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# ON THE MONOIDAL INVARIANCE OF THE COHOMOLOGICAL DIMENSION OF HOPF ALGEBRAS

JULIEN BICHON

ABSTRACT. We discuss the question of whether the global dimension is a monoidal invariant for Hopf algebras, in the sense that if two Hopf algebras have equivalent monoidal categories of comodules, then their global dimensions should be equal. We provide several positive new answers to this question, under various assumptions of smoothness, cosemisimplicity or finite dimension. We also discuss the comparison between the global dimension and the Gerstenhaber-Schack cohomological dimension in the cosemisimple case, obtaining equality in the case the latter is finite. One of our main tools is the new concept of twisted separable functor.

## 1. INTRODUCTION

A classical invariant of an algebra  $A$  is its (right) global dimension

$$\text{r.gldim}(A) = \max \{ \text{pd}_A(M), M \in \mathcal{M}_A \} \in \mathbb{N} \cup \{ \infty \}$$

where for a (right)  $A$ -module  $M$ ,  $\text{pd}_A(M)$  stands for its projective dimension, i.e. the smallest possible length for a resolution of  $M$  by projective  $A$ -modules.

The global dimension is a key ingredient in the analysis of certain geometric properties of discrete groups [9, 14], and often serves as a good analogue of the dimension of a smooth affine variety. However in some noncommutative situations, it is better to replace it by the Hochschild cohomological dimension, which has similar geometric significance, and is defined by :

$$\begin{aligned} \text{cd}(A) &= \max \{ n : H^n(A, M) \neq 0 \text{ for some } A - \text{bimodule } M \} \in \mathbb{N} \cup \{ \infty \} \\ &= \min \{ n : H^{n+1}(A, M) = 0 \text{ for any } A - \text{bimodule } M \} \\ &= \text{pd}_{A\mathcal{M}_A}(A) \end{aligned}$$

where  $H^*(A, -)$  denotes Hochschild cohomology and  $\text{pd}_{A\mathcal{M}_A}(A)$  is the projective dimension of  $A$  in the category of  $A$ -bimodules.

Indeed, for example if  $A = A_1(k)$  is the first Weyl algebra ( $k$  is, as in all the paper, an algebraically closed field), we have  $\text{r.gldim}(A_1(k)) = 1$  (in characteristic zero) and  $\text{cd}(A_1(k)) = 2$ , while  $A_1(k)$  should definitively be considered as a 2-dimensional object.

When  $A$  is a Hopf algebra, it is well-known that we have

$$\text{r.gldim}(A) = \text{pd}_A(k_\varepsilon) = \text{cd}(A) = \text{l.gldim}(A) = \text{pd}_A({}_\varepsilon k)$$

where  $k_\varepsilon$  and  ${}_\varepsilon k$  denote the respective right and left trivial  $A$ -modules, and  $\text{l.gldim}(A)$  is the left global dimension. See [26] for the equalities at the extreme left and right, and, for example, [20] for the other equality.

A general classical problem is whether the global dimension or the Hochschild cohomological dimension remain preserved under various kind of “deformations” of  $A$ , and the question we are particularly interested in, originally asked in [6] and suggested by examples studied in [5], is the following one.

**Question 1.1.** *If  $A$  and  $B$  are Hopf algebras having equivalent linear tensor categories of comodules, do we have  $\text{cd}(A) = \text{cd}(B)$ ?*

Notice that the word “tensor” is important in the above question, since this is what captures information about the algebra structure inside the category of comodules. Dropping it would make the question meaningless, as shown by the example of group algebras: if two group algebras have equivalent categories of comodules, the only conclusion, in lack of additional information, is that the groups have the same cardinality.

Partial positive answers to Question 1.1 were provided in [6, 7] when  $A, B$  are cosemisimple with antipode satisfying  $S^4 = \text{id}$ , and by Wang, Yu and Zhang in [39], when  $A$  is twisted Calabi-Yau and  $B$  is homologically smooth.

The aim of this paper is to provide several new positive answers to Question 1.1. The following theorem summarizes our contributions.

**Theorem 1.2.** *Let  $A, B$  be Hopf algebras that have equivalent linear tensor categories of comodules:  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ . We have  $\text{cd}(A) = \text{cd}(B)$  if one of the following conditions holds:*

- (1)  $A, B$  are homologically smooth and have bijective antipode;
- (2)  $A, B$  are cosemisimple and  $\text{cd}(A), \text{cd}(B)$  are finite;
- (3)  $A, B$  are finite-dimensional and  $\text{char}(k) = 0$  or  $\text{char}(k) = p > d^{\frac{\varphi(d)}{2}}$  where  $d = \dim(A)$ , or  $A^*$  is unimodular.

While the proof of the first statement (Theorem 8.1) is obtained by carefully inspecting arguments in [42, 39] and the third one (Theorem 8.3) is a rather direct consequence of previous results [24, 16, 1], the main effort in this paper is in proving the second statement (Theorem 4.6). Removing the assumption  $S^4 = \text{id}$  from [7] (with instead the finiteness assumption on cohomological dimensions) enables us to compute cohomological dimension in a number of new situations, see Section 7 for examples regarding universal cosovereign Hopf algebras and free wreath products.

Our method for proving (2) in Theorem 1.2 is based on the fact that if  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$  as above, results by Schauenburg [35] ensures that there exists an  $A$ - $B$  Galois object  $R$ , and then on proving that  $\text{cd}(A) = \text{cd}(R) = \text{cd}(B)$ . For this, one notices that  $\text{cd}(A) = \text{pd}_{R\mathcal{M}_R^B}(R)$ , the projective dimension of  $R$  in the category of  $R$ -bimodules inside  $B$ -comodules, and then the main question is to compare  $\text{pd}_{R\mathcal{M}_R^B}(R)$  and  $\text{pd}_{R\mathcal{M}_R}(R) = \text{cd}(R)$ . The main ingredient in this comparison is a twisted averaging trick, Lemma 4.2, that we believe to be quite non-straightforward. The averaging lemma leads to the concept of twisted separable functor we define in Section 3, a generalization of the notion of separable functor introduced in [30].

Our initial idea to tackle Question 1.1 was, in [6], to use an auxiliary cohomological dimension for the Hopf algebra  $A$ , the Gestenhaver-Schack cohomological dimension, defined by

$$\text{cd}_{\text{GS}}(A) = \max\{n : \text{Ext}_{\mathcal{YD}_A^A}^n(k, V) \neq 0 \text{ for some } V \in \mathcal{YD}_A^A\} \in \mathbb{N} \cup \{\infty\}$$

where  $\mathcal{YD}_A^A$  is the category of Yetter-Drinfeld modules over  $A$  and  $k$  is the trivial Yetter-Drinfeld module. It was shown in [6, Theorem 5.6, Corollary 5.7] that  $\text{cd}(A) \leq \text{cd}_{\text{GS}}(A)$  and that if  $A, B$  are Hopf algebras with  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ , then

$$\max(\text{cd}(A), \text{cd}(B)) \leq \text{cd}_{\text{GS}}(A) = \text{cd}_{\text{GS}}(B)$$

Therefore, comparing  $\text{cd}(A)$  and  $\text{cd}_{\text{GS}}(A)$  can be a key step towards answers to Question 1.1. In this direction, we obtain the following result.

**Theorem 1.3.** *Let  $A$  be a cosemisimple Hopf algebra. If  $\text{cd}_{\text{GS}}(A)$  is finite, we have  $\text{cd}(A) = \text{cd}_{\text{GS}}(A)$ .*

The above theorem has, as a corollary, a weak form of (2) in Theorem 1.2, which is probably sufficient in dealing with numerous examples. Again the method of proof is based on a twisted averaging trick and uses an appropriate twisted separable functor.

We expect that the equality  $\text{cd}(A) = \text{cd}_{\text{GS}}(A)$  holds for any cosemisimple Hopf algebra, but as already pointed in [6], it cannot hold for any Hopf algebra over any field, as we see by taking a semisimple non cosemisimple Hopf algebra over a field of positive characteristic, so we asked there whether the equality was true in characteristic zero. Etingof pointed out that it does not hold in characteristic zero even for the very simple example  $A = k[x]$  with  $x$  primitive. Hence we have now the following question.

**Question 1.4.** *What are the Hopf algebras such that  $\text{cd}(A) = \text{cd}_{\text{GS}}(A)$ ?*

The paper is organized as follows. Section 2 collects preliminary notations and results on cosemisimple Hopf algebras. Section 3 introduces the notion of twisted separable functor. After some preliminary material on categories of bimodules inside categories of comodules, Part (2) of Theorem 1.2 is proved in Section 4. Section 5 provides the proof of Theorem 1.3, together with the necessary material on Yetter-Drinfeld modules. Section 6 studies the behaviour of Gerstenhaber-Schack cohomological dimension under Hopf subalgebras in the cosemisimple case. Section 7 is devoted to applications to some examples. The final Section 8 reviews what is known about Question 1.1 outside the cosemisimple case, in the smooth case and in the finite-dimensional situation, providing the proofs of (1) and (3) in Theorem 1.2. The reader only interested in those results might go directly to this section.

**Notations and conventions.** We work over an algebraically closed field  $k$ . We assume that the reader is familiar with the theory of Hopf algebras and their tensor categories of comodules, as e.g. in [17, 22, 29], and with the basics of homological algebra [9, 40]. If  $A$  is a Hopf algebra, as usual,  $\Delta$ ,  $\varepsilon$  and  $S$  stand respectively for the comultiplication, counit and antipode of  $A$ . We use Sweedler's notations in the standard way. The category of right  $A$ -comodules is denoted  $\mathcal{M}^A$ , the category of right  $A$ -modules is denoted  $\mathcal{M}_A$ , etc... The trivial (right)  $A$ -module is denoted  $k_\varepsilon$ . The set of  $A$ -module morphisms (resp.  $A$ -comodule morphisms) between two  $A$ -modules (resp. two  $A$ -comodules)  $V$  and  $W$  is denoted  $\text{Hom}_A(V, W)$  (resp.  $\text{Hom}^A(V, W)$ ).

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## 2. COSEMISIMPLE HOPF ALGEBRAS

In this section we collect some preliminary facts and notations on cosemisimple Hopf algebras. Recall that a Hopf algebra is cosemisimple if and only if it admits a Haar integral, i.e. a linear map  $h : A \rightarrow k$  such that for any  $a \in A$ , we have

$$h(a_{(1)})a_{(2)} = h(a) = h(a_{(2)})a_{(1)} \quad \text{and} \quad h(1) = 1$$

The proof of the semisimplicity of the category of comodules from the existence of a Haar integral is a consequence of the following averaging construction, that we record for future use.

