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Linear-time algorithms for three domination-based separation problems in block graphs

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Abstract

The problems of determining minimum identifying, locating-dominating or open locating-dominating codes are special search problems that are challenging both from a theoretical and a computational point of view, even for several graph classes where other in general hard problems are easy to solve, like bipartite graphs or chordal graphs. Hence, a typical line of attack for these problems is to determine minimum codes of special graphs. In this work we study the problem of determining the cardinality of minimum such codes in block graphs (that are diamond-free chordal graphs). We present linear-time algorithms for these problems, as a generalization of a linear-time algorithm proposed by Auger in 2010 for identifying codes in trees. Thereby, we provide a subclass of chordal graphs for which all three problems can be solved in linear time.

1 Introduction

For a graph G that models a facility or a multiprocessor network, detection devices can be placed at its vertices to locate an intruder (like a fire, a thief or a saboteur) or a faulty processor. Depending on the features of the detection devices (to detect an intruder only if it is present at the vertex v where the detector is installed and/or also at a vertex adjacent to v), different dominating sets can be used to determine the optimal distribution of the detection devices in G . In the following, we study three problems arising in this context which all have been actively studied during the last decade, see e.g. the bibliography maintained by Lobstein [16].

Let $G = (V, E)$ be a graph. The (open) neighborhood of a vertex u is the set $N(u)$ of all vertices of G adjacent to u , and $N[u] = \{u\} \cup N(u)$ is the closed neighborhood of u .

Identifying codes A subset $C \subseteq V$ is an *identifying code* (for short: *ID-code*) of G if

- $N[u] \cap C \neq \emptyset$ for all $u \in V$ (domination),
- $N[u] \cap C \neq N[v] \cap C$ for all $u, v \in V$ (separation),

see Figure 1(a) for an example. Identifying codes were introduced by Karpovsky et al. [15]. There it was noted that not every graph G admits an identifying code, i.e. is *identifiable*: this holds if and only if there are no true twins in G , i.e., there is no pair of distinct vertices $u, v \in V$ with $N[u] = N[v]$ [15]. On the other hand, for every identifiable graph, its whole vertex set trivially forms an identifying code.

The *identifying code number* $\gamma_{ID}(G)$ of a graph G is the minimum cardinality of an identifying code of G . Determining $\gamma_{ID}(G)$ is in general NP-hard [10] and remains hard for several graph classes where other in general hard problems are easy to solve, including bipartite graphs [10] and two classes of chordal graphs, namely split graphs and interval graphs [12].

Hence, typical lines of attack for the identifying code problem are to determine minimum identifying codes of special graphs. Closed formulas for the exact value of $\gamma_{ID}(G)$ have been found so far only for restricted graph families (e.g. for paths and cycles [6], for stars [13], for complete multipartite graphs [1] and some subclasses of split graphs [3]). A linear-time algorithm to determine $\gamma_{ID}(G)$ if G is a tree was provided by Auger [5].

Locating-dominating codes A subset $C \subseteq V$ is a *locating-dominating code* (for short: *LD-code*) of G if

- $N[u] \cap C \neq \emptyset$ for all $u \in V$,
- $N(u) \cap C \neq N(v) \cap C$ for all $u, v \in V - C$,

see Figure 1(b) for an example. Locating-dominating codes were introduced by Slater [20, 21]. By definition, every graph has a locating-dominating code (as its whole vertex set trivially forms an *LD-code*).

The *locating-dominating number* $\gamma_{LD}(G)$ of a graph G is the minimum cardinality of a locating-dominating code of G . Determining $\gamma_{LD}(G)$ is in general NP-hard [10] and even remains hard for bipartite graphs [10]. This result is extended to planar bipartite unit disk graphs in [17].

Hence, a typical line of attack for the *LD-code* problem is again to determine minimum *LD-codes* of special graphs. Closed formulas for the exact value of $\gamma_{LD}(G)$ have been found so far for restricted graph families as e.g. paths [21], cycles [6], stars, complete multipartite graphs and thin suns [4]. Bounds for the *LD-number* of trees were provided in [8], characterizations of trees with unique minimum locating-dominating sets were found in [7]. A linear-time algorithm to determine $\gamma_{LD}(G)$ if G is a tree was provided by Slater in [20].

Open locating-dominating codes A subset $C \subseteq V$ is an *open locating-dominating code* (for short: *OLD-code*) of G if

- $N(u) \cap C \neq \emptyset$ for all $u \in V$ (open-domination),
- $N(u) \cap C \neq N(v) \cap C$ for all $u, v \in V$ (open-separation),

see Figure 1(c) for an example. Open locating-dominating codes were introduced by Seo and Slater [22]. There it was noted that not every graph G admits an *OLD-code*: this holds if and only if there are neither isolated vertices nor false

twins in G (i.e., no pair of distinct vertices $u, v \in V$ with $N(u) = N(v)$) [22]. On the other hand, the whole vertex set trivially forms an *OLD*-code of any false twin-free graph.

The *open locating-dominating number* $\gamma_{OLD}(G)$ of a graph G is the minimum cardinality of an *OLD*-code of G . Determining $\gamma_{OLD}(G)$ is in general NP-hard [22] and remains NP-hard for perfect elimination bipartite graphs and APX-complete for chordal graphs with maximum degree 4 [18]. Closed formulas for the exact value of $\gamma_{OLD}(G)$ have been found so far only for restricted graph families such as cliques and paths [22], some algorithmic aspects are discussed in [19].

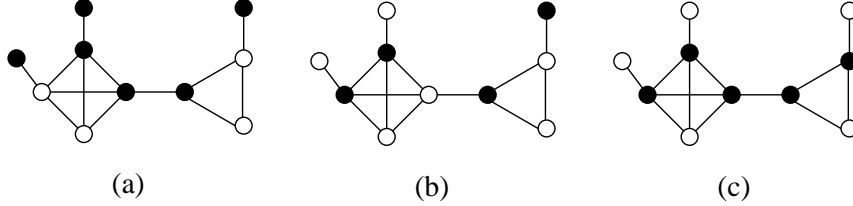


Figure 1: A block graph where the black vertices form a minimum (a) *ID*-code, (b) *LD*-code, (c) *OLD*-code.

More results on all three problems are listed in [16]. Concerning the relation of the three studied domination numbers, it is immediate from the definitions that

$$\gamma_{LD}(G) \leq \min\{\gamma_{ID}(G), \gamma_{OLD}(G)\}$$

as every *ID*-code and every *OLD*-code satisfy the conditions for locating-domination.

In this paper, we determine the identifying code, locating-dominating and open locating-dominating numbers of block graphs. A *block graph* is a graph in which every maximal 2-connected subgraph (block) is a clique (see Figure 1). Block graphs are precisely the diamond-free chordal graphs [9] resp. those chordal graphs in which every two maximal cliques have at most one vertex in common [14].

We present linear-time algorithms for the three studied problems on block graphs, as a generalization of the linear-time algorithm by Auger [5] for $\gamma_{ID}(G)$ on trees.

In fact, trees are exactly the block graphs with clique size 2 and constitute exactly the intersection of block graphs with bipartite graphs.

Note that a tree is identifiable (i.e. has no true twins) whenever it is different from the clique of size 2, whereas a block graph B admits an identifying code if and only if each maximal clique K of B satisfies that all vertices in K , except at most one, have a neighbor that is not in $V(K)$.

In [22] it was observed that a tree has an *OLD*-code (i.e. has no false twins) if and only if no two vertices with degree one (called endpoints) have a common neighbor (called support vertex). It is straightforward to see that this condition carries over to block graphs. Also, a block graph B admits an *OLD*-code if and only if no vertex in G has more than one neighbor with degree one.

Hence, in contrary to Auger's algorithm for identifying codes in trees, our algorithms have to take into account that the block graph given as input might not admit an *ID*-code or an *OLD*-code.

In Section 2, we provide basic definitions and notations necessary for what follows. In Section 3, we present our general algorithmic framework for determining the three studied domination numbers in block graphs, and give the details and proofs for finding minimum *ID*-codes, *LD*-codes and *OLD*-codes in Sections 4, 5 and 6, resp. We close with some concluding remarks and lines of future research. The results on *ID*-codes were presented without proofs in [2].

2 Basic definitions and notations

In order to provide linear-time algorithms that compute $\gamma_{ID}(B)$, $\gamma_{LD}(B)$ and $\gamma_{OLD}(B)$ of a block graph B , we adopt the following notation from [5]. Let $G = (V, E)$ be a graph and $v \in V$. Then $C \subseteq V$ is a *v-almost X-code* of G where $X \in \{ID, LD, OLD\}$, if the code C is an X -code for the induced subgraph $G - \{v\}$ of G . Moreover, we say that C satisfies the property

- *ID* (*LD*, *OLD*) if C is an *ID* (*LD* or *OLD* resp.)-code in G ,
- *CO* if $v \in C$,
- *ADJ* if v has a neighbour in C ,
- *FN* if v has a neighbour w with $N[w] \cap C = \{v\}$,
- *OFN* if v has a neighbour w with $N(w) \cap C = \{v\}$,
- \bar{P} if C does not satisfy property $P \in \{ID, LD, OLD, CO, ADJ, FN, OFN\}$.

In properties *FN* and *OFN* the vertex w is called the *favored neighbor* of v . Let $\mathcal{P} = \{ID, LD, OLD, CO, ADJ, FN, OFN, \bar{ID}, \bar{LD}, \bar{OLD}, \bar{CO}, \bar{ADJ}, \bar{FN}, \bar{OFN}\}$ and $X \in \{ID, LD, OLD\}$. We denote by $\gamma_{P_1, \dots, P_k}(v, G)$ the function that returns the minimum size of a *v-almost X-code* in G satisfying the properties $\{P_1, \dots, P_k\} \subset \mathcal{P}$ or ∞ if no such code exists.

We observe that there exist dependency relationships between these properties. In fact, if a *v-almost X-code* C satisfies *FN* then it satisfies *CO*, if C satisfies *ID* or *LD* then it satisfies *CO* or *ADJ*, if C satisfies *ID*, *CO* and *FN* then it must satisfy *ADJ* and finally, if C satisfies *OLD* then it must satisfy *ADJ*. As a consequence we observe the following:

Remark 1 *There is no v-almost X-code in G that satisfies the following sets of properties:*

- | | |
|--|--|
| • ID, CO, \bar{ADJ} and FN , | • LD, \bar{CO}, \bar{ADJ} and \bar{FN} , |
| • ID, \bar{CO}, ADJ and FN , | • OLD, CO, \bar{ADJ} and OFN , |
| • ID, \bar{CO}, \bar{ADJ} and FN , | • OLD, CO, \bar{ADJ} and \bar{OFN} , |
| • ID, \bar{CO}, \bar{ADJ} and \bar{FN} , | • OLD, \bar{CO}, \bar{ADJ} and OFN , |
| • LD, \bar{CO}, ADJ and FN , | • OLD, \bar{CO}, \bar{ADJ} and \bar{OFN} , |
| • LD, \bar{CO}, \bar{ADJ} and FN , | • OLD, \bar{CO}, ADJ and OFN . |

Let $X \in \{ID, LD, OLD\}$. If $v \in V(G)$, we denote by $\gamma_X(v, G)$ the minimum size of a *v-almost X code*. From Remark 1 it follows that

$$\gamma_{ID}(v, G) = \min \begin{cases} \gamma_{ID, CO, ADJ, FN}(v, G), \gamma_{ID, CO, ADJ, \bar{FN}}(v, G) \\ \gamma_{ID, CO, \bar{ADJ}}(v, G), \gamma_{ID, \bar{CO}, ADJ}(v, G) \end{cases} \quad (1)$$

$$\gamma_{LD}(v, G) = \min\{\gamma_{LD,CO}(v, G), \gamma_{LD,\overline{CO}}(v, G)\} \quad (2)$$

$$\gamma_{OLD}(v, G) = \min\{\gamma_{OLD,CO,OFN}(v, G), \gamma_{OLD,CO,\overline{OFN}}(v, G), \gamma_{OLD,\overline{CO}}(v, G)\} \quad (3)$$

In the particular case of a block graph, we observe the following. If B is a block graph, $v_1 \in V(B)$ and K a maximal clique with $V(K) = \{v_1, \dots, v_k\}$ then deleting all the edges in K yields k block subgraphs, say B_1, B_2, \dots, B_k , containing v_1, v_2, \dots, v_k , respectively. In the sequel, it is therefore convenient to use the following notation:

Notation 2 Let B be a block graph, v_1 a chosen vertex and K a maximal clique with vertices v_1, \dots, v_k , i.e., containing v_1 . We denote by B_1, \dots, B_k the block graphs, containing v_1, \dots, v_k respectively, obtained from B after deletion of the edges in K .

