

1 **SUPPLEMENTARY MATERIALS: A Variational Model Dedicated to Joint**  
2 **Segmentation, Registration and Atlas Generation for Shape Analysis\***

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6 **SM1. Detailed proof of Theorem 2.6.**

7 *Proof.* The proof follows the calculus of variations arguments. We first have that  $\mathcal{F}_1$  is  
8 bounded below by 0 and that  $\mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) = \frac{1}{M} \sum_{i=1}^M (\|T_i - \bar{T}_i\|_{L^2(\Omega)}^2 + \|T_i - \bar{T}\|_{L^2(\Omega)}^2) <$   
9  $+\infty$  for  $\theta_R = (1, 0, \dots, 0)$ ,  $\theta_{T_i} = (1, 0, \dots, 0)$ ,  $\forall i = 1, \dots, M$ ,  $\varphi_i = \text{Id}$ ,  $\forall i = 1, \dots, M$  and  
10 with  $\bar{T}_i = \frac{\int_{\Omega} T_i dx}{\text{meas}(\Omega)}$  and  $\bar{T} = \frac{1}{M} \sum_{i=1}^M \frac{\int_{\Omega} T_i dx}{\text{meas}(\Omega)}$ . Therefore, the infimum exists and is finite.

11 Let  $(\theta_R^k, \{\theta_{T_i}^k, \varphi_i^k\}_{i=1}^M)_k \in \mathcal{U} \times \mathcal{U}^M \times \hat{\mathcal{W}}^M$  such that  $\theta_{T_i,l}^k \circ \varphi_i^k - \theta_{R,l}^k \in BV(\Omega)$  for all  $l \in$   
12  $\{1, \dots, N\}$ , and for all  $i \in \{1, \dots, M\}$  be a minimizing sequence such that  $\lim_{k \rightarrow +\infty} \mathcal{F}_1(\theta_R^k,$   
13  $\{\theta_{T_i}^k, \varphi_i^k\}_{i=1}^M) = \inf \mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) < +\infty$ . Hence there exists  $K \in \mathbb{N}$  such that  $\forall k \in \mathbb{N}$ ,  
14  $k \geq K \Rightarrow \mathcal{F}_1(\theta_R^k, \{\theta_{T_i}^k, \varphi_i^k\}_{i=1}^M) \leq \inf \mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) + 1 < +\infty$ . From now on, we consider  
15  $k \geq K$ .

16 One thus has

$$\begin{aligned} & \frac{1}{M} \sum_{i=1}^M \left( \frac{\gamma_T}{2} \sum_{l=1}^N TV(\theta_{T_i,l}^k) + \frac{\gamma_R}{2} \sum_{l=1}^N TV(\theta_{R,l}^k) + a_1 \|\nabla \varphi_i\|_{L^4(\Omega, M_3(\mathbb{R}))}^4 - 9a_1 \text{meas}(\Omega) \right. \\ & + a_2 \|\text{Cof} \nabla \varphi_i\|_{L^4(\Omega, M_3(\mathbb{R}))}^4 - 9a_2 \text{meas}(\Omega) + \frac{a_3}{2} \|\det \nabla \varphi_i\|_{L^2(\Omega)}^2 - (a_3 + a_4) \text{meas}(\Omega) \\ & \left. + \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}}(\nabla \varphi_i) + \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta\}}((\nabla \varphi_i)^{-1}) \right) \\ & \leq \mathcal{F}_1(\theta_R^k, \{\theta_{T_i}^k, \varphi_i^k\}_{i=1}^M) \leq \inf \mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) + 1 < +\infty. \end{aligned}$$

22 We deduce from this inequality that  $\forall i = 1, \dots, M$ ,

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**SM1**

- 23 •  $(\varphi_i^k)$  is uniformly bounded according to  $k$  in  $W^{1,4}(\Omega)$  by using the generalized Poincaré's  
24 inequality and the fact that  $\varphi_i^k = \text{Id}$  on  $\partial\Omega$ .  
25 •  $(\nabla\varphi_i^k)$  is uniformly bounded according to  $k$  in  $L^\infty(\Omega, M_3(\mathbb{R}))$ .  
26 •  $((\nabla\varphi_i^k)^{-1})$  is uniformly bounded according to  $k$  in  $L^\infty(\Omega, M_3(\mathbb{R}))$ .  
27 •  $(\text{Cof}\nabla\varphi_i^k)$  is uniformly bounded according to  $k$  in  $L^4(\Omega, M_3(\mathbb{R}))$ .  
28 •  $(\det\nabla\varphi_i^k)$  is uniformly bounded according to  $k$  in  $L^2(\Omega)$ .  
29 •  $\forall l = 1, \dots, N$ ,  $(\theta_{T_i,l}^k)$  is uniformly bounded according to  $k$  in  $BV(\Omega)$  since  $(TV(\theta_{T_i,l}^k))$   
30 is uniformly bounded according to  $k$  and  $\|\theta_{T_i,l}^k\|_{L^1(\Omega)} \leq \text{meas}(\Omega)$  for all  $k \in \mathbb{N}$ .  
31 •  $\forall l = 1, \dots, N$ ,  $(\theta_{R,l}^k)$  is uniformly bounded according to  $k$  in  $BV(\Omega)$  since  $(TV(\theta_{R,l}^k))$   
32 is uniformly bounded according to  $k$  and  $\|\theta_{R,l}^k\|_{L^1(\Omega)} \leq \text{meas}(\Omega)$  for all  $k \in \mathbb{N}$ .  
33 Since we have assumed enough regularity on the boundary of  $\Omega$ , we can apply [?, Theorem  
34 136] and extract subsequences still denoted  $(\theta_{T_i,l}^k)$  and  $(\theta_{R,l}^k)$  such that

$$35 \quad \theta_{T_i,l}^k \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{T_i,l} \in BV(\Omega) \text{ in } L^1(\Omega),$$

$$36 \quad \theta_{R,l}^k \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{R,l} \in BV(\Omega) \text{ in } L^1(\Omega).$$

As the convergence is strong in  $L^1(\Omega)$ , then we can extract subsequences of  $(\theta_{T_i,l}^k)_k$  and  $(\theta_{R,l}^k)_k$  still denoted  $(\theta_{T_i,l}^k)_k$  and  $(\theta_{R,l}^k)_k$  such that  $\theta_{T_i,l}^k \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{T_i,l}$  almost everywhere in  $\Omega$  and  $\theta_{R,l}^k \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{R,l}$  almost everywhere in  $\Omega$  and thus  $\bar{\theta}_{T_i,l} \in BV(\Omega, \{0,1\})$  and  $\bar{\theta}_{R,l} \in BV(\Omega, \{0,1\})$ . Besides,  $\sum_{l=1}^N \theta_{T_i,l}^k(x) = 1 \xrightarrow[k \rightarrow +\infty]{} \sum_{l=1}^N \bar{\theta}_{T_i,l}(x) = 1$  almost everywhere in  $\Omega$  and  $\sum_{l=1}^N \theta_{R,l}^k(x) = 1 \xrightarrow[k \rightarrow +\infty]{} \sum_{l=1}^N \bar{\theta}_{R,l}(x) = 1$  so that  $\bar{\theta}_{T_i} \in \mathcal{U}$  and  $\bar{\theta}_R \in \mathcal{U}$ .

There also exist subsequences still denoted  $(\varphi_i^k)_k$  such that

$$\varphi_i^k \xrightarrow[k \rightarrow +\infty]{} \bar{\varphi}_i \text{ in } W^{1,4}(\Omega, \mathbb{R}^3).$$

- 38 By continuity of the trace operator, we deduce that  $\bar{\varphi}_i \in \text{Id} + W_0^{1,4}(\Omega, \mathbb{R}^3)$ . We can also  
39 extract subsequences still denoted  $(\text{Cof}\nabla\varphi_i^k)_k$  and  $(\det\nabla\varphi_i^k)_k$  such that

$$40 \quad \text{Cof}\nabla\varphi_i^k \xrightarrow[k \rightarrow +\infty]{} \bar{X}_i \text{ in } L^4(\Omega, M_3(\mathbb{R})),$$

$$41 \quad \det\nabla\varphi_i^k \xrightarrow[k \rightarrow +\infty]{} \bar{\delta}_i \text{ in } L^2(\Omega),$$

- 43 and by [?, Theorem VI.3.3], we deduce that  $\bar{X}_i = \text{Cof}\nabla\bar{\varphi}_i$  and  $\bar{\delta}_i = \det\nabla\bar{\varphi}_i$ .  
44 There exist subsequences still denoted  $(\nabla\varphi_i^k)$  and  $((\nabla\varphi_i^k)^{-1})$  such that

$$45 \quad (\nabla\varphi_i^k)^{-1} \xrightarrow[k \rightarrow +\infty]{} \nabla\bar{\varphi}_i \text{ in } L^\infty(\Omega, M_3(\mathbb{R})),$$

$$46 \quad \nabla\varphi_i^k \xrightarrow[k \rightarrow +\infty]{} \nabla\bar{\varphi}_i \text{ in } L^\infty(\Omega, M_3(\mathbb{R})),$$

48 by uniqueness of the weak limit in  $L^4(\Omega, M_3(\mathbb{R}))$  and the continuous embedding of  $L^\infty(\Omega, M_3(\mathbb{R}))$   
 49 into  $L^4(\Omega, M_3(\mathbb{R}))$ , and thus  $\bar{\varphi}_i \in \text{Id} + W_0^{1,\infty}(\Omega, \mathbb{R}^3)$ .  
 50 Let us now prove that  $u_i = (\nabla \bar{\varphi}_i)^{-1}$  for each  $i$ . For all  $p \in L^1(\Omega, M_3(\mathbb{R}))$ , we have that

$$\begin{aligned}
 51 \quad \int_{\Omega} ((\nabla \varphi_i^k)^{-1} \nabla \bar{\varphi}_i - \text{I}_3) : p \, dx &= \int_{\Omega} ((\nabla \varphi_i^k)^{-1} \nabla \bar{\varphi}_i - u_i \nabla \bar{\varphi}_i + u_i \nabla \bar{\varphi}_i - u_i \nabla \varphi_i^k \\
 52 \quad &+ u_i \nabla \varphi_i^k - (\nabla \varphi_i^k)^{-1} \nabla \varphi_i^k) : p \, dx, \\
 53 \quad &= \int_{\Omega} (((\nabla \varphi_i^k)^{-1} - u_i) \nabla \bar{\varphi}_i) : p \, dx + \int_{\Omega} (u_i (\nabla \bar{\varphi}_i - \nabla \varphi_i^k)) : p \, dx \\
 54 \quad &+ \int_{\Omega} ((u_i - (\nabla \varphi_i^k)^{-1}) \nabla \varphi_i^k) : p \, dx, \\
 55 \quad &= \int_{\Omega} ((\nabla \varphi_i^k)^{-1} - u_i) : p (\nabla \bar{\varphi}_i)^T \, dx + \int_{\Omega} (\nabla \bar{\varphi}_i - \nabla \varphi_i^k) : u_i^T p \, dx \\
 56 \quad &+ \int_{\Omega} (u_i - (\nabla \varphi_i^k)^{-1}) : p (\nabla \varphi_i^k)^T \, dx. \\
 57 \quad &
 \end{aligned}$$

58 Since  $\nabla \bar{\varphi}_i \in L^\infty(\Omega, M_3(\mathbb{R}))$ ,  $u_i \in L^\infty(\Omega, M_3(\mathbb{R}))$  and for all  $k \in \mathbb{N}$ ,  $\|\nabla \varphi_i^k\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq$   
 59  $\alpha$ , then we have that  $p (\nabla \bar{\varphi}_i)^T \in L^1(\Omega, M_3(\mathbb{R}))$  so  $\int_{\Omega} ((\nabla \varphi_i^k)^{-1} - u_i) : p (\nabla \bar{\varphi}_i)^T \, dx \xrightarrow{k \rightarrow +\infty}$   
 60  $0$ ,  $u_i^T p \in L^1(\Omega, M_3(\mathbb{R}))$  so  $\int_{\Omega} (\nabla \bar{\varphi}_i - \nabla \varphi_i^k) : u_i^T p \, dx \xrightarrow{k \rightarrow +\infty} 0$ , and  $|\int_{\Omega} (u_i - (\nabla \varphi_i^k)^{-1}) :$   
 61  $p (\nabla \varphi_i^k)^T \, dx| \leq \alpha |\int_{\Omega} (u_i - (\nabla \varphi_i^k)^{-1}) : p' \, dx| \xrightarrow{k \rightarrow +\infty} 0$  with  $p' \in L^1(\Omega, M_3(\mathbb{R}))$ . Therefore,  
 62 we get  $(\nabla \varphi_i^k)^{-1} \nabla \bar{\varphi}_i \xrightarrow{k \rightarrow +\infty} \text{I}_3$  in  $L^\infty(\Omega, M_3(\mathbb{R}))$ . But we know that  $(\nabla \varphi_i^k)^{-1} \xrightarrow{k \rightarrow +\infty} u_i$  in  
 63  $L^\infty(\Omega, M_3(\mathbb{R}))$ , so that  $(\nabla \varphi_i^k)^{-1} \nabla \bar{\varphi}_i \xrightarrow{k \rightarrow +\infty} u_i \nabla \bar{\varphi}_i = \text{I}_3$  in  $L^\infty(\Omega, M_3(\mathbb{R}))$  and  $u_i = (\nabla \bar{\varphi}_i)^{-1}$ .  
 64 Since  $\theta_{T_i,l}^k \xrightarrow{k \rightarrow +\infty} \bar{\theta}_{T_i,l}$  almost everywhere in  $\Omega$ , then  $T_i \theta_{T_i,l}^k \xrightarrow{k \rightarrow +\infty} T_i \bar{\theta}_{T_i,l}$  almost everywhere  
 65 in  $\Omega$ . Furthermore,  $|\theta_{T_i,l}^k| \leq 1 \in L^1(\Omega)$  almost everywhere in  $\Omega$  as  $\Omega$  is bounded, and  
 66  $|T_i \theta_{T_i,l}^k| \leq \|T_i\|_{L^\infty(\Omega)} \in L^1(\Omega)$  almost everywhere in  $\Omega$ . We can thus apply the dominated  
 67 convergence theorem and get

$$\begin{aligned}
 68 \quad \int_{\Omega} \theta_{T_i,l}^k \, dx &\xrightarrow{k \rightarrow +\infty} \int_{\Omega} \bar{\theta}_{T_i,l} \, dx, \\
 69 \quad \int_{\Omega} \theta_{T_i,l}^k T_i \, dx &\xrightarrow{k \rightarrow +\infty} \int_{\Omega} \bar{\theta}_{T_i,l} T_i \, dx, \\
 70 \quad c_{T_i,l}^k &\xrightarrow{k \rightarrow +\infty} \bar{c}_{T_i,l}. \\
 71 \quad &
 \end{aligned}$$

72 As  $T_i \in W^{1,\infty}(\Omega)$ , we denote  $\kappa_i$  the Lipschitz constant related to  $T_i$ . Then the chain rule  
 73 applies and we have that  $T_i \circ \varphi_i^k \in W^{1,\infty}(\Omega)$ ,  $T_i \circ \bar{\varphi}_i \in W^{1,\infty}(\Omega)$  and  $\|T_i \circ \varphi_i^k - T_i \circ \bar{\varphi}_i\|_{C^0(\Omega)} \leq$   
 74  $\kappa_i \|\varphi_i^k - \bar{\varphi}_i\|_{C^0(\Omega)} \xrightarrow{k \rightarrow +\infty} 0$  using the Sobolev compact embedding  $W^{1,\infty}(\Omega) \hookrightarrow C^0(\Omega)$ . Therefore,  
 75 up to a subsequence,  $\theta_{R,l}^k T_i \circ \varphi_i^k \xrightarrow{k \rightarrow +\infty} \bar{\theta}_{R,l} T_i \circ \bar{\varphi}_i$  almost everywhere in  $\Omega$ ,  $\theta_{R,l}^k \xrightarrow{k \rightarrow +\infty} \bar{\theta}_{R,l}$   
 76 almost everywhere in  $\Omega$  with  $|\theta_{R,l}^k T_i \circ \varphi_i^k| \leq \|T_i\|_{L^\infty(\Omega)} \in L^1(\Omega)$  and  $|\theta_{R,l}^k| \leq 1 \in L^1(\Omega)$  almost

77 everywhere in  $\Omega$ . Thus the dominated convergence theorem applies and we get

$$\begin{aligned}
 78 \quad & \frac{1}{M} \sum_{i=1}^M \int_{\Omega} \theta_{R,l}^k T_i \circ \varphi_i^k dx \xrightarrow{k \rightarrow +\infty} \frac{1}{M} \sum_{i=1}^M \int_{\Omega} \bar{\theta}_{R,l} T_i \circ \bar{\varphi}_i dx, \\
 79 \quad & \int_{\Omega} \theta_{R,l}^k dx \xrightarrow{k \rightarrow +\infty} \int_{\Omega} \bar{\theta}_{R,l} dx, \\
 80 \quad & c_{R,l}^k \xrightarrow{k \rightarrow +\infty} \bar{c}_{R,l}. \\
 81
 \end{aligned}$$

82 We also know that  $W_{Op}$  is continuous and convex. If  $\psi_n \xrightarrow{n \rightarrow +\infty} \psi$  in  $W^{1,4}(\Omega, \mathbb{R}^3)$ , then  
 83  $\nabla \psi_n \xrightarrow{n \rightarrow +\infty} \nabla \psi$  in  $L^4(\Omega, M_3(\mathbb{R}))$  and one can extract a subsequence still denoted  $(\nabla \psi_n)$  such  
 84 that  $\nabla \psi_n \xrightarrow{n \rightarrow +\infty} \nabla \psi$  almost everywhere in  $\Omega$ . If  $\alpha_n \xrightarrow{n \rightarrow +\infty} \alpha$  in  $L^4(\Omega, M_3(\mathbb{R}))$ , then one can  
 85 extract a subsequence still denoted  $(\alpha_n)$  such that  $\alpha_n \xrightarrow{n \rightarrow +\infty} \alpha$  almost everywhere in  $\Omega$ . If  
 86  $\delta_n \xrightarrow{n \rightarrow +\infty} \delta$  in  $L^2(\Omega)$  then one can extract a subsequence still denoted  $(\delta_n)$  such that  $\delta_n \xrightarrow{n \rightarrow +\infty} \delta$   
 87 almost everywhere in  $\Omega$ . Then by continuity of  $W_{Op}$ , we get that  $W_{Op}(\nabla \psi_n(x), \alpha_n(x), \delta_n(x))$   
 88  $\xrightarrow{n \rightarrow +\infty} W_{Op}(\nabla \psi(x), \alpha(x), \delta(x))$  almost everywhere in  $\Omega$ . We then apply Fatou's lemma and get  
 89  $\liminf_{n \rightarrow +\infty} \int_{\Omega} W_{Op}(\nabla \psi_n(x), \alpha_n(x), \delta_n(x)) dx \geq \int_{\Omega} W_{Op}(\nabla \psi(x), \alpha(x), \delta(x)) dx$ . As  $W_{Op}$  is convex,  
 90 so is  $\int_{\Omega} W_{Op}(\xi(x), \alpha(x), \delta(x)) dx$  and we can apply [?, Corollaire III.8]. Therefore  $\int_{\Omega} W_{Op}(\xi(x),$   
 91  $\alpha(x), \delta(x)) dx$  is also weakly lower semicontinuous in  $L^4(\Omega, M_3(\mathbb{R})) \times L^4(\Omega, M_3(\mathbb{R})) \times L^2(\Omega)$ .  
 92 We then deduce that

$$93 \quad +\infty > \liminf_{k \rightarrow +\infty} \int_{\Omega} W_{Op}(\nabla \varphi_i^k, \text{Cof} \nabla \varphi_i^k, \det \nabla \varphi_i^k) dx \geq \int_{\Omega} W_{Op}(\nabla \bar{\varphi}_i, \text{Cof} \nabla \bar{\varphi}_i, \det \nabla \bar{\varphi}_i) dx.$$

95 Let  $\bar{q} = q \frac{1+s}{q+s} = 4 \frac{1+10}{4+10} = \frac{44}{14} > 3$ . We then compute

$$\begin{aligned}
 96 \quad & \int_{\Omega} \|(\nabla \bar{\varphi}_i)^{-1}\|_F^{\bar{q}} \det \nabla \bar{\varphi}_i dx = \int_{\Omega} \left\| \frac{\text{Cof} \nabla \bar{\varphi}_i^T}{\det \nabla \bar{\varphi}_i} \right\|_F^{\bar{q}} \det \nabla \bar{\varphi}_i dx, \\
 97 \quad & = \int_{\Omega} \|\text{Cof} \nabla \bar{\varphi}_i\|_F^{\bar{q}} (\det \nabla \bar{\varphi}_i)^{1-\bar{q}} dx, \\
 98 \quad & \leq \|\text{Cof} \nabla \bar{\varphi}_i\|_{L^4(\Omega)}^{\bar{q}} \|(\det \nabla \bar{\varphi}_i)^{-1}\|_{L^{10}(\Omega)}^{1-\bar{q}} < +\infty, \\
 99
 \end{aligned}$$

100 from Hölder's inequality and the finiteness of  $\int_{\Omega} W_{Op}(\nabla \bar{\varphi}_i, \text{Cof} \nabla \bar{\varphi}_i, \det \nabla \bar{\varphi}_i) dx$ . Thus Ball's  
 101 conditions are satisfied [?, Theorems 1 and 2] and  $\bar{\varphi}_i$  are bi-Hölder homeomorphisms.

