Q_T)}.$$

Since $\rho_3^{-1}(0,s) \leqslant \rho^{-1}(x,0,s)$ for all $x \in \Omega$, the third term of the right hand-side (18) is easily bounded as follows:

$$\|\nabla p_1(0)\|^2_{L^2(\Omega)} = \|\rho_3^{-1}(0,s)\nabla p(0)\|^2_{L^2(\Omega)} \leqslant \|\rho^{-1}(0,s)\nabla p(0)\|^2_{L^2(\Omega)}.$$

The inequality (17) then follows from the definition of p_1, the estimate (18) and the Carleman inequality (16). □

2.2 Application to controllability

In the next section, we see how Lemma 1 and 2 imply control results for the controllability problem (7)–(8).

This part is merely classical, and the results presented below differs only slightly from the ones in [1, Theorem 2.6] and [13, Theorem 3.3]. Still, we present it below with some details, as it is an essential part of our argument and requires to be adequately adapted from [1, Theorem 2.6] and [13, Theorem 3.3]

In particular, we will give a construction which, in dimension d, provides a control $v \in L^{p_d}(q_T)$, where p_d is given by

$$p_d = \begin{cases} +\infty & \text{if} \quad d = 1, \\ \text{any number} \in [2,\infty) & \text{if} \quad d = 2, \\ \dfrac{2(d+2)}{d-2} & \text{if} \quad d \geqslant 3. \end{cases} \tag{19}$$

Before going further, for $s \geqslant s_0$, let us introduce the bilinear form

$$(p,q)_{P,s} := \int_{Q_T} \rho^{-2}(s)L^\star p\, L^\star q + s^3\int_{q_T}\rho_0^{-2}(s)p\, q$$

where $L^\star q := -\partial_t q - \Delta q$ defined for $p, q \in L^2(0,T;H^2\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$.

It is easily seen that $(\cdot,\cdot)_{P,s}$ is a scalar product on $P = L^2(0,T;H^2\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$. We thus introduce P_s as the completion of $L^2(0,T;H^2\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$ for the norm $\|\cdot\|_{P,s}$ associated with this scalar product, which therefore endows P_s with an Hilbert structure.

Besides, by density arguments, (16) remains true for all $s \geqslant s_0$ and for all $p \in P_s$.

We can now state the main result of this section.

Theorem 5. *For $s \geqslant s_0$, $B \in L^2(\rho_0(s), Q_T)$ and $u_0 \in L^2(\Omega)$, there exists a unique function $p_s \in P_s$, depending linearly on (B, u_0) such that*

$$(p_s, q)_{P,s} = \int_\Omega u_0 q(0) + \int_{Q_T} Bq, \quad \forall q \in P_s. \tag{20}$$

Then $v_s = -s^3 \rho_0^{-2}(s) p_s 1_\omega$ is a control function for (7), the corresponding controlled trajectory is given by $z_s = \rho^{-2}(s) L^\star p_s$, and the operator $\Lambda_s^0 : (B, u_0) \mapsto z_s$ is linear and continuous from $L^2(\rho_0(s), Q_T) \times L^2(\Omega)$ to $L^2(\rho(s), Q_T)$.

Furthermore, we have the following estimates, for some constant $C_1 = C_1(\Omega, \omega, T) > 0$ which does not depend on s:

$$\begin{aligned} \|\rho(s)\, z_s\|_{L^2(Q_T)} + s^{-1}\|\rho_1(s)\nabla z_s\|_{L^2(Q_T)^d} + s^{-1}\|\rho_1(s) z_s\|_{L^\infty(0,T;L^2(\Omega))} + s^{-3/2}\|\rho_0(s)\, v_s\|_{L^2(q_T)} \\ \leqslant C_1\big(s^{-3/2}\|\rho_0(s)B\|_{L^2(Q_T)} + s^{-1}e^{c_2 s}\|u_0\|_{L^2(\Omega)}\big) \end{aligned} \tag{21}$$

with $c_2 := \|\varphi(\cdot, 0)\|_\infty$. Eventually, $v_s \in L^{p_d}(q_T)$, and for some constant

$$\|v_s\|_{L^{p_d}(q_T)} \leqslant C\big(\|\rho_0(s)B\|_{L^2(Q_T)} + e^{c_2 s}\|u_0\|_{L^2(\Omega)}\big). \tag{22}$$

Proof. For simplicity, we divide the proof of Theorem 5 in several lemmas.

We first analyze the solvability of the equation (20).

Lemma 3. *Under the assumptions of Theorem 5, there exists a unique $p_s \in P_s$ solution of (20).*

This function p_s satisfies the following estimate (with $c_2 = \|\varphi(\cdot, 0)\|_\infty$)

$$\|p_s\|_{P_s} \leqslant C\big(s^{-3/2}\|\rho_0(s)\, B\|_{L^2(Q_T)} + s^{-1}e^{c_2 s}\|u_0\|_{L^2(\Omega)}\big). \tag{23}$$

for some constant $C = C(\Omega, \omega, T) > 0$.

Finally, p_s depends linearly on the couple (B, u_0).

Proof of Lemma 3. Since $(\cdot, \cdot)_{P_s}$ is a scalar product on P_s, Lemma 3 is a consequence of the fact that the right hand-side of (20) corresponds to a linear continuous form on P_s.

The linear map $L_1 : P_s \to \mathbb{R}$, $q \mapsto \int_{Q_T} Bq$ is continuous since $\rho(s)B \in L^2(Q_T)$. Indeed, for all $q \in P_s$

$$\Big|\int_{Q_T} Bq\Big| \leqslant \Big(\int_{Q_T} |\rho_0(s)B|^2\Big)^{1/2}\Big(\int_{Q_T} |\rho_0^{-1}(s)q|^2\Big)^{1/2}$$

and since from (16) $\Big(\int_{Q_T} |\rho_0^{-1}(s)q|^2\Big)^{1/2} \leqslant Cs^{-3/2}\|q\|_{P_s}$, it follows that

$$|L_1(q)| = \Big|\int_{Q_T} Bq\Big| \leqslant Cs^{-3/2}\|\rho_0(s)B\|_{L^2(Q_T)}\|q\|_{P_s}.$$

Thus L_1 is continuous on P_s.

From (16) we deduce that the linear map $L_2 : P_s \to \mathbb{R}$, $q \mapsto \int_\Omega u_0 q(0)$ is continuous. Indeed, using $s \geqslant 1$, we obtain for all $q \in P_s$ that:

$$\begin{aligned} |L_2(q)| &= s^{-1}e^{c_2 s}\|u_0\|_{L^2(\Omega)} s e^{-c_2 s}\|q(0)\|_{L^2(\Omega)} \\ &\leqslant s^{-1}e^{c_2 s}\|u_0\|_{L^2(\Omega)} s\|q(0)e^{-s\varphi(x,0)}\|_{L^2(\Omega)} = s^{-1}e^{c_2 s}\|u_0\|_{L^2(\Omega)} s\|\rho^{-1}(s,0)q(0)\|_{L^2(\Omega)} \\ &\leqslant Cs^{-1}e^{c_2 s}\|u_0\|_{L^2(\Omega)}\|q\|_{P_s}. \end{aligned}$$

Accordingly, the right hand-side of (20) corresponds to a linear continuous form on P_s and Riesz representation theorem gives the existence of a unique $p_s \in P_s$ solving (20), which additionally satisfies (23). □

We then establish the link between the equation (20) and the control problem (7)–(8).

Lemma 4. *Let $s \geqslant s_0$ and $p_s \in P_s$ be the unique solution of (20) given in Lemma 3 and define (z_s, v_s) by*

$$z_s = \rho^{-2}(s) L^\star p_s \quad \text{and} \quad v_s = -s^3 \rho_0^{-2}(s) p_s 1_\omega. \tag{24}$$

Then z_s solves (7)-(8) with $v = v_s$, $\rho(s) z_s \in L^2(Q_T)$, $\rho_0(s) v_s \in L^2(q_T)$, and satisfies the following estimate

$$\|\rho(s)\, z_s\|_{L^2(Q_T)} + s^{-3/2} \|\rho_0(s)\, v_s\|_{L^2(q_T)} \leqslant C_1 \big(s^{-3/2} \|\rho_0(s) B\|_{L^2(Q_T)} + s^{-1} e^{c_2 s} \|u_0\|_{L^2(\Omega)} \big). \tag{25}$$

Proof of Lemma 4. From the definition of P_s, $\rho(s) z_s \in L^2(Q_T)$ and $\rho_0(s) v_s \in L^2(q_T)$, and (25) immediately follows from (23).

In view of (20), (z_s, v_s) satisfies, for all $q \in L^2(0,T; H^2 \cap H_0^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$,

$$\int_{Q_T} z_s L^\star q = \int_{Q_T} v_s 1_\omega \, q + \int_\Omega u_0 q(0) + \int_{Q_T} B q. \tag{26}$$

Therefore, z_s is a solution of (7)-(8) corresponding to the choice $v = v_s$ in the sense of transposition. Since $(z_s, v_s) \in L^2(Q_T) \times L^2(q_T)$, by uniqueness of solutions of (7)-(8) in the sense of transpositions, z_s also solves (7)-(8) in the weak sense. □

Remark 1. *Following [19], it is not difficult to check that the functions z_s and v_s provided by Lemma 4 can also be characterized as the unique minimizer of the functional J_s defined as*

$$J_s(z,v) = \frac{s^3}{2} \int_{Q_T} \rho^2(s) |z|^2 + \frac{1}{2} \int_{q_T} \rho_0^2(s) |v|^2 \tag{27}$$

over the set

$$\Big\{ (z,v) : \rho(s) z \in L^2(Q_T),\ \rho_0(s) v \in L^2(q_T),\ (z, v1_\omega) \text{ solves (7)-(8) in the transposition sense} \Big\}.$$

The next lemma gives additional estimates on the trajectory z_s given by (4), based on the fact that it solves a parabolic equation with a source term in $L^2(Q_T)$ and an initial datum in $L^2(\Omega)$, thus belonging to the space $L^2(0,T; H_0^1(\Omega)) \cap H^1(0,T; H^{-1}(\Omega))$, and strongly inspired by [1, Theorem 2.6] in which the initial datum was zero (see also [13, Theorem 3.3], where similar results are obtained):

Lemma 5. *With the notations and assumptions of Lemma 4, we further have that $\rho_1(s) z_s \in L^\infty(0,T; L^2(\Omega))$, $\nabla z_s \in L^2(\rho_1(s), Q_T)^d$, and there exists a constant $C^\star$ independent of $s \geqslant s_0$ such that*

$$\|\rho_1(s) z_s\|_{L^\infty(0,T;L^2(\Omega))} + \|\rho_1(s) \nabla z_s\|_{L^2(Q_T)^d} \leqslant C^\star \big(s^{-1/2} \|\rho_0(s) B\|_{L^2(Q_T)} + e^{c_2 s} \|u_0\|_{L^2(\Omega)} \big). \tag{28}$$

Proof of Lemma 5. Since $z_s \in L^2(0,T; H_0^1(\Omega))$, $\partial_t z_s \in L^2(0,T; H^{-1}(\Omega))$ and $\rho_1(s) \in \mathcal{C}^\infty([0,T[; \mathcal{C}^\infty(\overline{\Omega}))$, multiplying (7) by $\rho_1^2(s) z_s$ and integrating by part we obtain in $\mathcal{D}'(0,T)$

$$\begin{aligned} &\frac{1}{2} \partial_t \int_\Omega |z_s|^2 \rho_1^2(s) + \int_\Omega \rho_1^2(s) |\nabla z_s|^2 \\ &\qquad = \int_\omega v_s \rho_1^2(s) z_s + \int_\Omega B \rho_1^2(s) z_s + \int_\Omega |z_s|^2 \rho_1(s) \partial_t \rho_1(s) - 2 \int_\Omega \rho_1(s) z_s \nabla \rho_1(s) \cdot \nabla z_s. \end{aligned} \tag{29}$$