Remark 3 Let $X \in \{ID, LD, OLD\}$, B be a block graph, $V(K) = \{v_1, \dots, v_k\}$. If C is a v_1 -almost X -code in B then $C_i = C \cap V(B_i)$ is a v_i -almost X -code in B_i for all $i \in \{1, 2, \dots, k\}$.

Lemma 4 Let $X \in \{ID, LD, OLD\}$, B be a block graph, $V(K) = \{v_1, \dots, v_k\}$. If C_i is a v_i -almost X -code in B_i for $i \in \{1, 2, \dots, k\}$, and $C = \bigcup_{i=1}^k C_i$ then the vertices in $V' = \bigcup_{i=1}^k (V(B_i) - \{v_i\})$ are dominated and separated by C .

Proof. Observe that u, v are trivially separated if $d(u, v) > 2$. Consider $u, v \in V'$ such that $d(u, v) \leq 2$. Hence $u, v \in V(B_i) - \{v_i\}$ for some $i \in \{1, 2, \dots, k\}$. By assumption C_i is a v_i -almost X -code in B_i for all $i \in \{1, 2, \dots, k\}$, the vertices in V' are dominated and separated by C . \square

Lemma 5 Let $X \in \{ID, LD, OLD\}$, B be a block graph, $V(K) = \{v_1, \dots, v_k\}$. If C is a v_1 -almost X -code in B and $C_i = C \cap V(B_i) \forall i \in \{1, 2, \dots, k\}$ then,

- (i) if $X = ID$ and C satisfies ID then there exists at most one $i \in \{1, 2, \dots, k\}$ such that C_i satisfies \overline{ADJ} ,
- (ii) if $X = LD$ (resp. OLD) and C satisfies LD (resp. OLD) then there exists at most one $i \in \{1, 2, \dots, k\}$ such that C_i satisfies \overline{ADJ} and \overline{CO} ,
- (iii) if $X = ID$ then there exists at most one $i \in \{2, \dots, k\}$ such that C_i satisfies \overline{ADJ} ,
- (iv) if $X = LD$ (resp. OLD) then there exists at most one $i \in \{2, \dots, k\}$ such that C_i satisfies \overline{ADJ} and \overline{CO} .

Proof. Consider the maximal clique K with $V(K) = \{v_1, v_2, \dots, v_k\}$. If there is $j \in \{2, \dots, k\}$ such that C_1 and C_j satisfy \overline{ADJ} then $N[v_j] \cap C = N[v_1] \cap C = V(K) \cap C$. Then C clearly does not satisfy ID , and (i) is proved. If in addition the same pair satisfies \overline{CO} then $N(v_j) \cap C = N(v_1) \cap C = V(K) \cap C$ and C is neither an LD - nor an OLD -code, proving (ii). Suppose that C is a v_j -almost X -code with $X \in \{ID, LD, OLD\}$, then using the previous items the lemma follows. \square

3 An algorithm for determining the minimum X -code of block graphs

Let B be a block graph and $v \in V(B)$, in order to obtain the X -number of B , with $X \in \{ID, LD, OLD\}$, we will prove that we need to compute 46 functions that are based on the formulas (1), (2) and (3). The first 27 functions will be called *main functions* and are given in Table 1 and the latter 21 *auxiliary functions* in Table 2. Sometimes it is necessary to point out the particular problem we are dealing with. In such cases, we write $\gamma_{(P_1, \dots, P_k)_X}(v, G)$ when properties P_1, \dots, P_k must be satisfied for the problem X , with $X \in \{ID, LD, OLD\}$. Observe that if $B = \{v\}$ the values of the functions are straightforward and are also given in Table 1 and 2.

Name	Function	Initial value $f_j(v, \{v\})$
f_1	$\gamma_{ID, CO, ADJ, FN}$	∞
f_2	$\gamma_{ID, CO, ADJ, \overline{FN}}$	∞
f_3	$\gamma_{ID, CO, \overline{ADJ}}$	1
f_4	$\gamma_{ID, \overline{CO}, ADJ}$	∞
f_5	$\gamma_{(CO, ADJ, FN)_{ID}}$	∞
f_6	$\gamma_{(CO, ADJ, \overline{FN})_{ID}}$	∞
f_7	$\gamma_{(CO, \overline{ADJ}, FN)_{ID}}$	∞
f_8	$\gamma_{(CO, \overline{ADJ}, \overline{FN})_{ID}}$	1
f_9	$\gamma_{(\overline{CO}, ADJ)_{ID}}$	∞
f_{10}	$\gamma_{(\overline{CO}, \overline{ADJ})_{ID}}$	0
f_{11}	$\gamma_{LD, CO}$	1
f_{12}	$\gamma_{LD, \overline{CO}}$	∞
f_{13}	$\gamma_{(CO, ADJ, FN)_{LD}}$	∞
f_{14}	$\gamma_{(CO, ADJ, \overline{FN})_{LD}}$	∞
f_{15}	$\gamma_{(CO, \overline{ADJ}, FN)_{LD}}$	∞
f_{16}	$\gamma_{(CO, \overline{ADJ}, \overline{FN})_{LD}}$	1
f_{17}	$\gamma_{(\overline{CO}, ADJ)_{LD}}$	∞
f_{18}	$\gamma_{(\overline{CO}, \overline{ADJ})_{LD}}$	0
f_{19}	$\gamma_{OLD, CO, OFN}$	∞
f_{20}	$\gamma_{OLD, CO, \overline{OFN}}$	∞
f_{21}	$\gamma_{OLD, \overline{CO}}$	∞
f_{22}	$\gamma_{CO, ADJ, OFN}$	∞
f_{23}	$\gamma_{CO, ADJ, \overline{OFN}}$	∞
f_{24}	$\gamma_{CO, \overline{ADJ}, OFN}$	∞
f_{25}	$\gamma_{CO, \overline{ADJ}, \overline{OFN}}$	1
f_{26}	$\gamma_{(\overline{CO}, ADJ)_{OLD}}$	∞
f_{27}	$\gamma_{(\overline{CO}, \overline{ADJ})_{OLD}}$	0

Table 1: List of main functions together with their initial values $f_j(v, \{v\})$.

According to the notation introduced in Table 1, we can rewrite the formulas (1), (2) and (3).

$$\begin{aligned}
\gamma_{ID}(v, B) &= \min \{f_1(v, B), f_2(v, B), f_3(v, B), f_4(v, B)\} \\
\gamma_{LD}(v, B) &= \min \{f_{11}(v, B), f_{12}(v, B)\} \\
\gamma_{OLD}(v, B) &= \min \{f_{19}(v, B), f_{20}(v, B), f_{21}(v, B)\}
\end{aligned} \tag{4}$$

Below, we present the algorithms XB for $X \in \{ID, LD, OLD\}$ that find $\gamma_X(v, B)$ applying the equations above.

Name	Function	Initial value $f_j(v, \{v\})$
f_{28}	$\gamma_{(CO, \overline{ADJ})_{ID}} = \min\{f_7, f_8\}$	1
f_{29}	$\gamma_{(CO, ADJ)_{ID}} = \min\{f_5, f_6\}$	∞
f_{30}	$\gamma_{(CO, FN)_{ID}} = \min\{f_5, f_7\}$	∞
f_{31}	$\gamma_{(CO, \overline{FN})_{ID}} = \min\{f_6, f_8\}$	1
f_{32}	$\gamma_{\overline{CO}_{ID}} = \min\{f_9, f_{10}\}$	0
f_{33}	$\gamma_{ADJ_{ID}} = \min\{f_5, f_6, f_9\}$	∞
f_{34}	$\gamma_{ID, CO, ADJ} = \min\{f_1, f_2\}$	∞
f_{35}	$\gamma_{(CO \vee ADJ)_{LD}} = \min\{f_{13}, f_{14}, f_{15}, f_{16}, f_{17}\}$	1
f_{36}	$\gamma_{(CO, \overline{FN})_{LD}} = \min\{f_{14}, f_{16}\}$	1
f_{37}	$\gamma_{(CO, FN)_{LD}} = \min\{f_{13}, f_{15}\}$	∞
f_{38}	$\gamma_{\overline{CO}_{LD}} = \min\{f_{17}, f_{18}\}$	0
f_{39}	$\gamma_{CO_{LD}} = \min\{f_{13}, f_{14}, f_{15}, f_{16}\}$	1
f_{40}	$\gamma_{OLD, CO} = \min\{f_{19}, f_{20}\}$	∞
f_{41}	$\gamma_{(CO, ADJ)_{OLD}} = \min\{f_{22}, f_{23}\}$	∞
f_{42}	$\gamma_{(CO, \overline{ADJ})_{OLD}} = \min\{f_{24}, f_{25}\}$	1
f_{43}	$\gamma_{CO_{OLD}} = \min\{f_{22}, f_{23}, f_{24}, f_{25}\}$	1
f_{44}	$\gamma_{ADJ_{OLD}} = \min\{f_{22}, f_{23}, f_{26}\}$	∞
f_{45}	$\gamma_{(CO \vee ADJ)_{OLD}} = \min\{f_{22}, f_{23}, f_{24}, f_{25}, f_{26}\}$	1
f_{46}	$\gamma_{CO, OFN} = \min\{f_{22}, f_{24}\}$	∞
f_{47}	$\gamma_{CO, \overline{OFN}} = \min\{f_{23}, f_{25}\}$	1
f_{48}	$\gamma_{\overline{CO}_{OLD}} = \min\{f_{26}, f_{27}\}$	0

Table 2: List of auxiliary functions together with their initial values $f_j(v, \{v\})$.

Algorithm 6 (Algorithm XB)

Input: a connected block graph B and its list of maximal cliques.

Output: $\gamma_X(B)$.

```

1: randomly select a vertex  $v_1$  and call  $RXB(v_1, B)$ ;
2(If  $X = ID$ ):
  return  $\gamma_{ID}(v_1, B) = \min\{f_1(v_1, B), f_2(v_1, B), f_3(v_1, B), f_4(v_1, B)\}$ .
2(If  $X = LD$ ):
  return  $\gamma_{LD}(v_1, B) = \min\{f_{11}(v_1, B), f_{12}(v_1, B)\}$ .
2(If  $X = OLD$ ):
  return  $\gamma_{OLD}(v_1, B) = \min\{f_{19}(v_1, B), f_{20}(v_1, B), f_{21}(v_1, B)\}$ .

```

Algorithm RXB chooses a maximal clique K with vertices $\{v_1, \dots, v_k\}$ and either returns the initial function values if $K = \{v_1\}$ or deletes all edges of K and calls recursively $RXB(v_i, B_i)$ for all so-obtained components B_i of $B - E(K)$ to compute the lists of functions LX_i for (v_i, B_i) with $i \in \{1, \dots, k\}$ according to the problem $X \in \{ID, LD, OLD\}$.

In the following sections, we use this common algorithmic scheme for all 3 studied problems and specify which of the main and auxiliary functions are relevant for the particular problem and how we can compute them in all 3 cases.

Algorithm 7 (Algorithm RXB)

Input: a block graph B , its list of maximal cliques and $v_1 \in V(B)$. Output: list L of the values of the main functions f_j on (v_1, B) corresponding to problem X .
1: if v_1 has degree 0 in B then 2: initialize LX (corresponding to problem X); 3: else 4: let K be a maximal clique with $V(K) = \{v_1, \dots, v_k\}$ and delete its edges; 5: let B_1, \dots, B_k be the remaining block graphs, resp., containing v_1, \dots, v_k ; 6: let $LX_i = RXB(v_i, B_i)$ for all $i \in \{1, \dots, k\}$ corresponding to problem X ; 7: compute the main functions on (v_1, B) from LX_i for all $i \in \{1, \dots, k\}$ corresponding to problem X . 8: end if 9: return the list LX of the values of the main functions f_j on (v_1, B) corresponding to problem X .

4 A linear-time algorithm for the identifying code number of B

Let $v_1 \in V(B)$ and let k be the order of a maximal clique containing v_1 . We will see that the algorithm $RIDB$ needs to compute the main functions $f_j(v_1, B)$ with $j \in \{1, \dots, 10\}$. Actually, for $i \in \{1, \dots, k\}$ we need the list LID_i that consists in functions $f_j(v_i, B_i)$ for $j \in \{1, \dots, 10\}$. In order to do so we need the auxiliary functions $f_j(v_i, B_i)$ for $j \in \{28, \dots, 34\}$.

To see the correctness of our algorithm, let us prove first a technical lemma.