**Proposition 2.1.** *Let  $V, W$  be right  $A$ -comodules over a cosemisimple Hopf algebra  $A$ , and let  $f : V \rightarrow W$  be a linear map. The map*

$$\begin{aligned} \mathbf{M}_{V,W}(f) : V &\longrightarrow W \\ v &\longmapsto h(f(v_{(0)})(1)S(v_{(1)}))f(v_{(0)})_{(0)} \end{aligned}$$

*is a morphism of comodules, with  $M(f) = f$  if and only if  $f$  is a morphism of comodules and with, for any morphisms of comodules  $\alpha : V' \rightarrow V$  and  $\beta : W \rightarrow W'$ ,  $\beta \circ \mathbf{M}_{V,W}(f) \circ \alpha = \mathbf{M}_{V',W'}(\beta \circ f \circ \alpha)$ . The above construction therefore defines a projection  $\mathbf{M}_{V,W} : \text{Hom}(V, W) \rightarrow \text{Hom}^A(V, W)$ , that we call the averaging with respect to  $V$  and  $W$ .*

The Haar integral is not a trace in general, but satisfies a KMS type property, discovered by Woronowicz [41] in the setting of compact quantum groups.

**Theorem 2.2.** *Let  $A$  be a cosemisimple Hopf algebra with Haar integral  $h$ . There exists a convolution invertible linear map  $\psi : A \rightarrow k$ , called a modular functional on  $A$ , satisfying the following conditions:*

- $S^2 = \psi * \text{id} * \psi^{-1}$ ;
- $\sigma := \psi * \text{id} * \psi$  is an algebra automorphism of  $A$ ;
- we have  $h(ab) = h(b\sigma(a))$  for any  $a, b \in A$ .

The proof relies on the orthogonality relations, whose first occurrence is due to Larson [23], and were completed by Woronowicz [41]. In all the treatment we are aware of [22, 31], the setting is over the field of complex numbers, but inspecting the proof shows that it is valid for any cosemisimple Hopf algebra over any algebraically closed field.

To conclude the section, we introduce a last piece of notation. If  $A$  is a cosemisimple Hopf algebra  $A$  with Haar integral  $h$  and modular functional  $\psi$ , we denote by  $\theta$  the algebra automorphism of  $A$  defined by  $\theta = \psi^2 * \text{id}$ .

### 3. TWISTED SEPARABLE FUNCTORS

In this section we introduce the notion of twisted separable functor, as follows.

**Definition 3.1.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be some categories. We say that a functor  $F : \mathcal{C} \rightarrow \mathcal{D}$  is *twisted separable* if there exist

- (1) an autoequivalence  $\Theta$  of the category  $\mathcal{D}$ ;
- (2) a generating subclass  $\mathcal{F}$  of objects of  $\mathcal{C}$  (i.e. for every object  $V$  of  $\mathcal{C}$ , there exists an object  $P$  of  $\mathcal{F}$  together with an epimorphism  $P \rightarrow V$ ) together with, for any object  $P$  of  $\mathcal{F}$ , an isomorphism  $\theta_P : F(P) \rightarrow \Theta F(P)$ ;
- (3) a natural morphism  $\mathbf{M}_{-, -} : \text{Hom}_{\mathcal{D}}(F(-), \Theta F(-)) \rightarrow \text{Hom}_{\mathcal{C}}(-, -)$  such that for any object  $P$  of  $\mathcal{F}$ , we have  $M_{P,P}(\theta_P) = \text{id}_P$ .

The naturality condition above means that for any morphisms  $\alpha : V' \rightarrow V$ ,  $\beta : W \rightarrow W'$  in  $\mathcal{C}$  and any morphism  $f : F(V) \rightarrow \Theta F(W)$  in  $\mathcal{D}$ , we have

$$\beta \circ \mathbf{M}_{V,W}(f) \circ \alpha = \mathbf{M}_{V',W'}(\Theta F(\beta) \circ f \circ F(\alpha))$$

When  $\mathcal{F}$  is the whole class of objects of  $\mathcal{C}$ , the autoequivalence  $\Theta$  is the identity and the isomorphisms  $\theta_P$  all are the identity, we get the notion of separable functor from [30], which is known to be provide a convenient setting for various types of generalized Maschke theorems, see [11]. A basic example of a separable functor is, when  $A$  is a cosemisimple Hopf algebra, the forgetful functor  $\mathcal{M}^A \rightarrow \text{Vec}_k$ : this is the content of Proposition 2.1.

Our motivation to introduce the present notion of twisted separable functor is the following result.

**Proposition 3.2.** *Let  $\mathcal{C}$  and  $\mathcal{D}$  be abelian categories having enough projective objects, and let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be a functor. Assume that the following conditions hold:*

- (1) *the functor  $F$  is exact and preserves projective objects;*
- (2) *the functor  $F$  is twisted separable and  $\mathcal{F}$ , the corresponding class of objects of  $\mathcal{C}$ , contains a generating subclass  $\mathcal{F}_0$  consisting of projective objects.*

*Then, for any object  $V$  of  $\mathcal{C}$  such that  $\text{pd}_{\mathcal{C}}(V)$  is finite, we have  $\text{pd}_{\mathcal{C}}(V) = \text{pd}_{\mathcal{D}}(F(V))$ .*

As usual, the notation  $\text{pd}_{\mathcal{C}}(V)$  refers to the projective dimension of the object  $V$ , i.e. the smallest length of a resolution of  $V$  by projective objects in  $\mathcal{C}$ , with, as well

$$\text{pd}_{\mathcal{C}}(V) = \max\{n : \text{Ext}_{\mathcal{C}}^n(V, W) \neq 0 \text{ for some object } W \text{ in } \mathcal{C}\}$$

We begin with some preliminaries.

**Lemma 3.3.** *Let  $\mathcal{C}$  be an abelian category having enough projective objects, and let  $\mathcal{F}_0$  be a generating subclass of  $\mathcal{C}$  consisting of projective objects. If  $\text{pd}_{\mathcal{C}}(V)$  is finite, we have*

$$\text{pd}_{\mathcal{C}}(V) = \max\{n : \text{Ext}_{\mathcal{C}}^n(V, F) \neq 0 \text{ for some object } F \text{ in } \mathcal{F}_0\}$$

*Proof.* Every object  $X$  fits into an exact sequence  $0 \rightarrow W \rightarrow F \rightarrow X \rightarrow 0$  with  $F$  an object of  $\mathcal{F}_0$ , hence projective. The result is thus obtained via a classical argument: if  $n = \text{pd}_{\mathcal{C}}(V)$ , the long Ext exact sequence gives that the functor  $\text{Ext}_{\mathcal{C}}^n(V, -)$  is right exact, and hence  $\text{Ext}_{\mathcal{C}}^n(V, F) \neq 0$  for some object  $F$  of  $\mathcal{F}_0$ .  $\square$

**Lemma 3.4.** *Assume we are in the setting of Proposition 3.2. For any objects  $X, W$  of  $\mathcal{C}$ , we have a morphism*

$$\text{Ext}_{\mathcal{D}}^*(F(X), \Theta F(W)) \longrightarrow \text{Ext}_{\mathcal{C}}^*(X, W)$$

*which is surjective if  $W$  is an object of  $\mathcal{F}$ .*

*Proof.* Start with a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \rightarrow 0$$

of  $X$  by objects in  $\mathcal{C}$ . Since the functor  $F$  is exact and preserves projectives, we get a projective resolution

$$\cdots \longrightarrow F(P_n) \xrightarrow{F(d_n)} F(P_{n-1}) \xrightarrow{F(d_{n-1})} \cdots \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(d_0)} F(X) \rightarrow 0$$

of  $F(X)$  in  $\mathcal{D}$ . For all  $i \geq 0$ , we have, by the naturality assumption, commutative diagrams

$$\begin{array}{ccc} \text{Hom}_{\mathcal{D}}(F(P_i), \Theta F(W)) & \xrightarrow{-\circ F(d_{i+1})} & \text{Hom}_{\mathcal{D}}(F(P_{i+1}), \Theta F(W)) \\ \downarrow \mathbf{M}_{P_i, W} & & \downarrow \mathbf{M}_{P_{i+1}, W} \\ \text{Hom}_{\mathcal{C}}(P_i, W) & \xrightarrow{-\circ d_{i+1}} & \text{Hom}_{\mathcal{C}}(P_{i+1}, W) \end{array}$$

that induce a morphism of complexes

$$\widetilde{\mathbf{M}} : \text{Hom}_{\mathcal{D}}(F(P_*), \Theta F(W)) \rightarrow \text{Hom}_{\mathcal{C}}(P_*, W)$$

and hence a morphism between the corresponding cohomologies:

$$H^*(\widetilde{\mathbf{M}}) : \text{Ext}_{\mathcal{D}}^*(F(X), \Theta F(W)) \longrightarrow \text{Ext}_{\mathcal{C}}^*(X, W)$$

Assume now that  $W$  is an object of  $\mathcal{F}$ , and let  $f \in \text{Hom}_{\mathcal{C}}(P_i, W)$ . We have

$$\mathbf{M}_{P_i, W}(\theta_W \circ F(f)) = M_{W, W}(\theta_W) \circ f = f$$

and if moreover  $f \circ d_{i+1} = 0$ , we have also  $\theta_W \circ F(f) \circ F(d_{i+1}) = 0$ . This shows that  $H^*(\widetilde{\mathbf{M}})$  is surjective.  $\square$

*Remark 3.5.* Assume, as the setting of Proposition 3.2 allows us to, that in the proof of the previous lemma, we have started with a projective resolution

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} X \rightarrow 0$$

of  $X$  by objects in  $\mathcal{F}$ . Then, for  $f \in \text{Hom}_{\mathcal{C}}(P_i, W)$ , we have

$$\mathbf{M}_{P_i, W}(\Theta(F(f)) \circ \theta_{P_i}) = f \circ M_{P_i, P_i}(\theta_{P_i}) = f$$

This shows that the morphism of complexes  $\widetilde{\mathbf{M}} : \text{Hom}_{\mathcal{D}}(F(P_*), \Theta F(W)) \rightarrow \text{Hom}_{\mathcal{C}}(P_*, W)$  is surjective in general. However, since we see no reason that  $\Theta F(f) \circ \theta_{P_i} \circ F(d_{i+1}) = 0$ , we cannot conclude that the corresponding morphism in cohomology is surjective without our assumption on  $W$ .