102 By the weak-\* lower semicontinuity of  $\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))}$ , we have that  $\|\nabla \bar{\varphi}_i\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq$   
 103  $\liminf_{k \rightarrow +\infty} \|\nabla \varphi_i^k\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha$  and  $\|(\nabla \bar{\varphi}_i)^{-1}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \liminf_{k \rightarrow +\infty} \|(\nabla \varphi_i^k)^{-1}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta$

104 so that  $\mathbf{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}}(\nabla \bar{\varphi}_i) = 0 \leq \liminf_{k \rightarrow +\infty} \mathbf{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}}(\nabla \varphi_i^k)$  and  $\mathbf{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta\}}$   
 105  $((\nabla \bar{\varphi}_i)^{-1}) = 0 \leq \liminf_{k \rightarrow +\infty} \mathbf{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta\}}((\nabla \varphi_i^k)^{-1})$ . The same reasoning applies for each

106  $i = 1, \dots, M$ .

107 We also have

$$108 \quad \|\nabla(\bar{\varphi}_i^{-1})\|_{L^\infty(\Omega, M_3(\mathbb{R}))} = \|(\nabla \bar{\varphi}_i)^{-1}(\bar{\varphi}_i^{-1})\|_{L^\infty(\Omega, M_3(\mathbb{R}))} = \|(\nabla \bar{\varphi}_i)^{-1}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta < +\infty,$$

110 since  $\bar{\varphi}_i$  is an homeomorphism from  $\Omega$  to  $\text{Id}(\Omega) = \Omega$ , according to Ball's results, meaning that  
 111  $\bar{\varphi}_i^{-1}$  is a one to one mapping from  $\Omega$  to  $\Omega$ . So, using Poincaré-Wirtinger inequality, we deduce  
 112 that  $\bar{\varphi}_i$  is a bi-Lipschitz homeomorphism. The same reasoning applies for the sequence  $(\varphi_i^k)_k$ .

113 We now have that  $\forall i \in \{1, \dots, M\}, \forall l \in \{1, \dots, N\}$ ,

- 114 •  $\theta_{T_i,l}^k \xrightarrow{k \rightarrow +\infty} \bar{\theta}_{T_i,l}$  in  $L^1(\Omega)$  with  $\bar{\theta}_{T_i,l} \in BV(\Omega, \{0, 1\})$ .
- 115 •  $\varphi_i^k \xrightarrow{k \rightarrow +\infty} \bar{\varphi}_i$  in  $W^{1,4}(\Omega, \mathbb{R}^3)$  with  $(\varphi_i^k)$  bi-Lipschitz homeomorphisms ( $\varphi_i^k \in \mathcal{C}^{0,\alpha}(\Omega, \mathbb{R}^3)$ ,  
 116  $(\varphi_i^k)^{-1} \in \mathcal{C}^{0,\alpha'}(\Omega, \mathbb{R}^3)$ ),  $\bar{\varphi}_i$  bi-Lipschitz homeomorphisms ( $\bar{\varphi}_i \in \mathcal{C}^{0,\alpha}(\Omega, \mathbb{R}^3)$ ,  $(\bar{\varphi}_i)^{-1} \in$   
 117  $\mathcal{C}^{0,\alpha'}(\Omega, \mathbb{R}^3)$ ) and  $\varphi_i^k \xrightarrow{k \rightarrow +\infty} \bar{\varphi}_i$  in  $\mathcal{C}^{0,\alpha}(\Omega, \mathbb{R}^3)$ ,  $\alpha < \frac{1}{4}$  and  $\alpha' < 1 - \frac{3}{4}$  using the Sobolev  
 118 embeddings properties.

119 We then can prove that  $\varphi_i^k \circ \bar{\varphi}_i^{-1} \xrightarrow{k \rightarrow +\infty} \text{Id}$  in  $\mathcal{C}^{0,\alpha''}(\Omega, \mathbb{R}^3)$  with  $\alpha'' = \alpha\alpha'$  for all  $i \in \{1, \dots, M\}$   
 120 and all  $l \in \{1, \dots, N\}$ . Indeed one has

$$\begin{aligned}
 121 \quad & \sup_{x \in \Omega} |\varphi_i^k \circ \bar{\varphi}_i^{-1}(x) - \bar{\varphi}_i \circ \bar{\varphi}_i^{-1}(x)| + \sup_{x \neq y} \frac{|\varphi_i^k \circ \bar{\varphi}_i^{-1}(x) - \varphi_i^k \circ \bar{\varphi}_i^{-1}(y) - \bar{\varphi}_i \circ \bar{\varphi}_i^{-1}(x) + \bar{\varphi}_i \circ \bar{\varphi}_i^{-1}(y)|}{|x - y|^{\alpha''}} \\
 122 \quad & \leq \sup_{x \in \Omega} |\varphi_i^k(x) - \bar{\varphi}_i(x)| + \sup_{x \neq y} \frac{|\varphi_i^k \circ \bar{\varphi}_i^{-1}(x) - \varphi_i^k \circ \bar{\varphi}_i^{-1}(y) - \bar{\varphi}_i \circ \bar{\varphi}_i^{-1}(x) + \bar{\varphi}_i \circ \bar{\varphi}_i^{-1}(y)|}{|\bar{\varphi}_i^{-1}(x) - \bar{\varphi}_i^{-1}(y)|^\alpha} \\
 123 \quad & \cdot \frac{|\bar{\varphi}_i^{-1}(y) - \bar{\varphi}_i^{-1}(x)|^\alpha}{|x - y|^{\alpha''}} \text{ as } \bar{\varphi}_i^{-1} \text{ is a one to one mapping from } \Omega \text{ to } \Omega, \\
 124 \quad & \leq \sup_{x \in \Omega} |\varphi_i^k(x) - \bar{\varphi}_i(x)| + \sup_{x \neq y} \frac{|\varphi_i^k(x) - \varphi_i^k(y) - \bar{\varphi}_i(x) + \bar{\varphi}_i(y)|}{|x - y|^\alpha} \cdot \sup_{x \neq y} \frac{|\bar{\varphi}_i^{-1}(y) - \bar{\varphi}_i^{-1}(x)|^\alpha}{|x - y|^{\alpha''}}, \\
 125 \quad & \leq (1 + \|\bar{\varphi}_i^{-1}\|_{\mathcal{C}^{0,\alpha'}(\Omega, \mathbb{R}^3)}^\alpha) \|\varphi_i^k - \bar{\varphi}_i\|_{\mathcal{C}^{0,\alpha}(\Omega, \mathbb{R}^3)} \xrightarrow{k \rightarrow +\infty} 0. \quad \blacksquare
 \end{aligned}$$

127 Let us now prove that  $\theta_{T_i,l}^k \circ \varphi_i^k \xrightarrow{k \rightarrow +\infty} \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i$  in  $L^1(\Omega)$  based on the proof of [?, Lemma 5.1].

128 Since  $\varphi_i^k \circ \bar{\varphi}_i^{-1} \xrightarrow{k \rightarrow +\infty} \text{Id}$  in  $\mathcal{C}^{0,\alpha''}(\Omega)$  and so uniformly, we can write

$$129 \quad \forall \varepsilon > 0, \exists K \in \mathbb{N}, \forall k \in \mathbb{N}, k \geq K \Rightarrow \sup_{x \in \Omega} |\varphi_i^k \circ \bar{\varphi}_i^{-1}(x) - x| \leq \varepsilon.$$

131 Also, given  $\eta > 0$ , by Egorov's theorem, there exists a set  $S_\eta$  with Lebesgue measure  
 132  $\text{meas}(S_\eta) \leq \eta$  such that  $\theta_{T_i,l}^k \xrightarrow{k \rightarrow +\infty} \bar{\theta}_{T_i,l}$  uniformly in  $\Omega \setminus S_\eta$ .

133 Now, we choose  $K$  large enough such that  $\forall k \in \mathbb{N}, k \geq K$  then

$$\begin{aligned}
 134 \quad & |\varphi_i^k \circ \bar{\varphi}_i(x) - x| \leq \eta \text{ in } \Omega, \\
 135 \quad & |\theta_{T_i,l}^k(x) - \bar{\theta}_{T_i,l}(x)| \leq \eta \text{ in } \Omega \setminus S_\eta.
 \end{aligned}$$

137 We also denote  $\mathcal{O} = \{x \in \Omega \mid \bar{\theta}_{T_i,l}(x) = 1\}$ ,  $(\mathcal{O} \setminus S_\eta)_\eta = \{x \in \mathcal{O} \setminus S_\eta \mid \text{dist}(x, \partial(\mathcal{O} \setminus S_\eta)) > \eta\}$ ,  
 138  $\mathcal{O}' = \{x \in \Omega \mid \text{dist}(x, \partial\mathcal{O}) \leq \eta\}$ . Then  $\forall x \in (\mathcal{O} \setminus S_\eta)_\eta$ ,  $\varphi_i^k \circ \bar{\varphi}_i^{-1}(x) \in \mathcal{O} \setminus S_\eta$  and  $\forall x \in$   
 139  $(\Omega \setminus (S_\eta \cup \mathcal{O}))_\eta$ ,  $\varphi_i^k \circ \bar{\varphi}_i^{-1}(x) \in \Omega \setminus (S_\eta \cup \mathcal{O})$ . We then have

$$140 \quad \int_{\Omega} |\theta_{T_i,l}^k \circ \varphi_i^k - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i| dx = \int_{\Omega} \frac{|\theta_{T_i,l}^k \circ \varphi_i^k \circ \bar{\varphi}_i^{-1}(x) - \bar{\theta}_{T_i,l}(x)|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx,$$

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$$\begin{aligned}
 &\leq \int_{(\mathcal{O} \setminus S_\eta)_\eta} \frac{|\theta_{T_i,l}^k \circ \varphi_i^k \circ \bar{\varphi}_i^{-1}(x) - 1|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx + \int_{(\Omega \setminus (\mathcal{O} \cup S_\eta))_\eta} \frac{|\theta_{T_i,l}^k \circ \varphi_i^k \circ \bar{\varphi}_i^{-1}(x)|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx \\
 &+ \int_{S_\eta \cup S_\eta^\eta \cup \mathcal{O} \cup \Omega^\eta} \frac{|\theta_{T_i,l}^k \circ \varphi_i^k \circ \bar{\varphi}_i^{-1}(x) - \bar{\theta}_{T_i,l}(x)|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx, \\
 &\leq \eta \int_{(\mathcal{O} \setminus S_\eta)_\eta} \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx + \eta \int_{(\Omega \setminus (\mathcal{O} \cup S_\eta))_\eta} \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx \\
 &+ 2 \int_{S_\eta \cup S_\eta^\eta \cup \mathcal{O} \cup \Omega^\eta} \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx, \\
 &\leq (\eta \text{meas}(\Omega)^{\frac{10}{11}} + \eta \text{meas}(\Omega)^{\frac{10}{11}} + 2(\text{meas}(S_\eta) + \text{meas}(S_\eta^\eta) + \text{meas}(\mathcal{O}^\eta) + \text{meas}(\Omega^\eta))^{\frac{10}{11}}) \\
 &\left( \int_{\Omega} (\det \nabla \bar{\varphi}_i(x))^{-10} dx \right)^{\frac{1}{11}}, \\
 &\leq (2\eta \text{meas}(\Omega)^{\frac{10}{11}} + 2\left(\frac{4\pi}{3}\eta^3(\text{meas}(\partial S_\eta) + \text{meas}(\partial \mathcal{O}) + \text{meas}(\partial \Omega)) + \eta\right)^{\frac{10}{11}}) \|\det \nabla \bar{\varphi}_i(x)^{-1}\|_{L^{10}(\Omega)}^{\frac{10}{11}},
 \end{aligned}$$

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152 with  $\|\det \nabla \bar{\varphi}_i(x)^{-1}\|_{L^{10}(\Omega)}^{\frac{10}{11}} < +\infty$ . By letting  $\eta$  tend to 0, we obtain that  $\theta_{T_i,l}^k \circ \varphi_i^k \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i$  in  $L^1(\Omega)$ , for all  $i \in \{1, \dots, M\}$  and for all  $l \in \{1, \dots, N\}$ .

153 Since  $\theta_{T_i,l}^k \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{T_i,l}$  in  $L^1(\Omega)$  and  $\theta_{R,l}^k \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{R,l}$  in  $L^1(\Omega)$  and  $\theta_{T_i,l}^k \circ \varphi_i^k - \theta_{R,l}^k \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \bar{\theta}_{R,l}$  for all  $l \in \{1, \dots, N\}$  and for all  $i \in \{1, \dots, M\}$ , we deduce by the lower

155 semicontinuity of the total variation that  $\frac{1}{M} \sum_{i=1}^M \sum_{l=1}^N \frac{\gamma_T}{2} TV(\bar{\theta}_{T_i,l}) + \frac{\gamma_R}{2} TV(\bar{\theta}_{R,l}) + \frac{\lambda}{2} TV(\bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \bar{\theta}_{R,l}) \leq \liminf_{k \rightarrow +\infty} \frac{1}{M} \sum_{i=1}^M \sum_{l=1}^N \frac{\gamma_T}{2} TV(\theta_{T_i,l}^k) + \frac{\gamma_R}{2} TV(\theta_{R,l}^k) + \frac{\lambda}{2} TV(\theta_{T_i,l}^k \circ \varphi_i^k - \theta_{R,l}^k)$ .

157 Since from what precedes  $\theta_{T_i,l}^k (c_{T_i,l}^k - T_i)^2 \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{T_i,l} (\bar{c}_{T_i,l} - T_i)^2$  almost everywhere in  $\Omega$  and  $\theta_{R,l}^k (c_{R,l}^k - T_i \circ \varphi_i^k)^2 \xrightarrow[k \rightarrow +\infty]{} \bar{\theta}_{R,l} (\bar{c}_{R,l} - T_i \circ \bar{\varphi}_i)^2$  almost everywhere in  $\Omega$  with  $|\theta_{T_i,l}^k (c_{T_i,l}^k - T_i)^2| \leq 4\|T_i\|_{L^\infty(\Omega)}^2$  and  $|\theta_{R,l}^k (c_{R,l}^k - T_i \circ \varphi_i^k)^2| \leq 4\sum_{i=1}^M \|T_i\|_{L^\infty(\Omega)}^2$ , we can apply the dominated convergence

158 theorem and get  $\int_{\Omega} \theta_{T_i,l}^k (c_{T_i,l}^k - T_i)^2 dx \xrightarrow[k \rightarrow +\infty]{} \int_{\Omega} \bar{\theta}_{T_i,l} (\bar{c}_{T_i,l} - T_i)^2 dx$  and  $\int_{\Omega} \theta_{R,l}^k (c_{R,l}^k - T_i \circ \varphi_i^k) dx \xrightarrow[k \rightarrow +\infty]{} \int_{\Omega} \bar{\theta}_{R,l} (\bar{c}_{R,l} - T_i \circ \bar{\varphi}_i)^2 dx$ .

159 By combining all the results, we get that

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$$\mathcal{F}_1(\bar{\theta}_R, \{\bar{\theta}_{T_i}, \bar{\varphi}_i\}_{i=1}^M) \leq \liminf_{k \rightarrow +\infty} \mathcal{F}_1(\theta_R^k, \{\theta_{T_i}^k, \varphi_i^k\}_{i=1}^M) < +\infty.$$

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166 By finiteness of the functional, we deduce that  $\bar{\theta}_R \in \mathcal{U}$ ,  $\bar{\theta}_{T_i} \in \mathcal{U}$  for all  $i \in \{1, \dots, M\}$  and  $\bar{\varphi}_i \in \hat{\mathcal{W}}$  for all  $i \in \{1, \dots, M\}$  with  $\bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \bar{\theta}_{R,l} \in BV(\Omega)$  for all  $l \in \{1, \dots, N\}$  and for all  $i \in \{1, \dots, M\}$  and there exists a minimizer to our problem. ■

169 **SM2. Detailed proof of Theorem 3.1.**

170 *Proof.* Let  $(\gamma_j)_{j \geq 0}$  be an increasing sequence of positive real numbers such that  $\lim_{j \rightarrow +\infty} \gamma_j =$   
 171  $+\infty$ . Let  $(\{\varphi_{i,k_j}, \theta_{T_i,k_j}, V_{i,k_j}, W_{i,k_j}\}_{i=1}^M, (\theta_{\tilde{T}_i,l,k_j})_{i=1,\dots,M, l=1,\dots,N}, \theta_{R,k_j})$  be a minimizing sequence of this  
 172 problem  $\mathcal{F}_{1,\gamma}$  for  $\gamma = \gamma_j$ . By definition, there exists  $((\theta_{R,n}, \{\theta_{T_i,n}, \varphi_{i,n}\}_{i=1}^M) \in \mathcal{U} \times \mathcal{U}^M \times \hat{\mathcal{W}}^M$   
 173 such that  $\theta_{T_i,l,n} \circ \varphi_{i,n} - \theta_{R,l,n} \in BV(\Omega, \{-1, 0, 1\})$  for any  $l \in \{1, \dots, N\}$  and all  $i = 1, \dots, M,$   
 174 and

$$\begin{aligned} 175 & \mathcal{F}_1(\theta_{R,n}, \{\varphi_{i,n}, \theta_{T_i,n}\}_{i=1}^M) \\ 176 & = \mathcal{F}_{1,\gamma_j}(\{\varphi_{i,n}, \theta_{T_i,n}, \nabla \varphi_{i,n}, (\nabla \varphi_{i,n})^{-1}\}_{i=1}^M, (\theta_{T_i,l,n} \circ \varphi_{i,n} - \theta_{R,l,n})_{i=1,\dots,M, l=1,\dots,N}, \theta_{R,n}) \\ 177 & \leq \inf \mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) + \frac{1}{n} < +\infty. \end{aligned}$$