Let us estimate each term of the right side of this inequality. We have $\partial_t \rho_1(s) = -\frac{\partial_t \theta}{\theta} \rho_1(s) + s \frac{\partial_t \theta}{\theta} \varphi \rho_1(s)$ and thus, since from the definition of φ, $\varphi \leqslant C\theta$ and $\left| \frac{\partial_t \theta}{\theta} \varphi \right| \leqslant C s \theta^2$ and since $s \geqslant 1$, $\theta \geqslant 1$

and $\rho(s)=\theta\rho_1(s)$ we deduce that, on $[0,T)$:

$$\Big|\int_\Omega |z_s|^2\rho_1(s)\partial_t\rho_1(s)\Big| \leqslant \int_\Omega \Big|\frac{\partial_t\theta}{\theta}\Big||\rho_1(s)z_s|^2 + s\int_\Omega \Big|\frac{\partial_t\theta}{\theta}\varphi\Big||\rho_1(s)z_s|^2 \leqslant Cs^2\int_\Omega \rho^2(s)|z_s|^2.$$

On the other hand

$$\nabla\rho_1(s)=\theta^{-1}\nabla\rho(s)\leqslant Cs\rho(s),$$

and thus,

$$\begin{aligned}\Big|\int_\Omega \rho_1(s)z_s\nabla\rho_1(s)\cdot\nabla z_s\Big| &\leqslant C\int_\Omega s|\rho(s)z_s|\,|\rho_1(s)\nabla z_s| \\ &\leqslant Cs^2\int_\Omega|\rho(s)z_s|^2+\frac12\int_\Omega|\rho_1(s)\nabla z_s|^2.\end{aligned}$$

Finally, since $\rho_1^2(s)=\theta^{-1/2}\rho_0(s)\rho(s)$ and $\theta^{-1/2}\leqslant 1$, we infer that

$$\Big|\int_\omega v_s\rho_1^2(s)z_s\Big|\leqslant\Big|\int_\omega\rho_0(s)v_s\theta^{-1/2}\rho(s)z_s\Big|\leqslant\|\rho_0(s)v_s\|_{L^2(\omega)}\|\rho(s)z_s\|_{L^2(\Omega)}$$

and $|\int_\Omega B\rho_1^2(s)z_s|\leqslant\|\rho_0(s)B\|_{L^2(\Omega)}\|\rho(s)z_s\|_{L^2(\Omega)}$. Thus (29) implies that

$$\partial_t\int_\Omega\rho_1^2(s)|z_s|^2+\int_\Omega\rho_1^2(s)|\nabla z_s|^2\leqslant Cs^2\|\rho(s)z_s\|^2_{L^2(\Omega)}+\Big(\|\rho_0(s)v_s\|_{L^2(\omega)}+\|\rho_0(s)B\|_{L^2(\Omega)}\Big)\|\rho(s)z_s\|_{L^2(\Omega)}$$

and therefore, since $\|\rho_1(0,s)u_0\|^2_{L^2(\Omega)}\leqslant e^{2c_2s}\|u_0\|^2_{L^2(\Omega)}$, we get, for all $t\in[0,T)$,

$$\begin{aligned}\Big(\int_\Omega\rho_1^2(s)|z_s|^2\Big)(t)+\int_{Q_t}\rho_1^2(s)|\nabla z_s|^2 &\leqslant Cs^2\|\rho(s)z_s\|^2_{L^2(Q_T)} \\ &\quad+\big(\|\rho_0(s)v_s\|_{L^2(q_T)}+\|\rho_0(s)B\|_{L^2(Q_T)}\big)\|\rho(s)z_s\|_{L^2(Q_T)}+e^{2c_2s}\|u_0\|^2_{L^2(\Omega)}.\end{aligned}$$

Since $s\geqslant 1$, using (25) we obtain, for all $t\in[0,T)$,

$$\Big(\int_\Omega\rho_1^2(s)|z_s|^2\Big)(t)+\int_{Q_t}\rho_1^2(s)|\nabla z_s|^2\leqslant C\big(s^{-1}\|\rho_0(s)B\|^2_{L^2(Q_T)}+e^{2c_2s}\|u_0\|^2_{L^2(\Omega)}\big)$$

which gives (28). □

The next result is slightly less classical and proves that the control v_s produced by Lemma 4 enjoys some nice integrability property.

Lemma 6. *Let $s\geqslant s_0$. Let v_s be given by Lemma 4. Then $v_s\in L^{p_d}(q_T)$ (recall the definition of p_d in (19)) and satisfies, for some $C>0$,*

$$\|v_s\|_{L^{p_d}(q_T)}\leqslant C(\|\rho_0(s)B\|_{L^2(Q_T)}+e^{c_2s}\|u_0\|_{L^2(\Omega)}).$$

Proof of Lemma 6. From estimates (16)–(17), for all $p\in L^2(0,T;H^2\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$,

$$\|\rho_3^{-1}(s)p\|_{H^1(0,T;L^2(\Omega))}+\|\rho_3^{-1}(s)p\|_{L^2(0,T;H^2(\Omega))}\leqslant C\|p\|_{P,s}.$$

Therefore, by density, this estimate is still true for all $p\in P_s$. Thus, by interpolation, for all $p\in P_s$, $\rho_3^{-1}p\in H^\tau(0,T;H^{2(1-\tau)}(\Omega))$ for all $\tau\in[0,1]$. In particular, with the choice $\tau=2/(d+2)$, in dimension $d\geqslant 3$, using Sobolev's embedding theorem, we get that $H^\tau(0,T)$ and $H^{2(1-\tau)}(\Omega)$ respectively embeds in $L^{p_d}(0,T)$ and $L^{p_d}(\Omega)$. Accordingly, we get, for all $p\in P_s$,

$$\|\rho_3^{-1}p\|_{L^{p_d}(Q_T)}\leqslant C\|p\|_{P_s}.$$

In dimension 1 and 2, this estimate also holds with p_d as in (19): indeed, one can simply use that $L^2(0,T;H^2\cap H_0^1(\Omega))\cap H^1(0,T;L^2(\Omega))$ embeds into $L^\infty(0,T;H_0^1(\Omega))$ and that $H_0^1(\Omega)$ embeds into $L^\infty(\Omega)$ in dimension 1, and in any $L^p(\Omega)$ with $p<\infty$ when $d=2$.

Let $p_s \in P_s$ be the unique solution of (20) given in Lemma 3, $z_s = \rho^{-2}(s)L^\star p_s$ and $v_s = -s^3\rho_0^{-2}(s)p_s 1_\omega$. Then $\rho_3^{-1}(s)p_s 1_\omega = -s^{-3}\theta^{-3}\rho^2(s)\rho_3^{-1}(s)v_s$. We then have, using $s \geqslant 1$ and (23):

$$\|s^{-3}\theta^{-3}\rho^2\rho_3^{-1}v_s\|_{L^{p_d}(q_T)} \leqslant C\|p_s\|_P \leqslant C(\|\rho_0(s)B\|_{L^2(Q_T)} + e^{c_2 s}\|u_0\|_{L^2(\Omega)}). \tag{30}$$

Now, to conclude the proof of Lemma 6, we simply check that there exists $C_3 > 0$ such that $s^{-3}\theta^{-3}\rho^2\rho_3^{-1} \geqslant C_3$. From the definition of ρ and ρ_3 we have

$$\rho^2(s)\rho_3^{-1}(s) = e^{2s\varphi}e^{-\frac{3}{2}s\varphi_*} \geqslant e^{s\theta(\frac{1}{2}\lambda e^{12\lambda} - 2e^{\lambda\hat{\psi}})} \geqslant e^{cs\theta}$$

for some positive constant $c > 0$, thus making obvious the existence of a positive C_3 such that for all $s \geqslant 1$, $s^{-3}\theta^{-3}\rho^2\rho_3^{-1} \geqslant C_3$. □

We can now conclude easily Theorem 5 by putting together Lemma 3, Lemma 4, Lemma 5 and Lemma 6. □

2.3 Additional properties of the controlled trajectories given by Theorem 5

In this section, we provide two properties of the controlled trajectory given by Theorem 5, which will be useful in our fixed point argument.

The first one concerns the integrability of the controlled trajectory. To better understand the origin of this property, we first state two classical results on the heat equation:

Lemma 7 (Maximal regularity in L^q class)**.** *[31, Theorem 9.1 p.341] Let $q \in (1,\infty)$. For all $F \in L^q(Q_T)$, the unique weak solution z_F of*

$$\begin{cases} \partial_t z_F - \Delta z_F = F \quad in \quad Q_T, \\ z_F = 0 \text{ on } \Sigma_T, \quad z_F(\cdot,0) = 0 \text{ in } \Omega \end{cases} \tag{31}$$

satisfies $z_F \in L^q(0,T;W^{2,q}(\Omega)) \cap W^{1,q}(0,T;L^q(\Omega))$ and

$$\|z_F\|_{L^q(0,T;W^{2,q}(\Omega))\cap W^{1,q}(0,T;L^q(\Omega))} \leqslant C\|F\|_{L^q(Q_T)}. \tag{32}$$

As an immediate corollary of Lemma 7, from [31, Lemma 3.4 p.83], $z \in \mathcal{C}^0([0,T];W^{2-\frac{2}{q},q}(\Omega))$ and therefore from the Sobolev-Gagliardo-Nirenberg Theorem (see [12, Theorems 6.7 and 8.2] and [6, Corollary 9.14 p. 284]) and since, for $q > \frac{d+2}{2}$, $L^q(0,T;W^{2,q}(\Omega)) \cap W^{1,q}(0,T;L^q(\Omega)) \hookrightarrow \mathcal{C}^0(\overline{Q_T})$ is compact (see [32, Corollary 8 p. 90 and Lemma 12 p. 91], we easily get the following result:

Corollary 1. *With the notations of Lemma 7, if $q > q_d$, where q_d is defined by*

$$q_d = \frac{d+2}{2}, \tag{33}$$

there exists a constant $C > 0$ such that for all $F \in L^q(Q_T)$, the solution z_F of (31) satisfies

$$\|z_F\|_{L^\infty(Q_T)} \leqslant C\|F\|_{L^q(Q_T)}. \tag{34}$$

Furthermore, if $(F_n)_{n\in\mathbb{N}}$ is a bounded sequence of $L^q(Q_T)$, $(z_{F_n})_{n\in\mathbb{N}}$ is compact in $\mathcal{C}^0(\overline{Q_T})$.