*Proof of Proposition 3.2.* Let  $V$  be an object of  $\mathcal{C}$ , and let

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} V \rightarrow 0$$

be a projective resolution of  $V$ . Since the functor  $F$  is exact and preserves projectives, we get a projective resolution

$$\cdots \longrightarrow F(P_n) \xrightarrow{F(d_n)} F(P_{n-1}) \xrightarrow{F(d_{n-1})} \cdots \xrightarrow{F(d_2)} F(P_1) \xrightarrow{F(d_1)} F(P_0) \xrightarrow{F(d_0)} F(V) \rightarrow 0$$

of  $F(V)$  in  $\mathcal{D}$ . This shows that  $\text{pd}_{\mathcal{D}}(F(V)) \leq \text{pd}_{\mathcal{C}}(V)$ . To prove the converse inequality, we can assume that  $n = \text{pd}_{\mathcal{D}}(F(V))$  is finite. We then have in particular  $\text{Ext}_{\mathcal{D}}^{n+1}(F(V), \Theta F(P)) = \{0\}$  for any object  $P$  in  $\mathcal{F}$ , and by Lemma 3.4, we have  $\text{Ext}_{\mathcal{C}}^{n+1}(V, P) = \{0\}$  as well. Hence, assuming that  $\text{pd}_{\mathcal{C}}(V)$  is finite, Lemma 3.3 shows that  $\text{pd}_{\mathcal{C}}(V) \leq n$ , concluding the proof.  $\square$

In this paper we will not develop any more theory on twisted separable functors, and will focus on applications of Proposition 3.2.

#### 4. COMODULE ALGEBRAS AND EQUIVARIANT BIMODULE CATEGORIES

Our aim in this section is to prove part (2) of Theorem 1.2. We begin with some preliminaries on categories of bimodules in categories of comodules. Let  $A$  be a Hopf algebra, let  $R$  be a right  $A$ -comodule algebra ( $R$  is an algebra in the monoidal category  $\mathcal{M}^A$ ) and let  ${}_R\mathcal{M}_R^A$  be the category of  $A$ -bimodules in the category  $\mathcal{M}^A$ : the objects are the  $A$ -comodules  $V$  with an  $R$ -bimodule structure having the Hopf bimodule compatibility conditions

$$(x \cdot v)_{(0)} \otimes (x \cdot v)_{(1)} = x_{(0)} \cdot v_{(0)} \otimes x_{(1)} v_{(1)}, \quad (v \cdot x)_{(0)} \otimes (v \cdot x)_{(1)} = v_{(0)} \cdot x_{(0)} \otimes v_{(1)} x_{(1)}$$

for any  $x \in R$  and  $v \in V$ . The morphisms are the  $A$ -colinear and  $R$ -bilinear maps. The category  ${}_R\mathcal{M}_R^A$  is obviously abelian, and the tensor product of bimodules induces a monoidal structure on it.

The following basic property is certainly well-known, and a straightforward verification.

**Proposition 4.1.** *Let  $A$  be a Hopf algebra and let  $R$  be an  $A$ -comodule algebra.*

- (1) *The forgetful functor  $\Omega^A : {}_R\mathcal{M}_R^A \rightarrow \mathcal{M}^A$  has a left adjoint, which associates to a comodule  $V$  the object  $R \otimes V \otimes R$  whose bimodule structure is given by left and right multiplication of  $R$  and whose comodule structure is the tensor product of the underlying comodules.*

- (2) The forgetful functor  $\Omega_R : {}_R\mathcal{M}_R^A \rightarrow {}_R\mathcal{M}_R$  has a right adjoint, which associates to an  $R$ -bimodule  $V$  the object  $V \otimes A$  whose  $R$ -bimodule structure is given by

$$x \cdot (v \otimes a) = x_{(0)} \cdot v \otimes x_{(1)}a, \quad (v \otimes a) \cdot x = v \cdot x_{(0)} \otimes ax_{(1)}$$

and whose  $A$ -comodule structure is induced by the comultiplication of  $A$ .

Objects in  ${}_R\mathcal{M}_R^A$  that are images of the above left adjoint functor are called free, they are indeed free as bimodules. Any object in  ${}_R\mathcal{M}_R^A$  is a quotient of a free object. The following are direct consequences of the standard properties of pairs of adjoint functors.

- (1) The category  ${}_R\mathcal{M}_R^A$  has enough injective objects, since  ${}_R\mathcal{M}_R$  has, and we have, for any object  $V$  in  ${}_R\mathcal{M}_R^A$  and any  $R$ -bimodule  $W$ :

$$\text{Ext}_{{}_R\mathcal{M}_R}^*(\Omega_R(V), W) \simeq \text{Ext}_{{}_R\mathcal{M}_R^A}^*(V, W \otimes A)$$

- (2) If  $\mathcal{M}^A$  has enough projective objects (in which case one says that  $A$  is co-Frobenius), so has  ${}_R\mathcal{M}_R^A$ . In that case, the previous isomorphism ensures that for any object  $V$  in  ${}_R\mathcal{M}_R^A$ , we have

$$\text{pd}_{{}_R\mathcal{M}_R}(\Omega_R(V)) \leq \text{pd}_{{}_R\mathcal{M}_R^A}(V)$$

- (3) If  $A$  is cosemisimple, then  ${}_R\mathcal{M}_R^A$  has enough projective objects, and the projective objects are the direct summands of the free ones.

We now present our key averaging lemma for bimodules. If  $R$  is an  $A$ -comodule algebra over a cosemisimple Hopf algebra  $A$ , we denote by  $\rho$  the algebra automorphism of  $R$  defined by  $\rho = \text{id} * \psi^{-2}$ , i.e.  $\rho(x) = \psi^{-2}(x_{(1)})x_{(0)}$ , with  $\psi$  a modular functional as in Section 2.

**Lemma 4.2.** *Let  $V, W$  be objects of  ${}_R\mathcal{M}_R^A$ , where  $A$  is a cosemisimple Hopf algebra  $A$  and  $R$  is a right  $A$ -comodule algebra. If  $f : V \rightarrow W$  is a linear map satisfying*

$$f(v \cdot x) = f(v) \cdot x \text{ and } f(x \cdot v) = \rho(x) \cdot f(v)$$

for any  $v \in V$  and  $x \in R$ , then  $\mathbf{M}_{V,W}(f) : V \rightarrow W$  is a morphism in the category  ${}_R\mathcal{M}_R^A$ .

*Proof.* We already know that  $\mathbf{M}_{V,W}(f) : V \rightarrow W$  is colinear and there remains to prove that  $\mathbf{M}_{V,W}(f)$  is left and right  $R$ -linear as well. Let  $v \in V$  and  $x \in R$ . We have, using our condition on  $f$  and the compatibility between the comodule and right module structure:

$$\begin{aligned} \mathbf{M}_{V,W}(f)(v \cdot x) &= h(f((v \cdot x)_{(0)})_{(1)}S((v \cdot x)_{(1)}))f((v \cdot x)_{(0)})_{(0)} \\ &= h(f(v_{(0)} \cdot x_{(0)})_{(1)}S(v_{(1)}x_{(1)}))f(v_{(0)} \cdot x_{(0)})_{(0)} \\ &= h((f(v_{(0)}) \cdot x_{(0)})_{(1)}S(v_{(1)}x_{(1)}))f(v_{(0)}) \cdot x_{(0)} \\ &= h((f(v_{(0)})_{(1)}x_{(1)}S(v_{(1)}x_{(2)}))f(v_{(0)})_{(0)} \cdot x_{(0)} \\ &= h((f(v_{(0)})_{(1)}x_{(1)}S(x_{(2)})S(v_{(1)}))f(v_{(0)})_{(0)} \cdot x_{(0)} \\ &= \mathbf{M}_{V,W}(f)(v) \cdot x \end{aligned}$$

Hence  $f$  is right  $R$ -linear. We also have, using our condition on  $f$  and the compatibility between the comodule and left module structure:

$$\begin{aligned} \mathbf{M}_{V,W}(f)(x \cdot v) &= h(f((x \cdot v)_{(0)})_{(1)}S((x \cdot v)_{(1)}))f((x \cdot v)_{(0)})_{(0)} \\ &= h((f(x_{(0)} \cdot v_{(0)})_{(1)}S(x_{(1)}v_{(1)}))f(x_{(0)} \cdot v_{(0)})_{(0)} \\ &= \psi^{-2}(x_{(1)})h((x_{(0)} \cdot f(v_{(0)}))_{(1)}S(x_{(2)}v_{(1)}))x_{(0)} \cdot f(v_{(0)})_{(0)} \\ &= \psi^{-2}(x_{(2)})h(x_{(1)}f(v_{(0)})_{(1)}S(x_{(3)}v_{(1)}))x_{(0)} \cdot f(v_{(0)})_{(0)} \end{aligned}$$

Using the properties of the modular functional, this gives:

$$\begin{aligned}
\mathbf{M}_{V,W}(f)(x \cdot v) &= \psi^{-2}(x_{(4)})h(f(v_{(0)})_{(1)}S(v_{(1)})S(x_{(5)})\psi(x_{(1)})x_{(2)}\psi(x_{(3)}))x_{(0)} \cdot f(v_{(0)})_{(0)} \\
&= h(f(v_{(0)})_{(1)}S(v_{(1)})S(x_{(4)})\psi(x_{(1)})x_{(2)}\psi^{-1}(x_{(3)}))x_{(0)} \cdot f(v_{(0)})_{(0)} \\
&= h(f(v_{(0)})_{(1)}v_{(1)}S(x_{(2)})S^2(x_{(1)}))x_{(0)} \cdot f(v_{(0)})_{(0)} \\
&= x \cdot \mathbf{M}_{V,W}(f)(v)
\end{aligned}$$

and this shows that  $\mathbf{M}_{V,W}(f)$  is left  $R$ -linear as well.  $\square$

**Lemma 4.3.** *Let  $V$  be a comodule over the cosemisimple Hopf algebra  $A$ , let  $R$  be an  $A$ -comodule algebra and consider the map  $\rho_V = \rho \otimes \text{id}_V \otimes \text{id}_R : R \otimes V \otimes R \rightarrow R \otimes V \otimes R$ . We have  $\mathbf{M}(\rho_V) = \text{id}_{R \otimes V \otimes R}$ , where  $\mathbf{M}$  stands for averaging with respect to  $R \otimes V \otimes R$ .*

*Proof.* It is immediate that  $\rho_V = \rho \otimes \text{id}_V \otimes \text{id}_R : R \otimes V \otimes R \rightarrow R \otimes V \otimes R$  satisfies the assumption of Lemma 4.2, hence  $\mathbf{M}(\rho_V)$  is left and right  $R$ -linear. Since it is clear that  $\mathbf{M}(\rho_V)(1 \otimes v \otimes 1) = 1 \otimes v \otimes 1$  for any  $v \in V$ , we get the result by the  $R$ -bilinearity of  $\mathbf{M}(\rho_V)$ .  $\square$

We now have all the ingredients to prove that for  $A$  cosemisimple and an  $A$ -comodule algebra  $R$ , the forgetful functor  $\Omega_R : {}_R\mathcal{M}_R^A \rightarrow {}_R\mathcal{M}_R$  is twisted separable, and hence the following result.