179 Then by definition of a minimizing sequence we get that

$$\begin{aligned} 180 & \forall \varepsilon > 0, \exists N(\varepsilon, \gamma_j) \in \mathbb{N}, \forall k \in \mathbb{N}, k \geq N(\varepsilon, \gamma_j) \Rightarrow \mathcal{F}_{1,\gamma_j}(\{\varphi_{i,k_j}, \theta_{T_i,k_j}, V_{i,k_j}, W_{i,k_j}\}_{i=1}^M, \\ 181 & (\theta_{\tilde{T}_i,l,k_j})_{i=1,\dots,M, l=1,\dots,N}, \theta_{R,k_j}) \leq \inf \mathcal{F}_{1,\gamma_j}(\{\varphi_i, \theta_{T_i}, V_i, W_i\}_{i=1}^M, (\theta_{\tilde{T}_i,l})_{i=1,\dots,M, l=1,\dots,N}, \theta_R) + \varepsilon, \\ 182 & \leq \mathcal{F}_1(\theta_{R,n}, \{\theta_{T_i,n}, \varphi_{i,n}\}_{i=1}^M) + \varepsilon, \\ 183 & = \mathcal{F}_{1,\gamma_j}(\{\varphi_{i,n}, \theta_{T_i,n}, \nabla \varphi_{i,n}, (\nabla \varphi_{i,n})^{-1}\}_{i=1}^M, (\theta_{T_i,l,n} \circ \varphi_{i,n} - \theta_{R,l,n})_{i=1,\dots,M, l=1,\dots,N}, \theta_{R,n}) + \varepsilon, \\ 184 & \leq \inf \mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) + \frac{1}{n} + \varepsilon < +\infty. \end{aligned}$$

186 Let us set in particular  $\varepsilon = \frac{1}{\gamma_j}$ , then there exists  $N_j \in \mathbb{N}$  such that for all  $k \in \mathbb{N}, k \geq N_j \Rightarrow$

$$\begin{aligned} 187 & \mathcal{F}_{1,\gamma_j}(\{\varphi_{i,k_j}, \theta_{T_i,k_j}, V_{i,k_j}, W_{i,k_j}\}_{i=1}^M, (\theta_{\tilde{T}_i,l,k_j})_{i=1,\dots,M, l=1,\dots,N}, \theta_{R,k_j}) \leq \inf \mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) + \frac{1}{n} + \frac{1}{\gamma_j} \\ 188 & \leq \inf \mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) + \frac{1}{n} + \frac{1}{\gamma_0} \\ 189 & < +\infty. \end{aligned}$$

191 So according to the previous inequality and by setting, for the sake of simplicity, for all  
 192  $i = 1, \dots, M,$

$$\begin{aligned} 193 & \varphi_{i,j} = \varphi_{i,N_j,j}, \\ 194 & V_{i,j} = V_{i,N_j,j}, \\ 195 & \theta_{T_i,j} = \theta_{T_i,N_j,j}, \\ 196 & \theta_{R,j} = \theta_{R,N_j,j}, \\ 197 & \theta_{\tilde{T}_i,l,j} = \theta_{\tilde{T}_i,l,N_j,j}, \\ 198 & W_{i,j} = W_{i,N_j,j}, \end{aligned}$$

200 we get that

- 201 •  $(\varphi_{i,j})$  is uniformly bounded according to  $j$  in  $W^{1,4}(\Omega)$  by using the generalized Poincaré's  
202 inequality since  $(V_{i,j})$  is uniformly bounded according to  $j$  in  $L^4(\Omega, M_3(\mathbb{R}))$ , for all  
203  $i = 1, \dots, M$ .
- 204 •  $(V_{i,j})$  is uniformly bounded according to  $j$  in  $L^4(\Omega, M_3(\mathbb{R}))$  and in  $L^\infty(\Omega, M_3(\mathbb{R}))$ , for  
205 all  $i = 1, \dots, M$ .
- 206 •  $(\theta_{T_i,l,j})$  is uniformly bounded according to  $j$  in  $BV(\Omega)$  since  $\|\theta_{T_i,l,j}\|_{L^1(\Omega)} \leq \text{meas}(\Omega) <$   
207  $+\infty$  and  $(TV(\theta_{T_i,l,j}))$  is uniformly bounded according to  $j$  for all  $l \in \{1, \dots, N\}$ , and  
208 for all  $i = 1, \dots, M$ .
- 209 •  $(\theta_{R,l,j})$  is uniformly bounded according to  $j$  in  $BV(\Omega)$  since  $\|\theta_{R,l,j}\|_{L^1(\Omega)} \leq \text{meas}(\Omega) <$   
210  $+\infty$  and  $(TV(\theta_{R,l,j}))$  is uniformly bounded according to  $j$  for all  $l \in \{1, \dots, N\}$ .
- 211 •  $(\theta_{\tilde{T}_i,l,j})$  is uniformly bounded according to  $j$  in  $BV(\Omega)$  since  $\forall l \in \{1, \dots, N\}$  and  
212 all  $i = 1, \dots, M$ ,  $\|\theta_{\tilde{T}_i,l,j}\|_{L^1(\Omega)} \leq \text{meas}(\Omega)$  and  $(TV(\theta_{\tilde{T}_i,l,j}))$  is uniformly bounded  
213 according to  $j$ .
- 214 •  $(\text{Cof}V_{i,j})$  is uniformly bounded according to  $j$  in  $L^4(\Omega, M_3(\mathbb{R}))$  for all  $i = 1, \dots, M$ .
- 215 •  $(\det V_{i,j})$  is uniformly bounded according to  $j$  in  $L^2(\Omega)$  for all  $i = 1, \dots, M$ .
- 216 •  $(W_{i,j})$  is uniformly bounded according to  $j$  in  $L^\infty(\Omega, M_3(\mathbb{R}))$  and in  $L^2(\Omega, M_3(\mathbb{R}))$  since  
217  $(V_{i,j})$  is uniformly bounded according to  $j$  in  $L^4(\Omega, M_3(\mathbb{R}))$  and thus in  $L^2(\Omega, M_3(\mathbb{R}))$ ,  
218 for all  $i = 1, \dots, M$ .

219 So, up to subsequences, we get that

- 220  $\varphi_{i,j} \rightharpoonup \bar{\varphi}_i$  in  $W^{1,4}(\Omega, \mathbb{R}^3)$ , for all  $i = 1, \dots, M$ ,
- 221  $V_{i,j} \rightharpoonup \bar{V}_i$  in  $L^4(\Omega, M_3(\mathbb{R}))$ , and  $V_{i,j} \xrightarrow{*} \bar{V}_i$  in  $L^\infty(\Omega, M_3(\mathbb{R}))$ , for all  $i = 1, \dots, M$ ,
- 222 by uniqueness of the weak limit in  $L^4(\Omega, M_3(\mathbb{R}))$  and by the continuous embedding of  
223  $L^\infty(\Omega, M_3(\mathbb{R})) \subset L^4(\Omega, M_3(\mathbb{R}))$ ,
- 224  $\text{Cof}V_{i,j} \rightharpoonup \bar{X}_i$  in  $L^4(\Omega, M_3(\mathbb{R}))$ , for all  $i = 1, \dots, M$ ,
- 225  $\det V_{i,j} \rightharpoonup \bar{\delta}_i$  in  $L^2(\Omega)$ , for all  $i = 1, \dots, M$ ,
- 226  $W_{i,j} \rightharpoonup \bar{W}_i$  in  $L^2(\Omega, M_3(\mathbb{R}))$ , and  $W_{i,j} \xrightarrow{*} \bar{W}_i$  in  $L^\infty(\Omega, M_3(\mathbb{R}))$ , for all  $i = 1, \dots, M$ ,
- 227 by uniqueness of the weak limit in  $L^2(\Omega, M_3(\mathbb{R}))$  and by the continuous embedding of  
228  $L^\infty(\Omega, M_3(\mathbb{R})) \subset L^2(\Omega, M_3(\mathbb{R}))$ ,
- 229  $\theta_{T_i,l,j} \longrightarrow \bar{\theta}_{T_i,l}$  in  $L^1(\Omega)$ , with  $\bar{\theta}_{T_i,l} \in BV(\Omega)$ ,  $\forall l \in \{1, \dots, N\}$ ,  $\forall i \in \{1, \dots, M\}$ ,
- 230  $\theta_{R,l,j} \longrightarrow \bar{\theta}_{R,l}$  in  $L^1(\Omega)$ , with  $\bar{\theta}_{R,l} \in BV(\Omega)$ ,  $\forall l \in \{1, \dots, N\}$ ,
- 231  $\theta_{\tilde{T}_i,l,j} \longrightarrow \bar{\theta}_{\tilde{T}_i,l}$  in  $L^1(\Omega)$ , with  $\bar{\theta}_{\tilde{T}_i,l} \in BV(\Omega)$ ,  $\forall l \in \{1, \dots, N\}$ ,  $\forall i \in \{1, \dots, M\}$ .
- 232

Since the convergence is strong in  $L^1(\Omega)$  then it is also almost everywhere up to subsequences and we deduce that  $\bar{\theta}_{T_i,l} \in BV(\Omega, \{0, 1\})$ ,  $\bar{\theta}_{R,l} \in BV(\Omega, \{0, 1\})$  and  $\bar{\theta}_{\tilde{T}_i,l} \in BV(\Omega, \{-1, 0, 1\})$  for all  $l \in \{1, \dots, N\}$  and all  $i \in \{1, \dots, M\}$  and  $\sum_{l=1}^M \bar{\theta}_{R,l}(x) = 1$  almost everywhere in  $\Omega$ ,



$\sum_{l=1}^M \bar{\theta}_{T_i, l}(x) = 1$  almost everywhere in  $\Omega$  for all  $i = 1, \dots, M$ , so that  $\bar{\theta}_R \in \mathcal{U}$  and  $\bar{\theta}_{T_i} \in \mathcal{U}$  for each  $i = 1, \dots, M$ .

Let us set  $z_{i,j,l} = \theta_{\bar{T}_i, l, j} - (\theta_{T_i, l, j} \circ \varphi_{i,j} - \theta_{R, l, j}) \xrightarrow{j \rightarrow +\infty} 0$  in  $L^1(\Omega)$  for all  $l \in \{1, \dots, N\}$  and all  $i \in \{1, \dots, M\}$ ,  $w_{i,j} = V_{i,j} - \nabla \varphi_{i,j} \xrightarrow{j \rightarrow +\infty} 0$  in  $L^4(\Omega, M_3(\mathbb{R}))$  for all  $i = 1, \dots, M$ , and  $y_{i,j} = W_{i,j} - V_{i,j}^{-1} \xrightarrow{j \rightarrow +\infty} 0$  in  $L^2(\Omega, M_3(\mathbb{R}))$ , for all  $i = 1, \dots, M$ .

Since strong convergence implies weak convergence, we have that

$$\forall i = 1, \dots, M, \forall \Phi \in L^{\frac{4}{3}}(\Omega, M_3(\mathbb{R})), \int_{\Omega} (V_{i,j} - \nabla \varphi_{i,j}) : \Phi \, dx \xrightarrow{j \rightarrow +\infty} 0.$$

But we also know that  $\int_{\Omega} V_{i,j} : \Phi \, dx \xrightarrow{j \rightarrow +\infty} \int_{\Omega} \bar{V}_i : \Phi \, dx$ , and we get  $\nabla \varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \bar{V}_i$  in  $L^4(\Omega, M_3(\mathbb{R}))$ , for all  $i = 1, \dots, M$ . By uniqueness of the weak limit in  $L^4(\Omega, M_3(\mathbb{R}))$ , we have that  $\nabla \bar{\varphi}_i = \bar{V}_i \in L^\infty(\Omega, M_3(\mathbb{R}))$  for each  $i = 1, \dots, M$ .

We also have that

$$\forall i \in \{1, \dots, M\}, \forall \Phi \in L^2(\Omega, M_3(\mathbb{R})), \int_{\Omega} W_{i,j} - V_{i,j}^{-1} : \Phi \, dx \xrightarrow{j \rightarrow +\infty} 0.$$

233 But we know that  $\int_{\Omega} W_{i,j} : \Phi \, dx \xrightarrow{j \rightarrow +\infty} \int_{\Omega} \bar{W}_i : \Phi \, dx$ , and we get  $V_{i,j}^{-1} \xrightarrow{j \rightarrow +\infty} \bar{W}_i$  in  $L^2(\Omega, M_3(\mathbb{R}))$ .

234 Let us now show that  $\bar{W}_i = \bar{V}_i^{-1} = \nabla \bar{\varphi}_i^{-1}$ , for all  $i = 1, \dots, M$ . We have that  $\forall \Phi \in$   
 235  $L^2(\Omega, M_3(\mathbb{R}))$ :

$$\begin{aligned} 236 \quad & \int_{\Omega} V_{i,j}^{-1} \bar{V}_i - I : \Phi \, dx = \int_{\Omega} V_{i,j}^{-1} \bar{V}_i - \bar{W}_i \bar{V}_i + \bar{W}_i \bar{V}_i - \bar{W}_i V_{i,j} + \bar{W}_i V_{i,j} - V_{i,j}^{-1} V_{i,j} : \Phi \, dx, \\ 237 \quad & = \int_{\Omega} (V_{i,j}^{-1} - \bar{W}_i) \bar{V}_i : \Phi \, dx + \int_{\Omega} \bar{W}_i (\bar{V}_i - V_{i,j}) : \Phi \, dx + \int_{\Omega} (\bar{W}_i - V_{i,j}^{-1}) V_{i,j} : \Phi \, dx, \\ 238 \quad & = \int_{\Omega} (V_{i,j}^{-1} - \bar{W}_i) : \Phi \bar{V}_i^T \, dx + \int_{\Omega} (\bar{V}_i - V_{i,j}) : \bar{W}_i^T \Phi \, dx + \int_{\Omega} (\bar{W}_i - V_{i,j}^{-1}) : \Phi V_{i,j}^T \, dx, \\ 239 \end{aligned}$$

240 with  $\Phi \bar{V}_i^T \in L^2(\Omega, M_3(\mathbb{R}))$  since  $\bar{V}_i \in L^\infty(\Omega, M_3(\mathbb{R}))$  so that  $\int_{\Omega} (V_{i,j}^{-1} - \bar{W}_i) : \Phi \bar{V}_i^T \, dx \xrightarrow{j \rightarrow +\infty} 0$ ,

241  $\bar{W}_i^T \Phi \in L^2(\Omega, M_3(\mathbb{R})) \subset L^{\frac{4}{3}}(\Omega, M_3(\mathbb{R}))$  since  $\bar{W}_i \in L^\infty(\Omega, M_3(\mathbb{R}))$  so that  $\int_{\Omega} (V_{i,j} - \bar{V}_i) : \bar{W}_i^T \Phi \, dx \xrightarrow{j \rightarrow +\infty} 0$ , and  $\int_{\Omega} (\bar{W}_i - V_{i,j}^{-1}) : \Phi V_{i,j}^T \, dx \leq \alpha \int_{\Omega} (\bar{W}_i - V_{i,j}^{-1}) : \Phi' \, dx$  with  $\Phi' \in L^2(\Omega)$

242 since  $\forall j \in \mathbb{N}, \forall i \in \{1, \dots, M\}, \|V_{i,j}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha$  so that  $\int_{\Omega} (\bar{W}_i - V_{i,j}^{-1}) : \Phi V_{i,j}^T \, dx \xrightarrow{j \rightarrow +\infty}$

243 0. We therefore obtain that  $V_{i,j}^{-1} \bar{V}_i \xrightarrow{j \rightarrow +\infty} I_3$  in  $L^2(\Omega, M_3(\mathbb{R}))$  and since  $V_{i,j}^{-1} \xrightarrow{j \rightarrow +\infty} \bar{W}_i$

244 in  $L^2(\Omega, M_3(\mathbb{R}))$ , we deduce that  $\bar{W}_i \bar{V}_i = I_3$  and consequently  $\bar{W}_i = \bar{V}_i^{-1} = (\nabla \bar{\varphi}_i)^{-1} \in$   
 245  $L^\infty(\Omega, M_3(\mathbb{R}))$ , for all  $i = 1, \dots, M$ .

246 Now, we consider  $V_{i,j} = w_{i,j} + \nabla \varphi_{i,j}$  and  $\text{Cof} V_{i,j} = \text{Cof}(w_{i,j} + \nabla \varphi_{i,j})$ .

247 For the purpose of illustration since the same reasoning applies to other indices, we focus on

249 the component of the first row, first column :

$$\begin{aligned}
 250 \quad (\text{Cof}V_{i,j})_{11} &= \left( w_{i,22} + \frac{\partial \varphi_{i,2}}{\partial x_2} \right) \left( w_{i,33} + \frac{\partial \varphi_{i,3}}{\partial x_3} \right) - \left( w_{i,23} + \frac{\partial \varphi_{i,2}}{\partial x_3} \right) \left( w_{i,32} + \frac{\partial \varphi_{i,3}}{\partial x_2} \right), \\
 251 \quad &= w_{i,22}w_{i,33} - w_{i,23}w_{i,32} + \frac{\partial \varphi_{i,3}}{\partial x_3} \frac{\partial \varphi_{i,2}}{\partial x_2} - \frac{\partial \varphi_{i,3}}{\partial x_2} \frac{\partial \varphi_{i,2}}{\partial x_3} + w_{i,22} \frac{\partial \varphi_{i,3}}{\partial x_3} + w_{i,33} \frac{\partial \varphi_{i,2}}{\partial x_2} \\
 252 \quad &- w_{i,23} \frac{\partial \varphi_{i,3}}{\partial x_2} - w_{i,32} \frac{\partial \varphi_{i,2}}{\partial x_3}, \\
 253 \quad &= (\text{Cof}w_{i,j})_{11} + (\text{Cof}\nabla\varphi_{i,j})_{11} + w_{i,22} \frac{\partial \varphi_{i,3}}{\partial x_3} + w_{i,33} \frac{\partial \varphi_{i,2}}{\partial x_2} - w_{i,23} \frac{\partial \varphi_{i,3}}{\partial x_2} - w_{i,32} \frac{\partial \varphi_{i,2}}{\partial x_3}. \\
 254
 \end{aligned}$$

255 Let us set  $d_{i,j} = (\text{Cof}w_{i,j})_{11} + w_{i,22} \frac{\partial \varphi_{i,3}}{\partial x_3} + w_{i,33} \frac{\partial \varphi_{i,2}}{\partial x_2} - w_{i,23} \frac{\partial \varphi_{i,3}}{\partial x_2} - w_{i,32} \frac{\partial \varphi_{i,2}}{\partial x_3} \in L^2(\Omega)$  by  
256 generalized Hölder's inequality. Then using Hölder's inequality, one can prove that

$$\begin{aligned}
 257 \quad \|d_{i,j}\|_{L^2(\Omega)} &\leq \|w_{i,22}\|_{L^4(\Omega)} \left\| \frac{\partial \varphi_{i,3}}{\partial x_3} \right\|_{L^4(\Omega)} + \|w_{i,33}\|_{L^4(\Omega)} \left\| \frac{\partial \varphi_{i,2}}{\partial x_2} \right\|_{L^4(\Omega)} \\
 258 \quad &+ \|w_{i,23}\|_{L^4(\Omega)} \left\| \frac{\partial \varphi_{i,3}}{\partial x_2} \right\|_{L^4(\Omega)} + \|w_{i,32}\|_{L^4(\Omega)} \left\| \frac{\partial \varphi_{i,2}}{\partial x_3} \right\|_{L^4(\Omega)} \\
 259 \quad &+ \|w_{i,22}\|_{L^4(\Omega)} \|w_{i,33}\|_{L^4(\Omega)} + \|w_{i,23}\|_{L^4(\Omega)} \|w_{i,32}\|_{L^4(\Omega)}, \\
 260 \quad &\leq (\|w_{i,22}\|_{L^4(\Omega)}^4 + \|w_{i,33}\|_{L^4(\Omega)}^4 + \|w_{i,23}\|_{L^4(\Omega)}^4 + \|w_{i,32}\|_{L^4(\Omega)}^4)^{\frac{1}{4}} \\
 261 \quad &(\left\| \frac{\partial \varphi_{i,2}}{\partial x_3} \right\|_{L^4(\Omega)}^{\frac{4}{3}} + \left\| \frac{\partial \varphi_{i,3}}{\partial x_3} \right\|_{L^4(\Omega)}^{\frac{4}{3}} + \left\| \frac{\partial \varphi_{i,2}}{\partial x_2} \right\|_{L^4(\Omega)}^{\frac{4}{3}} + \left\| \frac{\partial \varphi_{i,3}}{\partial x_2} \right\|_{L^4(\Omega)}^{\frac{4}{3}})^{\frac{3}{4}} \\
 262 \quad &+ (\|w_{i,22}\|_{L^4(\Omega)}^4 + \|w_{i,33}\|_{L^4(\Omega)}^4 + \|w_{i,23}\|_{L^4(\Omega)}^4 + \|w_{i,32}\|_{L^4(\Omega)}^4)^{\frac{1}{4}} \\
 263 \quad &(\|w_{i,22}\|_{L^4(\Omega)}^{\frac{4}{3}} + \|w_{i,33}\|_{L^4(\Omega)}^{\frac{4}{3}} + \|w_{i,23}\|_{L^4(\Omega)}^{\frac{4}{3}} + \|w_{i,32}\|_{L^4(\Omega)}^{\frac{4}{3}})^{\frac{3}{4}}, \\
 264 \quad &\leq c_4 c_2^2 \|w_{i,j}\|_{L^4(\Omega, M_3(\mathbb{R}))} \|\nabla\varphi_{i,j}\|_{L^4(\Omega, M_3(\mathbb{R}))} + c_4 c_2^2 \|w_{i,j}\|_{L^4(\Omega, M_3(\mathbb{R}))}^2. \\
 265
 \end{aligned}$$

Indeed, thanks to the property of equivalence of norms in finite dimension, there exist  $(c_1, c_2, c_3, c_4) \in \mathbb{R}^4$  such that  $\forall x \in \mathbb{R}^9$ ,  $c_1 \|x\|_2 \leq \|x\|_4 \leq c_2 \|x\|_2$  and  $c_3 \|x\|_4 \leq \|x\|_{\frac{4}{3}} \leq c_4 \|x\|_4 \leq c_4 c_2 \|x\|_2$ .