Accordingly, using (22), an interesting $L^\infty(Q_T)$ estimate on the controlled trajectory given by Theorem 5 can be proved when $u_0 \in L^\infty(\Omega)$ and $B \in L^q(Q_T)$ for $q > q_d$ when $p_d > q_d$, i.e. $d \leqslant 5$:

Theorem 6. *Within the setting of Theorem 5, when $d \leqslant 5$, if we further assume that $B \in L^q(Q_T)$ for some $q > q_d$, and $u_0 \in L^\infty(\Omega)$, we have the following additional estimate: there exists a constant $C_q > 0$ such that the controlled trajectory provided by Theorem 5 belongs to $L^\infty(Q_T)$ and satisfies:*

$$\|z_s\|_{L^\infty(Q_T)} \leqslant C_q(\|B\|_{L^q(Q_T)} + \|\rho_0(s)B\|_{L^2(Q_T)} + e^{c_2 s}\|u_0\|_{L^\infty(\Omega)}). \tag{35}$$

Proof. According to Theorem 5, the control v_s provided by Theorem 5 belongs to $L^{p_d}(Q_T)$ and its $L^{p_d}(Q_T)$ norm is bounded by $C\left(\|\rho_0(s)B\|_{L^2(Q_T)} + e^{c_2 s}\|u_0\|_{L^2(\Omega)}\right)$. Since $d \leqslant 5$, $p_d > q_d$, and we can then use Corollary 1 to estimate the solution z_F of (31) corresponding to the choice $F = B + v_s 1_\omega$:

$$\|z_F\|_{L^\infty(Q_T)} \leqslant C(\|B\|_{L^q(Q_T)} + \|\rho_0(s)B\|_{L^2(Q_T)} + e^{c_2 s}\|u_0\|_{L^2(\Omega)}). \tag{36}$$

We then write that the controlled trajectory z_s given by Theorem 5 as $z_F + z_{u_0}$, where z_{u_0} is the solution of

$$\begin{cases} \partial_t z_{u_0} - \Delta z_{u_0} = 0 \quad \text{in} \quad Q_T, \\ z_{u_0} = 0 \text{ on } \Sigma_T, \quad z_{u_0}(\cdot, 0) = u_0 \text{ in } \Omega, \end{cases} \tag{37}$$

for which the maximum principle immediately yields $z_{u_0} \in L^\infty(Q_T)$ and

$$\|z_{u_0}\|_{L^\infty(Q_T)} \leqslant \|u_0\|_{L^\infty(\Omega)}. \tag{38}$$

Combining the estimates (36) and (38) easily gives the estimate (35). □

3 A controllability result for (1) with $f \in \mathcal{C}^0(\mathbb{R})$: proof of the first item of Theorem 4

The goal of this section is to prove the first item of Theorem 4, in a slightly more precise form, namely:

Theorem 7. *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $1 \leqslant d \leqslant 5$. There exists $\beta^\star > 0$ (given in (50) afterwards) such that if the function $f \in \mathcal{C}^0(\mathbb{R})$ satisfies $(\mathbf{H_2})$, for all $u_0 \in L^\infty(\Omega)$, there exists a control $v \in L^{p_d}(q_T)$ and a solution $y \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$ of (1) associated with $v1_\omega$ such that $y(\cdot,T) = 0$. Moreover, there exists $s \geqslant 1$ such that $(y,v) \in L^2(\rho(s),Q_T) \times L^2(\rho_0(s),q_T)$, where ρ and ρ_0 are respectively defined in (12) and (15).*

Remark 2. *We emphasize that, contrary to Theorem 1, we do not assume that f is locally Lipschitz continuous. In particular, we do not assume a priori the existence of local solutions in time for (1).*

Remark 3. *We also emphasize that the controllability time is arbitrary. However, the specific structure of the control v_s together with lower bound on s forces the control to act from the beginning so as to prevent the solution to blow up.*

For $s \geqslant s_0$ a parameter to be fixed later, for all $\widehat{y} \in L^2(\rho_0(s),Q_T) \cap L^\infty(Q_T)$, we solve the linearized null controllability problem

$$\begin{cases} \partial_t y - \Delta y = v1_\omega - f(\widehat{y}) \quad \text{in} \quad Q_T, \\ y = 0 \text{ on } \Sigma_T, \ y(.,0) = u_0 \text{ in } \Omega, \\ y(\cdot,T) = 0 \text{ in } \Omega \end{cases} \tag{39}$$

and we will prove the existence of a fixed point of

$$\Lambda_s : \widehat{y} \ \mapsto y, \qquad \text{(equivalently, with the notation of Theorem 5, } \Lambda_s(\widehat{y}) = \Lambda_s^0(-f(\widehat{y}), u_0)) \tag{40}$$

where y is the solution of the null controllability problem (39) associated with $v1_\omega$ given in Theorem 5 for $B = -f(\widehat{y})$.

In order to that, we will employ the Schauder fixed point theorem.

To be more precise, for $s \geqslant s_0$, we first introduce the Banach space $L^\infty(Q_T) \cap L^2(\rho_0(s),Q_T)$, endowed with the norm

$$|||y|||_s := \|y\|_{L^\infty(Q_T)} + \|y\|_{L^2(\rho_0(s),Q_T)}.$$

For $R > 0$ and $s \geqslant s_0$, we then introduce the class $\mathcal{C}_R(s)$, defined as the non empty closed convex set of $L^2(\rho_0(s), Q_T) \cap L^\infty(Q_T)$ given by

$$\mathcal{C}_R(s) = \Big\{ y \in L^\infty(Q_T), \quad \|y\|_{L^\infty(Q_T)} \leqslant R, \quad \|\rho_0(s) y\|_{L^2(Q_T)} \leqslant R^{1/2} \Big\}. \tag{41}$$

Our goal will then be to prove that:

- the map Λ_s is well-defined on $L^2(\rho_0(s), Q_T) \cap L^\infty(Q_T)$; see Section 3.1;
- If $\beta^\star > 0$ is small enough in $(\mathbf{H_2})$, there exists s and R large enough so that $\Lambda_s(\mathcal{C}_R(s)) \subset \mathcal{C}_R(s)$; see Section 3.2;
- $\Lambda_s(\mathcal{C}_R(s))$ is compact in $\mathcal{C}_R(s)$ for the topology induced by $|||\cdot|||_s$; see Section 3.3;
- Λ_s is continuous on $\mathcal{C}_R(s)$ for the topology induced by $|||\cdot|||_s$; see Section 3.4.

Accordingly, by Schauder fixed point theorem, Λ_s will have a fixed point on $\mathcal{C}_R(s)$, and this fixed point will provide the controlled trajectory y of Theorem 7, see Section 3.5.

3.1 The map Λ_s is well-defined on $L^2(\rho_0(s), Q_T) \cap L^\infty(Q_T)$

To properly define the map Λ_s, we need to check the following lemma:

Lemma 8. *Let $d \leqslant 5$ and $2 \leqslant q \leqslant +\infty$. Under the assumptions of Theorem 7. For all $s \geqslant s_0$, $\widehat{y} \in L^2(\rho_0(s), Q_T) \cap L^\infty(Q_T)$, $f(\widehat{y}) \in L^2(\rho_0(s), Q_T) \cap L^q(Q_T)$ and*

$$\|\rho_0(s) f(\widehat{y})\|_{L^2(Q_T)} \leqslant \big(\alpha + \beta^\star \ln_+ \big(\|\widehat{y}\|_{L^\infty(Q_T)}\big)\big)^{3/2} \|\rho_0(s)\widehat{y}\|_{L^2(Q_T)}, \tag{42}$$

$$\|f(\widehat{y})\|_{L^q(Q_T)} \leqslant \big(\alpha + \beta^\star \ln_+ \big(\|\widehat{y}\|_{L^\infty(Q_T)}\big)\big)^{3/2} \|\rho_0(s)\widehat{y}\|^{2/q}_{L^2(Q_T)} \|\widehat{y}\|^{1-2/q}_{L^\infty(Q_T)}. \tag{43}$$

Proof. Assumption $(\mathbf{H_2})$ immediately implies (42).

To estimate $f(\widehat{y})$ in $L^q(Q_T)$, we write that $\|f(\widehat{y})\|_{L^q(Q_T)} \leqslant \|f(\widehat{y})\|^\theta_{L^2(Q_T)} \|f(\widehat{y})\|^{1-\theta}_{L^\infty(Q_T)}$ where $\dfrac{1}{q} = \dfrac{\theta}{2} + \dfrac{1-\theta}{\infty}$, that is $\theta = \dfrac{2}{q}$. Since $\rho_0 \geqslant 1$, we obviously have from (42) that

$$\|f(\widehat{y})\|_{L^2(Q_T)} \leqslant \|\rho_0 f(\widehat{y})\|_{L^2(Q_T)} \leqslant \big(\alpha + \beta^\star \ln_+ \big(\|\widehat{y}\|_{L^\infty(Q_T)}\big)\big)^{3/2} \|\rho_0(s)\widehat{y}\|_{L^2(Q_T)},$$

while we easily have

$$\|f(\widehat{y})\|_{L^\infty(Q_T)} \leqslant \big(\alpha + \beta^\star \ln_+ \big(\|\widehat{y}\|_{L^\infty(Q_T)}\big)\big)^{3/2} \|\widehat{y}\|_{L^\infty(Q_T)}.$$

These last estimates easily give (43). □

An immediate consequence of Lemma 8, Theorem 5 and Theorem 6 is the following one:

Proposition 1. *Under the assumptions of Theorem 7, the map Λ_s defined in (40) as the solution of the null-controllability problem (39) using Theorem 5 is well-defined for $s \geqslant s_0$ on $L^2(\rho_0(s), Q_T) \cap L^\infty(Q_T)$, and for $\widehat{y} \in L^2(\rho_0(s), Q_T) \cap L^\infty(Q_T)$, $y = \Lambda_s(\widehat{y})$ satisfies the following estimates:*

$$\begin{aligned}\|\rho(s)\, y\|_{L^2(Q_T)} + s^{-1}\|\rho_1(s)\nabla y\|_{L^2(Q_T)^d} + s^{-1}\|\rho_1(s) y\|_{L^\infty(0,T;L^2(\Omega))} \\ \leqslant C_1 s^{-3/2}\big(\alpha + \beta^\star \ln_+\big(\|\widehat{y}\|_{L^\infty(Q_T)}\big)\big)^{3/2}\|\rho_0(s)\widehat{y}\|_{L^2(Q_T)} + C_1 s^{-1} e^{c_2 s}\|u_0\|_{L^2(\Omega)}, \quad (44)\end{aligned}$$

and for $q > \max\{q_d, 2\}$, we have

$$\begin{aligned}\|y\|_{L^\infty(Q_T)} \leqslant C_q\big(\alpha + \beta^\star \ln_+\big(\|\widehat{y}\|_{L^\infty(Q_T)}\big)\big)^{3/2}\|\rho_0(s)\widehat{y}\|^{2/q}_{L^2(Q_T)}\|\widehat{y}\|^{1-2/q}_{L^\infty(Q_T)} \\ + C_q\big(\alpha + \beta^\star \ln_+\big(\|\widehat{y}\|_{L^\infty(Q_T)}\big)\big)^{3/2}\|\rho_0(s)\widehat{y}\|_{L^2(Q_T)} + C_q e^{c_2 s}\|u_0\|_{L^\infty(\Omega)}. \quad (45)\end{aligned}$$

In particular, Λ_s maps $L^2(\rho_0(s), Q_T) \cap L^\infty(Q_T)$ into itself.

Furthermore, the control v provided by Theorem 5 satisfies

$$\|v\|_{L^{p_d}(q_T)} \leqslant C\bigg(\big(\alpha + \beta^\star \ln_+\big(\|\widehat{y}\|_{L^\infty(Q_T)}\big)\big)^{3/2}\|\rho_0(s)\widehat{y}\|_{L^2(Q_T)} + e^{c_2 s}\|u_0\|_{L^2(\Omega)}\bigg). \quad (46)$$

3.2 A stable class for suitable choices of parameters

Our next goal is to show the following result:

Lemma 9. *Under the assumptions of Theorem 7, if $\beta^\star$ in $(\mathbf{H_1})$ is small enough, there exist s and $R > 0$ such that*

$$\Lambda_s(\mathcal{C}_R(s)) \subset \mathcal{C}_R(s), \quad (47)$$

where the class $\mathcal{C}_R(s)$ is given in (41).

Proof. Of course, the proof relies on the estimates given by Proposition 1.

We fix $q > \max\{q_d, 2\}$ with q finite. Then, for $s \geqslant s_0$ and $\widehat{y} \in \mathcal{C}_R(s)$, we have from (44) (recalling $s \geqslant s_0 \geqslant 1$ and the inequality $\rho_0 \leqslant \rho$) that $y = \Lambda_s(\widehat{y})$ satisfies

$$\|\rho_0(s)\, y\|_{L^2(Q_T)} \leqslant \|\rho(s)\, y\|_{L^2(Q_T)} \leqslant C_1 s^{-3/2}\big(\alpha + \beta^\star \ln_+\big(R\big)\big)^{3/2} R^{1/2} + C_1 e^{c_2 s}\|u_0\|_{L^2(\Omega)}.$$

On the other hand, for $s \geqslant s_0$ and $\widehat{y} \in \mathcal{C}_R(s)$, from (45), $y = \Lambda_s(\widehat{y})$ satisfies

$$\|y\|_{L^\infty(Q_T)} \leqslant C_q\big(\alpha + \beta^\star \ln_+\big(R\big)\big)^{3/2}(R^{1-1/q} + R^{1/2}) + C_q e^{c_2 s}\|u_0\|_{L^\infty(\Omega)}.$$

In view of these two estimates, it is natural to impose the relation

$$s = \frac{1}{4c_2}\ln_+\big(R\big), \quad (48)$$

taking R large enough so that $s \geqslant s_0$.