Lemma 8 *Consider a block graph B and $V(K) = \{v_1, v_2, \dots, v_k\}$. Let C_i be a v_i -almost identifying code in B_i for $i \in \{1, 2, \dots, k\}$ and $C = \bigcup_{i=1}^k C_i$ and*

$$V' = \bigcup_{i=1}^k (V(B_i) - \{v_i\}).$$

- (1) *Let $v \in V'$ and $v_j \in V(K)$ such that $d(v, v_j) = 1$. Then v_j and v are dominated and separated by C if there is $i \in \{1, 2, \dots, k\}$ $i \neq j$ such that C_i is CO.*
- (2) *Let $v \in V'$ and $v_j \in V(K)$ with $d(v, v_j) = 2$. Then v_j and v are dominated and separated by C if C_j satisfies CO or there is $i \in \{1, 2, \dots, k\}$ $i \neq j$ such that C_i is CO and $v \notin V(B_i)$.*
- (3) *Let $v_i, v_j \in V(K)$ with $i \neq j$. Then v_i and v_j are dominated and separated by C if either C_i or C_j is ADJ.*

Proof. If $v \in V'$ and $v_j \in V(K)$ such that $d(v, v_j) = 1$ then it is clear that $v \in V(B_j)$. If there is $i \neq j$ such that C_i is CO then $v_i \in N[v_j] - N[v]$ and (1) follows.

Let $v \in V(B_r) - \{v_r\}$ for some $r \in \{1, 2, \dots, k\}$ and $j \in \{1, 2, \dots, k\}$ such that $d(v_j, v) = 2$. Clearly $v_j \in N[v_r] - N[v]$. If $v_j \in C$ then (2) follows. Otherwise suppose that there is $i \neq j$ such that $v_i \in C$ and $i \neq r$. Hence $v_i \in N[v_j] - N[v]$ and this completes the proof of (2).

It is easy to observe that if C_i (C_j) is ADJ then there is $w \in (N[v_i] - N[v_j])$ ($w \in N[v_j] - N[v_i]$) with $w \in V(B_i) \cap C_i$ ($w \in V(B_j) \cap C_j$) and the proof is complete. \square

The following result shows how to compute $f_1(v_1, B)$ for a given $v_1 \in V(B)$.

Theorem 9 Consider a block graph B and $V(K) = \{v_1, v_2, \dots, v_k\}$. Let C be a code in B and $C_i = C \cap V(B_i)$ for all $i \in \{1, 2, \dots, k\}$. Then C is a v_1 -almost identifying code in B with properties ID , CO , ADJ , FN if and only if C_i is a v_i -almost identifying code in B_i for all $i \in \{1, 2, \dots, k\}$, and one of the following sets of assertions is satisfied:

- (i) C_1 satisfies ID , CO , ADJ and \overline{FN} , there exists $j \neq 1$ such that C_j satisfies \overline{CO} and \overline{ADJ} , and C_i satisfies \overline{CO} and ADJ for all $i \neq 1, j$.
- (ii) C_1 satisfies CO , ADJ and FN , there exists $j \neq 1$ such that C_j satisfies CO and \overline{ADJ} , and C_i satisfies ADJ for all $i \neq 1, j$.
- (iii) If $k \geq 3$, C_1 satisfies CO , ADJ and FN , there exists $h \neq 1$ such that C_h satisfies CO , there exists $j \neq 1$ such that C_j satisfies \overline{CO} and \overline{ADJ} , and C_i satisfies ADJ for all $i \neq j$.
- (iv) C_1 satisfies CO and FN , there exists $h \neq 1$ such that C_h satisfies CO , and C_i satisfies ADJ for all $i \neq 1$.
- (v) C_1 satisfies ID , CO , ADJ and FN , and C_i satisfies \overline{CO} and ADJ for all $i \neq 1$.

Proof. Let C be a v_1 -almost identifying code in B with properties ID , CO , ADJ and FN . Observe that by Remark 3, C_i is a v_i -almost identifying code in B_i for all i . Also, C_1 satisfies CO since C satisfies CO and $v_1 \in V(B_1)$. Since C is FN there is only one $w \in V(B) - C$ such that $N[w] \cap C = \{v_1\}$. We analyze the different cases that may occur (see Figure 2).

If $w = v_j$ for some $j \in \{2, \dots, k\}$ then C_1 is \overline{FN} , C_i is \overline{CO} for all $i \neq 1$ and C_j is \overline{ADJ} . Then by Lemma 5 and the fact that C is ID it follows that C_i is ADJ for all $i \neq j$. Since $N(v_1) \cap C \cap V(B_i) = \emptyset$ for all $i \neq 1$ and C is ID , C_1 is ID and (i) is proved.

Now, if $w \in V(B_1)$ then C_1 is FN . Since C is ADJ there is $v \in N(v_1) \cap C$.

If $V(K) \cap C = \{v_1\}$ then $v \in V(B_1)$ and C_1 is ADJ , C_i is \overline{CO} for all $i \neq 1$. Since C is ID and C_1 is FN it follows that C_i is ADJ for all $i \neq 1$ and C_1 is ID . Then (v) follows.

If $|V(K) \cap C| \geq 2$ then there is $h \neq 1$ such that C_h is CO . Hence, if C_i is ADJ for all $i \neq 1$ then we obtain (iv). But, if C_h is \overline{ADJ} , Lemma 5 implies that C_i is ADJ for all $i \neq h$ which proves (ii). In addition if there is $j \neq h$ such that C_j is \overline{ADJ} and \overline{CO} again from Lemma 5 C_i is ADJ for all $i \neq j$ which gives us (iii).

Conversely, let C_i be a v_i -almost identifying code in B_i for all $i \in \{1, \dots, k\}$ satisfying one of the sets of properties from (i) to (v) and let $C = \bigcup_{i=1}^k C_i$ (see Figure 2).

We will prove that C is a v_1 -almost identifying code in B with the properties ID , CO , ADJ and FN .

It is easy to see that C is CO and ADJ since C_1 is CO and either C_1 satisfies ADJ or C_h is CO with $h \neq 1$ in each one of the statements (i) to (v). Also, it is easy to verify that each of the statements implies that C is FN .

Then it remains to prove that C is ID , i.e., we need to prove that given $u, v \in V(B)$ with $d(u, v) \leq 2$ they are dominated and separated by C .

From Lemma 4 if $u, v \in \bigcup_{i=1}^k (V(B_i) - \{v_i\})$ the theorem follows.

Consider $u, v \in V(B)$ with $d(u, v) = 2$ such that $u \in V(K)$. Then $v \in V(B_i) - \{v_i\}$ for some i and $u = v_r$ for some r . If $r = 1$, by using Lemma 8 (2) and the fact that C_1 is CO in any of the statements (i) to (v), it follows that v_1 and v are dominated and separated by C .

Now, let $r \neq 1$. If $i \neq 1$ again Lemma 8 (2) together with the fact that $v_1 \in C$ imply that v_r and v are dominated and separated by C .

When, $r \neq 1$ and $i = 1$, statements (ii) to (iv) say that there is $h \neq 1$ with $v_h \in C$ and Lemma 8 (2) ensures that v_r and v are separated and dominated by C . If statement (i) holds then C_1 is FN and then there is $w \in N[v] \cap C \cap V(B_1)$ and $w \notin N[v_r]$. If statement (v) is satisfied then C_r is ADJ . In both cases v and v_r are dominated and separated by C .

If $u, v \in V(B)$ with $d(u, v) = 1$ such that $u \in V(K)$ and $v \notin V(K)$ then $v \in V(B_i) - \{v_i\}$ and $u = v_i$ for some i . If $i = 1$ we see that statements (ii) to (iv) ensure there is $h \neq 1$ such that C_h is CO while statements (i) and (v) state that C_1 is ID . Hence Lemma 8 (1) implies that they are dominated and separated. In case $i \neq 1$ all statements imply that C_1 is CO and the same lemma proves that u and v are dominated and separated by C .

Finally, if $u = v_i$ and $v = v_j$ with $i \neq j$ then either C_i is ADJ or C_j is ADJ but not both. Hence Lemma 8 (3) completes the proof. \square

In all the pictures used as illustrations for the proofs, black dots represent vertices in the corresponding code, white dots depict vertices out of the code and crosses represent vertices that may be or may not be in the code.

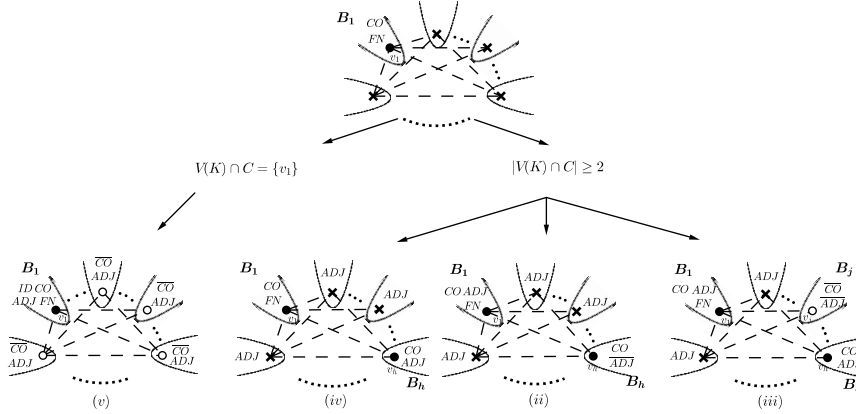


Figure 2: Illustration for the proof of Theorem 9, cases (ii) to (v).

From Theorem 9 we obtain the following:

Corollary 10 *With the notation of Theorem 9, we have the following equalities:*

- if $k = 2$, $f_1(v_1, B) = \min \begin{cases} f_2(v_1, B_1) + f_{10}(v_2, B_2) \\ f_5(v_1, B_1) + f_{28}(v_2, B_2) \\ f_{30}(v_1, B_1) + f_{29}(v_2, B_2) \\ f_1(v_1, B_1) + f_9(v_2, B_2) \end{cases}$
- if $k \geq 3$, then $f_1(v_1, B)$ equals
$$\min \begin{cases} f_2(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{10}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_9(v_i, B_i) \right\} \\ f_5(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{28}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{33}(v_i, B_i) \right\} \\ f_5(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{10}(v_j, B_j) + f_{29}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{33}(v_i, B_i) \right\} \\ f_{30}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{29}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{33}(v_i, B_i) \right\} \\ f_1(v_1, B_1) + \sum_{i=2}^k f_9(v_i, B_i) \end{cases}$$

In a similar way, the remaining main functions $f_j(v_1, B)$ for $j \in \{2, \dots, 10\}$ can be obtained. We give them in Tables 3, 4, 5 and 6.

Function	Function
$f_1(v_1, B) = \min \begin{cases} f_2(v_1, B_1) + f_{10}(v_2, B_2) \\ f_5(v_1, B_1) + f_{28}(v_2, B_2) \\ f_{30}(v_1, B_1) + f_{29}(v_2, B_2) \\ f_1(v_1, B_1) + f_9(v_2, B_2) \end{cases}$	$f_6(v_1, B) = \min \begin{cases} f_{31}(v_1, B_1) + f_{28}(v_2, B_2) \\ f_{31}(v_1, B_1) + f_{29}(v_2, B_2) \\ f_6(v_1, B_1) + f_9(v_2, B_2) \end{cases}$
$f_2(v_1, B) = \min \begin{cases} f_6(v_1, B_1) + f_{28}(v_2, B_2) \\ f_{31}(v_1, B_1) + f_{29}(v_2, B_2) \\ f_2(v_1, B_1) + f_9(v_2, B_2) \end{cases}$	$f_7(v_1, B) = \min \begin{cases} f_7(v_1, B_1) + f_9(v_2, B_2) \\ f_8(v_1, B_1) + f_{10}(v_2, B_2) \end{cases}$
$f_3(v_1, B) = f_3(v_1, B_1) + f_9(v_2, B_2)$	$f_8(v_1, B) = f_8(v_1, B_1) + f_9(v_2, B_2)$
$f_4(v_1, B) = \min \begin{cases} f_9(v_1, B_1) + f_3(v_2, B_2) \\ f_{10}(v_1, B_1) + f_2(v_2, B_2) \\ f_9(v_1, B_1) + f_{34}(v_2, B_2) \\ f_4(v_1, B_1) + f_4(v_2, B_2) \end{cases}$	$f_9(v_1, B) = \min \begin{cases} f_{32}(v_1, B_1) + f_3(v_2, B_2) \\ f_{32}(v_1, B_1) + f_{34}(v_2, B_2) \\ f_9(v_1, B_1) + f_4(v_2, B_2) \end{cases}$
$f_5(v_1, B) = \min \begin{cases} f_6(v_1, B_1) + f_{10}(v_2, B_2) \\ f_{30}(v_1, B_1) + f_{28}(v_2, B_2) \\ f_{30}(v_1, B_1) + f_{29}(v_2, B_2) \\ f_5(v_1, B_1) + f_9(v_2, B_2) \end{cases}$	$f_{10}(v_1, B) = f_{10}(v_1, B_1) + f_4(v_2, B_2)$

Table 3: Case $|V(K)| = 2$ for *RIDB*.