As  $(\varphi_{i,j})$  is uniformly bounded according to  $j$  in  $W^{1,4}(\Omega, \mathbb{R}^3)$  and as  $w_{i,j} \xrightarrow{j \rightarrow +\infty} 0$  strongly in  $L^4(\Omega)$ , then

$$\|d_{i,j}\|_{L^2(\Omega)} \xrightarrow{j \rightarrow +\infty} 0,$$

and

$$\int_{\Omega} (\text{Cof}V_{i,j})_{11} \Phi \, dx = \int_{\Omega} d_{i,j} \Phi \, dx + \int_{\Omega} (\text{Cof}\nabla\varphi_{i,j})_{11} \Phi \, dx \xrightarrow{j \rightarrow +\infty} \int_{\Omega} (\text{Cof}\nabla\bar{\varphi}_i)_{11} \Phi \, dx, \quad \forall \Phi \in L^2(\Omega),$$

266  $\forall i \in \{1, \dots, M\}$ . Indeed, from [?, Theorem 8.20], as  $\varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \bar{\varphi}_i$  in  $W^{1,4}(\Omega, \mathbb{R}^3)$ , then  
267  $\text{Cof}\nabla\varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \text{Cof}\nabla\bar{\varphi}_i$  in  $L^2(\Omega, M_3(\mathbb{R}))$ . Thus  $\text{Cof}V_{i,j} \xrightarrow{j \rightarrow +\infty} \text{Cof}\nabla\bar{\varphi}_i$  in  $L^2(\Omega)$  and by  
268 uniqueness of the weak limit in  $L^2(\Omega)$ ,  $\bar{X}_i = \text{Cof}\nabla\bar{\varphi}_i$  and  $\text{Cof}V_{i,j} \xrightarrow{j \rightarrow +\infty} \text{Cof}\nabla\bar{\varphi}_i$  in  $L^4(\Omega)$ .

269 We then have from the generalized Hölder's inequality :

$$270 \quad \det V_{i,j} = V_{i,11}(\text{Cof} V_{i,j})_{11} + V_{i,21}(\text{Cof} V_{i,j})_{21} + V_{i,31}(\text{Cof} V_{i,j})_{31} \in L^{\frac{4}{3}}(\Omega),$$

$$271 \quad = (w_{i,11} + \frac{\partial \varphi_{i,1}}{\partial x_1})(\text{Cof} \nabla \varphi_{i,j})_{11} + d_{i,j} + \text{similar components} .$$

273 Then  $\forall \Phi \in L^4(\Omega)$ , and for all  $i = 1, \dots, M$ ,

$$274 \quad \int_{\Omega} \det V_{i,j} : \Phi \, dx = \int_{\Omega} w_{i,11}(\text{Cof} \nabla \varphi_{i,j})_{11} \Phi \, dx + \int_{\Omega} w_{i,11} d_{i,j} \Phi \, dx + \int_{\Omega} \frac{\partial \varphi_{i,1}}{\partial x_1} d_{i,j} \Phi \, dx$$

$$275 \quad + \int_{\Omega} \frac{\partial \varphi_{i,1}}{\partial x_1} (\text{Cof} \nabla \varphi_{i,j})_{11} \Phi \, dx + \text{similar components}.$$

277 But we have for all  $i = 1, \dots, M$ ,

$$278 \quad \left| \int_{\Omega} w_{i,11}(\text{Cof} \nabla \varphi_{i,j})_{11} \Phi \, dx \right| \leq \|w_{i,11}\|_{L^4(\Omega)} \|\text{Cof} \nabla \varphi_{i,j}\|_{L^2(\Omega)} \|\Phi\|_{L^4(\Omega)},$$

$$279 \quad \left| \int_{\Omega} w_{i,11} d_{i,j} \Phi \, dx \right| \leq \|w_{i,11}\|_{L^4(\Omega)} \|d_{i,j}\|_{L^2(\Omega)} \|\Phi\|_{L^4(\Omega)},$$

$$280 \quad \left| \int_{\Omega} \frac{\partial \varphi_{i,1}}{\partial x_1} d_{i,j} \Phi \, dx \right| \leq \left\| \frac{\partial \varphi_{i,1}}{\partial x_1} \right\|_{L^4(\Omega)} \|d_{i,j}\|_{L^2(\Omega)} \|\Phi\|_{L^4(\Omega)},$$

282 with  $\|w_{i,11}\|_{L^4(\Omega)} \xrightarrow{j \rightarrow +\infty} 0$ ,  $\|d_{i,j}\|_{L^2(\Omega)} \xrightarrow{j \rightarrow +\infty} 0$  and  $\|\text{Cof} \nabla \varphi_{i,j}\|_{L^2(\Omega)}$  uniformly bounded since  
 283  $\text{Cof} \nabla \varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \text{Cof} \nabla \bar{\varphi}_i$  in  $L^4(\Omega, M_3(\mathbb{R}))$  and thus in  $L^2(\Omega, M_3(\mathbb{R}))$ ,  $\left\| \frac{\partial \varphi_{i,1}}{\partial x_1} \right\|_{L^4(\Omega)}$  uniformly  
 284 bounded since  $\varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \bar{\varphi}_i$  in  $W^{1,4}(\Omega)$  and  $\|\Phi\|_{L^4(\Omega)}$  uniformly bounded.

285 Since  $\varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \bar{\varphi}_i$  in  $W^{1,4}(\Omega, \mathbb{R}^3)$ , then  $\det \nabla \varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \det \nabla \bar{\varphi}_i$  in  $L^{\frac{4}{3}}(\Omega)$  and  $\int_{\Omega} \frac{\partial \varphi_{i,1}}{\partial x_1}$   
 286  $(\text{Cof} \nabla \varphi_{i,j})_{11} \Phi \, dx \xrightarrow{j \rightarrow +\infty} \int_{\Omega} \frac{\partial \bar{\varphi}_{i,1}}{\partial x_1} (\text{Cof} \nabla \bar{\varphi}_i)_{11} \Phi \, dx$ , for all  $i = 1, \dots, M$ .

287 Finally,  $\int_{\Omega} \det V_{i,j} \Phi \, dx \xrightarrow{j \rightarrow +\infty} \int_{\Omega} \det \nabla \bar{\varphi}_i \Phi \, dx$ ,  $\forall \Phi \in L^4(\Omega)$  and so  $\det V_{i,j} \xrightarrow{j \rightarrow +\infty} \det \nabla \bar{\varphi}_i$  in  
 288  $L^{\frac{4}{3}}(\Omega)$  and by uniqueness of the weak limit,  $\bar{\delta}_i = \det \nabla \bar{\varphi}_i$  and  $\det V_{i,j} \xrightarrow{j \rightarrow +\infty} \det \nabla \bar{\varphi}_i$  in  $L^2(\Omega)$ ,  
 289 for all  $i = 1, \dots, M$ .

290

We know that  $W_{Op}$  is convex and continuous. If  $\psi_n \xrightarrow{n \rightarrow +\infty} \psi$  in  $L^4(\Omega, M_3(\mathbb{R}))$  and we can extract a subsequence still denoted  $(\psi_n)$  such that  $\nabla \psi_n \xrightarrow{n \rightarrow +\infty} \nabla \psi$  almost everywhere on  $\Omega$ . If  $\alpha_n \xrightarrow{n \rightarrow +\infty} \bar{\alpha}$  in  $L^4(\Omega, M_3(\mathbb{R}))$ , then we can extract a subsequence still denoted  $(\alpha_n)$  such that  $\alpha_n \xrightarrow{n \rightarrow +\infty} \bar{\alpha}$  almost everywhere on  $\Omega$ . If  $\delta_n \xrightarrow{n \rightarrow +\infty} \bar{\delta}$  in  $L^2(\Omega)$ , then we can extract a subsequence still denoted  $(\delta_n)$  such that  $\delta_n \xrightarrow{n \rightarrow +\infty} \bar{\delta}$  almost everywhere on  $\Omega$ . Then by continuity of  $W_{Op}$ , we get that  $W_{Op}(\psi_n, \alpha_n, \delta_n) \xrightarrow{j \rightarrow +\infty} W_{Op}(\psi, \bar{\alpha}, \bar{\delta})$  almost everywhere on  $\Omega$ . Then by applying Fatou's lemma, we have that  $\liminf_{n \rightarrow +\infty} \int_{\Omega} W_{Op}(\psi_n, \alpha_n, \delta_n) \, dx \geq \int_{\Omega} W_{Op}(\psi, \bar{\alpha}, \bar{\delta}) \, dx$ . Since  $W_{Op}$  is convex, so is  $\int_{\Omega} W_{Op}(\xi, \alpha, \delta) \, dx$  and [?, Corollaire III.8 p.38] applies. Therefore

$\int_{\Omega} W_{Op}(\xi, \alpha, \delta) dx$  is also weakly lower semicontinuous in  $L^4(\Omega, M_3(\mathbb{R})) \times L^4(\Omega, M_3(\mathbb{R})) \times L^2(\Omega)$  and we deduce that  $+\infty > \liminf_{j \rightarrow +\infty} \int_{\Omega} W_{Op}(V_{i,j}, \text{Cof}V_{i,j}, \det V_{i,j}) dx \geq \int_{\Omega} W_{Op}(\nabla \bar{\varphi}_i, \text{Cof} \nabla \bar{\varphi}_i, \det \nabla \bar{\varphi}_i) dx$ .

Using the same arguments as in the previous proof, we get more regularity of  $\bar{\varphi}_i$ , that is to say,  $\bar{\varphi}_i$  is a bi-Lipschitz homeomorphism for each  $i = 1, \dots, M$ .

By the weak-\* lower semicontinuity of  $\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))}$ , we have that  $\|\nabla \bar{\varphi}_i\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \liminf_{j \rightarrow +\infty} \|V_{i,j}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha$ ,  $\|(\nabla \bar{\varphi}_i)^{-1}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \liminf_{j \rightarrow +\infty} \|W_{i,j}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta$  and  $\mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}}(\nabla \bar{\varphi}_i) \leq \liminf_{j \rightarrow +\infty} \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}}(V_{i,j})$ ,  $\mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta\}}((\nabla \bar{\varphi}_i)^{-1}) \leq \liminf_{j \rightarrow +\infty} \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta\}}(W_{i,j})$ .

We also know that  $\varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \bar{\varphi}_i$  in  $\mathcal{C}^{0,\alpha}(\Omega)$ ,  $\bar{\varphi}_i^{-1} \in \mathcal{C}^{0,\alpha'}(\Omega)$  and  $\varphi_{i,j} \circ \bar{\varphi}_i^{-1} \xrightarrow{j \rightarrow +\infty} \text{Id}$  in  $\mathcal{C}^{0,\alpha''}(\Omega)$  with  $\alpha'' = \alpha' \alpha$ ,  $\alpha < \frac{1}{4}$  and  $\alpha' < 1 - \frac{3}{q}$ , using the same arguments as in the previous proof. Let us prove that  $\theta_{T_i,l,j} \circ \varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i$  in  $L^1(\Omega)$  for all  $i = 1, \dots, M$ .

Since  $\varphi_{i,j} \circ \bar{\varphi}_i^{-1} \xrightarrow{j \rightarrow +\infty} \text{Id}$  in  $\mathcal{C}^{0,\alpha''}(\Omega)$ , then it converges uniformly and so

$$\forall \varepsilon > 0, \exists k \in \mathbb{N}, \forall j \in \mathbb{N}, j \geq k \Rightarrow \sup_{x \in \Omega} |\varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x) - x| \leq \varepsilon.$$

291 Also given  $\eta > 0$ , by Egorov's theorem, there exists a set  $S_\eta$  with Lebesgue measure  $\text{meas}(S_\eta) \leq$   
292  $\eta$  such that  $\theta_{T_i,l,j} \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{T_i,l}$  uniformly on  $\Omega \setminus S_\eta$ .

293 Now we choose  $K \in \mathbb{N}$  large enough such that  $\forall j \in \mathbb{N}$ , if  $j \geq K$ , then

$$\begin{aligned} 294 & |\varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x) - x| \leq \eta, \text{ on } \Omega, \\ 295 & |\theta_{T_i,l,j}(x) - \bar{\theta}_{T_i,l}(x)| \leq \eta, \text{ on } \Omega \setminus S_\eta. \end{aligned}$$

297 Then by denoting  $\mathcal{O}_i = \{x \in \Omega : \bar{\theta}_{T_i,l}(x) = 1\}$ ,

$$\begin{aligned} 298 & \|\theta_{T_i,l,j} \circ \varphi_{i,j} - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i\|_{L^1(\Omega)} = \int_{\bar{\varphi}_i^{-1}(\mathcal{O}_i)} |\theta_{T_i,l,j} \circ \varphi_{i,j}(x) - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i(x)| dx \\ 299 & + \int_{\bar{\varphi}_i^{-1}(\Omega \setminus \mathcal{O}_i)} |\theta_{T_i,l,j} \circ \varphi_{i,j}(x) - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i(x)| dx, \\ 300 & = \int_{\mathcal{O}_i} \frac{|\theta_{T_i,l,j} \circ \varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x) - 1|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx + \int_{\Omega \setminus \mathcal{O}_i} \frac{|\theta_{T_i,l,j} \circ \varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x)|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx, \\ 301 & \leq \int_{(\mathcal{O}_i \setminus (S_\eta))_\eta} \frac{|\theta_{T_i,l,j} \circ \varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x) - 1|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx + \int_{((\Omega \setminus \mathcal{O}_i) \setminus (S_\eta))_\eta} \frac{|\theta_{T_i,l,j} \circ \varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x)|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx \\ 302 & + \int_{S_\eta} \frac{|\theta_{T_i,l,j} \circ \varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x) - \bar{\theta}_{T_i,l}(x)|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx + \int_{\mathcal{O}_i^\eta \cup S_\eta^\eta \cup \Omega^\eta} \frac{|\theta_{T_i,l,j} \circ \varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x) - \bar{\theta}_{T_i,l}(x)|}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx, \\ 303 & \end{aligned}$$

304 with  $(\mathcal{O}_i \setminus S_\eta)_\eta = \{x \in \mathcal{O}_i \setminus S_\eta \mid \text{dist}(x, \partial(\mathcal{O}_i \setminus S_\eta)) > \eta\}$ ,  $S_\eta^\eta = \{x \in \Omega \mid \text{dist}(x, \partial S_\eta) \leq \eta\}$ . So  
305  $\forall x \in (\mathcal{O}_i \setminus (S_\eta))_\eta$ ,  $\varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x) \in \mathcal{O}_i \setminus S_\eta$ , and  $\forall x \in ((\Omega \setminus \mathcal{O}_i) \setminus S_\eta)_\eta$ ,  $\varphi_{i,j} \circ \bar{\varphi}_i^{-1}(x) \in (\Omega \setminus \mathcal{O}_i) \setminus S_\eta$ .

306 Then

$$\begin{aligned}
 307 \quad \|\theta_{T_i,l,j} \circ \varphi_{i,j} - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i\|_{L^1(\Omega)} &\leq \int_{(\mathcal{O}_i \setminus S_\eta)_\eta} \frac{\eta}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx \\
 308 &+ \int_{((\Omega \setminus \mathcal{O}_i) \setminus S_\eta)_\eta} \frac{\eta}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx \\
 309 &+ \int_{S_\eta} \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx \\
 310 &+ \int_{\mathcal{O}_i^\eta \cup S_\eta^\eta \cup \Omega^\eta} \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))} dx, \\
 311
 \end{aligned}$$

312 taking into account that  $\theta_{T_i,l,j} \in BV(\Omega, \{0, 1\})$  and  $\bar{\theta}_{T_i,l} \in BV(\Omega, \{0, 1\})$ . Using Hölder's  
 313 inequality, we get

$$\begin{aligned}
 314 \quad \|\theta_{T_i,l,j} \circ \varphi_{i,j} - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i\|_{L^1(\Omega)} &\leq \eta \left( \int_\Omega dx \right)^{\frac{s}{s+1}} \left( \int_\Omega \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))^{s+1}} dx \right)^{\frac{1}{s+1}} \\
 315 &+ \eta \left( \int_\Omega dx \right)^{\frac{s}{s+1}} \left( \int_\Omega \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))^{s+1}} dx \right)^{\frac{1}{s+1}} \\
 316 &+ \left( \int_{S_\eta} dx \right)^{\frac{s}{s+1}} \left( \int_\Omega \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))^{s+1}} dx \right)^{\frac{1}{s+1}} \\
 317 &+ \left( \int_{S_\eta^\eta \cup \mathcal{O}_i^\eta \cup \Omega^\eta} dx \right)^{\frac{s}{s+1}} \left( \int_\Omega \frac{1}{\det \nabla \bar{\varphi}_i(\bar{\varphi}_i^{-1}(x))^{s+1}} dx \right)^{\frac{1}{s+1}}, \\
 318 \quad \|\theta_{T_i,l,j} \circ \varphi_{i,j} - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i\|_{L^1(\Omega)} &\leq \eta (\text{meas}(\Omega))^{\frac{s}{s+1}} \left( \int_\Omega \frac{1}{\det \nabla \bar{\varphi}_i(x)^s} dx \right)^{\frac{1}{s+1}} \\
 319 &+ \eta (\text{meas}(\Omega))^{\frac{s}{s+1}} \left( \int_\Omega \frac{1}{\det \nabla \bar{\varphi}_i(x)^s} dx \right)^{\frac{1}{s+1}} \\
 320 &+ (\text{meas}(S_\eta))^{\frac{s}{s+1}} \left( \int_\Omega \frac{1}{\det \nabla \bar{\varphi}_i(x)^s} dx \right)^{\frac{1}{s+1}} \\
 321 &+ (\text{meas}(S_\eta^\eta) + \text{meas}(\mathcal{O}_i^\eta) + \text{meas}(\Omega^\eta))^{\frac{s}{s+1}} \left( \int_\Omega \frac{1}{\det \nabla \bar{\varphi}_i(x)^s} dx \right)^{\frac{1}{s+1}}. \\
 322
 \end{aligned}$$