Indeed, with this relation, the two above estimates yield that for $\widehat{y} \in \mathcal{C}_R(s)$, $y = \Lambda_s(y)$ satisfies:

$$\begin{aligned}\|\rho_0(s)\, y\|_{L^2(Q_T)} &\leqslant C_1(4c_2)^{3/2}\left(\frac{\alpha}{\ln_+(R)} + \beta^\star\right)^{3/2} R^{1/2} + C_1 R^{1/4}\|u_0\|_{L^2(\Omega)}, \\ \|y\|_{L^\infty(Q_T)} &\leqslant C_q\big(\alpha + \beta^\star \ln_+\big(R\big)\big)^{3/2}(R^{1-1/q} + R^{1/2}) + C_q R^{1/4}\|u_0\|_{L^\infty(\Omega)}.\end{aligned}$$

Therefore, if $(2C_1)^{2/3}4c_2\beta^\star < 1$, that is if $\beta^\star$ is small enough, for R large enough, we have

$$\begin{cases} C_1(4c_2)^{3/2}\left(\dfrac{\alpha}{\ln_+(R)}+\beta^\star\right)^{3/2} \leqslant \dfrac{1}{2}, \\ C_1R^{1/4}\|u_0\|_{L^2(\Omega)} \leqslant \dfrac{1}{2}R^{1/2}. \\ C_q\big(\alpha+\beta^\star\ln_+\left(R\right)\big)^{3/2}(R^{1-1/q}+R^{1/2}) \leqslant \dfrac{1}{2}R, \\ C_qR^{1/4}\|u_0\|_{L^\infty(\Omega)} \leqslant \dfrac{1}{2}R. \end{cases} \tag{49}$$

According to the previous estimates, we have thus proved that if $(2C_1)^{2/3}4c_2\beta^\star < 1$, imposing the relation (48), and taking R large enough so that $s \geqslant s_0$ and R satisfies (49), for all $\widehat{y} \in \mathcal{C}_R(s)$, $y = \Lambda_s(\widehat{y})$ belongs to $\mathcal{C}_R(s)$. □

Remark 4. *The smallness condition on $\beta^\star$ is explicit:*

$$\beta^\star < \frac{1}{4c_2(2C_1)^{2/3}}, \tag{50}$$

where the constants C_1 and c_2 are the constants appearing in Theorem 5.

Remark 5. *In fact, the above proof shows that, provided we impose the relation (48), $\mathcal{C}_R(s)$ is stable for Λ_s for any $R \geqslant R_0$ (equivalently $s \geqslant s_0$) for a suitably large R_0 (equivalently s_0). Furthermore, with the above choices, in view of (49), the lower bound R_0 depends on $\|u_0\|_{L^\infty(\Omega)}$ as a power of $\|u_0\|_{L^\infty(\Omega)}$, so that the lower bound s_0 can be chosen as depending logarithmically on $\|u_0\|_{L^\infty(\Omega)}$.*

3.3 $\Lambda_s(\mathcal{C}_R(s))$ is a relatively compact subset of $\mathcal{C}_R$ for the norm $|||\cdot|||_s$.

Proposition 2. *Under the assumptions of Lemma 9, $\Lambda_s(\mathcal{C}_R(s))$ is a relatively compact subset of $\mathcal{C}_R(s)$ for the norm $|||\cdot|||_s$.*

Proof. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of $\Lambda_s(\mathcal{C}_R)$ bounded for the norm $|||\cdot|||_s$. We have to prove that there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ of $(y_n)_{n\in\mathbb{N}}$ and $y \in \mathcal{C}_R$ such that $y_{n_k} \to y$ in $L^2(\rho_0(s), Q_T)$ and $y_{n_k} \to y$ in $L^\infty(Q_T)$.

First, since $\Lambda_s(\mathcal{C}_R(s)) \subset \mathcal{C}_R(s)$, $(y_n)_{n\in\mathbb{N}}$ is a bounded sequence of the Hilbert space $L^2(\rho_0(s), Q_T)$, there exists a subsequence $(y_{n_k})_{k\in\mathbb{N}}$ and $\widetilde{y} \in L^2(\rho_0(s), Q_T)$ such that $(y_{n_k})_{k\in\mathbb{N}}$ weakly converge to $\widetilde{y}$ in $L^2(\rho_0(s), Q_T)$, and therefore $\|\widetilde{y}\|_{L^2(\rho_0(s),Q_T)} \leqslant R^{1/2}$.

Furthermore, since $(y_{n_k})_{k\in\mathbb{N}}$ is bounded in $L^\infty(Q_T)$, extracting another subsequence if necessary, which we still denote the same for simplicity, the sequence $(y_{n_k})_{k\in\mathbb{N}}$ $\star$-weakly converges to $\widetilde{y}$ in $L^\infty(Q_T)$ and we also have $\|\widetilde{y}\|_{L^\infty(Q_T)} \leqslant R$.

On the other hand, there exists $(\widehat{y}_{n_k})_{k\in\mathbb{N}}$ a sequence of $\mathcal{C}_R(s)$ such that $y_{n_k} = \Lambda_s(\widehat{y}_{n_k})$ for all $k \in \mathbb{N}$, meaning that there exists $v_{n_k} \in L^{p_d}(q_T)$ (recall (46)) such that

$$\begin{cases} \partial_t y_{n_k} - \Delta y_{n_k} = v_{n_k}1_\omega - f(\widehat{y}_{n_k}) \quad \text{in} \quad Q_T, \\ y_{n_k} = 0 \text{ on } \Sigma_T,\ y_{n_k}(.,0) = u_0 \text{ in } \Omega. \end{cases}$$

In particular, each y_{n_k} can be decomposed as $y_{n_k} = z_{F_{n_k}} + z_{u_0}$ where $z_{F_{n_k}}$ solves (31) with $F_{n_k} = v_{n_k}1_\omega - f(\widehat{y}_{n_k})$ and z_{u_0} as in (37).

Since $(\widehat{y}_{n_k})_{k\in\mathbb{N}}$ belongs to $\mathcal{C}_R(s)$, from (43), $(f(\widehat{y}_{n_k}))_{k\in\mathbb{N}}$ is bounded in $L^\infty(Q_T)$, and from (46), $(v_{n_k}1_\omega)_{k\in\mathbb{N}}$ is bounded in $L^{p_d}(Q_T)$. Since $d \leqslant 5$, $p_d > q_d$, and thus, according to Corollary 1, there exists a subsequence, also denoted by $(z_{F_{n_k}})_{k\in\mathbb{N}}$ and $z \in \mathcal{C}^0(\overline{Q_T})$ such that $(z_{F_{n_k}})_{k\in\mathbb{N}}$ converges to z in $\mathcal{C}^0(\overline{Q_T})$ as $k \to \infty$.

Since z_{u_0} is independent of n and belongs to $L^\infty(Q_T)$ by the maximum principle, $(y_{n_k})_{k\in\mathbb{N}}$ strongly converges to $y = z + z_{u_0}$ in $L^\infty(Q_T)$. By uniqueness of the limit in $\mathcal{D}'(Q_T)$, $\widetilde{y} = y$.

Furthermore, it is easy to check from the above arguments that a.e in $t \in [0,T]$, $y_{n_k}(t) \to y(t)$ strongly in $L^\infty(\Omega)$.

To finish the proof, it suffices to prove that $y_{n_k} \to y$ in $L^2(\rho_0(s), Q_T)$. Since $(y_{n_k})_{k\in\mathbb{N}}$ weakly converge to y in $L^2(\rho_0(s), Q_T)$ it suffices to prove that $\|y_{n_k}\|_{L^2(\rho_0(s),Q_T)} \to \|y\|_{L^2(\rho_0(s),Q_T)}$.

From the previous step, we have the strong convergence $y_{n_k}(t) \to y(t)$ a.e in $t \in [0,T]$ in $L^\infty(\Omega)$ and thus, since for all $t \in [0,T)$, $\rho_0(\cdot,t,s) \in \mathcal{C}^0(\overline{\Omega})$, $\rho_0(\cdot,t,s)y_{n_k}(t) \to \rho_0(\cdot,t,s)y(t)$ in $L^2(\Omega)$ a.e in $t \in [0,T)$.

Moreover we deduce from (44) and the fact that $(\widehat{y}_{n_k})_{k\in\mathbb{N}}$ is a sequence in $\mathcal{C}_R(s)$ that there exists a constant $C > 0$ such that for every $k \in \mathbb{N}$ $\|\rho_1(s)y_{n_k}(t)\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C$. Since $\rho_0 \leqslant \rho_1$, this obviously implies that $\sup_{k\in\mathbb{N}} \|\rho_0(s)y_{n_k}(t)\|_{L^\infty(0,T;L^2(\Omega))} < \infty$.

We then deduce from the Lebesgue dominated convergence theorem that $\|\rho_0(s)y_{n_k}\|_{L^2(\Omega)} \to \|\rho_0(s)y\|_{L^2(\Omega)}$ in $L^2(0,T)$, which gives the strong convergence $y_{n_k} \to y$ in $L^2(\rho_0(s), Q_T)$ as $k \to \infty$. Therefore $\Lambda_s(\mathcal{C}_R(s))$ is a relatively compact subset of $\mathcal{C}_R(s)$ for the norm $|||\cdot|||_s$. □

3.4 $\Lambda_s : \mathcal{C}_R(s) \to \mathcal{C}_R(s)$ is a continuous mapping for the norm $|||\cdot|||_s$.

We have the following result:

Proposition 3. *Under the asumption of Lemma 9, $\Lambda_s : \mathcal{C}_R(s) \to \mathcal{C}_R(s)$ is a continuous mapping for the norm $|||\cdot|||_s$.*

Proof. For all $n \in \mathbb{N}^\star$, let $(\widehat{y}_n)_{n\in\mathbb{N}^\star}$ be a sequence of $\mathcal{C}_R(s)$ and $\widehat{y}_0 \in \mathcal{C}_R(s)$ such that $\widehat{y}_n \to \widehat{y}_0$ for the norm $|||\cdot|||_s$.

Setting $y_n = \Lambda_s(\widehat{y}_n)$ for all $n \in \mathbb{N}$, let us prove that $y_n \to y_0$ for the norm $|||\cdot|||_s$.

First, recall that $\Lambda_s(\widehat{y}_n) = \Lambda_s^0(-f(\widehat{y}_n), u_0)$, where Λ_s^0 is the operator given in Theorem 5, which is continuous from $L^2(\rho_0(s), Q_T) \times L^2(\Omega)$ to $L^2(\rho(s), Q_T)$.

Accordingly, if we manage to show that the sequence $(f(\widehat{y}_n))_{n\in\mathbb{N}^\star}$ weakly converges to $f(\widehat{y}_0)$ in $L^2(\rho_0(s), Q_T)$, the sequence (y_n) would weakly converge to y_0 in $L^2(\rho(s), Q_T)$ as $n \to \infty$, and thus in $L^2(\rho_0(s), Q_T)$ since $\rho_0 \leqslant \rho$. Furthermore, from the compactness of $\Lambda_s(\mathcal{C}_R(s))$ for the topology induced by $|||\cdot|||_s$ (proved in Proposition 2), this would entail the continuity of Λ_s on $\mathcal{C}_R(s)$ for the topology induced by $|||\cdot|||_s$.

Therefore, it is sufficient to prove that the sequence $(f(\widehat{y}_n))_{n\in\mathbb{N}^\star}$ weakly converges to $f(\widehat{y}_0)$ in $L^2(\rho_0(s), Q_T)$ as $n \to \infty$.