From the formulas in Tables 3, 4, 5 and 6, it is immediate to see that for each of the ten functions f_j with $j \in \{1, \dots, 10\}$, we can compute $f_j(v_1, B)$ from $LID_i(v_i, B_i)$ for all $i \in \{1, \dots, k\}$ in time $O(k)$, where k is the order of a maximal clique containing vertex v .

Theorem 11 *Algorithm IDB computes in linear time $\gamma_{ID}(B)$ of an identifiable block graph B or returns ∞ if no identifying code exists in B .*

Proof. In what follows we call $g(B)$ the number of operations the algorithm needs to solve the problem on graph B . The proof is by induction on the number of vertices of graph B . If $V(B) = \{v\}$ then using the initial values we get that $g(B) \in O(1)$. Assume that for every block graph with less than n vertices the result holds and let B be a block graph of order n .

Let $v \in V(B)$ and consider a maximal clique K containing v . Then, we erase all the edges in K and obtain $|V(K)| = k$ block subgraphs called B_i with $i = 1, \dots, k$. From induction hypothesis $g(B_i) \in O(n_i + m_i)$ where $n_i = |V(B_i)|$ and $m_i = |E(B_i)|$ for $i = 1, \dots, k$. Then, in order to obtain $\gamma_{ID}(B)$ we need $f_j(v, B)$ for $j \in \{1, \dots, 10\}$. As we have mentioned above, this can be computed in time $O(k)$. Then $g(B) \in O(n + m)$ and the proof is complete. \square

5 A linear-time algorithm for determining the locating-dominating number of B

We will see that the algorithm *RLDB* needs to compute the main functions $f_j(v_1, B)$ with $j \in \{11, \dots, 18\}$. In fact, for $i \in \{1, \dots, k\}$ we need the list LLD_i that consists in functions $f_j(v_i, B_i)$ for $j \in \{11, \dots, 18\}$, and in order to do so we need also the auxiliary functions $f_j(v_i, B_i)$ for $j \in \{35, \dots, 39\}$.

Theorem 12 *Consider a block graph B and $V(K) = \{v_1, v_2, \dots, v_k\}$. Let C be a code in B and $C_i = C \cap V(B_i)$ for all $i \in \{1, 2, \dots, k\}$. C is a v_1 -almost locating-dominating code in B having properties LD , CO if and only if C_i is a v_i -almost locating-dominating code in B_i for all $i \in \{1, 2, \dots, k\}$ and one of the following statements is satisfied.*

- (i) C_1 is CO , \overline{FN} , there exists $j \neq 1$ such that C_j is \overline{CO} and \overline{ADJ} , and C_i is \overline{CO} and ADJ for all $i \neq 1, j$.
- (ii) If $k \geq 3$, C_1 is CO , there exists $h \neq 1$ such that C_h is CO , there exists $j \neq 1, h$ such that C_j is \overline{CO} and \overline{ADJ} , and C_i is CO or ADJ for all $i \neq j, h$.
- (iii) C_1 is CO , and C_i is CO or ADJ for all i .

Proof. Let C be a v_1 -almost LD -code satisfying LD and CO .

From Remark 3, C_i is a v_i -almost LD -code in B_i for all $i \in \{1, \dots, k\}$. Also C_1 is CO , since C is CO and $v_1 \in V(B_1)$. In addition, Lemma 5 ensures that there is at most one $j \neq 1$ such that C_j is \overline{CO} and \overline{ADJ} .

Let $j \neq 1$ such that C_j is \overline{CO} and \overline{ADJ} . If C_i is \overline{CO} for all $i \neq 1$, C_i is ADJ for all $i \neq 1$ and as C is LD , C_1 is \overline{FN} and (i) holds.

Now, let $h \neq 1, j$ such that C_h is CO . Then for all $i \neq 1, j, h$, C_i is CO or ADJ and (ii) holds.

Finally, if C_i is CO or ADJ for all $i \neq 1$ then (iii) holds.

Conversely, let C_i be a v_i -almost LD -code in B_i for all i satisfying one of the statements from (i) to (iii) and let $C = \bigcup_{i=1}^k C_i$. We will prove that C is a v_1 -almost LD -code in B with properties LD and CO .

It is easy to see that C is CO since C_1 is CO in every case. Then to prove that C is LD we need to verify that $u, v \in V(B)$, $d(u, v) \leq 2$ are dominated and separated by C .

From Lemma 4, $u, v \in V(B) - \bigcup_{i=1}^k \{v_i\}$ are dominated and separated by C .

Then, let $u, v \in V(B)$, $d(u, v) = 2$ and $u \in V(K)$. We have that $v \in V(B_i) - \{v_i\}$ for some i and $u = v_r$ for $r \neq i$ such that C_r is \overline{CO} .

If $i \neq 1$, u and v are dominated and separated by C since C_1 is CO .

Let $i = 1$. In case (i), as C_1 is \overline{FN} it exists $w \in N[v] \cap C - N(u)$, $w \neq v_1$. In the remaining cases, since C_r is \overline{CO} and it is ADJ or there exists $h \neq 1$ such that C_h is CO . Then u and v are separated and dominated.

Now, let $u, v \in V(B)$, $d(u, v) = 1$, $u \in V(K)$ and $v \notin V(K)$.

If $u = v_i$ for some i , then $v \in V(B_i) - \{v_i\}$ and C_i is \overline{CO} . Since $i \neq 1$ and C_1 is CO , u and v are separated and dominated.

Finally, let $u = v_i$ and $v = v_j$, $i, j \neq 1$ and C_i, C_j be \overline{CO} . In all the cases, from Lemma 5 at least one of C_i and C_j is ADJ . Then u and v are separated and dominated. \square

Corollary 13 *With the notation of Theorem 12, we have:*

- for $k = 2$, $f_{11}(v_1, B) = \min \begin{cases} f_{36}(v_1, B_1) + f_{18}(v_2, B_2) \\ f_{39}(v_1, B_1) + f_{35}(v_2, B_2) \end{cases}$
- for $k \geq 3$, $f_{11}(v_1, B)$ equals
$$\min \begin{cases} f_{36}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{18}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{17}(v_i, B_i) \right\} \\ f_{39}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{18}(v_j, B_j) + f_{39}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{35}(v_i, B_i) \right\} \\ f_{39}(v_1, B_1) + \sum_{i=2}^k f_{35}(v_i, B_i) \end{cases}$$

In a similar way, the remaining main functions $f_j(v_1, B)$ for $j \in \{12, \dots, 18\}$ can be obtained. We give them in Tables 7, 8, 9 and 10.

From the formulas given in Tables 7, 8, 9 and 10, it is immediate to see that for each of the eight functions f_j $j \in 11, \dots, 18$, we can compute $f_j(v_1, B)$ from $LLD_i(v_i, B_i)$ for all $i \in \{1, \dots, k\}$ in time $O(k)$.

The next result can be proved in a similar way as Theorem 11 was proved.

Theorem 14 *Algorithm LDB computes in linear time the locating-dominating number $\gamma_{LD}(B)$ of a block graph B .*

6 A linear-time algorithm for the open locating-dominating number of B

We will see that the algorithm *ROLDB* needs to compute the main functions $f_j(v_1, B)$ with $j \in \{19, \dots, 27\}$, and in order to do so we need for all $i \in \{1, \dots, k\}$ the main functions $f_j(v_i, B_i)$ for $j \in \{19, \dots, 27\}$ and the auxiliary functions $f_j(v_i, B_i)$ for $j \in \{40, \dots, 48\}$.

Theorem 15 *Consider a block graph B and $V(K) = \{v_1, v_2, \dots, v_k\}$. Let C be a code in B and $C_i = C \cap V(B_i)$ for all $i \in \{1, 2, \dots, k\}$. C is a v_1 -almost open locating-dominating code in B having properties OLD, CO and OFN if and only*

if C_i is a v_i -almost open locating-dominating code in B_i for all $i \in \{1, 2, \dots, k\}$ and one of the following sets of statements is satisfied.

- (i) C_1 is *OLD*, CO and \overline{OFN} , there exists $j \neq 1$ such that C_j is \overline{CO} and \overline{ADJ} , and C_i is \overline{CO} and ADJ for all $i \neq 1, j$.
- (ii) If $k \geq 3$, C_1 is CO , ADJ and \overline{OFN} , there exists $h \neq 1$ such that C_h is \overline{CO} , there exists $j \neq 1, h$ such that C_j is CO and \overline{ADJ} , C_i is \overline{CO} and ADJ and for all $i \neq 1, h, j$.
- (iii) If $k \geq 3$, C_1 is CO and \overline{OFN} , there exists $h \neq 1$ such that C_h is \overline{CO} , there exists $j \neq 1, h$ such that C_j is CO and \overline{ADJ} and \overline{OFN} , and C_i is \overline{CO} and ADJ for all $i \neq 1, j$.
- (iv) If $k \geq 3$, C_1 is CO , ADJ and OFN , there exists $j \neq 1$ such that C_j is \overline{CO} and \overline{ADJ} , there exists $h \neq 1, j$ such that C_h is CO and ADJ , and C_i is \overline{CO} and ADJ for all $i \neq 1, j, h$.
- (v) If $k \geq 3$, C_1 is CO and OFN , there exists $j \neq 1$ such that C_j is \overline{CO} and \overline{ADJ} , there exists $h \neq 1, j$ such that C_h is CO , ADJ and \overline{OFN} , and C_i is \overline{CO} and ADJ for all $i \neq 1, j, h$.
- (vi) If $k \geq 4$, C_1 is CO and OFN , there exists $j \neq 1$ such that C_j is \overline{CO} and \overline{ADJ} , there exist $h, l \neq 1, h \neq l$ such that C_h, C_l are CO , and C_i is CO or ADJ for all $i \neq 1, j, h, l$.
- (vii) C_1 is *OLD*, CO and OFN , and C_i is \overline{CO} and ADJ for all $i \neq 1$.
- (viii) C_1 is CO , ADJ and OFN , there exists $h \neq 1$ such that C_h is CO and ADJ , and C_i is \overline{CO} and ADJ for all $i \neq 1, h$.
- (ix) C_1 is CO and OFN , there exists $h \neq 1$ such that C_h is CO , ADJ and \overline{OFN} , and C_i is \overline{CO} and ADJ for all $i \neq 1, h$.
- (x) If $k \geq 3$, C_1 is CO and OFN , there exist $h, l \neq 1, h \neq l$ such that C_h, C_l are CO , and C_i is CO or ADJ for all $i \neq 1, h, l$.

Proof. Let C be a v_1 -almost *OLD*-code satisfying *OLD*, *CO* and *OFN*. From Remark 3, C_i is a v_i -almost *OLD*-code in B_i for all i . Also, C_1 is *CO*, since C is *CO* and $v_1 \in V(B_1)$. In addition, as C is *OFN*, there exists a unique $w \in V(B)$ such that $N(w) \cap C = \{v_1\}$.

Assume that $w = v_j$ for some $j \neq 1$, then C_1 is \overline{OFN} , C_i is \overline{CO} for all $i \neq 1, j$ and C_j is \overline{ADJ} . In addition, if C_j is \overline{CO} , then C_i is ADJ for $i \neq 1, j$. Since $N(v_1) \cap C \cap V(B_i) = \emptyset$ for all $i \neq 1$ and C is *OLD*, we have that C_1 is *OLD*, and (i) holds. On one hand, if C_j is CO , from Lemma 5 there exists at most one $h \neq 1, j$ such that C_h is \overline{ADJ} . On the other hand, since C is *OLD* the neighbors of v_j are open-separated from v_1 . Then, either C_1 is ADJ and (ii) is proved, or C_j is \overline{OFN} and (iii) holds. Now, if $w \in V(B_1)$ then C_1 is *OFN*. We have to analyze different cases.

Assume first that there exists $j \neq 1$ such that C_j is \overline{CO} and \overline{ADJ} (such a j is unique from Lemma 5). As C_1 is *OFN*, there exists $h \neq 1, j$ such that C_h is *CO*. If C_i is \overline{CO} for all $i \neq 1, h$, C_i is ADJ for all $i \neq 1, j, h$ since C is *OLD* v_j is open-separated from v_i . Besides, as C_1 is *OFN*, we have that C_h is ADJ and as C is *OLD*, v_1 is open-separated from the vertices in $N(v_h)$. Hence, either C_1 is ADJ and (iv) holds, or C_h is \overline{OFN} and (v) holds.

Assume now that there exists at least two $h, l \neq 1, j, h \neq l$ such that C_h and C_l are *CO*. Then C_i is *CO* or ADJ for all $i \neq 1, j, h, l$, proving (vi). Now, if C_i is *CO* or ADJ for all $i \neq 1$, we have three cases.