**Proposition 4.4.** *If  $A$  is a cosemisimple Hopf algebra and  $R$  is a right  $A$ -comodule algebra, we have  $\text{pd}_{{}_R\mathcal{M}_R^A}(V) = \text{pd}_{{}_R\mathcal{M}_R}(V)$  for any object  $V$  in  ${}_R\mathcal{M}_R^A$  such that  $\text{pd}_{{}_R\mathcal{M}_R^A}(V)$  is finite. In particular, if  $\text{pd}_{{}_R\mathcal{M}_R^A}(R)$  is finite, we have  $\text{pd}_{{}_R\mathcal{M}_R^A}(R) = \text{pd}_{{}_R\mathcal{M}_R}(R) = \text{cd}(R)$ .*

*Proof.* In order to show that the forgetful functor  $\Omega_R : {}_R\mathcal{M}_R^A \rightarrow {}_R\mathcal{M}_R$  is twisted separable, consider

- (1) the class  $\mathcal{F} = \mathcal{F}_0$  of free bimodules in  ${}_R\mathcal{M}_R^A$ ;
- (2) the autoequivalence  $\Theta$  of the category  ${}_R\mathcal{M}_R$  that associates to an  $R$ -bimodule  $W$  the  $R$ -bimodule  ${}_\rho W$  having  $W$  as underlying vector space and  $R$ -bimodule structure given by  $x \cdot' w \cdot' y = \rho(x) \cdot w \cdot y$ , and is trivial on morphisms;
- (3) for a free object  $R \otimes V \otimes R$ , the  $R$ -bimodule isomorphism  $\rho_V : R \otimes V \otimes R \rightarrow {}_\rho(R \otimes V \otimes R)$  in Lemma 4.3;
- (4) for objects  $V, W$  in  ${}_R\mathcal{M}_R^A$ , the averaging map

$$\mathbf{M}_{V,W} : \text{Hom}_{{}_R\mathcal{M}_R}(V, {}_\rho W) \rightarrow \text{Hom}_{{}_R\mathcal{M}_R^A}(V, W)$$

from Lemma 4.2.

It follows from Lemma 4.2, Lemma 4.3 and Proposition 2.1 that the functor  $\Omega_R : {}_R\mathcal{M}_R^A \rightarrow {}_R\mathcal{M}_R$  is indeed twisted separable. Moreover, as already said, the class  $\mathcal{F}$  of free objects consists of projective objects, the projective objects in  ${}_R\mathcal{M}_R^A$  are direct summands of free objects and hence are preserved by  $\Omega_R$ , which is clearly exact. Hence we are in the situation of Proposition 3.2, and we obtain the equality of projective dimensions.  $\square$

Recall that a left Galois object over a Hopf algebra  $A$  is a non-zero left  $A$ -comodule algebra  $R$  such that the canonical map

$$\begin{aligned}
R \otimes R &\longrightarrow A \otimes R \\
x \otimes y &\longmapsto x_{(-1)} \otimes x_{(0)}y
\end{aligned}$$

is bijective. Similarly a right Galois object over  $A$  is a right  $A$ -comodule algebra such that the obvious analogue of the previous canonical map is bijective.

Our main application of Proposition 4.4 is the following result.

**Theorem 4.5.** *Let  $A$  be Hopf algebra and let  $R$  be a left or right  $A$ -Galois object. If  $A$  is cosemisimple and  $\text{cd}(A)$  is finite, we have  $\text{cd}(A) = \text{cd}(R)$*

*Proof.* First recall [34] that it follows from the structure theorem for Hopf modules that the functor

$$\begin{aligned} {}_A\mathcal{M} &\longrightarrow {}_A\mathcal{M}_A^A \\ V &\longmapsto V \odot A \end{aligned}$$

is a monoidal equivalence of categories, where  $V \odot A$  is  $V \otimes A$  as a vector space, has the tensor product left  $A$ -module structure and the right module and comodule structures are induced by the multiplication and comultiplication of  $A$  respectively. This monoidal equivalence transforms the trivial module  ${}_\varepsilon k$  into the  $A$ -bimodule  $A$ .

Now let  $R$  be a left Hopf-Galois object over  $A$ . By the results in [35], there exists a Hopf algebra  $B$  such that  $R$  is an  $A$ - $B$  bi-Galois object and the cotensor product  $-\square_A R$  induces a monoidal equivalence  ${}_{\mathcal{M}^A} \simeq^{\otimes} {}_{\mathcal{M}^B}$  sending  $A$  to  $R$ , which in turn clearly induces an equivalence between the bimodule categories  ${}_A\mathcal{M}_A^A$  and  ${}_R\mathcal{M}_R^B$ . Composing with the equivalence at the beginning of the proof, we get a monoidal equivalence

$${}_A\mathcal{M} \simeq^{\otimes} {}_R\mathcal{M}_R^B$$

sending  ${}_\varepsilon k$  to  $R$ , so that  $\text{cd}(A) = \text{pd}_A({}_\varepsilon k) = \text{pd}_{{}_R\mathcal{M}_R^B}(R)$ . Hence if  $\text{cd}(A)$  is finite, we get that  $\text{pd}_{{}_R\mathcal{M}_R^B}(R)$  is finite as well, and assuming moreover that  $A$  is cosemisimple, we conclude by Proposition 4.4 that  $\text{cd}(A) = \text{pd}_{{}_R\mathcal{M}_R^B}(R) = \text{cd}(R)$ , as claimed.

If we start with a right Hopf-Galois object  $R$  over  $A$ , it is well-known that  $R^{\text{op}}$  is a left  $A$ -Galois object in a natural way (if the antipode of  $A$  is bijective, which is the case here since we assume cosemisimplicity), so that we can use the result for left  $A$ -Galois objects to conclude that  $\text{cd}(A) = \text{cd}(R)$  as well.  $\square$

**Theorem 4.6.** *Let  $A, B$  be Hopf algebras that have equivalent linear tensor categories of comodules:  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ . If  $A$  and  $B$  are cosemisimple and  $\text{cd}(A), \text{cd}(B)$  are finite, we have  $\text{cd}(A) = \text{cd}(B)$ .*

*Proof.* By the results in [35], such a monoidal equivalence arises from an  $A$ - $B$ -bi-Galois object  $R$ , via the cotensor product. Hence Theorem 4.5 ensures, under the finiteness assumption on  $\text{cd}(A)$  and  $\text{cd}(B)$ , that  $\text{cd}(A) = \text{cd}(R) = \text{cd}(B)$ .  $\square$

We finish the section by noticing that Proposition 4.4 can be strengthened in the case  $S^4 = \text{id}$ .

**Proposition 4.7.** *Let  $A$  be a cosemisimple Hopf algebra with  $S^4 = \text{id}$ , and let  $R$  be a right  $A$ -comodule algebra.*

- (1) *The forgetful functor  $\Omega_R : {}_R\mathcal{M}_R^A \rightarrow {}_R\mathcal{M}_R$  is separable. We thus have  $\text{pd}_{{}_R\mathcal{M}_R^A}(V) = \text{pd}_{{}_R\mathcal{M}_R}(V)$  for any object  $V$  in  ${}_R\mathcal{M}_R^A$ , and  $\text{pd}_{{}_R\mathcal{M}_R^A}(R) = \text{cd}(R)$ .*
- (2) *Let  $F : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$  be a monoidal equivalence with  $B$  satisfying  $S^4 = \text{id}$  as well. We then have, for the  $B$ -comodule algebra  $S = F(R)$ ,  $\text{cd}(R) = \text{cd}(S)$ .*
- (3) *Let  $F : \mathcal{M}^A \rightarrow \text{Vec}_k$  be a fibre functor. If  $\text{cd}(R)$  is finite, we have, for the algebra  $S = F(R)$ ,  $\text{cd}(R) = \text{cd}(S)$ .*

*Proof.* As in the proof of Lemma 4.2, using the properties of the modular functional, we see that for any  $a, x \in A$

$$h(S(a_{(1)})xa_{(2)}) = \psi^{-2}(a_{(2)})h(xa_{(3)}S^{-1}(a_{(1)}))$$

At  $x = 1$  this gives  $\varepsilon(a) = \psi^{-2}(a_{(2)})h(a_{(3)}S^{-1}(a_{(1)}))$ . If  $S^4 = \text{id}$ , then  $\psi^{-2}$  convolution commutes with the identity, hence we get  $\psi^{-2} = \varepsilon$ . Hence the automorphism  $\rho$  associated to an  $A$ -comodule algebra  $R$  is the identity, the autoequivalence  $\Theta$  in the proof of Proposition 4.4 is the identity, and the class  $\mathcal{F}$  is the class of all objects, and it follows that  $\Omega_R : {}_R\mathcal{M}_R^A \rightarrow {}_R\mathcal{M}_R$  is separable. The result about projective dimensions is then either well-known or follows from the obvious improvement of Proposition 4.4 in the separable case, having in mind that the conclusion of Lemma 3.4 now holds for any object.

A monoidal equivalence  $F : \mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$  induces, as before, an equivalence between the bimodule categories  ${}_R\mathcal{M}_R^A$  and  ${}_S\mathcal{M}_S^B$  for  $S = F(R)$ , sending  $R$  to  $S$ , and then the assumption  $S^4 = \text{id}$  on  $A$  and  $B$  ensures that  $\text{cd}(R) = \text{pd}_{{}_R\mathcal{M}_R^A}(R) = \text{pd}_{{}_S\mathcal{M}_S^B}(S) = \text{cd}(S)$ .

Start now with a fibre functor  $F : \mathcal{M}^A \rightarrow \text{Vec}_k$ , i.e. a  $k$ -linear monoidal exact faithful functor that commutes with colimits. Such a functor induces, by Tannaka-Krein duality (see e.g. [21, 17]) or by the results in [35], a monoidal equivalence  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$  for some Hopf algebra  $B$ , with as well a monoidal equivalence  ${}_R\mathcal{M}_R^A \simeq^{\otimes} {}_S\mathcal{M}_S^B$ . The assumption that  $S^4 = \text{id}$  for  $A$  then gives  $\text{pd}_{{}_R\mathcal{M}_R^A}(R) = \text{cd}(R)$ . Since  $\text{pd}_{{}_R\mathcal{M}_R^A}(R) = \text{pd}_{{}_S\mathcal{M}_S^B}(S)$ , Proposition 4.4 ensures, under the assumption that  $\text{cd}(R)$  is finite, that  $\text{pd}_{{}_S\mathcal{M}_S^B}(S) = \text{cd}(S)$ , and thus this gives the expected result.  $\square$

*Example 4.8.* Let  $\sigma : A \otimes A \rightarrow k$  be (Hopf, right) 2-cocycle on a Hopf algebra  $A$  (see [29]), i.e.  $\sigma$  is a convolution invertible linear map  $\sigma : A \otimes A \rightarrow k$  satisfying, for any  $a, b, c \in A$

$$\sigma(a, 1) = \varepsilon(a)1 = \sigma(1, a), \quad \sigma(a_{(2)}, b_{(2)})\sigma(a_{(1)}b_{(1)}, c) = \sigma(a, b_{(1)}c_{(1)})\sigma(b_{(2)}, c_{(2)})$$

If  $R$  is a right  $A$ -comodule algebra, we obtain a new (associative) algebra  $R_\sigma$  by letting

$$x \cdot y = \sigma(x_{(1)}, y_{(1)})x_{(0)}y_{(0)}$$

We then have  $\text{cd}(R) = \text{cd}(R_\sigma)$  if  $\text{cd}(R)$  is finite.