First, $\widehat{y}_n \to \widehat{y}_0$ in $L^\infty(Q_T)$ and f being continuous, it is uniformly continuous on $[-R, R]$. Therefore, the sequence $(f(\widehat{y}_n))_{n\in\mathbb{N}^\star}$ converges to $f(\widehat{y}_0)$ in $L^\infty(Q_T)$.

Secondly, since $(f(\widehat{y}_n))_{n\in\mathbb{N}^\star}$ is bounded in $L^2(\rho_0(s), Q_T)$ (recall (42) and that $(\widehat{y}_n)_{n\in\mathbb{N}^\star}$ is a sequence of $\mathcal{C}_R(s)$), there exists a subsequence $(f(\widehat{y}_{n_k}))_{k\in\mathbb{N}^\star}$ and $Y \in L^2(\rho_0(s), Q_T)$ such that $(f(\widehat{y}_{n_k}))_{k\in\mathbb{N}^\star}$ weakly converge to Y in $L^2(\rho_0(s), Q_T)$.

By uniqueness of the limit in $\mathcal{D}'(Q_T)$, $Y = f(\widehat{y}_0)$ and all the sequence $(f(\widehat{y}_n))_{n\in\mathbb{N}^\star}$ weakly converges to $f(\widehat{y}_0)$ in $L^2(\rho_0(s), Q_T)$. As said above, this suffices to conclude the proof of Proposition 3. □

3.5 Conclusion: proof of Theorem 7.

Taking $\beta^\star$ small enough so that Lemma 9 applies, with s and R given by Lemma 9, we can apply Schauder fixed point theorem to Λ_s on $\mathcal{C}_R(s)$. Indeed, we have:

- $\mathcal{C}_R(s)$ is a closed convex set of the Banach space $L^2(\rho_0(s), Q_T) \cap L^\infty(Q_T)$;
- $\mathcal{C}_R(s)$ is stable by Λ_s by Lemma 9;

- Λ_s is continuous on $\mathcal{C}_R(s)$ for the topology induced by $|||\cdot|||_s$ from Proposition 3;
- $\Lambda_s(\mathcal{C}_R(s))$ is compact for the topology induced by $|||\cdot|||_s$ from Proposition 2.

Therefore, we deduce from the Schauder fixed-point theorem that there exists a fixed point $y \in \mathcal{C}_R(s)$ of Λ_s. By construction of Λ_s, there exists $v_s \in L^2(\rho_0(s), q_T) \cap L^{p_d}(q_T)$ such that y is the solution of the null controllability problem (39) with $v = v_s$ and $\widehat{y} = y$.

4 A contracting property for the fixed point operator Λ_s: Proof of the second item of Theorem 4

In this section, under additional regularity assumptions on the nonlinear function f, we prove that the operator Λ_s used in the previous section to prove Theorem 7 using Schauder fixed point theorem is a contracting mapping leading to a constructive method to find its fixed point.

In this section, we assume that the nonlinear function f is locally Lipschitz-continuous and satisfies the hypothesis $(\mathbf{H_1'})$ with $\beta^\star$ as in (50).

Since $(\mathbf{H_1'})$ obviously implies $(\mathbf{H_2})$ with the same $\beta^\star$ (recall $f(0) = 0$), Theorem 7 obviously applies and we can follow the main steps of its proof. Therefore, we define Λ_s as in (40), $\mathcal{C}_R(s)$ as in (41), and we choose the parameters s and R such that Lemma 9 applies.

Our goal then is to check the following property:

Proposition 4. *Assume that f satisfies $(\mathbf{H_1'})$ with $\beta^\star$ defined in (50) and s and R chosen as in Lemma 9. Then, for $\widehat{y}_1, \widehat{y}_2 \in \mathcal{C}_R(s)$, we have*

$$\|\Lambda_s(\widehat{y}_2) - \Lambda_s(\widehat{y}_1)\|_{L^2(\rho_0(s),Q_T)} \leqslant \frac{1}{2}\|\widehat{y}_2 - \widehat{y}_1\|_{L^2(\rho_0(s),Q_T)}. \tag{51}$$

In particular, Λ_s is a contraction mapping from $\mathcal{C}_R(s)$ into itself for the weighted norm $L^2(\rho_0(s), Q_T)$.

Proof. Let $\widehat{y}_1, \widehat{y}_2 \in \mathcal{C}_R(s)$, and recall that for $i = 1, 2$, $\Lambda_s(\widehat{y}_i) = \Lambda_s^0(-f(\widehat{y}_i), u_0)$. Therefore, $\Lambda_s(\widehat{y}_2) - \Lambda_s(\widehat{y}_1) = \Lambda_s^0(f(\widehat{y}_1) - f(\widehat{y}_2), 0)$. According to Theorem 5, using that $\rho_0 \leqslant \rho$, we thus have:

$$\|\Lambda_s(\widehat{y}_2) - \Lambda_s(\widehat{y}_1)\|_{L^2(\rho_0(s),Q_T)} \leqslant C_1 s^{-3/2}\|f(\widehat{y}_2) - f(\widehat{y}_1)\|_{L^2(\rho_0(s),Q_T)}.$$

In particular, taking $\widehat{y}_i \in \mathcal{C}_R(s)$ for $i = 2$, we can use $(\mathbf{H_1'})$ to deduce

$$\|f(\widehat{y}_2) - f(\widehat{y}_1)\|_{L^2(\rho_0(s),Q_T)} \leqslant (\alpha + \beta^\star \ln_+(R))^{3/2}\|\widehat{y}_2 - \widehat{y}_1\|_{L^2(\rho_0(s),Q_T)}. \tag{52}$$

Now, with s and R as in Lemma 9, i.e. as in (48), we immediately derive

$$\|\Lambda_s(\widehat{y}_2) - \Lambda_s(\widehat{y}_1)\|_{L^2(\rho_0(s),Q_T)} \leqslant C_1(4c_2)^{3/2}\left(\frac{\alpha}{\ln_+(R)} + \beta^\star\right)^{3/2}\|\widehat{y}_2 - \widehat{y}_1\|_{L^2(\rho_0(s),Q_T)}. \tag{53}$$

In view of the constraint $(49)_{(1)}$ satisfied by s and R in Lemma 9, we immediately deduce (51). □

Remark 6. *Interestingly, estimate (53) underlines that the Lipschitz constant of Λ_s seems to get smaller as $R \to \infty$ (which is allowed, recall Remark 5), that is, in view of the relation (48), when $s \to \infty$, and approach the constant $C_1(4c_2\beta^*)^{3/2}$ at the limit. In particular, if one considers a semilinear function f which satisfies*

$$\lim_{|r|\to\infty} \frac{|f'(r)|}{\ln_+^{3/2}|r|} = 0,$$

then, for any $\varepsilon > 0$, we can guarantee that Λ_s gets ε-contractive by taking s large enough. This suggests that the speed of convergence of the sequence $(y_k)_{k\in\mathbb{N}}$ increases with s, a fact which we will illustrate numerically afterwards.

As as corollary of the previous result and the classical Banach-Picard's fixed point theorem, the contraction property of the operator Λ_s for $\beta^\star$ small enough given in (50) and s and R given by Lemma 9 allows to define a convergent sequence $(y_k, v_k)_{k\in\mathbb{N}}$ to a controlled pair for (1) and prove the following precise version of the second item of Theorem 4:

Theorem 8. *Let $d \leqslant 5$ and $u_0 \in L^\infty(\Omega)$. Assume that f is locally Lipschitz continuous and satisfies $(\mathbf{H_1'})$ with $\beta^\star$ given in (50), and let s and R chosen as in Lemma 9. Then, for any $y_0 \in \mathcal{C}_R(s)$, the sequence $(y_k)_{k\in\mathbb{N}^\star} \in \mathcal{C}_R(s)$ given by*

$$y_{k+1} = \Lambda_s(y_k), \quad k \in \mathbb{N}, \tag{54}$$

where Λ_s is defined as in (40) (relying on Λ_s^0 defined in Theorem 5) together with the corresponding sequence of controls $(v_k)_{k\in\mathbb{N}^\star}$ strongly converge in $L^2(\rho_0(s), Q_T) \times L^2(\rho_0(s), q_T)$ to a controlled pair solution for (1). Moreover, the convergence is at least linear.

Proof. The $L^2(\rho_0(s), Q_T)$ convergence of the sequence $(y_k)_{k\in\mathbb{N}}$ toward $y = \Lambda_s(y) \in \mathcal{C}_R(s)$ at a linear rate follows from the contracting property of Λ_s. From the definition of Λ_s in (40), let $v \in L^2(\rho_0(s), q_T)$ associated with y so that $y - y_k$ solves for all $k \in \mathbb{N}^\star$

$$\begin{cases} \partial_t(y - y_k) - \Delta(y - y_k) = (v - v_k)1_\omega - (f(y) - f(y_{k-1})) \quad \text{in} \quad Q_T, \\ y - y_k = 0 \text{ on } \Sigma_T, \ (y - y_k)(.,0) = 0 \text{ in } \Omega, \\ (y - y_k)(\cdot, T) = 0 \text{ in } \Omega. \end{cases} \tag{55}$$

Lemma 4 and estimate (52) then give that

$$\begin{aligned} \|\rho_0(s)\,(v - v_k)\|_{L^2(q_T)} &\leqslant C_1 \|\rho_0(s)(f(y) - f(y_{k-1}))\|_{L^2(Q_T)} \\ &\leqslant C_1(\alpha + \beta^\star \ln_+(R))^{3/2} \|\rho_0(s)(y - y_{k-1})\|_{L^2(Q_T)} \end{aligned}$$

and therefore the convergence at a linear rate in $L^2(\rho_0(s), q_T)$ of the sequence $(v_k)_{k\in\mathbb{N}^\star}$ toward the null control v for system (1). □

5 Numerical illustrations

We illustrate in this section our results of convergence, precisely Theorem 8 by computing a sequence $(y_k, v_k)_{k\in\mathbb{N}^\star}$ solution of

$$\begin{cases} \partial_t y_k - \Delta y_k = v_k 1_\omega - f(y_{k-1}), \\ y_k = 0 \text{ on } \Sigma_T, \ y_k(.,0) = u_0 \text{ in } \Omega, \\ y_k(.,T) = 0 \end{cases} \tag{56}$$

obtained through the solution of the variational formulation (20) with $B = -f(y_{k-1})$. We first provide some practical details of the algorithm then discuss some experiments in the one dimensional case performed with the software Freefem++ (see [22]).

Approximations of null controls for the (linear) heat equation is a delicate issue: we mention the seminal work [7] dealing with the control of minimal L^2-norm which is very oscillatory near the final time $t = T$ and therefore difficult to construct (see also [24, 30] where this is discussed at length). On the other hand, the introduction of Carleman weights in the cost functional J corresponding to the minimization of weighted L^2-norms (as in Remark 1) leads to robust method and strong convergent approximations with respect to the discretization parameter. We refer to [15, 16]. For each $k \in \mathbb{N}^\star$, the construction of the pair (y_k, v_k) is based on a suitable discretization of (20). Here, we display numerical evidences of the convergence as $k \to \infty$.

5.1 Construction of the sequence $(y_k, v_k)_{k\in\mathbb{N}}$

Based on Theorem 5, for $k \in \mathbb{N}$, the controlled trajectory y_k and the corresponding control function v_k are computed as follows.

For s large enough according to Remark 5, we first find $p_k \in P_s$ the solution of the variational formulation:

$$(p_k, \overline{p})_{P,s} = -\int_{Q_T} f(y_{k-1})\overline{p} + \int_\Omega u_0\overline{p}(0) \quad \forall \overline{p} \in P_s. \tag{57}$$

Then, we simply set

$$y_k = \rho^{-2}(s)L^\star p_k, \quad v_k = -s^3\rho_0^{-2}(s)p_k\, 1_\omega.$$

The numerical approximation of the variational formulation (57) (of second order in time and fourth order in space) has been discussed at length in [15]. In particular, a conformal parametrized approximation, say P_h of P_s, leads to a strong convergent approximation $p_{k,h}$ of p_k as the discretization parameter h goes to 0, i.e. $\|p_{k,h} - p_k\|_{P,s} \to 0$ as $h \to 0$. From $p_{k,h}$, an approximation of the controlled state is given by $(y_{k,h}, v_{k,h}) := (\rho^{-2}(s)L^\star p_{k,h}, -s^3\rho_0^{-2}(s)p_{k,h}\, 1_\omega)$.