323

324 With  $s = 10$  and the finiteness of  $\int_\Omega W_{Op}(\nabla \bar{\varphi}_i, \text{Cof} \nabla \bar{\varphi}_i, \det \nabla \bar{\varphi}_i) dx$ , there exists  $C \in \mathbb{R}_*^+$  such  
 325 that

$$\begin{aligned}
 326 \quad \|\theta_{T_i,l,j} \circ \varphi_{i,j} - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i\|_{L^1(\Omega)} &\leq \eta (\text{meas}(\Omega))^{\frac{s}{s+1}} C^{\frac{1}{s+1}} + \eta (\text{meas}(\Omega))^{\frac{s}{s+1}} C^{\frac{1}{s+1}} + (\text{meas}(S_\eta))^{\frac{s}{s+1}} \\
 327 &C^{\frac{1}{s+1}} + \left( \int_{\partial S_\eta} \int_{B(x,\eta)} dy dx + \int_{\partial \mathcal{O}_i} \int_{B(x,\eta)} dy dx + \int_{\partial \Omega} \int_{B(x,\eta)} dy dx \right)^{\frac{s}{s+1}} \\
 328 &C^{\frac{1}{s+1}}, \\
 329
 \end{aligned}$$

330

$$\begin{aligned}
 331 \quad & \|\theta_{T_i,l,j} \circ \varphi_{i,j} - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i\|_{L^1(\Omega)} \leq \eta (\text{meas}(\Omega))^{\frac{s}{s+1}} C^{\frac{1}{s+1}} \\
 332 \quad & \quad \quad \quad + \eta (\text{meas}(\Omega))^{\frac{s}{s+1}} C^{\frac{1}{s+1}} \\
 333 \quad & \quad \quad \quad + (\eta)^{\frac{s}{s+1}} C^{\frac{1}{s+1}} \\
 334 \quad & \quad \quad \quad + \left( \frac{4\pi}{3} \eta^3 (\text{meas}(\partial S_\eta) + \text{meas}(\partial \mathcal{O}_i) + \text{meas}(\partial \Omega)) \right)^{\frac{s}{s+1}} C^{\frac{1}{s+1}}. \\
 335 \quad &
 \end{aligned}$$

336 By letting  $\eta$  tend to 0 and so  $j$  to infinity, we finally get for all  $i = 1, \dots, M$

$$\begin{aligned}
 337 \quad & \|\theta_{T_i,l,j} \circ \varphi_{i,j} - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i\|_{L^1(\Omega)} \xrightarrow{j \rightarrow +\infty} 0. \\
 338 \quad &
 \end{aligned}$$

339 This is true for all  $l \in \{1, \dots, N\}$ .

340 Finally, for all  $i = 1, \dots, M$  and all  $l = 1, \dots, N$ :

$$\begin{aligned}
 341 \quad & \|\bar{\theta}_{\tilde{T}_i,l} - \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i + \bar{\theta}_{R,l}\|_{L^1(\Omega)} \leq \|\theta_{\tilde{T}_i,l,j} - \theta_{T_i,l,j} \circ \varphi_{i,j} + \theta_{R,l,j}\|_{L^1(\Omega)} \\
 342 \quad & \quad \quad \quad + \|\bar{\theta}_{\tilde{T}_i,l} - \theta_{\tilde{T}_i,l,j}\|_{L^1(\Omega)} + \|\bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \theta_{T_i,l,j} \circ \varphi_{i,j}\|_{L^1(\Omega)} \\
 343 \quad & \quad \quad \quad + \|\bar{\theta}_{R,l} - \theta_{R,l,j}\|_{L^1(\Omega)}, \\
 344 \quad & \quad \quad \quad \xrightarrow{j \rightarrow +\infty} 0, \\
 345 \quad &
 \end{aligned}$$

346 from what precedes. We deduce that  $\bar{\theta}_{\tilde{T}_i,l} = \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \bar{\theta}_{R,l}$  in  $L^1(\Omega)$  and so almost everywhere  
 347 in  $\Omega$  and  $\theta_{\tilde{T}_i,l,j} \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \bar{\theta}_{R,l}$  in  $L^1(\Omega)$  for all  $i = 1, \dots, M$  and all  $l = 1, \dots, N$ . By  
 348 the semicontinuity of the total variation we get that

$$\begin{aligned}
 349 \quad & \frac{\lambda}{2} \sum_{l=1}^N TV(\bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \bar{\theta}_{R,l}) \leq \frac{\lambda}{2} \liminf_{j \rightarrow +\infty} \sum_{l=1}^N TV(\theta_{\tilde{T}_i,l,j}). \\
 350 \quad &
 \end{aligned}$$

351 Furthermore, we have that  $\theta_{T_i,l,j} \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{T_i,l}$  in  $L^1(\Omega)$  and so almost everywhere in  $\Omega$  up  
 352 to a subsequence. Then  $\theta_{T_i,l,j} T_i \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{T_i,l} T_i$  almost everywhere in  $\Omega$  with  $|\theta_{T_i,l,j} T_i| \leq$   
 353  $\|T_i\|_{L^\infty(\Omega)} \in L^1(\Omega)$  since  $\Omega$  is bounded, and by the dominated convergence theorem one  
 354 gets  $\int_\Omega \theta_{T_i,l,j} T_i dx \xrightarrow{j \rightarrow +\infty} \int_\Omega \bar{\theta}_{T_i,l} T_i dx$ . Using the same reasoning, we get  $\int_\Omega \theta_{T_i,l,j} dx \xrightarrow{j \rightarrow +\infty}$   
 355  $\int_\Omega \bar{\theta}_{T_i,l} dx$ . As  $T_i \in W^{1,\infty}(\Omega)$ , we denote  $\kappa_i$  the Lipschitz constant related to  $T_i$ . Then the chain  
 356 rule applies and we have that  $T_i \circ \varphi_i^k \in W^{1,4}(\Omega)$ ,  $T_i \circ \bar{\varphi}_i \in W^{1,4}(\Omega)$  and  $\|T_i \circ \varphi_{i,j} - T_i \circ \bar{\varphi}_i\|_{C^0(\Omega)} \leq$   
 357  $\kappa_i \|\varphi_{i,j} - \bar{\varphi}_i\|_{C^0(\Omega)} \xrightarrow{j \rightarrow +\infty} 0$  using the Sobolev compact embedding  $W^{1,4}(\Omega) \hookrightarrow C^0(\Omega)$ . Therefore,  
 358 up to a subsequence  $\theta_{R,l,j} T_i \circ \varphi_{i,j} \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{R,l} T_i \circ \bar{\varphi}_i$  almost everywhere in  $\Omega$ ,  $\theta_{R,l,j} \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{R,l}$   
 359 almost everywhere in  $\Omega$  with  $|\theta_{R,l,j} T_i \circ \varphi_{i,j}| \leq \|T_i\|_{L^\infty(\Omega)} \in L^1(\Omega)$  and  $|\theta_{R,l,j}| \leq 1 \in L^1(\Omega)$

360 almost everywhere in  $\Omega$ . Thus the dominated convergence theorem applies and we get

$$\begin{aligned}
 361 \quad & \frac{1}{M} \sum_{i=1}^M \int_{\Omega} \theta_{R,l,j} T_i \circ \varphi_{i,j} \, dx \xrightarrow{j \rightarrow +\infty} \frac{1}{M} \sum_{i=1}^M \int_{\Omega} \bar{\theta}_{R,l} T_i \circ \bar{\varphi}_i \, dx, \\
 362 \quad & \int_{\Omega} \theta_{R,l,j} \, dx \xrightarrow{k \rightarrow +\infty} \int_{\Omega} \bar{\theta}_{R,l} \, dx, \\
 363 \quad & c_{R,l,j} \xrightarrow{k \rightarrow +\infty} \bar{c}_{R,l}.
 \end{aligned}$$

366 Then  $\forall l \in \{1, \dots, N\}$  and for all  $i = 1, \dots, M$ ,

$$\begin{aligned}
 367 \quad & \theta_{T_i,l,j} (c_{T_i,l,j} - T_i)^2 \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{T_i,l} (\bar{c}_{T_i,l} - T_i)^2, \\
 368 \quad & \theta_{R,l,j} (c_{R,l,j} - R)^2 \xrightarrow{j \rightarrow +\infty} \bar{\theta}_{R,l} (\bar{c}_{R,l} - R)^2,
 \end{aligned}$$

370 almost everywhere on  $\Omega$  with  $|\theta_{T_i,l,j} (c_{T_i,l,j} - T_i)^2| \leq 4 \|T_i\|_{L^\infty(\Omega)}^2 \in L^1(\Omega)$  and  $|\theta_{R,l,j} (c_{R,l,j} - R)^2| \leq 4 \|T_i\|_{L^\infty(\Omega)}^2 \in L^1(\Omega)$ . By applying the dominated convergence theorem, we have for  
 371 all  $l \in \{1, \dots, N\}$ :

$$\begin{aligned}
 373 \quad & \|\theta_{T_i,l,j} (c_{T_i,l,j} - T_i)^2\|_{L^1(\Omega)} \xrightarrow{j \rightarrow +\infty} \|\bar{\theta}_{T_i,l} (\bar{c}_{T_i,l} - T_i)^2\|_{L^1(\Omega)}, \\
 374 \quad & \|\theta_{R,l,j} (c_{R,l,j} - R)^2\|_{L^1(\Omega)} \xrightarrow{j \rightarrow +\infty} \|\bar{\theta}_{R,l} (\bar{c}_{R,l} - R)^2\|_{L^1(\Omega)}.
 \end{aligned}$$

376 By combining all the results, we finally get

$$\begin{aligned}
 377 \quad & \frac{1}{M} \sum_{i=1}^M \frac{\gamma_T}{2} \sum_{l=1}^N TV(\bar{\theta}_{T_i,l}) + \int_{\Omega} \sum_{l=1}^N \bar{\theta}_{T_i,l} (\bar{c}_{T_i,l} - T_i)^2 \, dx \\
 378 \quad & + \frac{\gamma_R}{2} \sum_{l=1}^N TV(\bar{\theta}_{R,l}) + \int_{\Omega} \sum_{l=1}^N \bar{\theta}_{R,l} (\bar{c}_{R,l} - R)^2 \, dx \\
 379 \quad & + \frac{\lambda}{2} \sum_{l=1}^N TV(\bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \bar{\theta}_{R,l}) + \int_{\Omega} W_{Op}(\nabla \bar{\varphi}_i, \text{Cof} \nabla \bar{\varphi}_i, \det \nabla \bar{\varphi}_i) \, dx \\
 380 \quad & + \mathbf{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}}(\nabla \bar{\varphi}_i) + \mathbf{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta\}}((\nabla \bar{\varphi}_i)^{-1}) \\
 381 \quad & \leq \liminf_{j \rightarrow +\infty} \mathcal{F}_{1,\gamma_j}(\{\varphi_{i,j}, \theta_{T_i,j}, V_{i,j}, W_{i,j}\}_{i=1}^M, \{\theta_{\bar{T}_i,l,j}\}_{l=1,\dots,N}^M, \theta_{R,j}), \\
 382 \quad & \leq \limsup_{j \rightarrow +\infty} \mathcal{F}_{1,\gamma_j}(\{\varphi_{i,j}, \theta_{T_i,j}, V_{i,j}, W_{i,j}\}_{i=1}^M, \{\theta_{\bar{T}_i,l,j}\}_{l=1,\dots,N}^M, \theta_{R,j}), \\
 383 \quad & \leq \limsup_{j \rightarrow +\infty} \inf \mathcal{F}_1(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M) + \frac{2}{\gamma_j} = \inf \mathcal{F}_i(\theta_R, \{\theta_{T_i}, \varphi_i\}_{i=1}^M).
 \end{aligned}$$

385 By finiteness of  $\int_{\Omega} W_{Op}(\nabla \bar{\varphi}_i, \text{Cof} \nabla \bar{\varphi}_i, \det \nabla \bar{\varphi}_i) \, dx$  and by continuity of the trace operator, we  
 386 have that  $\bar{\varphi}_i \in \hat{\mathcal{W}}$ . By strong convergence in  $L^1(\Omega)$  and so almost everywhere, we get that  
 387  $\bar{\theta}_{T_i} \in \mathcal{U}$  and  $\bar{\theta}_R \in \mathcal{U}$  with  $\bar{\theta}_{T_i,l} \circ \bar{\varphi}_i - \bar{\theta}_{R,l} \in BV(\Omega, \{-1, 0, 1\})$  for all  $l \in \{1, \dots, N\}$  and all  
 388  $i \in \{1, \dots, N\}$ . Therefore our decoupled problem (DP) converges to the initial problem (P)  
 389 as  $\gamma$  tends to infinity. ■

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**Algorithm SM3.1** Alternating scheme of resolution.

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1. Define  $k := 1$ ,  $T_i := i$ -th template image,  $\theta_{T_i} :=$  Potts segmentation of  $T_i$ ,

$$\theta_R := \text{Potts segmentation of } \frac{1}{M} \sum_{i=1}^M T_i, \quad \theta_{\tilde{T}_i} := \theta_{T_i} - \theta_R, \quad V_i = \begin{pmatrix} V_{i,11} & V_{i,12} \\ V_{i,21} & V_{i,22} \end{pmatrix} := I, \quad W_i :=$$

$$\begin{pmatrix} W_{i,11} & W_{i,12} \\ W_{i,21} & W_{i,22} \end{pmatrix} := I, \quad a_1, a_2, a_3, \lambda_T, \lambda_R, \gamma_T, \gamma_R, \gamma_1, \gamma_2, \gamma_3, nbIter, \alpha, \beta, U_i = (U_{i,1}, U_{i,2}) :=$$

0, displacements associated to the deformation  $\varphi_i$ , for  $i = 1, \dots, M$ .

**while**  $k < nbIter$  **do**

**if**  $k \% 10 == 0$  **then**

    2.1. Compute for  $i = 1, \dots, M$  the inverse deformation  $\varphi_i^{-1}$  using a Delaunay triangulation and linear interpolation.

    2.2. For  $i = 1, \dots, M$ , update  $\theta_{T_i}$  by solving the Potts model with algorithm in [?]:  
 $\inf_u \gamma_T \|\nabla u\|_{L^0(\Omega)} + \left\| \sqrt{\lambda_T + (\det \nabla \varphi_i)^{-1} \frac{\gamma_1}{2}} u - \frac{\lambda_T T_i + \frac{\gamma_1}{2} (\det \nabla \varphi_i)^{-1} (\theta_{\tilde{T}_i} \circ \varphi_i^{-1} + \theta_R \circ \varphi_i^{-1})}{\sqrt{\lambda_T + (\det \nabla \varphi_i)^{-1} \frac{\gamma_1}{2}}} \right\|_{L^2(\Omega)}^2$  end for 2.2.

    2.3. For  $i = 1, \dots, M$ , update  $\theta_{\tilde{T}_i}$  by solving the Potts model with algorithm in [?]:  
 $\inf_u \gamma_{\tilde{T}} \|\nabla u\|_{L^0(\Omega)} + \frac{\gamma_1}{2} \|(\theta_{T_i} \circ \varphi_i - \theta_R) - u\|_{L^2(\Omega)}^2$  end for 2.3.

    2.4. Update  $\theta_R$  by solving the Potts model with algorithm in [?]:  $\inf_u \gamma_R \|\nabla u\|_{L^0(\Omega)} +$   
 $(\lambda_R + \frac{\gamma_1}{2}) \|u - \frac{\frac{\lambda_R}{M} \sum_{i=1}^M T_i \circ \varphi_i + \frac{\gamma_1}{2M} \sum_{i=1}^M (\theta_{T_i} \circ \varphi_i - \theta_{\tilde{T}_i})}{\lambda_R + \frac{\gamma_1}{2}}\|_{L^2(\Omega)}^2$ .

**end if**

    2.5. For each  $i = 1, \dots, M$ , for each pixel  $(l, j)$ ,  $-c$  playing a role similar to the one of a step size in a gradient method— update  $V_i$  using the following equations:

$$\left\{ \begin{array}{l} temp_1(l, j) = V_{i,11}(l, j) + c \left( \frac{10a_3}{(\det V_i(l, j))^{11}} V_{i,22}(l, j) - 4a_1 V_{i,11}(l, j) \right. \\ \quad \|V_i(l, j)\|^2 - 2a_2 (\det V_i(l, j) - 1) V_{i,22}(l, j) + \gamma_2 \left( \frac{\partial \varphi_{i,1}}{\partial x}(l, j) \right. \\ \quad \left. - V_{i,11}(l, j) \right) - \gamma_3 \left( W_{i,11}(l, j) - \frac{V_{i,22}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,22}(l, j)^2}{(\det V_i(l, j))^2} \right) \\ \quad + \gamma_3 \left( W_{i,12}(l, j) + \frac{V_{i,12}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,22}(l, j) V_{i,12}(l, j)}{(\det V_i(l, j))^2} \right) \\ \quad + \gamma_3 \left( W_{i,21}(l, j) + \frac{V_{i,21}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,22}(l, j) V_{i,21}(l, j)}{(\det V_i(l, j))^2} \right) \\ \quad \left. - \gamma_3 \left( W_{i,22}(l, j) - \frac{V_{i,11}(l, j)}{\det V_i(l, j)} \right) \left( -\frac{1}{\det V_i(l, j)} + \frac{V_{i,11}(l, j) V_{i,22}(l, j)}{(\det V_i(l, j))^2} \right) \right), \\ V_{i,11}(l, j) = \begin{cases} -\alpha & \text{if } temp_1(l, j) < -\alpha, \\ temp_1(l, j) & \text{if } |temp_1(l, j)| \leq \alpha, \\ \alpha & \text{if } temp_1(l, j) > \alpha \end{cases}, \\ temp_2(l, j) = V_{i,12}(l, j) + c \left( \frac{10a_3}{(\det V_i(l, j))^{11}} (-V_{i,21}(l, j)) - 4a_1 (V_{i,12}(l, j)) \right. \\ \quad \|V_i(l, j)\|^2 + 2a_2 (\det V_i(l, j) - 1) (V_{i,21}(l, j)) + \gamma_2 \left( \frac{\partial \varphi_{i,1}}{\partial y}(l, j) \right. \\ \quad \left. - V_{i,12}(l, j) \right) + \gamma_3 \left( W_{i,11}(l, j) - \frac{V_{i,22}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,22}(l, j) V_{i,21}(l, j)}{(\det V_i(l, j))^2} \right) \\ \quad - \gamma_3 \left( W_{i,12}(l, j) + \frac{V_{i,12}(l, j)}{\det V_i(l, j)} \right) \left( \frac{1}{\det V_i(l, j)} + \frac{V_{i,21}(l, j) V_{i,12}(l, j)}{(\det V_i(l, j))^2} \right) \\ \quad - \gamma_3 \left( W_{i,21}(l, j) + \frac{V_{i,21}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,21}(l, j)^2}{(\det V_i(l, j))^2} \right) \\ \quad \left. + \gamma_3 \left( W_{i,22}(l, j) - \frac{V_{i,11}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,11}(l, j) V_{i,21}(l, j)}{(\det V_i(l, j))^2} \right) \right), \\ V_{i,12}(l, j) = \begin{cases} -\alpha & \text{if } temp_2(l, j) < -\alpha, \\ temp_2(l, j) & \text{if } |temp_2(l, j)| \leq \alpha, \\ \alpha & \text{if } temp_2(l, j) > \alpha \end{cases}, \end{array} \right.$$