If C_i is \overline{CO} for all $i \neq 1$, then C_i is ADJ for all $i \neq 1$ and as C is OLD and $N(v_1) \cap C \cap V(B_i) = \emptyset$ for all $i \neq 1$, we have that C_1 is OLD proving (vii). If there exists a unique $h \neq 1$ such that C_h is CO , then C_i is \overline{CO} and ADJ for all $i \neq 1, h$. In addition, as C_1 is OFN , we have that C_h is ADJ . Now, as C is OLD , v_1 and the vertices in $N(v_h)$ are open-separated, hence either C_1 is ADJ and (viii) holds, or C_h is \overline{OFN} and (ix) holds.

Finally, if C_h and C_l are CO for some $h, l \neq 1$, $h \neq l$, C_i is CO or ADJ for all $i \neq 1, h, l$ and (x) holds.

Conversely, let C_i be a v_i -almost OLD -code in B_i for all i and let $C = \bigcup_{i=1}^k C_i$.

We will prove that C is a v_i -almost OLD -code in B with properties OLD , CO and OFN . C is CO since C_1 is CO in all the sets of statements from (i) to (x). Also, it is immediate to check that C is OFN .

To prove that C is OLD we need to verify that if $u, v \in V(B)$, $d(u, v) \leq 2$ are open-dominated and open-separated by C . From Lemma 4, if $u, v \in V(B) - \bigcup_{i=1}^k \{v_i\}$, u and v are open-dominated and open-separated by C . Now, let $u, v \in V(B)$, $d(u, v) = 2$ and $u \in V(K)$. Then $v \in V(B_i) - \{v_i\}$ for some i and $u = v_r$ for some r .

Assume first that $r = 1$. If $i = 1$, in (i) and (vii) C_1 is OLD . In the remaining cases, there exists $h \neq 1$ such that C_h is CO . Hence, in any case, u and v are open-dominated and open-separated. Now, let $i \neq 1$. In (i), (ii), (iv), (vii) and (viii), C_1 is ADJ , thus u and v are open-dominated and open-separated. In (vi) and (x), $\{v_h, v_l\} \subseteq N(u) \cap C$, but at most one of v_h and v_l is in $N(v) \cap C$, thus u and v are open-dominated and open-separated. In (iii), if $i = j$, as C_j is \overline{OFN} there exists $w \neq v_j$ such that $w \in (N(v) \cap C) - N(u)$, otherwise, if $i \neq j$, $v_j \in (N(u) \cap C) - N(u)$, then u and v are open-dominated and open-separated.

In (v) and (ix), if $i = h$ as C_h is \overline{OFN} there exists $w \neq v_h$ such that $w \in (N(v) \cap C) - N(u)$, otherwise, if $i \neq h$, $v_h \in (N(u) \cap C) - N(v)$, then u and v are open-dominated and open-separated.

Finally, let $r \neq 1$. If $i \neq 1$, as C_1 is CO in all the cases from (i) to (x), it holds that $v_1 \in (N(u) \cap C) - N(v)$ and u and v are open-dominated and open-separated. If $i = 1$, in (i), (ii) and (iii), C_1 is \overline{OFN} and there exists $w \in (N(v) \cap C) - N(u) - \{v_1\}$ and u and v are open-dominated and open-separated. In the remaining cases, C_r is ADJ or there exists $h \neq 1, r$ such that $v_h \in N(u) \cap C$, thus u and v are open-dominated and open-separated. Now, let $u, v \in V(B)$, $d(u, v) = 1$, $u \in V(K)$ and $v \notin V(K)$. Then, $v \in V(B_i) - \{v_i\}$ for some i and $u = v_i$.

If $i = 1$, in (i) and (vii) C_1 is OLD . In the remaining cases there exists $h \neq 1$ such that C_h is CO , thus u and v are open-dominated and open-separated. If $i \neq 1$, in all the cases from (i) to (x) C_1 is CO and u and v are open-dominated and open-separated.

Finally, let $u = v_i$ and $v = v_j$ for $i \neq j$. In all the cases from (i) to (x), C_i and C_j are CO or ADJ . Then u and v are open-dominated and open-separated. \square

Corollary 16 *With the notation of Theorem 15, we have the following:*

- if $k = 2$, then $f_{19}(v_1, B) = \min \begin{cases} f_{20}(v_1, B_1) + f_{27}(v_2, B_2) \\ f_{19}(v_1, B_1) + f_{26}(v_2, B_2) \\ f_{22}(v_1, B_1) + f_{41}(v_2, B_2) \\ f_{46}(v_1, B_1) + f_{23}(v_2, B_2) \end{cases}$
- if $k = 3$, then $f_{19}(v_1, B)$ equals
$$\min \begin{cases} f_{20}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{23}(v_1, B_1) + \min\{f_{48}(v_2, B_2) + f_{42}(v_3, B_3), f_{42}(v_2, B_2) + f_{48}(v_3, B_3)\} \\ f_{47}(v_1, B_1) + \min\{f_{48}(v_2, B_2) + f_{25}(v_3, B_3), f_{25}(v_2, B_2) + f_{48}(v_3, B_3)\} \\ f_{22}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{41}(v_3, B_3), f_{41}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{46}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{23}(v_3, B_3), f_{23}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{19}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) \\ f_{22}(v_1, B_1) + \min\{f_{41}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{41}(v_3, B_3)\} \\ f_{46}(v_1, B_1) + \min\{f_{23}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{23}(v_3, B_3)\} \\ f_{46}(v_1, B_1) + f_{43}(v_2, B_2) + f_{43}(v_3, B_3) \end{cases}$$
- if $k \geq 4$, then $f_{19}(v_1, B)$ equals
$$\min \begin{cases} f_{20}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{27}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{26}(v_i, B_i) \right\} \\ f_{23}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{48}(v_h, B_h) + f_{42}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ f_{47}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{48}(v_h, B_h) + f_{25}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{26}(v_i, B_i) \right\} \\ f_{22}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ f_{46}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{23}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ f_{46}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ f_{19}(v_1, B_1) + \sum_{i=2}^k f_{26}(v_i, B_i) \\ f_{22}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{26}(v_i, B_i) \right\} \\ f_{46}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{23}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{26}(v_i, B_i) \right\} \\ f_{46}(v_1, B_1) + \min_{\substack{h, l=2, \dots, k \\ h \neq l}} \left\{ f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq h, l}}^k f_{45}(v_i, B_i) \right\} \end{cases}$$

In a similar way, the remaining main functions $f_j(v_1, B)$ for $j \in \{20, \dots, 27\}$ can be obtained, see Tables 11, 12, 13, 14, 15, 16 and 17.

From these formulas, it is immediate to see that for each of the nine functions f_j $j \in \{19, \dots, 27\}$, we can compute $f_j(v_1, B)$ from $LOLD_i(v_i, B_i)$ for all $i \in \{1, \dots, k\}$ in time $O(k)$.

The next result can be proved in a similar way as Theorem 11 was proved.

Theorem 17 *Algorithm OLDB computes in linear time $\gamma_{OLD}(B)$ of a block graph B (or returns ∞ if no open locating-dominating code exists in B).*

7 Concluding remarks

The three here studied domination problems are challenging both from a theoretical and a computational point of view and even remain hard for several graph classes where other in general hard problems are easy to solve, including bipartite graphs and chordal graphs.

In this paper, we present linear-time algorithms that find the identifying code, locating-dominating and open locating-dominating numbers of a given block graph, as a generalization of the linear-time algorithm proposed by Auger [5] for identifying codes in trees. Although our algorithms work in a similar way, they take into account the identifiable and open locating-dominating condition for block graphs and the recomposition steps are built by defining distinct functions accordingly.

Thus, we provide a subclass of chordal graphs for which all the three here studied domination problems can be solved in linear time. Moreover, recall that trees are exactly the block graphs with clique size 2 and constitute exactly the intersection of block graphs with bipartite graphs. Hence, our results provide in particular a linear-time algorithm that finds the *OLD*-code number of a tree.

Furthermore, note that our algorithms could be modified in order to obtain the studied code of minimum size, just by keeping track of the functions where the minimum values are attained. In addition, if B is a vertex-weighted block graph, the algorithms can be easily modified in order to return the minimum weighted identifying code number by just replacing in Table 1 and 2 the entry with value 1 by the weight corresponding to the vertex.

Finally, it is interesting whether similar ideas could be adapted for graph classes with a similar structure, e.g. for cacti (graphs in which every maximal 2-connected subgraph is an edge or a cycle) or for block-cacti (graphs in which every maximal 2-connected subgraph is a clique or a cycle).

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Function	
$f_1(v_1, B) = \min$	$\begin{cases} f_2(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_9(v_3, B_3), f_9(v_2, B_2) + f_{10}(v_3, B_3)\} \\ f_5(v_1, B_1) + \min \{f_{28}(v_2, B_2) + f_{33}(v_3, B_3), f_{33}(v_2, B_2) + f_{28}(v_3, B_3)\} \\ f_5(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_{29}(v_3, B_3), f_{29}(v_2, B_2) + f_{10}(v_3, B_3)\} \\ f_{30}(v_1, B_1) + \min \{f_{29}(v_2, B_2) + f_{33}(v_3, B_3), f_{33}(v_2, B_2) + f_{29}(v_3, B_3)\} \\ f_1(v_1, B_1) + f_9(v_2, B_2) + f_9(v_3, B_3) \end{cases}$
$f_2(v_1, B) = \min$	$\begin{cases} f_6(v_1, B_1) + \min \{f_{28}(v_2, B_2) + f_{33}(v_3, B_3), f_{33}(v_2, B_2) + f_{28}(v_3, B_3)\} \\ f_6(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_{29}(v_3, B_3), f_{29}(v_2, B_2) + f_{10}(v_3, B_3)\} \\ f_{31}(v_1, B_1) + \min \{f_{29}(v_2, B_2) + f_{33}(v_3, B_3), f_{33}(v_2, B_2) + f_{29}(v_3, B_3)\} \\ f_2(v_1, B_1) + f_9(v_2, B_2) + f_9(v_3, B_3) \end{cases}$
$f_3(v_1, B) = f_3(v_1, B_1) + f_9(v_2, B_2) + f_9(v_3, B_3)$	
$f_4(v_1, B) = \min$	$\begin{cases} f_9(v_1, B_1) + \min \{f_{28}(v_2, B_2) + f_{29}(v_3, B_3), f_{29}(v_2, B_2) + f_{28}(v_3, B_3)\} \\ f_9(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_2(v_3, B_3), f_2(v_2, B_2) + f_{10}(v_3, B_3)\} \\ f_9(v_1, B_1) + \min \{f_3(v_2, B_2) + f_9(v_3, B_3), f_9(v_2, B_2) + f_3(v_3, B_3)\} \\ f_{32}(v_1, B_1) + f_{29}(v_2, B_2) + f_{29}(v_3, B_3) \\ f_{10}(v_1, B_1) + \min \{f_2(v_2, B_2) + f_9(v_3, B_3), f_9(v_2, B_2) + f_2(v_3, B_3)\} \\ f_9(v_1, B_1) + \min \{f_{34}(v_2, B_2) + f_9(v_3, B_3), f_9(v_2, B_2) + f_{34}(v_3, B_3)\} \\ f_4(v_1, B_1) + f_4(v_2, B_2) + f_4(v_3, B_3) \end{cases}$
$f_5(v_1, B) = \min$	$\begin{cases} f_6(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_9(v_3, B_3), f_9(v_2, B_2) + f_{10}(v_3, B_3)\} \\ f_{30}(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_{29}(v_3, B_3), f_{29}(v_2, B_2) + f_{10}(v_3, B_3)\} \\ f_{30}(v_1, B_1) + \min \{f_{28}(v_2, B_2) + f_{33}(v_3, B_3), f_{33}(v_2, B_2) + f_{28}(v_3, B_3)\} \\ f_{30}(v_1, B_1) + \min \{f_{29}(v_2, B_2) + f_{33}(v_3, B_3), f_{33}(v_2, B_2) + f_{29}(v_3, B_3)\} \\ f_5(v_1, B_1) + f_9(v_2, B_2) + f_9(v_3, B_3) \end{cases}$
$f_6(v_1, B) = \min$	$\begin{cases} f_{31}(v_1, B_1) + \min \{f_{28}(v_2, B_2) + f_{33}(v_3, B_3), f_{33}(v_2, B_2) + f_{28}(v_3, B_3)\} \\ f_{31}(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_{29}(v_3, B_3), f_{29}(v_2, B_2) + f_{10}(v_3, B_3)\} \\ f_{31}(v_1, B_1) + \min \{f_{29}(v_2, B_2) + f_{33}(v_3, B_3), f_{33}(v_2, B_2) + f_{29}(v_3, B_3)\} \\ f_6(v_1, B_1) + f_9(v_2, B_2) + f_9(v_3, B_3) \end{cases}$
$f_7(v_1, B) = \min$	$\begin{cases} f_7(v_1, B_1) + f_9(v_2, B_2) + f_9(v_3, B_3) \\ f_8(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_9(v_3, B_3), f_9(v_2, B_2) + f_{10}(v_3, B_3)\} \end{cases}$
$f_8(v_1, B) = f_8(v_1, B_1) + f_9(v_2, B_2) + f_9(v_3, B_3)$	
$f_9(v_1, B) = \min$	$\begin{cases} f_{32}(v_1, B_1) + \min \{f_3(v_2, B_2) + f_9(v_3, B_3), f_9(v_2, B_2) + f_3(v_3, B_3)\} \\ f_{32}(v_1, B_1) + \min \{f_{10}(v_2, B_2) + f_2(v_3, B_3), f_2(v_2, B_2) + f_{10}(v_3, B_3)\} \\ f_{32}(v_1, B_1) + \min \{f_{28}(v_2, B_2) + f_{29}(v_3, B_3), f_{29}(v_2, B_2) + f_{28}(v_3, B_3)\} \\ f_{32}(v_1, B_1) + f_{29}(v_2, B_2) + f_{29}(v_3, B_3) \\ f_{32}(v_1, B_1) + \min \{f_{34}(v_2, B_2) + f_9(v_3, B_3), f_9(v_2, B_2) + f_{34}(v_3, B_3)\} \\ f_9(v_1, B_1) + f_4(v_2, B_2) + f_4(v_3, B_3) \end{cases}$
$f_{10}(v_1, B) = f_{10}(v_1, B_1) + f_4(v_2, B_2) + f_4(v_3, B_3)$	

Table 4: Case $|V(K)| = 3$ for *RIDB*.