As discussed in [15, 16], in order to solve (57), it is very convenient to preliminary perform the change of variable

$$m_k = \rho_0^{-1}(s)p_k, \quad z_k = \rho^{-1}(s)L^\star p_k$$

so that $z_k = \rho^{-1}(s)L^\star(\rho_0(s)m_k)$ and $y_k = \rho^{-1}(s)z_k$ and then replace the formulation (57) by the equivalent and well-posed following mixed formulation: find $(z_k, m_k, \eta_k) \in L^2(Q_T) \times \rho_0^{-1}P_s \times L^2(Q_T)$ solution of

$$\begin{cases} \displaystyle\int_{Q_T} z_k\overline{z} + s^3\int_{q_T} m_k\,\overline{m} + \int_{Q_T}(T-t)^{1/2}\eta_k\big(\overline{z} - \rho^{-1}(s)L^\star(\rho_0(s)\overline{m})\big) \\ \qquad\qquad = -\displaystyle\int_{Q_T}\rho_0(s)f(\rho^{-1}(s)z_{k-1})\overline{m} + \int_\Omega \rho_0(s,0)u_0\overline{m}(0) \quad \forall(\overline{m},\overline{z}) \in \rho^{-1}(s)P_s \times L^2(Q_T), \\ \displaystyle\int_{Q_T}(T-t)^{1/2}\overline{\eta}\big(z_k - \rho^{-1}(s)L^\star(\rho_0(s)m_k)\big) = 0, \quad \forall\overline{\eta} \in L^2(Q_T). \end{cases} \tag{58}$$

The variable η_k stands as a Lagrange multiplier for the constraint $z_k - \rho^{-1}(s)L^\star(\rho_0(s)m_k) = 0$ in Q_T.

We check the following equality for every $m \in \rho^{-1}(s)P_s$

$$-\rho^{-1}(s)L^\star(\rho_0(s)m) = (g_1(\theta,\varphi) + g_2(\theta,\varphi))m + \theta^{-3/2}(\partial_t m + \Delta m) + g_3(\theta,\varphi)\cdot\nabla m$$

with

$$\begin{cases} g_1(\theta,\varphi) := \rho^{-1}(s)\partial_t\rho_0(s) = \partial_t(\theta^{-3/2}) + \theta^{-3/2}s(\partial_t\varphi), \\ g_2(\theta,\varphi) := \theta^{-3/2}(s\Delta\varphi + s^2(\nabla\varphi)^2), \\ g_3(\theta,\varphi) := \rho^{-1}(s)\nabla\rho_0(s) = \theta^{-3/2}s\nabla\varphi \end{cases} \tag{59}$$

where φ and θ are defined in (10) and (9). We observe that g_2 is slightly singular like $(T-t)^{-1/2}$ for $t \geqslant T - T_1$ and thus justifies the introduction of the weight function $(T-t)^{1/2}$ in (58). Eventually, with $f(r) = c_f(\alpha + \beta\ln_+|r|)^{3/2}r$, we get

$$-\int_{Q_T}\rho_0(s)f(\rho^{-1}(s)z_{k-1})\overline{m} = -c_f\int_{Q_T}\theta^{-3/2}(\alpha + \beta\ln_+|\rho^{-1}(s)z_{k-1}|)^{3/2}z_{k-1}\overline{m}, \quad \forall\overline{m} \in \rho^{-1}(s)P_s.$$

In the experiments, we use the equivalent formulation (58) instead of (57) as it allows, first to eliminate the singularity of the coefficients for t close to T and second to obtain simultaneously the control and the controlled solution.

The sequence $(y_k, v_k)_{k\in\mathbb{N}}$ is initialized with $y^{-1} = 0$ (so that $f(y^{-1}) \equiv 0$ and the first iteration computes the control pair (y^0, v^0) corresponding to the controlled trajectory of the linear heat equation with initial datum u_0 and zero source term) and is computed until the following criterion is satisfied

$$\frac{\|\rho_0(s)(y_{k+1} - y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)y_k\|_{L^2(Q_T)}} \leqslant 10^{-6}. \tag{60}$$

We shall denote by $k^\star$ the lowest integer k for which (60) holds true.

Concerning the approximation of the formulation (58), we use a conformal space-time finite element method (as described in [15]). We consider a regular family $\mathcal{T} = \{\mathcal{T}_h; h > 0\}$ of triangulation of Q_T such that $\overline{Q_T} = \cup_{K\in\mathcal{T}_h} K$. The family $\mathcal{T}$ is indexed by $h = \max_{K\in\mathcal{T}_h} diam(K)$. The variable z_k and η_k are approximated with the space $P_h = \{p_h \in \mathcal{C}^0(\overline{Q_T}); p_h|_K \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\} \subset L^2(Q_T)$ where $\mathbb{P}_1(K)$ denotes the space of affine functions both in x and t. The variable m_k is approximated with the space $V_h = \{v_h \in \mathcal{C}^1(\overline{Q_T}); v_h|_K \in \mathbb{P}(K), \forall K \in \mathcal{T}_h\} \subset \mathcal{M}$ where $\mathbb{P}(K)$ denotes the composite Hsieh-Clough-Tocher $\mathcal{C}^1$ element (we refer to [9, page 356]). Figure 1 depicts a typical triangulation $\mathcal{T}_h$ of $\overline{Q_T}$ corresponding to the value $h \approx 0.0208$. Contrarily to (57), theoretical convergence results with respect to h for the approximated solution of (58) are not known. We refer to [29] for some numerical evidences of the robustness of the method with respect to the parameter h.

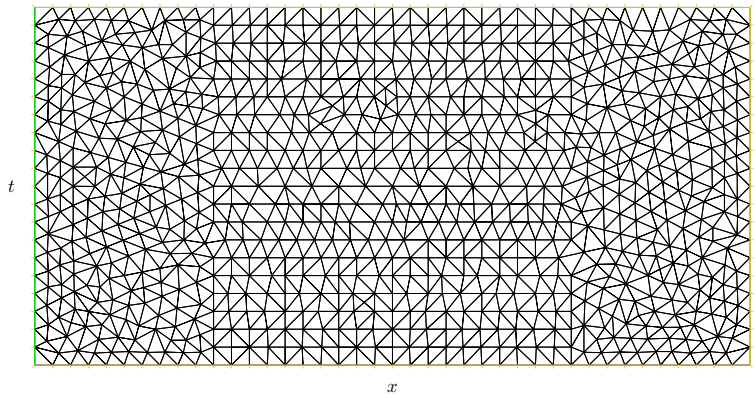


Figure 1: Regular space-time mesh of $Q_T := (0,1)\times(0,T)$ composed of 1744 triangles and 933 vertices; $h \approx 0.0208$.

Eventually, it should be noted that the corresponding finite dimensional linear system associated with (58) is independent of the iteration index k allowing a notable gain in term of computational time. An UMFPACK type solver is used.

5.2 Experiments

We present some numerical experiments in the one dimensional setting with $\Omega = (0,1)$. We consider simply connected interval $\omega \subset \Omega$. We take $T = 1/2$ and consider data for which the uncontrolled solution of (1) blows up before T. Moreover, in order to reduce the decay of the solution of (1) when $f \equiv 0$, we replace the term $-\Delta y$ in (1) by $-\nu\Delta y$ with $\nu = 10^{-1}$. The Carleman parameter λ is taken equal to one while the function $\widetilde{\psi} : \overline{\Omega} \to [0,1]$ is defined by

$$\widetilde{\psi}(x) = \frac{x(1-x)e^{-(x-c)^2}}{x_\star(1-x_\star)e^{-(x_\star-c)^2}}, \qquad c := x_\star - \frac{1-2x_\star}{2x_\star(1-x_\star)}$$

where $x_\star$ is the mid-point of ω (we refer to [15, Figure 1] for a plot of $\widetilde{\psi}$).

We consider the nonlinear even function f as follows

$$f(r) = c_f\big(\alpha + \beta \ln(1+|r|)\big)^{3/2} r$$

with $\alpha = \beta = 1$ and $c_f < 0$. We easily check that f satisfies $(\mathbf{H_1})$, $(\mathbf{H_1'})$ and also $(\overline{\mathbf{H}}_\mathbf{p})$ for every $p \in [0,1]$. In particular, $f'' \in L^\infty(Q_T)$. As for the initial condition to be controlled, we consider $u_0(x) = c_{u_0}\sin(\pi x)$ parametrized by $c_{u_0} > 0$.

We use a mesh composed of 29132 and 14807 triangles corresponding to $h \approx 1.17 \times 10^{-2}$. For $\omega = (0.2, 0.8)$, $c_{u_0} = 10$ and $c_f = -5$, Figure 2-left depicts the evolution of the relative error $\frac{\|\rho_0(s)(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)y_k\|_{L^2(Q_T)}}$ with respect to the parameter of iteration k for $s \in \{1,2,3,4\}$. In agreement with the theoretical part, the convergence is observed for s large enough, here $s \geqslant 2$. Moreover, the rate increases with s: the convergence is observed after $k^\star$ iterations equal to $48, 17, 13$ for $s = 2, 3$ and 4 respectively. Figure 2-right depicts the ratio $\frac{\|\rho_0(s)(\Lambda_s(y_k)-\Lambda_s(y_{k-1}))\|_{L^2(Q_T)}}{\|\rho_0(s)(y_k-y_{k-1})\|_{L^2(Q_T)}}$ (appearing in Proposition 4) highlighting the lack of contracting property of Λ_s for $s = 1$. Figure 3 depicts the evolution of the $L^2(\Omega)$ norm of the control and corresponding controlled solution with respect to the time variable for $s = 2, 3, 4$. As expected in view of the definition of the weights, large values of s concentrate the action of the control close to the initial time and leads to large $L^\infty(Q_T)$ norm of the control (see Table 1). Figure 4 and Figure 5 depict the control and corresponding controlled solution in Q_T for these values of s. We also observed that the initialization $(y_0, v_0) = (\rho^{-1}(s)u_0, 0)$ (valid since $u_0 \in H^2(\Omega)$) leads to the same limit of the algorithm.

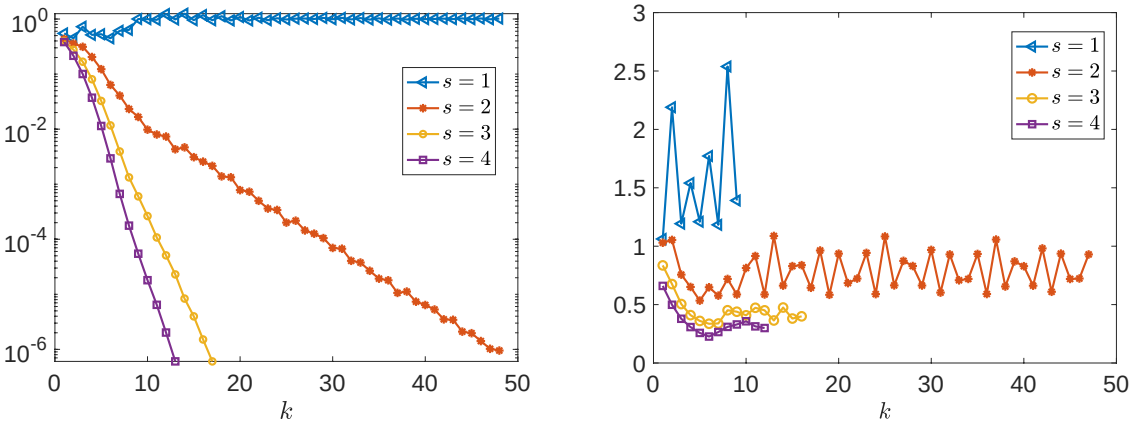


Figure 2: Relative error $\frac{\|\rho_0(s)(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)y_k\|_{L^2(Q_T)}}$ (**Left**) and $\frac{\|\rho_0(s)(y_{k+1}-y_k)\|_{L^2(Q_T)}}{\|\rho_0(s)(y_k-y_{k-1})\|_{L^2(Q_T)}}$ (**Right**) w.r.t. k for $s \in \{0,1,2,3\}$.