$$\begin{aligned}
 2.5. \quad & \left\{ \begin{aligned}
 temp_3(l, j) &= V_{i,21}(l, j) + c \left( \frac{10a_3}{(\det V_i(l, j))^{\Pi}} (-V_{i,12}(l, j)) - 4a_1(V_{i,21}(l, j)) \right. \\
 &\quad \left. \|V_i(l, j)\|^2 - 2a_2(\det V_i(l, j) - 1)(-V_{i,12}(l, j)) + \gamma_2 \left( \frac{\partial \varphi_{i,2}}{\partial x}(l, j) \right. \right. \\
 &\quad \left. \left. - V_{i,21}(l, j) \right) + \gamma_3 \left( W_{i,11}(l, j) - \frac{V_{i,22}(l, j)}{\det V_i(l, j)} \right) \right. \\
 &\quad \left. \left( \frac{V_{i,22}(l, j)V_{i,12}(l, j)}{(\det V_i(l, j))^2} \right) - \gamma_3 \left( W_{i,12}(l, j) + \frac{V_{i,12}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,12}(l, j)^2}{(\det V_i(l, j))^2} \right) \right. \\
 &\quad \left. - \gamma_3 \left( W_{i,21}(l, j) + \frac{V_{i,21}(l, j)}{\det V_i(l, j)} \right) \left( \frac{1}{\det V_i(l, j)} + \frac{V_{i,12}(l, j)V_{i,21}(l, j)}{(\det V_i(l, j))^2} \right) \right. \\
 &\quad \left. + \gamma_3 \left( W_{i,22}(l, j) - \frac{V_{i,11}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,11}(l, j)(V_{i,12}(l, j))}{(\det V_i(l, j))^2} \right) \right), \\
 V_{i,21}(l, j) &= \begin{cases} -\alpha & \text{if } temp_3(l, j) < -\alpha, \\ temp_3(l, j) & \text{if } |temp_3(l, j)| \leq \alpha, \\ \alpha & \text{if } temp_3(l, j) > \alpha \end{cases} \\
 temp_4(l, j) &= V_{i,22}(l, j) + c \left( \frac{10a_3}{(\det V_i(l, j))^{\Pi}} V_{i,11}(l, j) - 4a_1(V_{i,22}(l, j)) \right. \\
 &\quad \left. \|V_i(l, j)\|^2 - 2a_2(\det V_i(l, j) - 1)V_{i,11}(l, j) \right. \\
 &\quad \left. + \gamma_2 \left( \frac{\partial \varphi_{i,2}}{\partial y}(l, j) - V_{i,22}(l, j) \right) - \gamma_3 \left( W_{i,11}(l, j) - \frac{V_{i,22}(l, j)}{\det V_i(l, j)} \right) \right. \\
 &\quad \left. \left( -\frac{1}{\det V_i(l, j)} + \frac{V_{i,22}(l, j)V_{i,11}(l, j)}{(\det V_i(l, j))^2} \right) + \gamma_3 \left( W_{i,12}(l, j) \right. \right. \\
 &\quad \left. \left. + \frac{V_{i,12}(l, j)}{\det V_i(l, j)} \right) \left( \frac{V_{i,11}(l, j)V_{i,12}(l, j)}{(\det V_i(l, j))^2} \right) + \gamma_3 \left( W_{i,21}(l, j) + \frac{V_{i,21}(l, j)}{\det V_i(l, j)} \right) \right. \\
 &\quad \left. \left( \frac{V_{i,11}(l, j)V_{i,21}(l, j)}{(\det V_i(l, j))^2} \right) - \gamma_3 \left( W_{i,22}(l, j) - \frac{V_{i,11}(l, j)}{\det V_i(l, j)} \right) \right. \\
 &\quad \left. \left( \frac{(V_{i,11}(l, j))^2}{(\det V_i(l, j))^2} \right) \right), \\
 V_{i,22}(l, j) &= \begin{cases} -\alpha & \text{if } temp_4(l, j) < -\alpha, \\ temp_4(l, j) & \text{if } |temp_4(l, j)| \leq \alpha, \\ \alpha & \text{if } temp_4(l, j) > \alpha \end{cases} .
 \end{aligned} \right.
 \end{aligned}$$

end for 2.5.

2.6. For each  $i = 1, \dots, M$ , for each pixel  $(l, j)$ , update  $W_i$  with this closed form:

$$\left\{ \begin{aligned}
 W_{i,11}(l, j) &= \begin{cases} -\beta & \text{if } \frac{V_{i,22}(l, j)}{\det V_i(l, j)} < -\beta \\ \frac{V_{i,22}(l, j)}{\det V_i(l, j)} & \text{if } \left| \frac{V_{i,22}(l, j)}{\det V_i(l, j)} \right| \leq \beta \\ \beta & \text{if } \frac{V_{i,22}(l, j)}{\det V_i(l, j)} > \beta \end{cases} \\
 W_{i,12}(l, j) &= \begin{cases} -\beta & \text{if } \frac{-V_{i,12}(l, j)}{\det V_i(l, j)} < -\beta \\ \frac{-V_{i,12}(l, j)}{\det V_i(l, j)} & \text{if } \left| \frac{-V_{i,12}(l, j)}{\det V_i(l, j)} \right| \leq \beta \\ \beta & \text{if } \frac{-V_{i,12}(l, j)}{\det V_i(l, j)} > \beta \end{cases} \\
 W_{i,21}(l, j) &= \begin{cases} -\beta & \text{if } \frac{-V_{i,21}(l, j)}{\det V_i(l, j)} < -\beta \\ \frac{-V_{i,21}(l, j)}{\det V_i(l, j)} & \text{if } \left| \frac{-V_{i,21}(l, j)}{\det V_i(l, j)} \right| \leq \beta \\ \beta & \text{if } \frac{-V_{i,21}(l, j)}{\det V_i(l, j)} > \beta \end{cases} \\
 W_{i,22}(l, j) &= \begin{cases} -\beta & \text{if } \frac{V_{i,11}(l, j)}{\det V_i(l, j)} < -\beta \\ \frac{V_{i,11}(l, j)}{\det V_i(l, j)} & \text{if } \left| \frac{V_{i,11}(l, j)}{\det V_i(l, j)} \right| \leq \beta \\ \beta & \text{if } \frac{V_{i,11}(l, j)}{\det V_i(l, j)} > \beta \end{cases}
 \end{aligned} \right. \quad \text{end for 2.6.}$$

2.7. Solve for all  $i = 1, \dots, M$ , the Euler-Lagrange equation in  $U_i$  using an implicit finite difference scheme:  $0 = \gamma_1 \nabla \theta_{T_i} \circ \varphi_i (\theta_{T_i} \circ \varphi_i - \theta_R - \theta_{\tilde{T}_i}) + \lambda_R (T_i \circ \varphi_i - \theta_R) \nabla T_i \circ \varphi_i + \gamma_2 \begin{pmatrix} \text{div} V_{i,1} \\ \text{div} V_{i,2} \end{pmatrix}$ ,

where  $V_{i,j}$  stands for the  $j^{\text{th}}$  row of  $V_i$ , and  $\varphi_i = Id + U_i$ . end for 2.7.

2.8.  $k := k + 1$ .

end while

390 **SM3. Overall algorithm.**

391 **SM4. Detailed presentation of the first two methods to perform PCA.**

392 **SM4.1. First Approach : Linearisation around the Identity.** Let us recall the definition  
393 of the Cauchy-stress tensors and the Cauchy axiom relating, in a hyperelastic framework, the  
394 deformations and the forces applied on the material.

**Definition SM4.1 (Cauchy-stress tensor).** *In a hyperelastic framework, the Cauchy stress tensor in the reference configuration, is defined by*

$$\sigma^{ref}(x) = \frac{\partial W}{\partial F}(x, F),$$

395 where  $W$  is the stored energy function of the material. In a deformed configuration (char-  
396 acterised by the transformation  $\varphi$ ), the first Piola-Kirchhoff tensor is defined by  $\sigma(x) =$   
397  $\sigma^{ref}(\varphi^{-1}(x))\text{Cof}\nabla\varphi^{-1}(x)$ .

398 **Axiom SM4.2 (Cauchy axiom).** *In an equilibrium position the following relations hold*

$$\begin{aligned} 399 \quad \forall y \in \Gamma_1 \subset \partial\Omega, t(y, \vec{n}) &= \sigma^{ref}(y)\vec{n}, \\ 400 \quad \forall y \in \Omega, f(y) &= -\text{div} \sigma^{ref}(y), \end{aligned}$$

402 where  $t(y, \vec{n})$  is the pressure applied to the material at the boundary point  $y$  in the normal  
403 direction  $\vec{n}$ ,  $f(y)$  is the inner volumetric force applied at  $y \in \Omega$ .

404 **Remark SM4.3.** In our model, we aim at having  $\theta_{T_i} \circ \varphi_i$  close to  $\theta_R$ , and therefore  $\varphi_i$  is  
405 the deformation from  $\theta_R$  to  $\theta_{T_i}$ . The deformation from  $\theta_{T_i}$  to  $\theta_R$  is  $\varphi_i^{-1}$  and thanks to Ball's  
406 results [?], we have that

$$\begin{aligned} 407 \quad \int_{\Omega} W_{Op}(\nabla\varphi_i) dx &= \int_{\Omega} \tilde{W}(\nabla\varphi_i, \text{Cof}\nabla\varphi_i, \det\nabla\varphi_i) dx, \\ 408 \quad &= \int_{\Omega} \tilde{W}\left(\frac{\text{Cof}\nabla(\varphi_i^{-1})(\varphi_i)^T}{\det\nabla(\varphi_i^{-1})(\varphi_i)}, \frac{\nabla(\varphi_i^{-1})(\varphi_i)^T}{\det\nabla(\varphi_i^{-1})(\varphi_i)}, \frac{1}{\det\nabla(\varphi_i^{-1})(\varphi_i)}\right) dx, \\ 409 \quad &= \int_{\Omega} \det\nabla(\varphi_i^{-1})\tilde{W}\left(\frac{\text{Cof}\nabla(\varphi_i^{-1})^T}{\det\nabla(\varphi_i^{-1})}, \frac{\nabla(\varphi_i^{-1})^T}{\det\nabla(\varphi_i^{-1})}, \frac{1}{\det\nabla(\varphi_i^{-1})}\right) dx, \\ 410 \quad &= \int_{\Omega} \tilde{W}_{Op}(\nabla(\varphi_i^{-1})) dx, \\ 411 \end{aligned}$$

412 with  $\tilde{W}(\psi, \phi, \delta) = \begin{cases} a_1\|\psi\|^4 + a_2\|\phi\|^4 + a_3(\delta - 1)^2 + \frac{a_4}{\delta^{10}} - 3a_1 - 3a_2 - a_4 & \text{if } \delta > 0, \\ +\infty & \text{otherwise} \end{cases}$ . There-  
413 fore, the Cauchy-stress tensor applied to the reference configuration  $\theta_{T_i}$  associated to the  
414 deformation  $\varphi_i^{-1}$  to transform  $\theta_{T_i}$  into  $\theta_R$  is  $\sigma_i^{ref}(x) = \frac{\partial \tilde{W}_{Op}}{\partial F}(\nabla\varphi_i^{-1}(x))$ . The correspond-  
415 ing first Piola-Kirchhoff tensor applied to the deformed configuration  $\theta_R$  is given by  $\sigma_i =$   
416  $\frac{\partial \tilde{W}_{Op}}{\partial F}(\nabla\varphi_i^{-1}(\varphi_i))\text{Cof}\nabla\varphi_i$ . The inner volumetric forces applied to  $\theta_R$  that are necessary to  
417 deform  $\theta_{T_i}$  into  $\theta_R$  are given by  $-\text{div}\sigma_i$ .

418 Equipped with this material and the previous observation, we consider our mean configuration  
 419  $\theta_R$  as a free material (not pre-stressed) and look for the displacements  $v_i$  induced by a small  
 420 increase of the impact of  $\theta_{T_i}$  parameterised by  $\delta$  and minimising the following energy for each  
 421  $i = 1, \dots, M$ :

$$\begin{aligned}
 422 \quad \inf_{v_i \in \mathcal{V}} \left\{ \mathcal{F}_i^\delta(v_i) &= \int_{\Omega} \tilde{W}_{Op}((\mathbf{I} + \delta \nabla v_i)) \, dx - \delta^2 \int_{\Omega} \operatorname{div} \sigma_i : v_i \, dx, \\
 423 \quad &= \int_{\Omega} \tilde{W}_{Op}((\mathbf{I} + \delta \nabla v_i)) \, dx + \delta^2 \int_{\Omega} \sigma_i : \nabla v_i \, dx - \delta^2 \int_{\partial\Omega} \sigma_i v_i \vec{n}_{\partial\Omega} \, ds, \\
 424 \quad &= \int_{\Omega} \tilde{W}_{Op}((\mathbf{I} + \delta \nabla v_i)) \, dx + \delta^2 \int_{\Omega} \sigma_i : \nabla v_i \, dx \right\}, \\
 425
 \end{aligned}$$

426 with  $\mathcal{V} := \{\psi \in W_0^{1,4}(\Omega, \mathbb{R}^3) \mid \operatorname{Cof}(\mathbf{I} + \delta \nabla \psi) \in L^4(\Omega, M_3(\mathbb{R})), \det(\mathbf{I} + \delta \nabla \psi) \in L^2(\Omega), \frac{1}{\det(\mathbf{I} + \delta \nabla \psi)}$   
 427  $\in L^{10}(\Omega), \det(\mathbf{I} + \delta \psi) > 0$  a.e. on  $\Omega\}$ . The first term corresponds to the stored energy of the  
 428 chosen Ogden material, while the second component represents the work of the applied forces  
 429 with the following boundary conditions:  $v_i = 0$  on  $\partial\Omega$ . However, the solution of this problem  
 430 still lives in a nonlinear space and we thus propose to linearise the stored energy function  $\tilde{W}_{Op}$   
 431 around the identity. Before linearising the functional using a Taylor development, we introduce  
 432 the deformation Cauchy-Green tensor  $b_i = (\mathbf{I} + \delta v_i)^T (\mathbf{I} + \delta v_i) = \mathbf{I} + \delta(\nabla v_i^T + \nabla v_i) + \delta^2 \nabla v_i^T \nabla v_i =$   
 433  $\mathbf{I} + 2\delta\epsilon(v_i) + \delta^2 \nabla v_i^T \nabla v_i \approx \mathbf{I} + 2\delta\epsilon(v_i)$ . We first linearise the deformation tensor then the stored  
 434 energy function and will see that we come back to the linearised elasticity problem.

$$\begin{aligned}
 435 \quad \mathcal{F}_i^\delta(v_i) &= \int_{\Omega} \tilde{W}_{Op}(\mathbf{I} + \delta \nabla v_i) \, dx + \delta^2 \int_{\Omega} \nabla v_i : \sigma_i \, dx, \\
 436 \quad &= \int_{\Omega} \bar{W}_{Op}(b_i) + \delta^2 \int_{\Omega} \nabla v_i : \sigma_i \, dx, \\
 437 \quad &= \int_{\Omega} \bar{W}_{Op}(\mathbf{I} + 2\delta\epsilon(v_i)) + \delta^2 \int_{\Omega} \nabla v_i : \sigma_i \, dx, \\
 438 \quad &= \int_{\Omega} \bar{W}_{Op}(\mathbf{I}) \, dx + 2\delta \int_{\Omega} \frac{\partial \bar{W}_{Op}}{\partial b}(\mathbf{I}) : \epsilon(v_i) \, dx + 2\delta^2 \int_{\Omega} \frac{\partial^2 \bar{W}_{Op}}{\partial b^2}(\mathbf{I}) : \epsilon(v_i) : \epsilon(v_i) \, dx \\
 439 \quad &+ \delta^2 \int_{\Omega} \nabla v_i : \sigma_i \, dx, \\
 440
 \end{aligned}$$

441 with  $\bar{W}_{Op}(b) = a_1 \frac{I_b^2}{\sqrt{III_b^3}} + a_2 \frac{I_b^2}{\sqrt{III_b^3}} + a_4 \sqrt{III_b}^{11} + \frac{a_3}{\sqrt{III_b}} - 2a_3 + a_3 \sqrt{III_b} - (9a_1 + 9a_2 + a_4) \sqrt{III_b}$ ,  
 442  $I_b = \operatorname{Tr}(b) = \|\mathbf{I} + \delta v_i\|_F^2$ ,  $III_b = \frac{\operatorname{Tr}(b)^2 - \operatorname{Tr}(b^2)}{2} = \|\operatorname{Cof}(\mathbf{I} + \delta v_i)\|_F^2$  and  $III_b = \det b = \det(\mathbf{I} + \delta v_i)^2$ .  
 443 The detailed computations of the derivatives are given in [section SM5](#). For our configuration  
 444 to be in equilibrium, we need  $\frac{\partial \bar{W}_{Op}}{\partial b}|_{(I_b, III_b, III_b)=(3,3,1)} = 0$ , implying  $a_4 = \frac{12a_2 + 6a_1}{5}$ . We also  
 445 set  $\mu = 12a_1 + 12a_2 > 0$  and  $\lambda = 4(32a_1 + 74a_2 + \frac{a_3}{2}) > 0$ . Since  $\mathbf{I} \otimes \mathbf{I} : \epsilon(v) : \epsilon(v) = \operatorname{Tr}(\epsilon)^2$ ,  
 446  $\mathbf{I} : \epsilon(v) : \epsilon(v) = \bar{\mathbf{I}} : \epsilon(v) : \epsilon(v) = \operatorname{Tr}(\epsilon^2)$  by symmetry of  $\epsilon(v)$ , we finally obtain a linearised

447 elasticity functional with  $v_i \in H_0^1(\Omega, \mathbb{R}^3)$ :

$$\begin{aligned}
 448 \quad \mathcal{F}_i^\delta(v_i) &= \int_{\Omega} \tilde{W}_{Op}(\mathbf{I} + \delta v_i) dx + \delta^2 \int_{\Omega} \nabla v_i : \sigma_i dx, \\
 449 \quad &= \int_{\Omega} \bar{W}_{Op}(\mathbf{I}) dx + 2\delta \int_{\Omega} \frac{\partial \bar{W}_{Op}}{\partial b}(\mathbf{I}) : \epsilon(v_i) dx + 2\delta^2 \int_{\Omega} \frac{\partial^2 \bar{W}_{Op}}{\partial b^2}(\mathbf{I}) : \epsilon(v_i) : \epsilon(v_i) dx \\
 450 \quad &+ \delta^2 \int_{\Omega} \nabla v_i : \sigma_i dx, \\
 451 \quad &= 2\delta^2 \int_{\Omega} \mu \text{Tr}(\epsilon(v_i)^2) + \frac{\lambda}{2} \text{Tr}(\epsilon(v_i))^2 dx + \delta^2 \int_{\Omega} \sigma_i : \nabla v_i dx. \\
 452
 \end{aligned}$$

453 We now consider the following linearised elasticity problem (LEP)

$$454 \quad (\text{LEP}) \quad \inf_{v_i \in H_0^1(\Omega, \mathbb{R}^3)} \{F_i^\delta(v_i) = 2\delta^2 \int_{\Omega} \mu \text{Tr}(\epsilon(v_i)^2) + \frac{\lambda}{2} \text{Tr}(\epsilon(v_i))^2 dx + \delta^2 \int_{\Omega} \sigma_i : \nabla v_i dx\}.$$

456 By using the Lax-Milgram theorem, one can prove the following theorem.

457 **Theorem SM4.4 (Existence and uniqueness of a solution to problem (LEP)).** *There exists a*  
 458 *unique solution of (LEP) in the linear space  $H_0^1(\Omega, \mathbb{R}^2)$ .*

459 *Proof.* In order to prove the existence and uniqueness of a minimizer in  $H_0^1(\Omega)$ , we recall  
 460 the Lax-Milgram theorem :

461 **Theorem SM4.5 (Lax-Milgram).** *Let  $\mathcal{H}$  be a Hilbert space. Let  $a(., .)$  be a continuous bilinear*  
 462 *form on  $\mathcal{H} \times \mathcal{H}$  and coercive on  $\mathcal{H}$ . Let  $L(.)$  be a linear continuous form on  $\mathcal{H}$ . Then there*  
 463 *exists a unique element of  $\mathcal{H}$ ,  $u$ , such that for all  $v \in \mathcal{H}$ ,  $a(u, v) = L(v)$ .*  
 464 *Furthermore, if  $a(., .)$  is symmetric, then  $u$  minimizes  $J(v) = \frac{1}{2}a(v, v) - L(v)$ , that is to say*  
 465 *there exists a unique  $u \in \mathcal{H}$  such that  $J(u) = \min_{v \in \mathcal{H}} J(v)$ .*