*Proof.* The algebra  $R_\sigma$  is the image of  $R$  under the fibre functor  $\mathcal{M}^A \rightarrow \text{Vec}_k$  which has the forgetful functor as underlying functor and monoidal constraint  $V \otimes W \rightarrow V \otimes W$ ,  $v \otimes w \mapsto \sigma(v_{(1)}, w_{(1)})v_{(0)} \otimes w_{(0)}$ . The result is thus a consequence of Proposition 4.7.  $\square$

*Remark 4.9.* If the Hopf algebra  $A^\sigma$  (see [15, 35]) satisfies as well  $S^4 = \text{id}$ , we can conclude that  $\text{cd}(R) = \text{cd}(R_\sigma)$  without the finiteness assumption. This applies in particular, when  $A$  is a group algebra, to the 2-cocycle twisting of a group-graded algebra, which therefore has the same Hochschild cohomological dimension as the original algebra. This was probably well-known, but we are not aware of an explicit reference for this fact.

## 5. YETTER-DRINFELD MODULES OVER COSEMIMPLE HOPF ALGEBRAS

Our aim in this section is to prove Theorem 1.3. Recall that a (right-right) Yetter-Drinfeld module over a Hopf algebra  $A$  is a right  $A$ -comodule and right  $A$ -module  $V$  satisfying the condition,  $\forall v \in V, \forall a \in A$ ,

$$(v \cdot a)_{(0)} \otimes (v \cdot a)_{(1)} = v_{(0)} \cdot a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

The category of Yetter-Drinfeld modules over  $A$  is denoted  $\mathcal{YD}_A^A$ : the morphisms are the  $A$ -linear and  $A$ -colinear maps. The category  $\mathcal{YD}_A^A$  is obviously abelian, and, endowed with the usual tensor product of modules and comodules, is a tensor category, with unit the trivial Yetter-Drinfeld module, denoted  $k$ .

The forgetful functor  $\Omega^A : \mathcal{YD}_A^A \rightarrow \mathcal{M}^A$  has a left adjoint [10], the free Yetter-Drinfeld module functor, which sends a comodule  $V$  to the Yetter-Drinfeld module  $V \boxtimes A$ , which

as a vector space is  $V \otimes A$ , has the right module structure given by multiplication on the right, and right coaction given by

$$(v \otimes a)_{(0)} \otimes (v \otimes a)_{(1)} = v_{(0)} \otimes a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

A Yetter-Drinfeld module isomorphic to some  $V \boxtimes A$  as above is said to be free. Let us record the following facts, that are straightforward consequences of standard properties of pairs of adjoint functors.

- (1) Every Yetter-Drinfeld module is a quotient of a free Yetter-Drinfeld module. Indeed, for a Yetter-Drinfeld  $V$ , the  $A$ -module structure of  $V$  induces a surjective morphism  $\Omega^A(V) \boxtimes A \rightarrow V$ .
- (2) If the category  $\mathcal{M}^A$  has enough projective objects, then so has  $\mathcal{YD}_A^A$ .
- (3) If  $A$  is cosemisimple, then  $\mathcal{YD}_A^A$  has enough projective objects, and the projective objects are precisely the direct summands of the free Yetter-Drinfeld modules.

Similarly, the forgetful functor  $\Omega_A : \mathcal{YD}_A^A \rightarrow \mathcal{M}_A$  has a right adjoint [10], the cofree Yetter-Drinfeld module functor, which sends a module  $V$  to the Yetter-Drinfeld module  $V \# A$ , which as a vector space is  $V \otimes A$ , has the right comodule structure given by the comultiplication of  $A$  on the right, and right  $A$ -module structure given by

$$(v \otimes a) \cdot b = v \cdot b_{(2)} \otimes S(b_{(1)})ab_{(3)}$$

Again, as a consequence of general properties of adjoint functors, it follows that the category  $\mathcal{YD}_A^A$  has enough injective objects, since  $\mathcal{M}_A$  has.

Recall that we have defined the Gerstenhaber-Schack cohomological dimension of a Hopf algebra  $A$  by

$$\text{cd}_{\text{GS}}(A) = \max\{n : \text{Ext}_{\mathcal{YD}_A^A}^n(k, V) \neq 0 \text{ for some } V \in \mathcal{YD}_A^A\} \in \mathbb{N} \cup \{\infty\}$$

The name comes from the fact, proved in [37], that if  $V$  is a Yetter-Drinfeld over  $A$ , then  $\text{Ext}_{\mathcal{YD}_A^A}^*(k, V)$  is isomorphic with  $H_{\text{GS}}^*(A, V)$ , the Gerstenhaber-Schack cohomology of  $A$  with coefficients in  $V$  [19, 36].

Notice that since  $\mathcal{YD}_A^A$  has enough injective objects, the above Ext can be computed using injective resolutions of  $V$ , and if  $\mathcal{YD}_A^A$  has enough projective objects, using projective resolutions of  $k$  in  $\mathcal{YD}_A^A$ . Another consequence of general properties of pairs of adjoint functors is that we have, for any Yetter-Drinfeld module  $V$  and any  $A$ -module  $W$ , natural isomorphisms

$$\text{Ext}_A^*(\Omega_A(V), W) \simeq \text{Ext}_{\mathcal{YD}_A^A}^*(V, W \# A)$$

This is what proves that  $\text{cd}(A) \leq \text{cd}_{\text{GS}}(A)$  [6].

We now present an averaging lemma for Yetter-Drinfeld modules over cosemisimple Hopf algebras, in the same spirit as Lemma 4.2, which will be the key tool towards the proof of Theorem 1.3.

**Lemma 5.1.** *Let  $V, W$  be Yetter-Drinfeld modules over a cosemisimple Hopf algebra  $A$ . If  $f : V \rightarrow W$  is a linear map satisfying  $f(v \cdot a) = f(v) \cdot \theta(a)$  for any  $v \in V$  and  $a \in A$ , then  $\mathbf{M}_{V,W}(f) : V \rightarrow W$  is a morphism of Yetter-Drinfeld modules.*

*Proof.* We already know that  $\mathbf{M}_{V,W}(f) : V \rightarrow W$  is colinear and there remains to prove that  $\mathbf{M}_{V,W}(f)$  is  $A$ -linear as well. Let  $v \in V$  and  $a \in A$ . We have, using our condition

on  $f$  and the Yetter-Drinfeld property:

$$\begin{aligned}
\mathbf{M}_{V,W}(f)(v \cdot a) &= h(f((v \cdot a)_{(0)})_{(1)}S((v \cdot a)_{(1)}))f((v \cdot a)_{(0)})_{(0)} \\
&= h(f(v_{(0)} \cdot a_{(2)})_{(1)}S(S(a_{(1)})v_{(1)}a_{(3)}))f(v_{(0)} \cdot a_{(2)})_{(0)} \\
&= h((f(v_{(0)}) \cdot \theta(a_{(2)}))_{(1)}S(S(a_{(1)})v_{(1)}a_{(3)}))(f(v_{(0)}) \cdot \theta(a_{(2)}))_{(0)} \\
&= \psi^2(a_{(2)})h((f(v_{(0)}) \cdot a_{(3)})_{(1)}S(S(a_{(1)})v_{(1)}a_{(4)}))(f(v_{(0)}) \cdot a_{(3)})_{(0)} \\
&= \psi^2(a_{(2)})h(S(a_{(3)})f(v_{(0)})_{(1)}a_{(5)}S(S(a_{(1)})v_{(1)}a_{(6)}))f(v_{(0)})_{(0)} \cdot a_{(4)} \\
&= \psi^2(a_{(2)})h(S(a_{(3)})f(v_{(0)})_{(1)}a_{(5)}S(a_{(6)})S(v_{(1)})S^2(a_{(1)}))f(v_{(0)})_{(0)} \cdot a_{(4)} \\
&= \psi^2(a_{(2)})h(S(a_{(3)})f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)}))f(v_{(0)})_{(0)} \cdot a_{(4)}
\end{aligned}$$

Using the properties of the modular functional, and since  $\sigma \circ S = \sigma^{-1} = \psi^{-1} * S * \psi^{-1}$  because  $\sigma$  is an algebra map, this gives:

$$\begin{aligned}
\mathbf{M}_{V,W}(f)(v \cdot a) &= \psi^2(a_{(2)})h(f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})\sigma(S(a_{(3)}))f(v_{(0)})_{(0)} \cdot a_{(4)} \\
&= \psi^2(a_{(2)})h(f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})\psi^{-1}(a_{(3)})S(a_{(4)})\psi^{-1}(a_{(5)}))f(v_{(0)})_{(0)} \cdot a_{(6)} \\
&= h(f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})\psi(a_{(2)})S(a_{(3)})\psi^{-1}(a_{(4)}))f(v_{(0)})_{(0)} \cdot a_{(5)} \\
&= h(f(v_{(0)})_{(1)}S(v_{(1)})S^2(a_{(1)})S^3(a_{(2)}))f(v_{(0)})_{(0)} \cdot a_{(3)} \\
&= h(f(v_{(0)})_{(1)}S(v_{(1)}))f(v_{(0)})_{(0)} \cdot a = \mathbf{M}_{V,W}(f)(v) \cdot a
\end{aligned}$$

and this shows that  $\mathbf{M}_{V,W}(f)$  is  $A$ -linear.  $\square$

**Lemma 5.2.** *Let  $V$  be a right comodule over the cosemisimple Hopf algebra  $A$ , and consider the linear map  $\theta_V = \text{id}_V \otimes \theta : V \boxtimes A \rightarrow V \boxtimes A$ . We have  $\mathbf{M}(\theta_V) = \text{id}_{V \boxtimes A}$ , where  $\mathbf{M}(\theta_V)$  stands for  $\mathbf{M}_{V \boxtimes A, V \boxtimes A}(\theta_V)$ .*

*Proof.* It is immediate that  $\text{id}_V \otimes \theta : V \boxtimes A \rightarrow V \boxtimes A$  satisfies the assumption of Lemma 5.1, hence  $\mathbf{M}(\text{id}_V \otimes \theta)$  is  $A$ -linear. Since it is clear that  $\mathbf{M}(\text{id}_V \otimes \theta)(v \otimes 1) = v \otimes 1$  for any  $v \in V$ , we get the result by the  $A$ -linearity of  $\mathbf{M}(\text{id}_V \otimes \theta)$ .  $\square$

We now have all the ingredients to prove that for  $A$  cosemisimple, the forgetful functor  $\Omega_A : \mathcal{YD}_A^A \rightarrow \mathcal{M}_A$  is twisted separable, and hence the following result, which will have Theorem 1.3 as an immediate corollary.