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(q_T)}$	$\|\rho_0(s)v_{k^\star}\|_{L^2(q_T)}$	$\|v_{k^\star}\|_{L^\infty(q_T)}$	$k^\star$
2	2.43	80.50	58.24	208.52	297.56	48
3	1.415	86.53	51.30	463.69	414.93	17
4	1.108	173.17	52.83	1366.08	605.20	13
5	0.9307	429.07	57.04	4328.61	889.05	11

Table 1: $c_{u_0} = 10$; $c_f = -5$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s.

Table 2 provides some norms of the solution for $s = 3$ with respect to the fineness h of the triangular mesh used and highlights the stability of the approximation. Actually, the high degree (equal to 3) of the approximation induced by the composite finite element HCT makes the convergence of the approximation quite fast with respect to h. We also observe that the number of iterations to reach the convergence of the sequence $(y_k)_{k\geqslant 0}$ is independent of h.

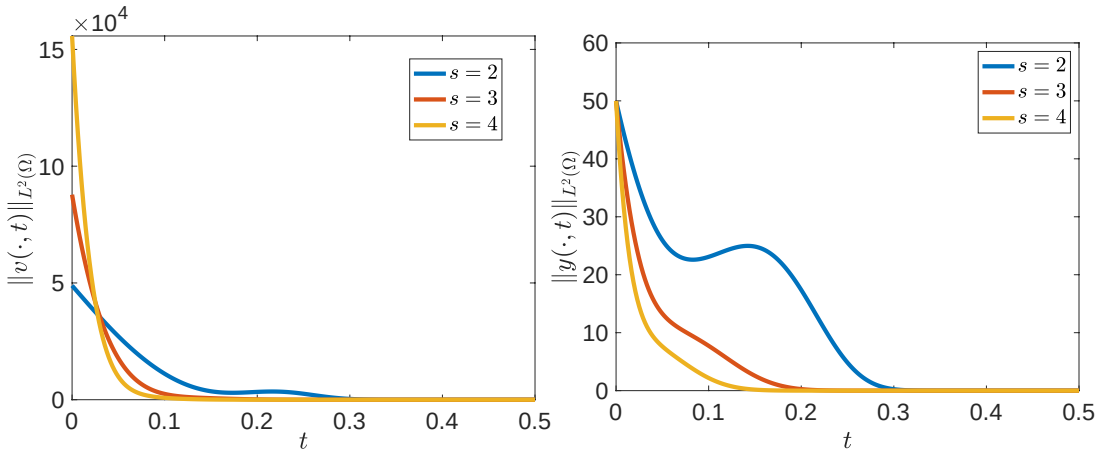


Figure 3: Evolution of $\|v_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ and $\|y_{k^\star}(\cdot,t)\|_{L^2(\Omega)}$ w.r.t. $t \in [0,T]$ for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{1,2,3\}$.

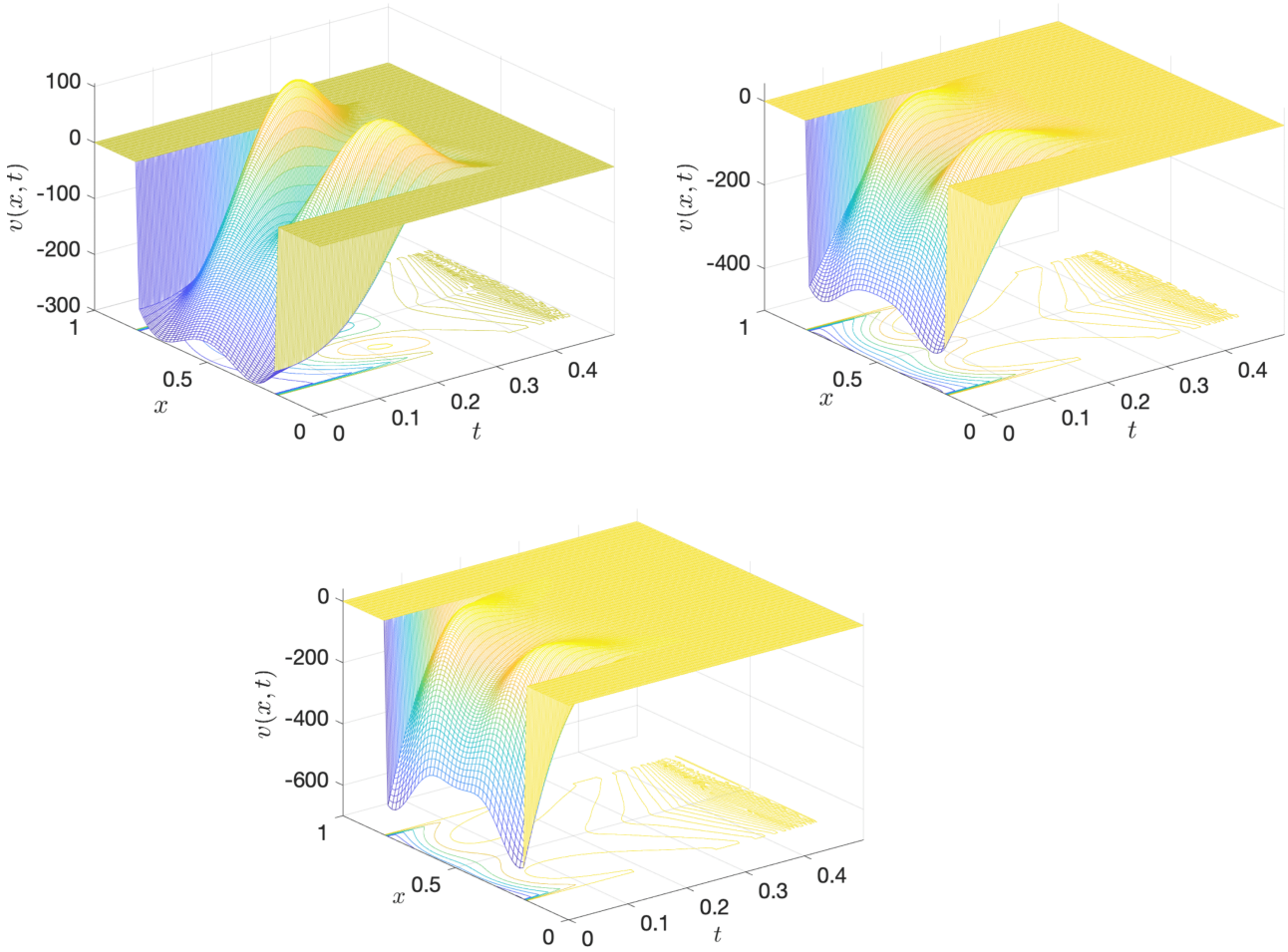


Figure 4: The control $v_{k^\star}$ in Q_T for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{1,2,3\}$.

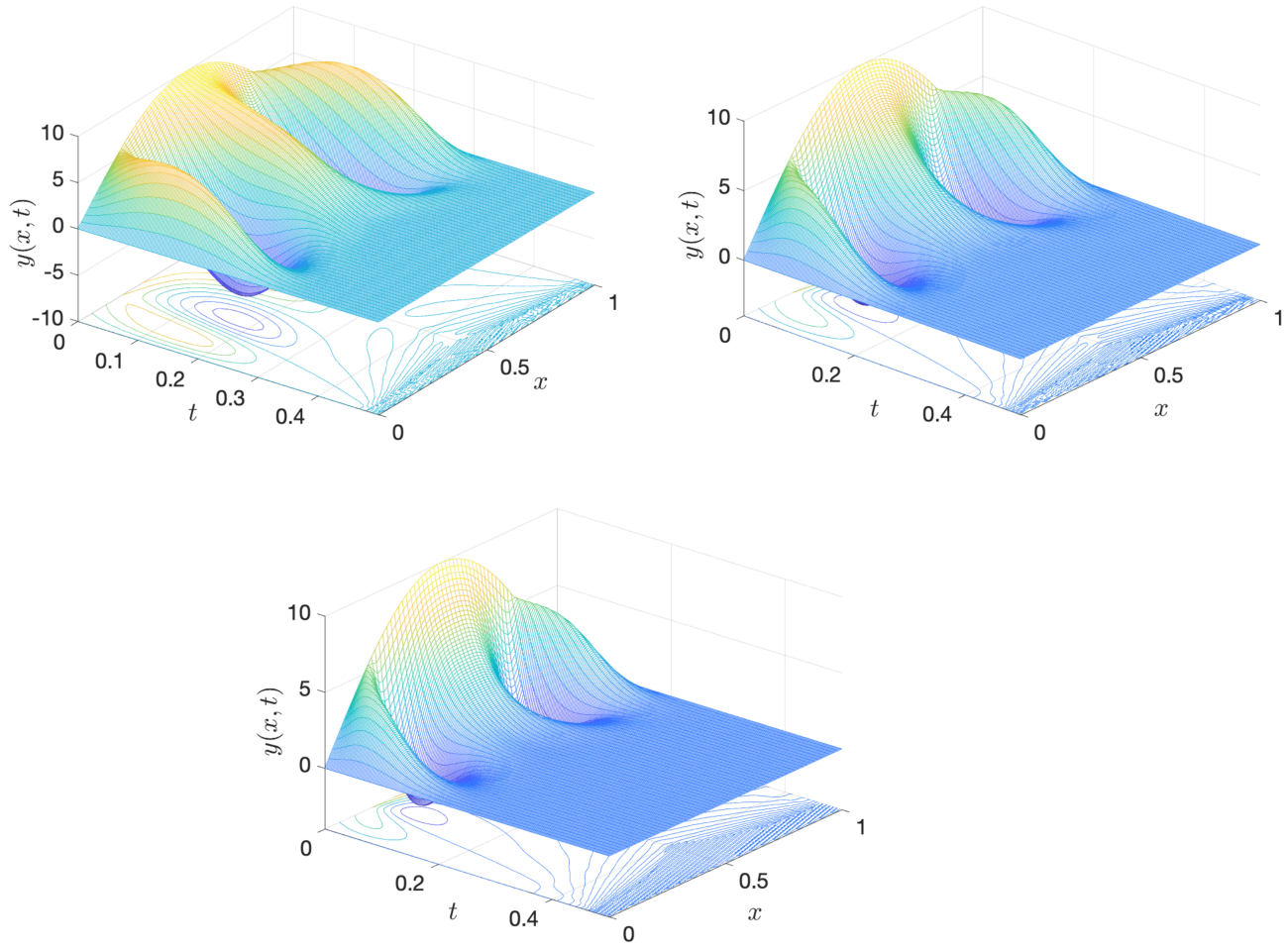


Figure 5: The controlled solution $y_{k^\star}$ in Q_T for $c_{u_0} = 10$, $c_f = -5$ and $s \in \{1, 2, 3\}$.

h	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(q_T)}$	$\|\rho_0(s)v_{k^\star}\|_{L^2(q_T)}$	$\|v_{k^\star}\|_{L^\infty(q_T)}$	$k^\star$
0.156205	1.47841	90.9285	51.4646	469.008	420.345	18
0.0760345	1.46148	87.9869	51.2379	465.822	419.42	17
0.044171	1.45521	87.0578	51.0243	464.527	416.886	17
0.0208981	1.45056	86.2678	51.0448	463.253	414.223	17
0.0117201	1.45203	86.5628	51.1068	463.723	415.114	17

Table 2: $c_{u_0} = 10$; $c_f = -5$; $s = 3$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. h.