Function	
$f_1(v_1, B) = \min$	$\begin{cases} f_2(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{10}(v_j, B_j) + \sum_{i=2, \dots, k, i \neq j} f_9(v_i, B_i) \right\} \\ f_5(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{28}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq j} f_{33}(v_i, B_i) \right\} \\ f_5(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{10}(v_j, B_j) + f_{29}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{33}(v_i, B_i) \right\} \\ f_{30}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{29}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{33}(v_i, B_i) \right\} \\ f_1(v_1, B_1) + \sum_{i=2}^k f_9(v_i, B_i) \end{cases}$
$f_2(v_1, B) = \min$	$\begin{cases} f_6(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{28}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{33}(v_i, B_i) \right\} \\ f_6(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{10}(v_j, B_j) + f_{29}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{33}(v_i, B_i) \right\} \\ f_{31}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{29}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{33}(v_i, B_i) \right\} \\ f_2(v_1, B_1) + \sum_{i=2}^k f_9(v_i, B_i) \end{cases}$
$f_3(v_1, B) = f_3(v_1, B_1) + \sum_{i=2}^k f_9(v_i, B_i)$	
$f_4(v_1, B) = \min$	$\begin{cases} f_9(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{10}(v_j, B_j) + f_{29}(v_h, B_h) + f_{29}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{33}(v_i, B_i) \right\} \\ f_9(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{28}(v_j, B_j) + f_{29}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{33}(v_i, B_i) \right\} \\ f_9(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{10}(v_j, B_j) + f_2(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_9(v_i, B_i) \right\} \\ f_9(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_3(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_9(v_i, B_i) \right\} \\ f_{32}(v_1, B_1) + \min_{h, l=2, \dots, k, h \neq l} \left\{ f_{29}(v_h, B_h) + f_{29}(v_l, B_l) + \sum_{i=2, \dots, k, i \neq h, l} f_{33}(v_i, B_i) \right\} \\ f_{10}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_2(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_9(v_i, B_i) \right\} \\ f_9(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{34}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_9(v_i, B_i) \right\} \\ \sum_{i=1}^k f_4(v_i, B_i) \end{cases}$
$f_5(v_1, B) = \min$	$\begin{cases} f_6(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{10}(v_j, B_j) + \sum_{i=2, \dots, k, i \neq j} f_9(v_i, B_i) \right\} \\ f_{30}(v_1, B_1) + \min_{j, h=2, \dots, k, j \neq h} \left\{ f_{10}(v_j, B_j) + f_{29}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq j, h} f_{33}(v_i, B_i) \right\} \\ f_{30}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{28}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{33}(v_i, B_i) \right\} \\ f_{30}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{29}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{33}(v_i, B_i) \right\} \\ f_5(v_1, B_1) + \sum_{i=2}^k f_9(v_i, B_i) \end{cases}$

Table 5: Case $|V(K)| \geq 4$ for *RIDB*.

Function	
$f_6(v_1, B) = \min$	$\left\{ \begin{aligned} &f_{31}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{29}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{33}(v_i, B_i) \right\} \\ &f_{31}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{10}(v_j, B_j) + f_{29}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{33}(v_i, B_i) \right\} \\ &f_{31}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{29}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{34}(v_i, B_i) \right\} \end{aligned} \right\}$
$f_7(v_1, B) = \min$	$\left\{ \begin{aligned} &f_6(v_1, B_1) + \sum_{i=2}^k f_9(v_i, B_i) \\ &f_7(v_1, B_1) + \sum_{i=2}^k f_9(v_i, B_i) \\ &f_8(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{10}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_9(v_i, B_i) \right\} \end{aligned} \right\}$
$f_8(v_1, B) = f_8(v_1, B_1) + \sum_{i=2}^k f_9(v_i, B_i)$	
$f_9(v_1, B) = \min$	$\left\{ \begin{aligned} &f_{32}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_3(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_9(v_i, B_i) \right\} \\ &f_{32}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{10}(v_j, B_j) + f_2(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_9(v_i, B_i) \right\} \\ &f_{32}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{10}(v_j, B_j) + f_{29}(v_h, B_h) + f_{29}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{33}(v_i, B_i) \right\} \\ &f_{32}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{28}(v_j, B_j) + f_{28}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{33}(v_i, B_i) \right\} \\ &f_{32}(v_1, B_1) + \min_{\substack{h, l=2, \dots, k \\ h \neq l}} \left\{ f_{29}(v_h, B_h) + f_{29}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq h, l}}^k f_{33}(v_i, B_i) \right\} \\ &f_{32}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{34}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_9(v_i, B_i) \right\} \\ &f_9(v_1, B_1) + \sum_{i=2}^k f_4(v_i, B_i) \end{aligned} \right\}$
$f_{10}(v_1, B) = f_{10}(v_1, B_1) + \sum_{i=2}^k f_4(v_i, B_i)$	

Table 6: Case $|V(K)| \geq 4$ for *RIDB*.

Function	Function
$f_{11}(v_1, B) = \min \begin{cases} f_{36}(v_1, B_1) + f_{18}(v_2, B_2) \\ f_{39}(v_1, B_1) + f_{35}(v_2, B_2) \end{cases}$	$f_{15}(v_1, B) = \min \begin{cases} f_{16}(v_1, B_1) + f_{18}(v_2, B_2) \\ f_{15}(v_1, B_1) + f_{17}(v_2, B_2) \end{cases}$
$f_{12}(v_1, B) = \min \begin{cases} f_{18}(v_1, B_1) + f_{36}(v_2, B_2) \\ f_{12}(v_1, B_1) + f_{12}(v_2, B_2) \\ f_{17}(v_1, B_1) + f_{39}(v_2, B_2) \end{cases}$	$f_{16}(v_1, B) = f_{16}(v_1, B_1) + f_{17}(v_2, B_2)$
$f_{13}(v_1, B) = \min \begin{cases} f_{14}(v_1, B_1) + f_{18}(v_2, B_2) \\ f_{13}(v_1, B_1) + f_{17}(v_2, B_2) \\ f_{37}(v_1, B_1) + f_{39}(v_2, B_2) \end{cases}$	$f_{17}(v_1, B) = \min \begin{cases} f_{17}(v_1, B_1) + f_{12}(v_2, B_2) \\ f_{38}(v_1, B_1) + f_{39}(v_2, B_2) \end{cases}$
$f_{14}(v_1, B) = \min \begin{cases} f_{14}(v_1, B_1) + f_{17}(v_2, B_2) \\ f_{36}(v_1, B_1) + f_{38}(v_2, B_2) \end{cases}$	$f_{18}(v_1, B) = f_{18}(v_1, B_1) + f_{12}(v_2, B_2)$

Table 7: Case $|V(K)| = 2$ for *RLDB*.

Function
$f_{11}(v_1, B) = \min \begin{cases} f_{36}(v_1, B_1) + \min \{f_{18}(v_2, B_2) + f_{17}(v_3, B_3), f_{17}(v_2, B_2) + f_{18}(v_3, B_3)\} \\ f_{39}(v_1, B_1) + \min \{f_{18}(v_2, B_2) + f_{39}(v_3, B_3), f_{39}(v_2, B_2) + f_{18}(v_3, B_3)\} \\ f_{39}(v_1, B_1) + f_{35}(v_2, B_2) + f_{35}(v_3, B_3) \end{cases}$
$f_{12}(v_1, B) = \min \begin{cases} f_{17}(v_1, B_1) + \min \{f_{18}(v_2, B_2) + f_{36}(v_3, B_3), f_{36}(v_2, B_2) + f_{18}(v_3, B_3)\} \\ f_{18}(v_1, B_1) + \min \{f_{36}(v_2, B_2) + f_{17}(v_3, B_3), f_{17}(v_2, B_2) + f_{36}(v_3, B_3)\} \\ f_{18}(v_1, B_1) + f_{39}(v_2, B_2) + f_{39}(v_3, B_3) \\ f_{12}(v_1, B_1) + f_{12}(v_2, B_2) + f_{12}(v_3, B_3) \\ f_{17}(v_1, B_1) + \min \{f_{39}(v_2, B_2) + f_{35}(v_3, B_3), f_{35}(v_2, B_2) + f_{39}(v_3, B_3)\} \end{cases}$
$f_{13}(v_1, B) = \min \begin{cases} f_{14}(v_1, B_1) + \min \{f_{18}(v_2, B_2) + f_{17}(v_3, B_3), f_{17}(v_2, B_2) + f_{18}(v_3, B_3)\} \\ f_{13}(v_1, B_1) + f_{17}(v_2, B_2) + f_{17}(v_3, B_3) \\ f_{37}(v_1, B_1) + \min \{f_{18}(v_2, B_2) + f_{39}(v_3, B_3), f_{39}(v_2, B_2) + f_{18}(v_3, B_3)\} \\ f_{37}(v_1, B_1) + \min \{f_{39}(v_2, B_2) + f_{35}(v_3, B_3), f_{35}(v_2, B_2) + f_{39}(v_3, B_3)\} \end{cases}$
$f_{14}(v_1, B) = \min \begin{cases} f_{36}(v_1, B_1) + \min \{f_{18}(v_2, B_2) + f_{39}(v_3, B_3), f_{39}(v_2, B_2) + f_{18}(v_3, B_3)\} \\ f_{14}(v_1, B_1) + f_{17}(v_2, B_2) + f_{17}(v_3, B_3) \\ f_{36}(v_1, B_1) + \min \{f_{39}(v_2, B_2) + f_{35}(v_3, B_3), f_{35}(v_2, B_2) + f_{39}(v_3, B_3)\} \end{cases}$
$f_{15}(v_1, B) = \min \begin{cases} f_{16}(v_1, B_1) + \min \{f_{18}(v_2, B_2) + f_{17}(v_3, B_3), f_{17}(v_2, B_2) + f_{18}(v_3, B_3)\} \\ f_{15}(v_1, B_1) + f_{17}(v_2, B_2) + f_{17}(v_3, B_3) \end{cases}$
$f_{16}(v_1, B) = f_{16}(v_1, B_1) + f_{17}(v_2, B_2) + f_{17}(v_3, B_3)$
$f_{17}(v_1, B) = \min \begin{cases} f_{17}(v_1, B_1) + f_{12}(v_2, B_2) + f_{12}(v_3, B_3) \\ f_{38}(v_1, B_1) + \min \{f_{18}(v_2, B_2) + f_{36}(v_3, B_3), f_{36}(v_2, B_2) + f_{18}(v_3, B_3)\} \\ f_{38}(v_1, B_1) + \min \{f_{39}(v_2, B_2) + f_{17}(v_3, B_3), f_{17}(v_2, B_2) + f_{39}(v_3, B_3)\} \\ f_{38}(v_1, B_1) + f_{39}(v_2, B_2) + f_{39}(v_3, B_3) \end{cases}$
$f_{18}(v_1, B) = f_{18}(v_1, B_1) + f_{12}(v_2, B_2) + f_{12}(v_3, B_3)$

Table 8: Case $|V(K)| = 3$ for *RLDB*.