**Proposition 5.3.** *If  $A$  is a cosemisimple Hopf algebra, we have  $\text{pd}_{\mathcal{YD}_A^A}(V) = \text{pd}_A(V)$  for any Yetter-Drinfeld module  $V$  such that  $\text{pd}_{\mathcal{YD}_A^A}(V)$  is finite.*

*Proof.* In order to show that the forgetful functor  $\Omega_A : \mathcal{YD}_A^A \rightarrow \mathcal{M}_A$  is twisted separable, consider

- (1) the class  $\mathcal{F} = \mathcal{F}_0$  of free Yetter-Drinfeld modules;
- (2) the autoequivalence  $\Theta$  of the category  $\mathcal{M}_A$  that associates to a right  $A$ -module  $W$  the  $A$ -module  $W_\theta$  having  $W$  as underlying vector space and  $A$ -module structure given by  $w \cdot a = w \cdot \theta(a)$ , and is trivial on morphisms;
- (3) for a free Yetter-Drinfeld module  $V \boxtimes A$ , the  $A$ -module isomorphism  $\theta_V : V \otimes A \rightarrow (V \otimes A)_\theta$  in Lemma 5.2.
- (4) for Yetter-Drinfeld modules  $V, W$ , the averaging map

$$\mathbf{M}_{V,W} : \text{Hom}_A(V, W_\theta) \rightarrow \text{Hom}_{\mathcal{YD}_A^A}(V, W)$$

from Lemma 5.1.

It follows from Lemma 5.1, Lemma 5.2 and Proposition 2.1 that the functor  $\Omega_A : \mathcal{YD}_A^A \rightarrow \mathcal{M}_A$  is indeed twisted separable. Moreover, as already said, the class  $\mathcal{F}$  of free Yetter-Drinfeld modules consists of projective objects, the projective objects in  $\mathcal{YD}_A^A$  are direct summands of free objects and hence are preserved by  $\Omega_A$ , which is exact. Hence we are in the situation of Proposition 3.2, and we obtain the equality of projective dimensions.  $\square$

*Proof of Theorem 1.3.* Let  $A$  be a cosemimple Hopf algebra. Since  $\text{cd}_{\text{GS}}(A) = \text{pd}_{\mathcal{YD}_A^A}(k)$  and  $\text{cd}(A) = \text{pd}_A(k_\varepsilon)$ , we have  $\text{cd}(A) = \text{cd}_{\text{GS}}(A)$  if  $\text{cd}_{\text{GS}}(A)$  is finite, by Proposition 5.3.  $\square$

We get the following weak form of Theorem 4.6, whose formulation is probably useful.

**Corollary 5.4.** *Let  $A, B$  be Hopf algebras such that  $\mathcal{M}^A \simeq^\otimes \mathcal{M}^B$ . If  $A$  and  $B$  are cosemisimple and  $\text{cd}_{\text{GS}}(A)$  is finite, we have  $\text{cd}(A) = \text{cd}(B)$ .*

As in Section 4, Proposition 5.3 can be strengthened when  $S^4 = \text{id}$ .

**Theorem 5.5.** *Let  $A$  be Hopf algebra. The forgetful functor  $\Omega_A : \mathcal{YD}_A^A \rightarrow \mathcal{M}_A$  is separable if and only if  $A$  is cosemisimple and  $S^4 = \text{id}$ , and in that case we have  $\text{pd}_{\mathcal{YD}_A^A}(V) = \text{pd}_A(V)$  for any Yetter-Drinfeld module  $V$ .*

*Proof.* If  $A$  is cosemisimple and  $S^4 = \text{id}$ , we see, as in the proof of Proposition 4.7, that the automorphism  $\theta$  of  $A$  is the identity, and that  $\Omega_A : \mathcal{YD}_A^A \rightarrow \mathcal{M}_A$  is indeed separable, and the assertion on projective dimensions, which was already proved in [7, Section 6], follows similarly.

Assume now that  $\Omega_A : \mathcal{YD}_A^A \rightarrow \mathcal{M}_A$  is separable. Since  $\Omega_A$  admits the right adjoint  $-\#A$ , the characterization of separability for functors that admit a right adjoint in [33] gives in particular an  $A$ -colinear and  $A$ -linear map

$$\eta : k\#A \rightarrow k \text{ with } \eta(1) = 1$$

By the  $A$ -colinearity and  $\eta(1) = 1$ , we have that  $\eta = h$  is a Haar integral on  $A$ , which is thus cosemisimple. The  $A$ -linearity of  $h$  gives, for any  $a, x \in A$ ,

$$h(S(a_{(1)})xa_{(2)}) = \varepsilon(a)h(x)$$

We have seen in the proof of Proposition 4.7 that for any  $a, x \in A$ ,

$$h(S(a_{(1)})xa_{(2)}) = \psi^{-2}(a_{(2)})h(xa_{(3)}S^{-1}(a_{(1)}))$$

Hence we have for any  $a, x \in A$

$$h(x(\varepsilon(a) - \psi^{-2}(a_{(2)})a_{(3)}S^{-1}(a_{(1)}))) = 0$$

The non-degeneracy of the Haar integral (which follows from the orthogonality relations) then gives, for any  $a \in A$

$$\varepsilon(a)1 = \psi^{-2}(a_{(2)})a_{(3)}S^{-1}(a_{(1)})$$

Hence applying  $\varepsilon$  gives  $\varepsilon = \psi^{-2}$ , and we thus have  $S^4 = \text{id}$ .  $\square$

## 6. HOPF SUBALGEBRAS AND COHOMOLOGICAL DIMENSION

Let  $B \subset A$  be a Hopf subalgebra. Under the assumption of faithful flatness of  $A$  as a  $B$ -module, which holds in many situations and in particular if  $A$  is cosemisimple [12], we have  $\text{cd}(B) \leq \text{cd}(A)$  [6, Proposition 3.1]. In this section we prove, in view of an example in the next section, an analogue inequality for Gerstenhaber-Schack cohomological dimension, in the cosemisimple case. Of course, if the conclusion of Theorem 1.3 was known to hold for any cosemisimple Hopf algebra, this would become trivial.

We begin with some results of independent interest. Recall [6] that a Yetter-Drinfeld module is said to be relative projective if it is a direct summand in a free one.

**Proposition 6.1.** *Let  $A$  be a Hopf algebra, let  $V$  be a Yetter-Drinfeld over  $A$  and let  $W$  be a right  $A$ -comodule. Then we have an isomorphism of Yetter-Drinfeld modules*

$$(\Omega^A(V) \otimes W) \boxtimes A \simeq V \otimes (W \boxtimes A)$$

*In particular, if  $P$  is a relative projective Yetter-Drinfeld module, so is the Yetter-Drinfeld module  $V \otimes P$ .*

*Proof.* The map

$$\begin{aligned} (\Omega^A(V) \otimes W) \boxtimes A &\longmapsto V \otimes (W \boxtimes A) \\ v \otimes w \otimes a &\longmapsto v \cdot a_{(1)} \otimes w \otimes a_{(2)} \end{aligned}$$

is easily seen to be a morphism of Yetter-Drinfeld modules, and its inverse is given by  $v \otimes w \otimes a \mapsto v \cdot S(a_{(1)}) \otimes w \otimes a_{(2)}$ . If  $P$  is relative projective, let  $W$  be a right  $A$ -comodule and  $Q$  be a Yetter-Drinfeld module such that  $W \boxtimes A \simeq P \oplus Q$ . We then have  $(V \otimes P) \oplus (V \otimes Q) \simeq V \otimes (W \boxtimes A) \simeq (\Omega^A(V) \otimes W) \boxtimes A$ , which proves that  $V \otimes P$  is relative projective.  $\square$

**Corollary 6.2.** *If  $A$  is a cosemisimple Hopf algebra, we have*

$$\begin{aligned} \text{cd}_{\text{GS}}(A) &= \text{pd}_{\mathcal{YD}_A^A}(k) = \max \left\{ n : \text{Ext}_{\mathcal{YD}_A^A}^n(k, V) \neq 0 \text{ for some } V \in \mathcal{YD}_A^A \right\} \\ &= \max \left\{ \text{pd}_{\mathcal{YD}_A^A}(V), V \in \mathcal{YD}_A^A \right\} = \max \left\{ n : \text{Ext}_{\mathcal{YD}_A^A}^n(V, W) \neq 0 \text{ for some } V, W \in \mathcal{YD}_A^A \right\} \\ &= \min \left\{ n : \text{Ext}_{\mathcal{YD}_A^A}^{n+1}(V, W) = 0 \text{ for any } V, W \in \mathcal{YD}_A^A \right\} \\ &= \max \left\{ \text{injd}_{\mathcal{YD}_A^A}(V), V \in \mathcal{YD}_A^A \right\} \end{aligned}$$

where  $\text{injd}_{\mathcal{YD}_A^A}$  is the injective dimension in the category  $\mathcal{YD}_A^A$ .