For larger data, the algorithm still converges if the parameter s is chosen sufficiently large. Table 3 provides for $s = 2$ and $c_{u_0} = 20$ some norms associated with the convergent sequence $(y_k, f_k)_{k\in\mathbb{N}}$ with respect to the amplitude c_f of the nonlinear function. The real $C_1(y, v)$ is defined by

$$C_1(y,v) := \frac{\|\rho(s)\, y\|_{L^2(Q_T)} + s^{-3/2}\|\rho_0(s)\, v\|_{L^2(q_T)}}{s^{-3/2}\|\rho_0(s) f(y)\|_{L^2(Q_T)} + s^{-1}e^{c_2 s}\|u_0\|_{L^2(\Omega)}}, \tag{61}$$

and is plotted in Table 3. The ratio $C_1(y, v)$ notably appears in the estimate (25) with $B = -f(y)$. The divergence of the sequence is observed for $c_f \leqslant -7$ suggesting that, for $s = 3$, the quantity $C_1(y_{k^\star}, v_{k^\star})$ is not uniformly bounded with respect to $|c_f|$. This does not contradict Theorem 8 where an upper bound is assumed on β. On the contrary, Table 4 and 5 suggest that the quantity $C_1(y_{k^\star}, v_{k^\star})$ associated with $(c_f, s) = (-2, 2)$ and $(c_f, s) = (-2, 3)$ is uniformly bounded with respect to the parameter c_{u_0}: this is in agreement with the uniform controllability of system (1) with respect to the initial data stated in Theorem 7.

c_f	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(q_T)}$	$\|\rho_0(s)v_{k^\star}\|_{L^2(q_T)}$	$\|v_{k^\star}\|_{L^\infty(q_T)}$	$C_1(y_{k^\star}, v_{k^\star})$	$k^\star$
6	1.5676	39.0933	18.4173	144.851	188.71	9.71×10^{-5}	18
5	1.63982	41.5574	21.5396	170.772	212.189	1.04×10^{-4}	15
4	1.71818	44.4218	25.3636	202.888	240.201	1.13×10^{-4}	13
3	1.80253	47.8002	30.0497	242.771	273.779	1.24×10^{-4}	11
2	1.89243	51.8744	35.7756	292.257	314.074	1.37×10^{-4}	9
1	1.98736	56.9667	42.7222	353.371	362.474	1.52×10^{-4}	6
0	2.08767	63.7034	51.053	428.191	420.662	1.73×10^{-4}	1
-1	2.19716	73.375	60.8936	518.678	491.01	2.01×10^{-4}	7
-2	2.32931	88.719	72.3206	626.54	576.832	2.42×10^{-4}	9
-3	2.52065	115.606	85.3784	753.626	682.763	3.09×10^{-4}	12
-4	2.85296	166.848	100.33	907.951	815.128	4.30×10^{-4}	16
-5	3.47542	272.771	119.009	1140.26	982.953	6.70×10^{-4}	25
-6	4.64025	523.143	148.453	1674.92	1198.33	1.22×10^{-3}	61

Table 3: $c_{u_0} = 20$; $s = 3$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. c_f.

Table 6 collects some norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s for the smaller support of control $\omega = (0.2, 0.6)$, $c_{u_0} = 10$ and $c_f = -2$ A smaller support yields to a larger constant $C_1 = C_1(\Omega, \omega, T)$ in (21) and larger norms for the control and corresponding controlled solution. For instance, for $s = 3$, we obtain $\|v_{k^\star}\|_{L^\infty(q_T)} \approx 1301.97$ with $\omega = (0.2, 0.6)$ and $\|v_{k^\star}\|_{L^\infty(q_T)} \approx 271.86$ with $\omega = (0.2, 0.8)$ (see the first line of Table 5). Accordingly, the number $k^\star$ of iterations to reach convergence increases when the size of ω is reduced.

We now consider the nonlinear function, for any $\eta \in (0, 1)$ and $\alpha \geqslant 0$

$$f(r) = c_f r|\cos(r^{-3/2+\eta})|\big(\alpha + \ln(1 + |r|)\big)^{3/2}, \quad c_f < 0. \tag{62}$$

c_{u_0}	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(q_T)}$	$\|\rho_0(s)v_{k^\star}\|_{L^2(q_T)}$	$\|v_{k^\star}\|_{L^\infty(q_T)}$	$C_1(y_{k^\star}, v_{k^\star})$	$k^\star$
10	1.54301	19.4769	30.6487	91.8124	170.047	0.0573655	11
100	17.4912	334.631	422.723	1323.21	2255.98	0.0857455	18
500	107.77	3234.76	2631.81	9152.24	13811.4	0.130185	29
1000	246.968	9077.9	5870.83	22622.8	30051.5	0.159445	38
1500	405.983	16828	9489.64	39571.9	47314.5	0.180076	48
2000	580.37	26235.4	13430	59621.2	65262.9	0.196388	55
3000	965.936	49498.3	22141.2	108020	102663	0.221715	77
4000	1391.49	78171.6	31830.1	166352	141531	0.241246	106
5000	1850.25	111855	42375.6	233684	181553	0.257167	141

Table 4: $c_f = -2$; $s = 2$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. c_{u_0}.

c_{u_0}	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(q_T)}$	$\|\rho_0(s)v_{k^\star}\|_{L^2(q_T)}$	$\|v_{k^\star}\|_{L^\infty(q_T)}$	$C_1(y_{k^\star}, v_{k^\star})$	$k^\star$
10	1.14031	41.0689	34.0273	292.739	271.86	2.25×10^{-4}	8
100	12.4721	576.084	420.613	3722.92	3381.68	3.07×10^{-4}	12
500	69.8357	4170.43	2443.36	22324.9	20055.3	4.28×10^{-4}	15
1000	149.215	10045.6	5213.5	48679.6	43195	5.05×10^{-4}	17
2000	322.25	24509.6	11144.3	107260	93000.8	6.05×10^{-4}	20
3000	507.998	41520.3	17405	171395	145648	6.77×10^{-4}	22
4000	703.063	60489.4	23901.8	239863	200195	7.34×10^{-4}	23
5000	905.632	81095	30586.9	311987	256227	7.83×10^{-4}	24
6000	1114.55	103128	37430.5	387325	313419	8.26×10^{-4}	25
7000	1329.02	126440	44412	465555	371629	8.65×10^{-4}	26
8000	1548.43	150917	51516.2	546433	430756	9.01×10^{-4}	28
9000	1772.31	176471	58731.6	629764	490642	9.34×10^{-4}	28
10000	2000.29	203029	66048.7	715388	551209	9.65×10^{-4}	30
20000	4455.89	513872	143566	1.67×10^6	1.18×10^6	1.20×10^{-3}	39

Table 5: $c_f = -2$; $s = 3$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. c_{u_0}.

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(q_T)}$	$\|\rho_0(s)v_{k^\star}\|_{L^2(q_T)}$	$\|v_{k^\star}\|_{L^\infty(q_T)}$	$k^\star$
1	5.00473	55.4739	117.728	144.256	1301.97	20
2	3.34631	94.7894	90.8444	358.58	1483.52	14
3	2.91028	237.695	109.246	1060	2948.23	12
4	2.87397	651.291	146.28	3690	5576.49	11

Table 6: $c_{u_0} = 10$; $c_f = -2$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s; $\omega = (0.2, 0.6)$

With $\alpha = 0$, the function f satisfies $(\mathbf{H_1})$, $(\mathbf{H_1'})$ but only $(\overline{\mathbf{H}}_\mathbf{p})$ for $p \in (0, \eta)$. $f'(r)$ behaves likes $\ln(1 + |r|)^{3/2}$ at infinity while in the neighborhood of zero, f' is bounded but highly oscillates. For $c_f = -5$, $c_{u_0} = 10$ and $\eta = 1/10$, Table 7 provides some norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. the parameter s leading to similar results than in the previous case. On the other hand, with $\alpha > 0$, the function f still satisfies $(\mathbf{H_1})$ but not $(\mathbf{H_1'})$ nor $(\overline{\mathbf{H}}_\mathbf{p})$. In particular, f' takes arbitrarily large values in the neighborhood of zero. The values $c_f = -5$, $c_{u_0} = 10$ do not lead to the convergence of the algorithm including for large values of s. In order to enhance oscillations for large values, we now consider the function

$$f(r) = c_f r |\cos(r^2)| \ln^{3/2}(1 + |r|), \quad c_f < 0 \tag{63}$$

which still satisfies $(\mathbf{H_1})$ but not $(\mathbf{H_1'})$ nor $(\overline{\mathbf{H}}_\mathbf{p})$. For small values of $|c_f|$, the method still produces convergent sequences: Table 8 reports some values for $c_f = -1$; observe that the number $k^\star$ increases

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(q_T)}$	$\|\rho_0(s)v_{k^\star}\|_{L^2(q_T)}$	$\|v_{k^\star}\|_{L^\infty(q_T)}$	$k^\star$
1	3.47589	35.7363	62.9227	67.9855	243.604	119
2	1.72679	25.1665	39.8201	120.101	214.439	20
3	1.22787	47.0941	39.477	339.374	308.99	12
4	0.981409	112.296	43.1533	1073.88	460.349	10

Table 7: $c_{u_0} = 10$; $c_f = -5$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s; f given by (62) with $\alpha = 0$.

from the case $s = 1$ and the case $s = 2$. For $c_f = -2$, we observe the convergence only for $s \geqslant 6$.

s	$\|y_{k^\star}\|_{L^2(Q_T)}$	$\|\rho(s)y_{k^\star}\|_{L^2(Q_T)}$	$\|v_{k^\star}\|_{L^2(q_T)}$	$\|\rho_0(s)v_{k^\star}\|_{L^2(q_T)}$	$\|v_{k^\star}\|_{L^\infty(q_T)}$	$k^\star$
1	2.16033	7.85245	15.7601	14.5863	47.9055	25
2	1.39592	14.5911	20.81	60.618	125.071	55
3	1.06386	33.145	27.0012	227.116	220.722	11
4	0.878876	85.2781	33.3402	815.691	354.915	9

Table 8: $c_{u_0} = 10$; $c_f = -1$; Norms of $(y_{k^\star}, v_{k^\star})$ w.r.t. s; f given by (63).

6 Concluding remarks

By introducing a different fixed point application than in the seminal paper of Fernández-Cara and Zuazua, we have derived under the same assumptions a simpler proof of the exact controllability for semilinear heat equations. The fixed point application involves a linearized heat equation with a right hand side but no potential. Within the Carleman functional setting introduced by Fursikov and Imanuvilov, precise estimates of the cost of observability with respect to the right hand side term allow to apply the Schauder fixed point theorem as soon as the Carleman parameter s is large enough. This is to our knowledge the first time that the analysis of the controllability is not based on the analysis of the cost of observability of the heat equation with potential. Moreover, assuming an additional asymptotic condition on the derivative of the nonlinear function, the fixed point application turns out to be contracting yielding to a constructive sequence of linear controls converging strongly and with a linear rate toward a control for the semilinear equation.

Numerical experiments illustrate this property and also that the contraction property is amplified as the Carleman parameter s increases. This construction is original in the multi-dimensional setting: it extends and simplify the one proposed in [28] based on a Newton type linearization and requiring as well the parameter s to be large enough to ensure a super-linear convergence. The numerical implementation is also much simpler. We also conjecture that the linearization introduced by Fernández-Cara and Zuazua also leads, for s large enough, to a contracting application but this remains to be done.

We emphasize that the method is mainly based on global Carleman estimates, which are by now well-known and employed. Therefore, the approach can very likely be extended to many others PDEs such as the wave equation, the burgers equation (addressed from a numerical perspective in [21, Part 1, Section 4]), the Navier-Stokes system (addressed in [17, section 4]), etc, for which such estimates are available, and may lead to constructive algorithms in semi-linear cases for these systems too.

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