Function	
$f_{11}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{36}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{18}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{17}(v_i, B_i) \right\} \\ & f_{39}(v_1, B_1) + \min_{\substack{j,h=2, \dots, k \\ j \neq h}} \left\{ f_{18}(v_j, B_j) + f_{39}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{35}(v_i, B_i) \right\} \\ & f_{39}(v_1, B_1) + \sum_{i=2}^k f_{35}(v_i, B_i) \end{aligned} \right\}$
$f_{12}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{17}(v_1, B_1) + \min_{\substack{j,h=2, \dots, k \\ j \neq h}} \left\{ f_{18}(v_j, B_j) + f_{36}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{17}(v_i, B_i) \right\} \\ & f_{17}(v_1, B_1) + \min_{\substack{j,h,l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{18}(v_j, B_j) + f_{39}(v_h, B_h) + f_{39}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{35}(v_i, B_i) \right\} \\ & f_{18}(v_1, B_1) + \min_{\substack{h=2, \dots, k \\ i \neq h}} \left\{ f_{36}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{17}(v_i, B_i) \right\} \\ & f_{18}(v_1, B_1) + \min_{\substack{h,l=2, \dots, k \\ h \neq l}} \left\{ f_{39}(v_h, B_h) + f_{39}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq h, l}}^k f_{35}(v_i, B_i) \right\} \\ & \sum_{i=1}^k f_{12}(v_i, B_i) \\ & f_{17}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{39}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{35}(v_i, B_i) \right\} \\ & f_{14}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{18}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{17}(v_i, B_i) \right\} \\ & f_{13}(v_1, B_1) + \sum_{i=2}^k f_{17}(v_i, B_i) \end{aligned} \right\}$
$f_{13}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{37}(v_1, B_1) + \min_{\substack{j,h=2, \dots, k \\ j \neq h}} \left\{ f_{18}(v_j, B_j) + f_{39}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{35}(v_i, B_i) \right\} \\ & f_{37}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{39}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{35}(v_i, B_i) \right\} \end{aligned} \right\}$
$f_{14}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{36}(v_1, B_1) + \min_{\substack{j,h=2, \dots, k \\ j \neq h}} \left\{ f_{18}(v_j, B_j) + f_{39}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{35}(v_i, B_i) \right\} \\ & f_{14}(v_1, B_1) + \sum_{i=2}^k f_{17}(v_i, B_i) \end{aligned} \right\}$
$f_{15}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{36}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{39}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{35}(v_i, B_i) \right\} \\ & f_{16}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{18}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{17}(v_i, B_i) \right\} \\ & f_{15}(v_1, B_1) + \sum_{i=2}^k f_{17}(v_i, B_i) \end{aligned} \right\}$
$f_{16}(v_1, B) = f_{16}(v_1, B_1) + \sum_{i=2}^k f_{17}(v_i, B_i)$	

Table 9: Case $|V(K)| \geq 4$ for *RLDB*.

Function	
$f_{17}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{17}(v_1, B_1) + \sum_{i=2}^k f_{12}(v_i, B_i) \\ & f_{38}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{18}(v_j, B_j) + f_{36}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{17}(v_i, B_i) \right\} \\ & f_{38}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{39}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{17}(v_i, B_i) \right\} \\ & f_{38}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{18}(v_j, B_j) + f_{39}(v_h, B_h) + f_{39}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{35}(v_i, B_i) \right\} \\ & f_{38}(v_1, B_1) + \min_{\substack{h, l=2, \dots, k \\ h \neq l}} \left\{ f_{39}(v_h, B_h) + f_{39}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq h, l}}^k f_{35}(v_i, B_i) \right\} \\ & \sum_{i=1}^k f_{12}(v_i, B_i) \end{aligned} \right\}$
$f_{18}(v_1, B) =$	$f_{18}(v_1, B_1) + \sum_{i=2}^k f_{12}(v_i, B_i)$

Table 10: Case $|V(K)| \geq 4$ for *RLDB*.

Function	Function
$f_{19}(v_1, B) = \min$	$f_{24}(v_1, B) = \min \left\{ \begin{aligned} & f_{20}(v_1, B_1) + f_{27}(v_2, B_2) \\ & f_{19}(v_1, B_1) + f_{26}(v_2, B_2) \\ & f_{22}(v_1, B_1) + f_{41}(v_2, B_2) \\ & f_{46}(v_1, B_1) + f_{23}(v_2, B_2) \end{aligned} \right\}$
$f_{20}(v_1, B) = \min$	$f_{25}(v_1, B) = f_{25}(v_1, B_1) + f_{26}(v_2, B_2)$
$f_{21}(v_1, B) = \min$	$f_{26}(v_1, B) = \min \left\{ \begin{aligned} & f_{20}(v_1, B_1) + f_{26}(v_2, B_2) \\ & f_{23}(v_1, B_1) + f_{41}(v_2, B_2) \\ & f_{47}(v_1, B_1) + f_{23}(v_2, B_2) \end{aligned} \right\}$
$f_{22}(v_1, B) = \min$	$f_{27}(v_1, B) = f_{27}(v_1, B_1) + f_{21}(v_2, B_2)$
$f_{23}(v_1, B) = \min$	

Table 11: Case $|V(K)| = 2$ for *ROLDB*.

Function	
$f_{19}(v_1, B) = \min$	$\begin{cases} f_{20}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{23}(v_1, B_1) + \min\{f_{48}(v_2, B_2) + f_{42}(v_3, B_3), f_{42}(v_2, B_2) + f_{48}(v_3, B_3)\} \\ f_{47}(v_1, B_1) + \min\{f_{48}(v_2, B_2) + f_{25}(v_3, B_3), f_{25}(v_2, B_2) + f_{48}(v_3, B_3)\} \\ f_{22}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{41}(v_3, B_3), f_{41}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{46}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{23}(v_3, B_3), f_{23}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{19}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) \\ f_{22}(v_1, B_1) + \min\{f_{41}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{41}(v_3, B_3)\} \\ f_{46}(v_1, B_1) + \min\{f_{23}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{23}(v_3, B_3)\} \\ f_{46}(v_1, B_1) + f_{43}(v_2, B_2) + f_{43}(v_3, B_3) \end{cases}$
$f_{20}(v_1, B) = \min$	$\begin{cases} f_{23}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{41}(v_3, B_3), f_{41}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{47}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{23}(v_3, B_3), f_{23}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{20}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) \\ f_{23}(v_1, B_1) + \min\{f_{41}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{41}(v_3, B_3)\} \\ f_{47}(v_1, B_1) + \min\{f_{23}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{23}(v_3, B_3)\} \\ f_{47}(v_1, B_1) + f_{43}(v_2, B_2) + f_{43}(v_3, B_3) \end{cases}$
$f_{21}(v_1, B) = \min$	$\begin{cases} f_{26}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{20}(v_3, B_3), f_{20}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{27}(v_1, B_1) + \min\{f_{20}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{20}(v_3, B_3)\} \\ f_{48}(v_1, B_1) + f_{41}(v_2, B_2) + f_{41}(v_3, B_3) \\ f_{48}(v_1, B_1) + f_{47}(v_2, B_2) + f_{47}(v_3, B_3) \\ f_{48}(v_1, B_1) + \min\{f_{43}(v_2, B_2) + f_{23}(v_3, B_3), f_{23}(v_2, B_2) + f_{43}(v_3, B_3)\} \\ f_{21}(v_1, B_1) + f_{21}(v_2, B_2) + f_{21}(v_3, B_3) \\ f_{26}(v_1, B_1) + \min\{f_{40}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{40}(v_3, B_3)\} \end{cases}$
$f_{22}(v_1, B) = \min$	$\begin{cases} f_{23}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{47}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{42}(v_3, B_3), f_{42}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{47}(v_1, B_1) + \min\{f_{42}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{42}(v_3, B_3)\} \\ f_{46}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{41}(v_3, B_3), f_{41}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{22}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) \\ f_{46}(v_1, B_1) + \min\{f_{41}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{41}(v_3, B_3)\} \\ f_{46}(v_1, B_1) + f_{43}(v_2, B_2) + f_{43}(v_3, B_3) \end{cases}$
$f_{23}(v_1, B) = \min$	$\begin{cases} f_{47}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{41}(v_3, B_3), f_{41}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{23}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) \\ f_{47}(v_1, B_1) + \min\{f_{41}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{41}(v_3, B_3)\} \\ f_{47}(v_1, B_1) + f_{43}(v_2, B_2) + f_{43}(v_3, B_3) \end{cases}$
$f_{24}(v_1, B) = \min$	$\begin{cases} f_{25}(v_1, B_1) + \min\{f_{27}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{27}(v_3, B_3)\} \\ f_{24}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) \end{cases}$
$f_{25}(v_1, B) = f_{25}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3)$	
$f_{26}(v_1, B) = \min$	$\begin{cases} f_{26}(v_1, B_1) + f_{21}(v_2, B_2) + f_{21}(v_3, B_3) \\ f_{48}(v_1, B_1) + \min\{f_{40}(v_2, B_2) + f_{26}(v_3, B_3), f_{26}(v_2, B_2) + f_{40}(v_3, B_3)\} \end{cases}$
$f_{27}(v_1, B) = f_{27}(v_1, B_1) + f_{21}(v_2, B_2) + f_{21}(v_3, B_3)$	

Table 12: Case $|V(K)| = 3$ for *ROLDB*.

Function	
$f_{19}(v_1, B) = \min$	$\left\{ \begin{aligned} &f_{20}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{27}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{26}(v_i, B_i) \right\} \\ &f_{23}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{48}(v_h, B_h) + f_{42}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{47}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{48}(v_h, B_h) + f_{25}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{22}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{46}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{23}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{46}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ &f_{19}(v_1, B_1) + \sum_{i=2}^k f_{26}(v_i, B_i) \\ &f_{22}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{46}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{23}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{46}(v_1, B_1) + \min_{\substack{h, l=2, \dots, k \\ h \neq l}} \left\{ f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq h, l}}^k f_{45}(v_i, B_i) \right\} \end{aligned} \right\}$
$f_{20}(v_1, B) = \min$	$\left\{ \begin{aligned} &f_{23}(v_1, B_1) + \min_{i, j, s=2, 3, 4, i \neq j, j \neq s, i \neq s} \{f_{26}(v_i, B_i) + f_{27}(v_j, B_j) + f_{41}(v_s, B_s)\} \\ &f_{47}(v_1, B_1) + \min_{i, j, s=2, 3, 4, i \neq j, j \neq s, i \neq s} \{f_{27}(v_i, B_i) + f_{23}(v_j, B_j) + f_{26}(v_s, B_s)\} \\ &f_{47}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{27}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{43}(v_i, B_i)\} \\ &f_{20}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) + f_{26}(v_4, B_4) \\ &f_{23}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{41}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{26}(v_i, B_i)\} \\ &f_{47}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{23}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{26}(v_i, B_i)\} \\ &f_{47}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{45}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{43}(v_i, B_i)\} \end{aligned} \right\}$
$f_{21}(v_1, B) = \min$	$\left\{ \begin{aligned} &f_{26}(v_1, B_1) + \min_{i, j, s=2, 3, 4, i \neq j, j \neq s, i \neq s} \{f_{27}(v_i, B_i) + f_{20}(v_j, B_j) + f_{26}(v_s, B_s)\} \\ &f_{26}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{27}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{41}(v_i, B_i)\} \\ &f_{26}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{27}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{47}(v_i, B_i)\} \\ &f_{26}(v_1, B_1) + \min_{i, j, s=2, 3, 4, i \neq j, j \neq s, i \neq s} \{f_{27}(v_i, B_i) + f_{23}(v_j, B_j) + f_{43}(v_s, B_s)\} \\ &f_{27}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{20}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{26}(v_i, B_i)\} \\ &f_{48}(v_1, B_1) + f_{43}(v_2, B_2) + f_{43}(v_3, B_3) + f_{43}(v_4, B_4) \\ &f_{48}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{45}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{41}(v_i, B_i)\} \\ &f_{48}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{45}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{47}(v_i, B_i)\} \\ &f_{48}(v_1, B_1) + \min_{i, j, s=2, 3, 4, i \neq j, j \neq s, i \neq s} \{f_{23}(v_i, B_i) + f_{43}(v_j, B_j) + f_{45}(v_s, B_s)\} \\ &f_{21}(v_1, B_1) + f_{21}(v_2, B_2) + f_{21}(v_3, B_3) + f_{21}(v_4, B_4) \\ &f_{26}(v_1, B_1) + \min_{j=2, 3, 4} \{f_{40}(v_j, B_j) + \sum_{i=2, 3, 4, i \neq j} f_{26}(v_i, B_i)\} \end{aligned} \right\}$

Table 13: Case $|V(K)| = 4$ for *ROLDB*.