*Proof.* The first two equalities have already been discussed. Let  $P_* \rightarrow k$  be resolution of  $k$  by projective objects, of length  $n = \text{pd}_{\mathcal{YD}_A^A}(k)$ . Since  $A$  is cosemisimple, the projective objects are the relative projectives, so if  $V$  is a Yetter-Drinfeld module, tensoring the above resolution with  $V$  yields, by Proposition 6.1, a length  $n$  resolution of  $V$  by projective objects. This gives the third equality, and the other ones then follow by classical arguments.  $\square$

Let  $B \subset A$  be a Hopf subalgebra. Recall [7] that there is a pair of adjoint functors

$$\begin{aligned} \mathcal{YD}_A^A &\longrightarrow \mathcal{YD}_B^B & \mathcal{YD}_B^B &\longrightarrow \mathcal{YD}_A^A \\ X &\longmapsto X^{(B)} & V &\longmapsto V \boxtimes_B A \end{aligned}$$

where

- (1) for a Yetter-Drinfeld module  $X$  over  $A$ ,  $X^{(B)} = \{x \in X \mid x_{(0)} \otimes x_{(1)} \in X \otimes B\}$  has the restricted  $B$ -module structure;
- (2) for a Yetter-Drinfeld module  $V$  over  $B$ ,  $V \boxtimes_B A$  is the induced module  $V \otimes_B A$ , with  $A$ -comodule structure given by

$$(v \otimes_B a)_{(0)} \otimes (v \otimes_B a)_{(1)} = v_{(0)} \otimes_B a_{(2)} \otimes S(a_{(1)})v_{(1)}a_{(3)}$$

**Lemma 6.3.** *Let  $B \subset A$  be a Hopf subalgebra, and assume that  $A$  is cosemisimple. Let  $V$  be a Yetter-Drinfeld module over  $B$ . Then  $V$  is isomorphic to a direct summand of  $(V \boxtimes_B A)^{(B)}$ .*

*Proof.* It is immediate to check that we have a morphism of Yetter-Drinfeld modules

$$i : V \rightarrow (V \boxtimes_B A)^{(B)}, \quad v \mapsto v \otimes_B 1$$

Assume now that  $A$  is cosemisimple. Then, by the proof of Theorem 2.1 in [12], there exists a sub- $B$ -bimodule  $T \subset A$ , which is as well a subcoalgebra, such that  $A = B \oplus T$ . Let  $E : A \rightarrow B$  be the corresponding projection:  $E(b) = b$  for  $b \in B$  and  $E(a) = 0$  for  $a \in T$ . By construction  $E$  is a  $B$ -bimodule map and a coalgebra map, and it is immediate to check that we have for any  $a \in A$

$$S(E(a)_{(1)}) \otimes E(a)_{(2)} \otimes E(a)_{(3)} = S(a_{(1)}) \otimes E(a_{(2)}) \otimes a_{(3)}$$

From this, we see that the map

$$(V \boxtimes_B A)^{(B)} \rightarrow V, \quad v \otimes_B a \rightarrow v \cdot E(a)$$

is a morphism of Yetter-Drinfeld modules. Since this map is clearly a retraction to  $i$ , this proves the lemma.  $\square$

We now have all the ingredients to prove the expected result.

**Proposition 6.4.** *Let  $B \subset A$  be a Hopf subalgebra. If  $A$  is cosemisimple, we have  $\text{cd}_{\text{GS}}(B) \leq \text{cd}_{\text{GS}}(A)$ .*

*Proof.* We can assume that  $\text{cd}_{\text{GS}}(A) = n$  is finite. Since  $A$  is cosemisimple, [12, Theorem 2.1] ensures that  $A$  is flat as a left  $B$ -module, and  $B \subset A$  is coflat. Hence, by [7, Proposition 3.3] we have

$$\text{Ext}_{\mathcal{YD}_A^A}^*(V \boxtimes_B A, X) \simeq \text{Ext}_{\mathcal{YD}_B^B}^*(V, X^{(B)})$$

for any Yetter-Drinfeld module  $X$  over  $A$ , and any Yetter-Drinfeld module  $V$  over  $B$ . Hence, for  $V = k$ , Corollary 6.2 yields

$$\text{Ext}_{\mathcal{YD}_B^B}^{n+1}(k, X^{(B)}) \simeq \text{Ext}_{\mathcal{YD}_A^A}^{n+1}(k \boxtimes_B A, X) = \{0\}$$

for any Yetter-Drinfeld module  $X$  over  $A$ . Lemma 6.3 ensures that any Yetter-Drinfeld module over  $B$  is a direct summand in one of type  $X^{(B)}$ , so we get  $\text{cd}_{\text{GS}}(B) \leq n$ , as required.  $\square$

## 7. EXAMPLES

We now use the previous results to examine some examples that were not covered by the literature.

**7.1. Universal cosovereign Hopf algebras.** In this subsection we complete some of the results of [7] on the cohomological dimension of the universal cosovereign Hopf algebras. Recall that for  $n \geq 2$  and  $F \in \text{GL}_n(k)$ , the algebra  $H(F)$  is the algebra generated by  $(u_{ij})_{1 \leq i, j \leq n}$  and  $(v_{ij})_{1 \leq i, j \leq n}$ , with relations:

$$uv^t = v^t u = I_n; \quad v F u^t F^{-1} = F u^t F^{-1} v = I_n,$$

where  $u = (u_{ij})$ ,  $v = (v_{ij})$  and  $I_n$  is the identity  $n \times n$  matrix. The algebra  $H(F)$  has a Hopf algebra structure defined by

$$\begin{aligned} \Delta(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj}, & \Delta(v_{ij}) &= \sum_k v_{ik} \otimes v_{kj}, \\ \varepsilon(u_{ij}) &= \varepsilon(v_{ij}) = \delta_{ij}, & S(u) &= v^t, \quad S(v) = F u^t F^{-1}. \end{aligned}$$

We refer the reader to [4, 7] for more information and background on the universal cosovereign Hopf algebras  $H(F)$ .

Recall [7] that we say that a matrix  $F \in \mathrm{GL}_n(k)$  is

- normalizable if  $\mathrm{tr}(F) \neq 0$  and  $\mathrm{tr}(F^{-1}) \neq 0$  or  $\mathrm{tr}(F) = 0 = \mathrm{tr}(F^{-1})$ ;
- generic if it is normalizable and the solutions of the equation  $q^2 - \sqrt{\mathrm{tr}(F)\mathrm{tr}(F^{-1})}q + 1 = 0$  are generic, i.e. are not roots of unity of order  $\geq 3$  (this property does not depend on the choice of the above square root);
- an asymmetry if there exists  $E \in \mathrm{GL}_n(k)$  such that  $F = E^t E^{-1}$ .

**Theorem 7.1.** *Let  $F \in \mathrm{GL}_n(k)$ ,  $n \geq 2$ . If  $F$  is an asymmetry or  $F$  is generic, we have  $\mathrm{cd}(H(F)) = 3$ .*

*Proof.* We know from [7, Theorem 2.1], that  $\mathrm{cd}(H(F)) = 3$  if  $F$  is an asymmetry and that  $\mathrm{cd}_{\mathrm{GS}}(H(F)) = 3$  if  $F$  is generic, in which case  $H(F)$  is cosemisimple [4], so Theorem 1.3 gives the result in that case.  $\square$

As an illustration of Theorem 4.5, consider, for  $E \in \mathrm{GL}_n(k)$  and  $F \in \mathrm{GL}_m(k)$ ,  $n, m \geq 2$ , the algebra  $H(E, F)$  presented by generators  $u_{ij}, v_{ij}$ ,  $1 \leq i \leq m, 1 \leq j \leq n$ , and relations

$$wv^t = I_m = vFu^tE^{-1} \quad ; \quad v^tu = I_n = Fu^tE^{-1}v.$$

**Theorem 7.2.** *If  $E, F$  are generic,  $\mathrm{tr}(E) = \mathrm{tr}(F)$  and  $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$ , then we have  $\mathrm{cd}(H(E, F)) = 3$ .*

*Proof.* The assumption  $\mathrm{tr}(E) = \mathrm{tr}(F)$  and  $\mathrm{tr}(E^{-1}) = \mathrm{tr}(F^{-1})$  ensures that  $H(E, F)$  is an  $H(E)$ - $H(F)$ -bi-Galois object [4]. Hence, since the genericity assumption ensures that  $H(E)$  cosemisimple and we know from the previous result that  $\mathrm{cd}(H(E))$  and  $\mathrm{cd}(H(F))$  are finite, the result follows from Theorem 4.5.  $\square$

**7.2. Free wreath products.** In this subsection we assume that the base field is  $k = \mathbb{C}$ , since the monoidal equivalences on which we rely [18, 25] were obtained in this framework. Before going to the general setting of Theorem 7.4, we feel it is probably worth to present a particular example. So for  $n, p \geq 1$ , consider, following the notation of [2], the algebra  $A_h^p(n)$  presented by generators  $u_{ij}$ ,  $1 \leq i, j \leq n$ , and relations

$$\sum_{j=1}^n u_{ij}^p = 1 = \sum_{j=1}^n u_{ji}^p, \quad u_{ij}u_{ik} = 0 = u_{ji}u_{ki}, \quad \text{for } k \neq j,$$

At  $p = 1$ ,  $A_h^1(n) = A_s(n)$ , the coordinate algebra of Wang's quantum permutation group [38]. In general  $A_h^p(n)$  is a Hopf algebra with [3]

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}, \quad \varepsilon(u_{ij}) = \delta_{ij}, \quad S(u_{ij}) = u_{ji}^{p-1}$$

The following result, for which the  $p = 1$  case was obtained in [6] (see [8] as well, where it is shown that  $A_s(n)$  is Calabi-Yau of dimension 3), will be a particular instance of the forthcoming Theorem 7.4.

**Theorem 7.3.** *We have, for  $p \geq 1$  and  $n \geq 4$ ,  $\mathrm{cd}(A_h^p(n)) = 3$ .*

Let  $A$  be a Hopf algebra, and consider  $A^{*n}$ , the free product algebra of  $n$  copies of  $A$ , which inherits a natural Hopf algebra structure such that the canonical morphisms  $\nu_i : A \rightarrow A^{*n}$  are Hopf algebras morphisms. The free wreath product  $A *_w A_s(n)$  [3] is the quotient of the algebra  $A^{*n} * A_s(n)$  by the two-sided ideal generated by the elements:

$$\nu_k(a)u_{ki} - u_{ki}\nu_k(a), \quad 1 \leq i, k \leq n, \quad a \in A.$$

The free wreath product  $A *_w A_s(n)$  admits a Hopf algebra structure given by

$$\Delta(u_{ij}) = \sum_{k=1}^n u_{ik} \otimes u_{kj}, \quad \Delta(\nu_i(a)) = \sum_{k=1}^n \nu_i(a_{(1)}) u_{ik} \otimes \nu_k(a_{(2)}),$$

$$\varepsilon(u_{ij}) = \delta_{ij}, \quad \varepsilon(\nu_i(a)) = \varepsilon(a), \quad S(u_{ij}) = u_{ji}, \quad S(\nu_i(a)) = \sum_{k=1}^n \nu_k(S(a)) u_{ki}.$$

When  $A$  is a compact Hopf algebra (i.e. arises from a compact quantum, we do not need the precise definition here), the free wreath product is as well a compact Hopf algebra. In that case the monoidal categories of comodules have been described for  $n \geq 4$  by Lemeux-Tarrago [25] in the case  $S^2 = \text{id}$  and Fima-Pittau [18] in general.

Taking  $A$  to be the group algebra  $\mathbb{C}[\mathbb{Z}/p\mathbb{Z}]$ , we have  $A_h^p(n) \simeq \mathbb{C}[\mathbb{Z}/p\mathbb{Z}] *_w A_s(n)$  by [3, Example 2.5], hence Theorem 7.3 is a particular instance of the following result.