466 We now set  $\mathcal{H} = H_0^1(\Omega, \mathbb{R}^3)$ ,  $a(u, v) = \int_{\Omega} 4\delta^2(\mu \epsilon(u) : \epsilon(v) + \frac{\lambda}{2} \text{Tr}(\epsilon(u)) \text{Tr}(\epsilon(v))) dx$  for all  
 467  $(u, v) \in H_0^1(\Omega, \mathbb{R}^3) \times H_0^1(\Omega, \mathbb{R}^3)$ , and  $L(v) = -\delta^2 \int_{\Omega} \sigma_i : \nabla v_i dx$  for all  $v \in H_0^1(\Omega, \mathbb{R}^3)$ .  $L$  is  
 468 linear on  $H_0^1(\Omega, \mathbb{R}^3)$  by linearity of the gradient operator and the scalar product in  $M_3(\mathbb{R})$ .  
 469 It is also continuous on  $H_0^1(\Omega, M_3(\mathbb{R}))$  by using Cauchy-Schwartz inequality and Poincaré's  
 470 inequality,  $\forall v \in H_0^1(\Omega, \mathbb{R}^3)$ :

$$471 \quad L(v) = \int_{\Omega} \sigma_i : \nabla v dx \leq \|\sigma_i\|_{L^2(\Omega, M_3(\mathbb{R}))} \|\nabla v\|_{L^2(\Omega, M_3(\mathbb{R}))} \leq \|\sigma_i\|_{L^2(\Omega, M_3(\mathbb{R}))} \|v\|_{H^1(\Omega, \mathbb{R}^3)},$$

473 with  $\|\sigma_i\|_{L^2(\Omega, M_3(\mathbb{R}))} < +\infty$  since we have proved earlier that  $\sigma_i \in L^2(\Omega, M_3(\mathbb{R}))$ .  
 474  $a(., .)$  is bilinear on  $H_0^1(\Omega, \mathbb{R}^3) \times H_0^1(\Omega, \mathbb{R}^3)$  by linearity of  $\epsilon$  coming from the linearity of the  
 475 gradient operator and the linearity of the scalar product in  $M_3(\mathbb{R})$  and the trace operator.  
 476  $a(., .)$  is also symmetric by symmetry of  $\epsilon(u)$  and  $\epsilon(v)$ .  $a(., .)$  is also continuous on  $H_0^1(\Omega, \mathbb{R}^3) \times$

477  $H_0^1(\Omega, \mathbb{R}^3)$  since we have :

$$\begin{aligned}
 478 \quad & \forall (u, v) \in H_0^1(\Omega, \mathbb{R}^3) \times H_0^1(\Omega, \mathbb{R}^3), a(u, v) = \int_{\Omega} 4\delta^2(\mu\epsilon(u) : \epsilon(v) + \frac{\lambda}{2}Tr(\epsilon(u))Tr(\epsilon(v))) dx, \\
 479 \quad & \leq 4\delta^2\mu\|\epsilon(u)\|_{L^2(\Omega, M_3(\mathbb{R}))}\|\epsilon(v)\|_{L^2(\Omega, M_3(\mathbb{R}))} + 2\delta^2\lambda\|Tr(\epsilon(u))\|_{L^2(\Omega)}\|Tr(\epsilon(v))\|_{L^2(\Omega)}, \\
 480 \quad & \leq 4\delta^2\mu\|\nabla u\|_{L^2(\Omega, M_3(\mathbb{R}))}\|\nabla v\|_{L^2(\Omega, M_3(\mathbb{R}))} + 2\delta^2\lambda\|\nabla u\|_{L^2(\Omega, M_3(\mathbb{R}))}\|\nabla v\|_{L^2(\Omega, M_3(\mathbb{R}))}, \\
 481 \quad & \leq (4\delta^2\mu + 2\delta^2\lambda)\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}, \text{ Poincaré's inequality.}
 \end{aligned}$$

483 We now need to prove that  $a(\cdot, \cdot)$  is coercive. Let  $v \in H_0^1(\Omega, \mathbb{R}^3)$ , then we have

$$484 \quad a(v, v) = \int_{\Omega} 4\delta^2(\mu\epsilon(v) : \epsilon(v) + \frac{\lambda}{2}Tr(\epsilon(v))^2) dx \geq 4\delta^2\mu\|\epsilon(v)\|_{L^2(\Omega, M_3(\mathbb{R}))}^2.$$

486 Since  $4\delta^2\mu > 0$ , we just need to prove the coercivity of  $\|\epsilon(v)\|_{L^2(\Omega, M_3(\mathbb{R}))}$ . To do so, we  
 487 use a reductio ad absurdum reasoning as in [?] and assume there exists a sequence  $(v_n) \in$   
 488  $H_0^1(\Omega, M_3(\mathbb{R}))$  such that  $\|\epsilon(v_n)\|_{L^2(\Omega, M_3(\mathbb{R}))} \xrightarrow{n \rightarrow +\infty} 0$  and  $\|v_n\|_{H_0^1(\Omega, \mathbb{R}^3)} = 1$ . As  $(v_n)$  is uniformly  
 489 bounded according to  $n$  in  $H^1(\Omega, \mathbb{R}^3)$ , we can extract a subsequence still denoted  $(v_n)$  such  
 490 that  $v_n \xrightarrow{n \rightarrow +\infty} v^*$  in  $H^1(\Omega, \mathbb{R}^3)$  and by continuity of the trace operator, we get that  $v^* \in$   
 491  $H_0^1(\Omega)$ . We also have that  $\epsilon(v_n) \xrightarrow{n \rightarrow +\infty} \epsilon(v^*)$  in  $L^2(\Omega, M_3(\mathbb{R}))$ . However, we have assumed  
 492 that  $\epsilon(v_n) \xrightarrow{n \rightarrow +\infty} 0$  in  $L^2(\Omega, M_3(\mathbb{R}))$  and so by uniqueness of the weak limit, we get that  
 493  $\epsilon(v^*) = 0$ . Then by [?, Lemma IV.3.3], it comes that  $v^*$  is an infinitesimal rigid displacement.  
 494 But a rigid displacement can only be null on a straight line if  $a \neq 0$ , nowhere if  $a \neq 0, b \neq 0$   
 495 or everywhere if  $a = b = 0$ . Since  $v^*$  is null on  $\partial\Omega$  with  $\text{meas}(\partial\Omega) > 0$ , then we are necessary  
 496 in the last case and so  $v^* = 0$ . Then Rellich theorem allows us to deduce that  $v_n \rightarrow 0$  in  
 497  $L^2(\Omega, \mathbb{R}^3)$ . Then by using Korn's inequality ([?, Theorem IV.3.2]), we get that  $\nabla v_n \xrightarrow{n \rightarrow +\infty} 0$   
 498 in  $L^2(\Omega, M_3(\mathbb{R}))$  so that  $v_n \xrightarrow{n \rightarrow +\infty} 0$  in  $H^1(\Omega, \mathbb{R}^3)$  which contradicts our second hypothesis.  
 499 Therefore,  $a(\cdot, \cdot)$  is coercive on  $H_0^1(\Omega, \mathbb{R}^3)$ . Lax Milgram's theorem applies and there exists a  
 500 unique solution to our linearised elasticity problem (LEP) in  $H_0^1(\Omega)$ . ■

501 This problem can be solved in the finite element approximation framework which results in  
 502 the resolution of a sparse high-dimensional linear system. A classical PCA is then performed  
 503 on the obtained displacement fields using the  $L^2(\Omega, \mathbb{R}^3)$  scalar product for the covariance  
 504 operator. We now address the second method based on the Cauchy-stress tensors.

505 **SM4.2. Second Approach : PCA on the Cauchy-stress Tensors.** This approach relies  
 506 on the following lemma.

507 **Lemma SM4.6.** *The Cauchy-stress tensors  $\sigma_i^{ref}$  and  $\sigma_i$  are in  $L^\infty(\Omega, M_3(\mathbb{R})) \subset L^2(\Omega, M_3(\mathbb{R}))$ .* ■

508 *Proof.* Let us compute  $\sigma_i^{ref}$  and  $\sigma_i$  and make sure they belong to  $L^\infty(\Omega, M_3(\mathbb{R}))$ . We first  
 509 recall that  $\sigma_i^{ref} = \frac{\partial \bar{W}_{Op}}{\partial F}(\nabla(\varphi_i^{-1})) = \frac{\partial}{\partial F}(\det F(a_1 \frac{\|Cof F\|_F^4}{\det F^4} + a_2 \frac{\|F\|_F^4}{\det F^4} + a_4 \det F^{10} - a_3(\frac{1}{\det F} - 1)^2 -$

510  $9a_1 - 9a_2 - a_4)(\nabla(\varphi_i^{-1}))$ . We compute the following terms :

$$511 \quad F = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix},$$

$$512 \quad \text{Cof} F = \begin{pmatrix} F_{22}F_{33} - F_{32}F_{23} & F_{31}F_{23} - F_{21}F_{33} & F_{21}F_{32} - F_{31}F_{22} \\ F_{32}F_{13} - F_{12}F_{33} & F_{11}F_{33} - F_{13}F_{31} & F_{31}F_{12} - F_{11}F_{32} \\ F_{12}F_{23} - F_{22}F_{13} & F_{21}F_{13} - F_{11}F_{23} & F_{11}F_{22} - F_{21}F_{12} \end{pmatrix},$$

$$513 \quad \det F = F_{11}(F_{22}F_{33} - F_{32}F_{23}) + F_{12}(F_{31}F_{23} - F_{21}F_{33}) + F_{13}(F_{21}F_{32} - F_{31}F_{22}),$$

$$514 \quad = F_{21}(F_{32}F_{13} - F_{12}F_{33}) + F_{22}(F_{11}F_{33} - F_{13}F_{31}) + F_{23}(F_{31}F_{12} - F_{11}F_{32}),$$

$$515 \quad = F_{31}(F_{12}F_{23} - F_{22}F_{13}) + F_{32}(F_{21}F_{13} - F_{11}F_{23}) + F_{33}(F_{11}F_{22} - F_{21}F_{12}),$$

$$516 \quad \frac{\partial \|F\|_F^2}{\partial F} = 2F,$$

$$517 \quad \left(\frac{\partial \|\text{Cof} F\|_F^2}{\partial F}\right)_1^T = 2 \begin{pmatrix} F_{33}\text{Cof} F_{22} - F_{32}\text{Cof} F_{23} - F_{23}\text{Cof} F_{32} + F_{22}\text{Cof} F_{33} \\ -F_{33}\text{Cof} F_{21} + F_{31}\text{Cof} F_{23} + F_{23}\text{Cof} F_{31} - F_{21}\text{Cof} F_{33} \\ F_{32}\text{Cof} F_{21} - F_{31}\text{Cof} F_{22} - F_{22}\text{Cof} F_{31} + F_{21}\text{Cof} F_{32} \end{pmatrix},$$

$$518 \quad \left(\frac{\partial \|\text{Cof} F\|_F^2}{\partial F}\right)_2^T = 2 \begin{pmatrix} -F_{31}\text{Cof} F_{12} + F_{32}\text{Cof} F_{13} + F_{13}\text{Cof} F_{32} - F_{12}\text{Cof} F_{33} \\ F_{33}\text{Cof} F_{11} - F_{31}\text{Cof} F_{13} - F_{13}\text{Cof} F_{31} + F_{11}\text{Cof} F_{33} \\ -F_{32}\text{Cof} F_{11} + F_{31}\text{Cof} F_{12} + F_{12}\text{Cof} F_{31} - F_{11}\text{Cof} F_{32} \end{pmatrix},$$

$$519 \quad \left(\frac{\partial \|\text{Cof} F\|_F^2}{\partial F}\right)_3^T = 2 \begin{pmatrix} F_{23}\text{Cof} F_{12} - F_{22}\text{Cof} F_{13} - F_{13}\text{Cof} F_{22} + F_{12}\text{Cof} F_{23} \\ -F_{23}\text{Cof} F_{11} + F_{21}\text{Cof} F_{13} + F_{13}\text{Cof} F_{21} - F_{11}\text{Cof} F_{23} \\ F_{22}\text{Cof} F_{11} - F_{21}\text{Cof} F_{12} - F_{12}\text{Cof} F_{21} + F_{11}\text{Cof} F_{22} \end{pmatrix},$$

$$520 \quad \frac{\partial \det F}{\partial F} = \text{Cof} F,$$

$$521 \quad \sigma_i^{ref} = 2a_1 \frac{\partial \|\text{Cof} F\|_F^2}{\partial F} \frac{\|\text{Cof} F\|_F^2}{\det F^3} - 3a_1 \frac{\|\text{Cof} F\|_F^4}{\det F^4} \frac{\partial \det F}{\partial F} - 9a_1 \frac{\partial \det F}{\partial F}$$

$$522 \quad + 2a_2 \frac{\partial \|F\|_F^2}{\partial F} \frac{\|F\|_F^2}{\det F^3} - 9a_2 \frac{\partial \det F}{\partial F} - 3a_2 \frac{\|F\|_F^4}{\det F^4} \frac{\partial \det F}{\partial F}$$

$$523 \quad - \frac{a_3}{\det F^2} \frac{\partial \det F}{\partial F} + a_3 \frac{\partial \det F}{\partial F} + 11a_4 \det F^{10} \frac{\partial \det F}{\partial F} - a_4 \frac{\partial \det F}{\partial F},$$

525

$$526 \quad = 2a_1 \frac{\|\text{Cof} F\|_F^2}{\det F^3} \frac{\partial \|\text{Cof} F\|_F^2}{\partial F} - 3a_1 \frac{\|\text{Cof} F\|_F^4}{\det F^4} \text{Cof} F - 9a_1 \text{Cof} F$$

$$527 \quad + 4a_2 \frac{\|F\|_F^2}{\det F^3} F - 9a_2 \text{Cof} F - 3a_2 \frac{\|F\|_F^4}{\det F^4} \text{Cof} F$$

$$528 \quad - \frac{a_3}{\det F^2} \text{Cof} F + a_3 \text{Cof} F + 11a_4 \det F^{10} \text{Cof} F - a_4 \text{Cof} F.$$

529

530 Since, for all  $i \in \{1, \dots, M\}$ ,  $\|\nabla \varphi_i\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha$  and  $\|(\nabla \varphi_i)^{-1}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta$ , we  
531 deduce that

$$533 \quad \|\nabla(\varphi_i^{-1})\|_{L^\infty(\Omega, M_3(\mathbb{R}))} = \|(\nabla \varphi_i)^{-1}(\varphi_i^{-1})\|_{L^\infty(\Omega, M_3(\mathbb{R}))} = \|(\nabla \varphi_i)^{-1}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \beta$$

534 and

$$\begin{aligned}
 535 \quad \|\nabla\varphi_i\|_{L^\infty(\Omega, M_3(\mathbb{R}))} &= \|(\nabla\varphi_i^{-1})^{-1}(\varphi_i)\|_{L^\infty(\Omega, M_3(\mathbb{R}))} = \|(\nabla\varphi_i^{-1})^{-1}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \\
 536 \quad &= \left\| \frac{\text{Cof}\nabla(\varphi_i)^{-1}}{\det\nabla(\varphi_i)^{-1}} \right\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha \\
 537
 \end{aligned}$$

538 so that  $\|\text{Cof}\nabla(\varphi_i)^{-1}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq 18\beta^2$ ,  $\|\det\nabla(\varphi_i)^{-1}\|_{L^\infty(\Omega)} \leq 6\beta^3$  and  $\|\text{Cof}\nabla\varphi_i\|_{L^\infty(\Omega, M_3(\mathbb{R}))}$  ■  
 539  $\leq 18\alpha^2$ ,  $\|\det\nabla\varphi_i\|_{L^\infty(\Omega)} = \left\| \frac{1}{\det\nabla(\varphi_i^{-1})(\varphi_i)} \right\|_{L^\infty(\Omega)} = \left\| \frac{1}{\det\nabla(\varphi_i)^{-1}} \right\|_{L^\infty(\Omega)} \leq 6\alpha^3$ . From these  
 540 observations, we deduce that  $\sigma_i^{ref} \in L^\infty(\Omega, M_3(\mathbb{R})) \subset L^2(\Omega, M_3(\mathbb{R}))$  and therefore  $\sigma_i \in$   
 541  $L^\infty(\Omega, M_3(\mathbb{R})) \subset L^2(\Omega, M_3(\mathbb{R}))$ . ■

542 We thus propose to perform a Principal Component Analysis on the tensors  $\sigma_i$  directly since  
 543 they belong to  $L^2(\Omega, M_3(\mathbb{R}))$  and following [?]. To do so, let  $V = L^2(\Omega, M_3(\mathbb{R}))$  and let  $\mathcal{V}$  be  
 544 a random variable with values in  $V$ . The associated covariance operator is thus defined by

$$545 \quad \mathcal{C}_{\mathcal{V}}(\sigma) = \mathbb{E}[\langle \sigma, \mathcal{V} - \mathbb{E}\mathcal{V} \rangle_V (\mathcal{V} - \mathbb{E}\mathcal{V})], \forall \sigma \in V.$$

547 Then, the corresponding empirical estimation of this operator is given by

$$548 \quad \hat{\mathcal{C}}(\sigma) = \frac{1}{M} \sum_{i=1}^M \langle \sigma, \sigma_i - \bar{\sigma} \rangle_V (\sigma_i - \bar{\sigma}), \sigma \in V, \bar{\sigma} = \frac{1}{M} \sum_{i=1}^M \sigma_i. \\
 549$$

550 We are now looking for the eigenvalues ( $\lambda_j$ ) and eigenvectors ( $\psi_j$ ) associated to this operator,  
 551 and then sort them in a decreasing order, i.e.  $\lambda_1 \geq \lambda_2 \geq \dots$ , so that ( $\psi_j$ ) is the  $j$ -th principal  
 552 component function. Since the number of observations is small with respect to the size of the  
 553 space, an appropriate choice for the base of  $V$  is given by the collection of observations  $\sigma_i$ .  
 554 We then aim to solve

$$\begin{aligned}
 555 \quad \frac{1}{M} \sum_{i=1}^M (\sigma_i - \bar{V}) \sum_{j=1}^M b_{ji} \langle (\sigma_i - \bar{\sigma}), (\sigma_j - \bar{\sigma}) \rangle_V &= \lambda_j \sum_{i=1}^M b_{ji} (\sigma_i - \bar{\sigma}), \\
 556 \quad \iff \frac{1}{M} W b_j &= \lambda_j b_j, \\
 557
 \end{aligned}$$

558 where  $W = (W_{ij})_{i,j} = (\langle \sigma_i - \bar{\sigma}, \sigma_j - \bar{\sigma} \rangle_V)_{i,j} \in M_M(\mathbb{R})$ . It results in finding the eigenvalues  
 559 and eigenvectors of the matrix  $W$ . Then the  $j$ -th PC function is given by  $\psi_j = \sum_{i=1}^M b_{ji} (\sigma_i - \bar{\sigma})$ .  
 560 In order to come back to the deformation space and to properly model the loaded configura-  
 561 tion, we compose the sought deformation with the previous deformations. We then solve the  
 562 following minimization problem with the last term ensuring that the sought displacements  $v_j$