Function	
$f_{22}(v_1, B) = \min$	$\begin{cases} f_{23}(v_1, B_1) + \min_{j=2,3,4} \{f_{27}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{26}(v_i, B_i)\} \\ f_{47}(v_1, B_1) + \min_{i,j,s=2,3,4, i \neq j, j \neq s, i \neq s} \{f_{26}(v_i, B_i) + f_{27}(v_j, B_j) + f_{42}(v_s, B_s)\} \\ f_{47}(v_1, B_1) + \min_{j=2,3,4} \{f_{42}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{26}(v_i, B_i)\} \\ f_{46}(v_1, B_1) + \min_{i,j,s=2,3,4, i \neq j, j \neq s, i \neq s} \{f_{26}(v_i, B_i) + f_{27}(v_j, B_j) + f_{41}(v_s, B_s)\} \\ f_{46}(v_1, B_1) + \min_{j=2,3,4} \{f_{27}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{43}(v_i, B_i)\} \\ f_{22}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) + f_{26}(v_4, B_4) \\ f_{46}(v_1, B_1) + \min_{j=2,3,4} \{f_{41}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{26}(v_i, B_i)\} \\ f_{46}(v_1, B_1) + \min_{j=2,3,4} \{f_{45}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{43}(v_i, B_i)\} \end{cases}$
$f_{23}(v_1, B) = \min$	$\begin{cases} f_{47}(v_1, B_1) + \min_{i,j,s=2,3,4, i \neq j, j \neq s, i \neq s} \{f_{26}(v_i, B_i) + f_{27}(v_j, B_j) + f_{41}(v_s, B_s)\} \\ f_{47}(v_1, B_1) + \min_{j=2,3,4} \{f_{27}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{43}(v_i, B_i)\} \\ f_{23}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) + f_{26}(v_4, B_4) \\ f_{47}(v_1, B_1) + \min_{j=2,3,4} \{f_{41}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{26}(v_i, B_i)\} \\ f_{47}(v_1, B_1) + \min_{j=2,3,4} \{f_{45}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{43}(v_i, B_i)\} \end{cases}$
$f_{24}(v_1, B) = \min$	$\begin{cases} f_{25}(v_1, B_1) + \min_{j=2,3,4} \{f_{27}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{26}(v_i, B_i)\} \\ f_{24}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) + f_{26}(v_4, B_4) \end{cases}$
$f_{25}(v_1, B) = f_{25}(v_1, B_1) + f_{26}(v_2, B_2) + f_{26}(v_3, B_3) + f_{26}(v_4, B_4)$	
$f_{26}(v_1, B) = \min$	$\begin{cases} f_{26}(v_1, B_1) + f_{21}(v_2, B_2) + f_{21}(v_3, B_3) + f_{21}(v_4, B_4) \\ f_{48}(v_1, B_1) + \min_{j=2,3,4} \{f_{40}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{26}(v_i, B_i)\} \\ f_{48}(v_1, B_1) + \min_{i,j,s=2,3,4, i \neq j, j \neq s, i \neq s} \{f_{26}(v_i, B_i) + f_{27}(v_j, B_j) + f_{20}(v_s, B_s)\} \\ f_{48}(v_1, B_1) + \min_{j=2,3,4} \{f_{48}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{41}(v_i, B_i)\} \\ f_{48}(v_1, B_1) + \min_{j=2,3,4} \{f_{48}(v_j, B_j) + \sum_{i=2,3,4, i \neq j} f_{47}(v_i, B_i)\} \\ f_{48}(v_1, B_1) + \min_{i,j,s=2,3,4, i \neq j, j \neq s, i \neq s} \{f_{23}(v_i, B_i) + f_{43}(v_j, B_j) + f_{48}(v_s, B_s)\} \\ f_{48}(v_1, B_1) + f_{43}(v_2, B_2) + f_{43}(v_3, B_3) + f_{43}(v_4, B_4) \end{cases}$
$f_{27}(v_1, B) = f_{27}(v_1, B_1) + f_{21}(v_2, B_2) + f_{21}(v_3, B_3) + f_{21}(v_4, B_4)$	

Table 14: Case $|V(K)| = 4$ for *ROLDB*.

Function	
$f_{19}(v_1, B) = \min$	$\left\{ \begin{aligned} &f_{20}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{27}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{26}(v_i, B_i) \right\} \\ &f_{23}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_h, B_h) + f_{42}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{23}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{42}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{26}(v_i, B_i) \right\} \\ &f_{47}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{25}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{26}(v_i, B_i) \right\} \\ &f_{22}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{46}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{23}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{46}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ &f_{19}(v_1, B_1) + \sum_{i=2}^k f_{26}(v_i, B_i) \\ &f_{22}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{46}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{23}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{46}(v_1, B_1) + \min_{\substack{h, l=2, \dots, k \\ h \neq l}} \left\{ f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq h, l}}^k f_{45}(v_i, B_i) \right\} \\ &f_{23}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ &f_{47}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{23}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \end{aligned} \right\}$
$f_{20}(v_1, B) = \min$	$\left\{ \begin{aligned} &f_{47}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ &f_{20}(v_1, B_1) + \sum_{i=2}^k f_{26}(v_i, B_i) \\ &f_{23}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{41}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{26}(v_i, B_i) \right\} \\ &f_{47}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{23}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{26}(v_i, B_i) \right\} \\ &f_{47}(v_1, B_1) + \min_{\substack{h, l=2, \dots, k \\ h \neq l}} \left\{ f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq h, l}}^k f_{45}(v_i, B_i) \right\} \end{aligned} \right\}$

Table 15: Case $|V(K)| \geq 5$ for *ROLDB*.

Function	
$f_{21}(v_1, B) =$	$\min \left\{ \begin{aligned} & f_{26}(v_1, B_1) + \min_{j, h=2, \dots, k, j \neq h} \left\{ f_{27}(v_j, B_j) + f_{20}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq j, h} f_{26}(v_i, B_i) \right\} \\ & f_{26}(v_1, B_1) + \min_{\substack{j, h, l, s=2, \dots, k \\ j \neq h, j \neq l, j \neq s \\ h \neq l, h \neq s, l \neq s}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + f_{43}(v_s, B_s) + \sum_{\substack{i=2 \\ i \neq j, h, l, s}}^k f_{45}(v_i, B_i) \right\} \\ & f_{26}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{41}(v_h, B_h) + f_{41}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ & f_{26}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{47}(v_h, B_h) + f_{47}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ & f_{26}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{23}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ & f_{26}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{20}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{26}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{43}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{h, l=2, \dots, k, h \neq l} \left\{ f_{41}(v_h, B_h) + f_{41}(v_l, B_l) + \sum_{i=2, \dots, k, i \neq h, l} f_{45}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{h, l=2, \dots, k, h \neq l} \left\{ f_{47}(v_h, B_h) + f_{47}(v_l, B_l) + \sum_{i=2, \dots, k, i \neq h, l} f_{45}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{h, l=2, \dots, k, h \neq l} \left\{ f_{43}(v_h, B_h) + f_{23}(v_l, B_l) + \sum_{i=2, \dots, k, i \neq h, l} f_{45}(v_i, B_i) \right\} \\ & \sum_{i=1}^k f_{21}(v_i, B_i) \\ & f_{26}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{40}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq h} f_{26}(v_i, B_i) \right\} \end{aligned} \right\}$
$f_{22}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{23}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{27}(v_j, B_j) + \sum_{i=2, \dots, k, i \neq j} f_{26}(v_i, B_i) \right\} \\ & f_{47}(v_1, B_1) + \min_{\substack{j, h=2, \dots, k \\ j \neq h}} \left\{ f_{27}(v_h, B_h) + f_{42}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j, h}}^k f_{26}(v_i, B_i) \right\} \\ & f_{47}(v_1, B_1) + \min_{j=2, \dots, k} \left\{ f_{42}(v_j, B_j) + \sum_{i=2, \dots, k, i \neq j} f_{26}(v_i, B_i) \right\} \\ & f_{46}(v_1, B_1) + \min_{j, h=2, \dots, k, j \neq h} \left\{ f_{27}(v_j, B_j) + f_{41}(v_h, B_h) + \sum_{i=2, \dots, k, i \neq j, h} f_{26}(v_i, B_i) \right\} \\ & f_{46}(v_1, B_1) + \min_{\substack{j, h, l=2, \dots, k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j, h, l}}^k f_{45}(v_i, B_i) \right\} \\ & f_{22}(v_1, B_1) + \sum_{i=2}^k f_{26}(v_i, B_i) \\ & f_{46}(v_1, B_1) + \min_{h=2, \dots, k} \left\{ f_{41}(v_j, B_j) + \sum_{i=2, \dots, k, i \neq h} f_{26}(v_i, B_i) \right\} \\ & f_{46}(v_1, B_1) + \min_{h, l=2, \dots, k, h \neq l} \left\{ f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{i=2, \dots, k, i \neq h, l} f_{45}(v_i, B_i) \right\} \end{aligned} \right\}$

Table 16: Case $|V(K)| \geq 5$ for *ROLDB*.

Function	
$f_{23}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{47}(v_1, B_1) + \min_{\substack{j,h=2,\dots,k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j,h}}^k f_{26}(v_i, B_i) \right\} \\ & f_{47}(v_1, B_1) + \min_{\substack{j,h,l=2,\dots,k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j,h,l}}^k f_{45}(v_i, B_i) \right\} \\ & f_{23}(v_1, B_1) + \sum_{i=2}^k f_{26}(v_i, B_i) \end{aligned} \right\}$
$f_{24}(v_1, B) = \min$	$\left\{ \begin{aligned} & f_{47}(v_1, B_1) + \min_{h=2,\dots,k} \left\{ f_{41}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{26}(v_i, B_i) \right\} \\ & f_{47}(v_1, B_1) + \min_{\substack{h,l=2,\dots,k \\ h \neq l}} \left\{ f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq h,l}}^k f_{45}(v_i, B_i) \right\} \\ & f_{25}(v_1, B_1) + \min_{j=2,\dots,k} \left\{ f_{27}(v_j, B_j) + \sum_{\substack{i=2 \\ i \neq j}}^k f_{26}(v_i, B_i) \right\} \\ & f_{24}(v_1, B_1) + \sum_{i=2}^k f_{26}(v_i, B_i) \end{aligned} \right\}$
$f_{25}(v_1, B) = f_{25}(v_1, B_1) + \sum_{i=2}^k f_{26}(v_i, B_i)$	
$f_{26}(v_1, B) =$	$\left\{ \begin{aligned} & f_{26}(v_1, B_1) + \sum_{i=1}^k f_{21}(v_i, B_i) \\ & f_{48}(v_1, B_1) + \min_{h=2,\dots,k} \left\{ f_{40}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq h}}^k f_{26}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{\substack{j,h=2,\dots,k \\ j \neq h}} \left\{ f_{27}(v_j, B_j) + f_{20}(v_h, B_h) + \sum_{\substack{i=2 \\ i \neq j,h}}^k f_{26}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{\substack{j,h,l=2,\dots,k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{48}(v_j, B_j) + f_{41}(v_h, B_h) + f_{41}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j,h,l}}^k f_{26}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{\substack{j,h,l=2,\dots,k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{48}(v_j, B_j) + f_{47}(v_h, B_h) + f_{47}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j,h,l}}^k f_{26}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{\substack{j,h,l=2,\dots,k \\ j \neq h, j \neq l, h \neq l}} \left\{ f_{48}(v_j, B_j) + f_{23}(v_h, B_h) + f_{43}(v_l, B_l) + \sum_{\substack{i=2 \\ i \neq j,h,l}}^k f_{26}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{\substack{h,l,s=2,\dots,k \\ h \neq l, h \neq s, l \neq s}} \left\{ f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + f_{43}(v_s, B_s) + \sum_{\substack{i=2 \\ i \neq j,h,l}}^k f_{45}(v_i, B_i) \right\} \\ & f_{48}(v_1, B_1) + \min_{\substack{j,h,l,s=2,\dots,k \\ j \neq h, j \neq l, j \neq s \\ h \neq l, h \neq s, l \neq s}} \left\{ f_{27}(v_j, B_j) + f_{43}(v_h, B_h) + f_{43}(v_l, B_l) + f_{43}(v_s, B_s) + \sum_{\substack{i=2 \\ i \neq j,h,l,s}}^k f_{45}(v_i, B_i) \right\} \end{aligned} \right\}$
$f_{27}(v_1, B) = f_{27}(v_1, B_1) + \sum_{i=2}^k f_{21}(v_i, B_i)$	

Table 17: Case $|V(K)| \geq 5$ for *ROLDB*.