**Theorem 7.4.** *We have  $\text{cd}(A *_w A_s(n)) = \max(\text{cd}(A), 3)$  for any compact Hopf algebra  $A$  such that  $\text{cd}(A) = \text{cd}_{\text{GS}}(A)$  and any  $n \geq 4$ .*

*Proof.* First notice that there is a Hopf algebra map  $\pi : A *_w A_s(n) \rightarrow A_s(n)$  such that  $\pi(u_{ij}) = u_{ij}$  and  $\pi(a) = \varepsilon(a)$ , hence  $A_s(n)$  stands as Hopf subalgebra of  $A *_w A_s(n)$ . We thus have, by [6, Proposition 3.1],  $3 = \text{cd}(A_s(n)) \leq \text{cd}(A *_w A_s(n))$ . Similarly the natural map  $A^{*n} \rightarrow A *_w A_s(n)$  has a retraction, and hence  $A^{*n}$  stands as left coideal  $*$ -subalgebra of  $A *_w A_s(n)$ . By the results in [13],  $A *_w A_s(n)$  is thus faithfully flat as  $A^{*n}$ -module, hence projective [28]. We then have, using [7, Corollary 5.3],  $\text{cd}(A) = \text{cd}(A^{*n}) \leq \text{cd}(A *_w A_s(n))$ , since restricting a resolution by projective  $A *_w A_s(n)$ -modules to  $A^{*n}$ -modules remains a projective resolution. Hence we have

$$\max(\text{cd}(A), 3) \leq \text{cd}(A *_w A_s(n))$$

The converse inequality obviously holds if  $\text{cd}(A)$  is infinite, hence we can assume that  $\text{cd}(A)$  is finite, and hence, in view of our assumption, that  $\text{cd}_{\text{GS}}(A)$  is finite.

The results in [18, 25] ensure the existence, for  $q$  satisfying  $q + q^{-1} = \sqrt{n}$ , of a monoidal equivalence between the category of comodules over  $A *_w A_s(n)$  and the category of comodules over a certain Hopf subalgebra  $H$  of the free product  $A * \mathcal{O}(SU_q(2))$ . We have, combining Proposition 6.4 and [7, Corollary 5.10]

$$\text{cd}_{\text{GS}}(H) \leq \text{cd}_{\text{GS}}(A * \mathcal{O}(SU_q(2))) = \max(\text{cd}_{\text{GS}}(A), \text{cd}_{\text{GS}}(\mathcal{O}(SU_q(2))))$$

Since  $\text{cd}_{\text{GS}}(\mathcal{O}(SU_q(2))) = 3$  by [5, 6], we get  $\text{cd}_{\text{GS}}(H) \leq \max(\text{cd}_{\text{GS}}(A), 3)$ , and since we assume that  $\text{cd}_{\text{GS}}(A)$  is finite, we get that  $\text{cd}_{\text{GS}}(H)$  is finite. Hence by Corollary 5.4 and Theorem 1.3, we get

$$\text{cd}(A *_w A_s(n)) = \text{cd}(H) = \text{cd}_{\text{GS}}(H) \leq \max(\text{cd}_{\text{GS}}(A), 3) = \max(\text{cd}(A), 3)$$

which concludes the proof.  $\square$

*Remark 7.5.* At  $n = 2$ , using the simple description of the free wreath product as a crossed coproduct in [3], it is not difficult to show directly that  $\text{cd}(A *_w A_s(2)) = \max(\text{cd}(A), 1)$  if  $A$  is non trivial.

*Remark 7.6.* Fima-Pittau [18] define more generally a free wreath product  $A *_w A_{\text{aut}}(R, \varphi)$ , for suitable pairs  $(R, \varphi)$  consisting of a finite-dimensional  $C^*$ -algebra and a faithful state, and prove a similar monoidal equivalence result, so that Theorem 7.4 should generalize to this setting.

## 8. QUESTION 1.1 OUTSIDE THE COSEMISIMPLE CASE

In this last section we prove the remaining assertions of Theorem 1.2.

**8.1. The smooth case.** In this subsection we remark that the results in [42, 39] enable us to have a positive answer to Question 1.1 only assuming that the Hopf algebras are homologically smooth, while [39, Theorem 2.4.5] assumed furthermore that one of the Hopf algebras is twisted Calabi-Yau, and then proved that the other one is twisted Calabi-Yau as well. Recall that an algebra is said to be homologically smooth if the trivial bimodule has a finite resolution by finitely generated projective bimodules. For Hopf algebras, this is equivalent to say that the trivial left or right  $A$ -module has a finite resolution by finitely generated projective modules.

**Theorem 8.1.** *Let  $A, B$  be Hopf algebras that have equivalent linear tensor categories of comodules:  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ . If  $A$  and  $B$  are homologically smooth and have bijective antipode, we have  $\text{cd}(A) = \text{cd}(B)$ .*

*Proof.* The assumption on the bijectivity of the antipodes enables us to use results in [42, 39]. By the results in [35] such a monoidal equivalence arises from an  $A$ - $B$ -bi-Galois object  $R$  ( $R$  is an  $A$ - $B$ -bicomodule algebra, is left Hopf-Galois over  $A$  and right Hopf-Galois over  $B$ ). Assuming that  $A$  is homologically smooth ensures that  $\text{cd}(A)$  is finite, so as in Lemma 3.3 we have  $\text{cd}(A) = \max\{n : \text{Ext}_{A\mathcal{M}}^n(\varepsilon k, F) \neq 0 \text{ for some free module } F\}$ , and that the functor  $\text{Ext}_{A\mathcal{M}}^*(\varepsilon k, -)$  commutes with direct colimits (see e.g. [9, Chapter VIII]), hence

$$\text{cd}(A) = \text{pd}_{A\mathcal{M}}(\varepsilon k) = \max\{n : \text{Ext}_{A\mathcal{M}}^n(\varepsilon k, A) \neq 0\}$$

The algebra  $R$  is homologically smooth since  $A$  is, by [42, Lemma 2.4], hence we have similarly

$$\text{cd}(R) = \text{pd}_{R\mathcal{M}_R}(R) = \max\{n : \text{Ext}_{R\mathcal{M}_R}^n(R, R \otimes R) \neq 0\}$$

Using again smoothness, we have by [42, Lemma 2.1, Lemma 2.2]

$$\text{Ext}_{R\mathcal{M}_R}^*(R, R \otimes R) \simeq \text{Ext}_{A\mathcal{M}}^n(\varepsilon k, {}_A A \otimes R) \simeq \text{Ext}_{A\mathcal{M}}^*(\varepsilon k, {}_A A) \otimes R$$

where the left  $A$ -module structure on  ${}_A A \otimes R$  is simply multiplying in  $A$  on the left. Hence  $\text{cd}(A) = \text{cd}(R)$ . That we have as well  $\text{cd}(B) = \text{cd}(R)$  follows in a symmetric manner from [39, Lemma 2.3.2.ii].  $\square$

*Remark 8.2.* The argument in the proof shows in fact that if  $A$  is a smooth Hopf algebra with bijective antipode, then for any left or right  $A$ -Galois object  $R$ , we have  $\text{cd}(A) = \text{cd}(R)$ . Notice that here, as the Weyl algebra example shows (which is a Galois object over  $k[x, y]$ ), the good dimension to consider is indeed the Hochschild cohomological dimension, and not the global dimension. Note also that a positive answer to the question of whether a Galois object has the same cohomological dimension as the Hopf algebra definitively needs some assumption on the Hopf algebra, as the example of the Taft algebra  $H_n$  shows, which admits the matrix algebra  $M_n(k)$  as a Galois object [27].

**8.2. The finite-dimensional situation.** Etingof pointed out that, as a step towards an answer to Question 1.1, one should understand the finite-dimensional case first. However, the situation is not so clear to us, and we only have the following partial result. Recall that a Hopf algebra  $A$  is said to be unimodular if there is a non-zero two-sided integral in  $A$ , i.e. there exists a non-zero  $t \in A$  such that  $ta = at = \varepsilon(a)t$  for any  $a$ . If  $A$  is cosemisimple and finite-dimensional, then  $A^*$  is unimodular.

**Theorem 8.3.** *Let  $A, B$  be finite-dimensional Hopf algebras such that  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$ . Then we have  $\text{cd}(A) = \text{cd}(B)$  if one of the following condition holds.*

- (1) The characteristic of  $k$  is zero, or satisfies  $p > d^{\frac{\varphi(d)}{2}}$ , where  $d = \dim(A)$ .
- (2)  $A^*$  is unimodular.

*Proof.* First notice that since a finite-dimensional Hopf algebra is self-injective (projective modules are injective), we have  $\text{cd}(A), \text{cd}(B) \in \{0, \infty\}$  and hence there are only few cases to consider. Moreover, for the Drinfeld double  $D(A)$ , we have  $\text{cd}(D(A)) = 0$  if and only if  $D(A)$  is semisimple, if and only if  $A$  is semisimple and cosemisimple [32, Proposition 7], and  $\text{cd}(D(A)) = \infty$  otherwise. Moreover, we have  $\text{cd}(D(A)) = \text{cd}(D(B))$  since our monoidal equivalence  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$  induces a monoidal equivalence between the monoidal centers of these categories (notice that  $\text{cd}(D(A)) = \text{cd}_{\text{GS}}(A)$ ).

If  $k$  has characteristic zero or satisfies  $p > d^{\frac{\varphi(d)}{2}}$ , then by [24, Theorem 3.3] and [16, Theorem 4.2] respectively, we have that  $A$  is semisimple if and only if  $A$  is semisimple and cosemisimple, if and only if  $\text{cd}(D(A)) = 0$ . Hence under one of these assumptions we have  $\text{cd}(A) = \text{cd}(B)$  because  $\text{cd}(D(A)) = \text{cd}(D(B))$ .

Since  $\mathcal{M}^A \simeq^{\otimes} \mathcal{M}^B$  and  $A, B$  are finite-dimensional, We have, by [35, Corollary 5.9],  $B \simeq A^\sigma$  for some Hopf 2-cocycle  $\sigma$ . At the dual level this means that  $B^* \simeq (A^*)^J$  for some Drinfeld twist  $J$ . Hence if  $\text{cd}(A) = 0$ , i.e.  $A$  is semisimple, we have that  $A^*$  is cosemisimple, and assuming that  $A^*$  is unimodular, we have that  $B^*$  is cosemisimple as well by [1, Corollary 3.6], and hence  $B$  is semisimple, so that  $\text{cd}(B) = 0$ , as needed. The assumption that  $A^*$  is unimodular is stable under Drinfeld twist since the multiplication does not change, thus  $B^*$  is unimodular as well, and hence we also have  $\text{cd}(B) = 0 \Rightarrow \text{cd}(A) = 0$ , concluding the proof.  $\square$

As we see in the proof of the previous theorem, a complete answer to Question 1.1 in the finite-dimensional case reduces to the question whether the class of finite-dimensional cosemisimple Hopf algebras is stable under Drinfeld twists. Remark 3.9 in [1] claimed that this is expected to be true, and would follow from a weak form of an important conjecture of Kaplansky saying that a finite-dimensional cosemisimple Hopf algebra is unimodular (the strong form says that a cosemisimple Hopf algebra satisfies  $S^2 = \text{id}$ ), but we are not aware of a proof since then.

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