563 are Lipschitz continuous so that the chain rule applies:

$$\begin{aligned}
 564 \quad \inf \left\{ \mathcal{F}(v_j) &= \frac{1}{M} \sum_{k=1}^M \int_{\Omega} \tilde{W}_{Op}(\nabla((\text{Id} - \delta v_j)^{-1} \circ \varphi_k^{-1})) dx - \delta^2 \int_{\Omega} \psi_j : \nabla v_j dx \right. \\
 565 \quad &+ \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega)} \leq \alpha\}} (\text{I} - \delta \nabla v_j), \\
 566 \quad &= \frac{1}{M} \sum_{k=1}^M \int_{\Omega} \tilde{W}_{Op}(\nabla(\varphi_k^{-1}) - \delta \nabla v_i(\varphi_k^{-1}) \nabla \varphi_k^{-1}) dx \\
 567 \quad &- \delta^2 \int_{\Omega} \psi_j : \nabla v_j dx + \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega)} \leq \alpha\}} (\text{I} - \delta \nabla v_j), \\
 568 \quad &= \frac{1}{M} \sum_{k=1}^M \int_{\Omega} \det \nabla \varphi_k \tilde{W}_{Op}(\nabla(\varphi_k^{-1})(\varphi_k) - \delta \nabla v_i \nabla \varphi_k^{-1}(\varphi_k)) dy \\
 569 \quad &- \delta^2 \int_{\Omega} \psi_j : \nabla v_j dx + \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega)} \leq \alpha\}} (\text{I} - \delta \nabla v_j), \\
 570 \quad &= \frac{1}{M} \sum_{k=1}^M \int_{\Omega} \det \nabla \varphi_k \tilde{W}_{Op}((\nabla \varphi_k)^{-1} - \delta \nabla v_i (\nabla \varphi_k)^{-1}) dy \\
 571 \quad \text{(P1)} \quad &- \delta^2 \int_{\Omega} \psi_j : \nabla v_j dx + \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega)} \leq \alpha\}} (\text{I} - \delta \nabla v_j) \left. \right\}. \\
 572
 \end{aligned}$$

573  $\tilde{W}_{Op}$  is continuous since  $\lim_{\det \xi \rightarrow 0^+} \tilde{W}_{Op}(\xi) = +\infty$ . Let us set  $\hat{\mathcal{W}}_1 = \{\psi \in W_0^{1,\infty}(\Omega, \mathbb{R}^3), \det(\text{I} -$   
574  $\delta \nabla \psi) > 0$  a.e. on  $\Omega, \|\text{I} - \delta \nabla \psi\|_{L^\infty(\Omega)} \leq \alpha, \forall k \in \{1, \dots, M\}, \frac{\|\text{Cof}((\nabla \varphi_k)^{-1} - \delta \nabla \psi (\nabla \varphi_k)^{-1})\|_F^4}{\det((\nabla \varphi_k)^{-1} - \delta \nabla \psi (\nabla \varphi_k)^{-1})^3} \in$   
575  $L^1(\Omega), \frac{\|(\text{I} - \delta \nabla \psi)(\nabla \varphi_k)^{-1}\|_F^4}{\det((\nabla \varphi_k)^{-1} - \delta \nabla \psi (\nabla \varphi_k)^{-1})^3} \in L^1(\Omega), \det((\nabla \varphi_k)^{-1} - \delta \nabla \psi (\nabla \varphi_k)^{-1}) \in L^1(\Omega)\}$ . The mini-  
576 mizer is searched for  $v_j \in \hat{\mathcal{W}}$ .

577 **Theorem SM4.7 (Existence of minimizers).** *One can prove that this problem (P1) admits*  
578 *at least one solution.*

579 *Proof.* We follow the classical steps of the direct method of the calculus of variations.

580 **1. Coercivity inequality :**

581 Let us first derive a coercivity inequality :

$$\begin{aligned}
 582 \quad \mathcal{F}(v) &\geq \frac{1}{M} \sum_{k=1}^M \int_{\Omega} \frac{1}{6\beta^3} (a_1 \frac{\|\text{Cof}((\nabla \varphi_k)^{-1} - \delta \nabla v (\nabla \varphi_k)^{-1})\|_F^4}{\det((\nabla \varphi_k)^{-1} - \delta \nabla v (\nabla \varphi_k)^{-1})^3} \\
 583 \quad &+ a_2 \frac{\|(\nabla \varphi_k)^{-1} - \delta \nabla v (\nabla \varphi_k)^{-1}\|_F^4}{\det((\nabla \varphi_k)^{-1} - \delta \nabla v (\nabla \varphi_k)^{-1})^3} + a_4 \det(\nabla \varphi_k^{-1} - \delta \nabla v_i (\nabla \varphi_k)^{-1})^{11} \\
 584 \quad &- (9a_1 + 9a_2 + a_4) 36\alpha^6 + \mathbb{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}} (\text{I} - \delta \nabla v), \\
 585
 \end{aligned}$$

586 since  $\Omega$  is bounded,  $\det((\nabla \varphi_k)^{-1} - \delta \nabla v (\nabla \varphi_k)^{-1}) = \det((\text{I} - \delta \nabla v)(\nabla \varphi_k)^{-1}) = \det(\text{I} -$   
587  $\delta \nabla v) \det((\nabla \varphi_k)^{-1}) \leq 36\alpha^6$ , and  $\|\det \nabla \varphi_k\|_{L^\infty(\Omega)} \leq 6\alpha^3$ ,  $\|\frac{1}{\det \nabla \varphi_k}\|_{L^\infty(\Omega)} \leq 6\beta^3$  so  
588 that for all  $x \in \Omega$ ,  $\frac{1}{6\beta^3} \leq \det \nabla \varphi_k(x) \leq 6\alpha^3$ . Furthermore, by taking  $v = 0$ , then



589  $\mathcal{F}(v) = \frac{1}{M} \int_{\Omega} \tilde{W}_{Op}(\nabla \varphi_k^{-1}) dx < +\infty$  from what precedes. Thus the functional is  
 590 proper and coercive so that the infimum exists and is finite.

591 **2. Convergence of a minimizing sequence :**

592 Let  $(v^n)$  be a minimizing sequence so that  $\lim_{n \rightarrow +\infty} \mathcal{F}(v^n) = \inf \mathcal{F}(v) < +\infty$  from what  
 593 precedes. So there exists  $M \in \mathbb{N}$  such that  $\forall n \in \mathbb{N}, n \geq M \Rightarrow \mathcal{F}(v^n) \leq \inf \mathcal{F}(v) + 1 <$   
 594  $+\infty$ . From now on, we will consider  $n \geq M$ .

595 From the previous coercivity inequality, one has:

- 596 •  $((I - \delta \nabla v^n)(\nabla \varphi_k)^{-1})$  is uniformly bounded according to  $n$  in  $L^4(\Omega, M_3(\mathbb{R}))$  for  
 597 all  $k \in \{1, \dots, M\}$ .
- 598 •  $(I - \delta \nabla v^n)$  and so  $(\nabla v^n)$  is uniformly bounded according to  $n$  in  $L^\infty(\Omega, M_3(\mathbb{R}))$ .
- 599 •  $(\text{Cof}((I - \delta v^n)(\nabla \varphi_k)^{-1})) = (\text{Cof}(I - \delta \nabla v^n) \text{Cof}((\nabla \varphi_k)^{-1})) = (\text{Cof}(I - \delta \nabla v^n)$   
 600  $(\text{Cof} \nabla \varphi_k)^{-1})$  is uniformly bounded according to  $n$  in  $L^4(\Omega, M_3(\mathbb{R}))$  for all  
 601  $k \in \{1, \dots, M\}$ .
- 602 •  $(\det((I - \delta \nabla v^n)(\nabla \varphi_k)^{-1})) = (\frac{\det(I - \delta \nabla v^n)}{\det \nabla \varphi_k})$  is uniformly bounded according to  
 603  $n$  in  $L^{11}(\Omega)$ .

604 There exists a subsequence still denoted  $((I - \delta \nabla v^n)(\nabla \varphi_k)^{-1})$  such that

$$\begin{aligned}
 605 \quad & (I - \delta \nabla v^n)(\nabla \varphi_k)^{-1} \rightharpoonup_{n \rightarrow +\infty} \alpha_k \text{ in } L^4(\Omega, M_3(\mathbb{R})), \forall k \in \{1, \dots, M\}, \\
 606 \quad & I - \delta \nabla v^n \rightharpoonup_{n \rightarrow +\infty} \alpha_k \nabla \varphi_k \text{ in } L^4(\Omega, M_3(\mathbb{R})), \forall k \in \{1, \dots, M\}, \\
 607 \quad & \nabla v^n \rightharpoonup_{n \rightarrow +\infty} \bar{V} = \frac{1}{\delta} (I - \alpha_k \nabla \varphi_k), \forall k \in \{1, \dots, M\}, \text{ in } L^4(\Omega, M_3(\mathbb{R})). \\
 608
 \end{aligned}$$

609 We thus deduce that  $(\nabla v^n)$  is uniformly bounded according to  $n$  in  $L^4(\Omega, M_3(\mathbb{R}))$  and  
 610 by Poincaré's inequality, we deduce that  $(v^n)$  is uniformly bounded according to  $n$  in  
 611  $W^{1,4}(\Omega, \mathbb{R}^3)$ . We can thus extract a subsequence still denoted  $(v^n)$  such that

$$\begin{aligned}
 612 \quad & v^n \rightharpoonup_{n \rightarrow +\infty} \bar{v} \text{ in } W^{1,4}(\Omega, \mathbb{R}^3). \\
 613
 \end{aligned}$$

614 Furthermore, we can extract a subsequence still denoted  $(\nabla v^n)$  such that :

$$\begin{aligned}
 615 \quad & \nabla v^n \xrightarrow{*}_{n \rightarrow +\infty} \nabla \bar{v} \text{ in } L^\infty(\Omega, M_3(\mathbb{R})), \\
 616
 \end{aligned}$$

617 by uniqueness of the weak limit in  $L^4(\Omega, M_3(\mathbb{R}))$  and the continuous embedding of  
 618  $L^\infty(\Omega, M_3(\mathbb{R}))$  into  $L^4(\Omega, M_3(\mathbb{R}))$ . Besides, using Poincaré-Wirtinger's inequality,  
 619 there exists a constant  $C$  depending only on  $\Omega$  such that

$$\begin{aligned}
 620 \quad & \|v^n\|_{L^\infty(\Omega, \mathbb{R}^3)} - \frac{\int_{\Omega} v^n dx}{\text{meas}(\Omega)} \leq \|v^n - \frac{\int_{\Omega} v^n dx}{\text{meas}(\Omega)}\|_{L^\infty(\Omega, \mathbb{R}^3)} \leq C \|\nabla v^n\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq C\alpha, \\
 621 \quad & \|v^n\|_{L^\infty(\Omega, \mathbb{R}^3)} \leq C\alpha + \frac{\int_{\Omega} v^n dx}{\text{meas}(\Omega)} \leq C\alpha + \frac{\|v^n\|_{L^1(\Omega, \mathbb{R}^3)}}{\text{meas}(\Omega)} \leq C\alpha + \frac{\|v^n\|_{L^4(\Omega, \mathbb{R}^3)}}{\text{meas}(\Omega)}, \\
 622 \quad & \leq C\alpha + \frac{C'}{\text{meas}(\Omega)} \|\nabla v^n\|_{L^4(\Omega, M_3(\mathbb{R}))}, \\
 623
 \end{aligned}$$

624 with  $C'$  a constant depending only on  $\Omega$  coming from Poincaré's inequality. Since,  
625  $(\nabla v^n)$  is uniformly bounded in  $L^4(\Omega, M_3(\mathbb{R}))$ , we deduce that  $(v^n)$  is uniformly bounded  
626 according to  $n$  in  $W^{1,\infty}(\Omega, \mathbb{R}^3)$  and there exists a subsequence still denoted  $(v^n)$  such  
627 that

$$628 \quad v^n \xrightarrow[n \rightarrow +\infty]{*} \bar{v} \text{ in } W^{1,\infty}(\Omega, \mathbb{R}^3),$$

630 by uniqueness of the weak limit in  $W^{1,4}(\Omega, \mathbb{R}^3)$  and the continuous embedding of  
631  $W^{1,\infty}(\Omega, \mathbb{R}^3)$  into  $W^{1,4}(\Omega, \mathbb{R}^3)$ . By continuity of the trace operator, we get  $\bar{v} \in$   
632  $W_0^{1,\infty}(\Omega, \mathbb{R}^3)$ .

633 We can also extract subsequences of  $(\text{Cof}(I - \delta \nabla v^n)(\text{Cof} \nabla \varphi_k)^{-1})$  and  $(\frac{\det(I - \delta \nabla v^n)}{\det \nabla \varphi_k})$   
634 still denoted  $(\text{Cof}(I - \delta \nabla v^n)(\text{Cof} \nabla \varphi_k)^{-1})$  and  $(\frac{\det(I - \delta \nabla v^n)}{\det \nabla \varphi_k})$  for all  $k \in \{1, \dots, M\}$ ,  
635 such that

$$636 \quad \text{Cof}(I - \delta \nabla v^n)(\text{Cof} \nabla \varphi_k)^{-1} \xrightarrow[n \rightarrow +\infty]{} H_k \text{ in } L^4(\Omega, M_3(\mathbb{R})),$$

$$637 \quad \text{Cof}(I - \delta \nabla v^n) \xrightarrow[n \rightarrow +\infty]{} H_k \text{Cof} \nabla \varphi_k \text{ in } L^4(\Omega, \mathbb{R}),$$

$$638 \quad \frac{\det(I - \delta \nabla v^n)}{\det \nabla \varphi_k} \xrightarrow[n \rightarrow +\infty]{} \delta_k \text{ in } L^{11}(\Omega),$$

$$639 \quad \det(I - \delta \nabla v^n) \xrightarrow[n \rightarrow +\infty]{} \delta_k \det \nabla \varphi_k \text{ in } L^{11}(\Omega).$$

641 By uniqueness of the weak limit, we have that  $H = H_k \text{Cof} \nabla \varphi_k$  and  $\delta = \delta_k \det \nabla \varphi_k$  for  
642 all  $k \in \{1, \dots, M\}$ . By [?, Theorem VI.3.3] and  $\text{Id} - \delta v^n \xrightarrow[n \rightarrow +\infty]{} \text{Id} - \delta \bar{v}$  in  $W^{1,4}(\Omega, \mathbb{R}^3)$ ,  
643 we have that  $H = \text{Cof}(I - \delta \nabla \bar{v})$  and  $\delta = \det(I - \delta \nabla \bar{v})$ .

644 **3. Lower semi-continuity :**  $\tilde{W}_{Op}$  is convex and continuous. Indeed, let  $f(x, y) =$   
645  $\frac{x^4}{y^3}$  for all  $(x, y) \in (\mathbb{R}^+)^2$ . Then the Hessian matrix of  $f$  is given by  $H(x, y) =$   
646  $\frac{12x^2}{y^3} \begin{pmatrix} 1 & -\frac{x}{y} \\ -\frac{x}{y} & \frac{x^2}{y^2} \end{pmatrix}$  and is positive semi-definite :  $\forall (a, b) \in \mathbb{R}^2, (a, b)H(x, y)(a, b)^T =$   
647  $\frac{12x^2}{y^3}(bx - ay)^2 \geq 0$ , for all  $(x, y) \in (\mathbb{R}^+)^2$ . Besides,  $\|\cdot\|_F$  is also convex and we have  
648  $\forall (A, B) \in M_3(\mathbb{R})^2, \forall (c, d) \in (\mathbb{R}^+)^2, \forall \lambda \in (0, 1), \frac{(\lambda A + (1-\lambda)B)\|_F^4}{(\lambda c + (1-\lambda)d)^3} \leq \frac{(\lambda \|A\|_F + (1-\lambda)\|B\|_F)^4}{(\lambda c + (1-\lambda)d)^3} =$   
649  $f(\lambda \|A\|_F + (1-\lambda)\|B\|_F, \lambda c + (1-\lambda)d) \leq \lambda f(\|A\|_F, c) + (1-\lambda)f(\|B\|_F, d) = \lambda \frac{\|A\|_F^4}{c^3} +$   
650  $(1-\lambda) \frac{\|B\|_F^4}{d^3}$ . If  $\psi_n \xrightarrow[n \rightarrow +\infty]{} \bar{\psi}$  in  $W^{1,4}(\Omega, \mathbb{R}^3)$  then  $\nabla \psi_n \xrightarrow[n \rightarrow +\infty]{} \nabla \bar{\psi}$  in  $L^4(\Omega, M_3(\mathbb{R}))$  and  
651 we can extract a subsequence still denoted  $(\nabla \psi_n)$  such that  $\nabla \psi_n \xrightarrow[n \rightarrow +\infty]{} \nabla \bar{\psi}$  almost  
652 everywhere in  $\Omega$ . If  $\alpha_n \xrightarrow[n \rightarrow +\infty]{} \bar{\alpha}$  in  $L^4(\Omega, M_3(\mathbb{R}))$  then we can extract a subsequence  
653 still denoted  $(\alpha_n)$  such that  $\alpha_n \xrightarrow[n \rightarrow +\infty]{} \bar{\alpha}$  almost everywhere in  $\Omega$ . If  $\delta_n \xrightarrow[n \rightarrow +\infty]{} \bar{\delta}$  in  
654  $L^{11}(\Omega)$ , then there exists a subsequence still denoted  $(\delta_n)$  such that  $\delta_n \xrightarrow[n \rightarrow +\infty]{} \bar{\delta}$  almost  
655 everywhere in  $\Omega$ . Then by continuity of  $\tilde{W}_{Op}$ , we get

$$656 \quad \tilde{W}_{Op}(\nabla \psi_n, \alpha_n, \delta_n) \xrightarrow[n \rightarrow +\infty]{} \tilde{W}_{Op}(\nabla \bar{\psi}, \bar{\alpha}, \bar{\delta}) \text{ almost everywhere in } \Omega.$$

657

658 Then, by applying Fatou's lemma, we have that

$$659 \quad \liminf_{n \rightarrow +\infty} \int_{\Omega} \det \nabla \varphi_k \tilde{W}_{Op}(\nabla \psi_n, \alpha_n, \delta_n) dx \geq \int_{\Omega} \det \nabla \varphi_k \tilde{W}_{Op}(\nabla \bar{\psi}, \bar{\alpha}, \bar{\delta}) dx.$$

660

661 Since  $\tilde{W}_{Op}$  is convex, so is  $\int_{\Omega} \det \nabla \varphi_k \tilde{W}_{Op}(\xi, \alpha, \delta) dx$ , and we can apply [?, Corollaire  
662 III.8] to get that  $\int_{\Omega} \det \nabla \varphi_k \tilde{W}_{Op}(\xi, \alpha, \delta) dx$  is lower semicontinuous in  $L^4(\Omega, M_3(\mathbb{R})) \times$   
663  $L^4(\Omega, M_3(\mathbb{R})) \times L^{11}(\Omega)$ . We deduce that

$$664 \quad +\infty > \liminf_{n \rightarrow +\infty} \int_{\Omega} \det \nabla \varphi_k \tilde{W}_{Op}((I - \delta \nabla v^n)(\nabla \varphi_k)^{-1}, \text{Cof}(I - \delta \nabla v^n)(\text{Cof} \nabla \varphi_k)^{-1},$$

$$665 \quad \frac{\det(I - \delta \nabla v^n)}{\det \nabla \varphi_k}) dx$$

$$666 \quad \geq \int_{\Omega} \det \nabla \varphi_k(x) \tilde{W}_{Op}((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1}, \text{Cof}(I - \delta \nabla \bar{v})(\text{Cof} \nabla \varphi_k)^{-1},$$

$$667 \quad \frac{\det(I - \delta \nabla \bar{v})}{\det \nabla \varphi_k}) dx.$$

668

By the weak-\* lower semi-continuity of the  $\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))}$ , we get that

$$\|I - \delta \nabla \bar{v}\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \liminf_{n \rightarrow +\infty} \|I - \nabla v^n\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha,$$

so that

$$\mathbf{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}}(I - \nabla \bar{v}) = 0 \leq \liminf_{n \rightarrow +\infty} \mathbf{1}_{\{\|\cdot\|_{L^\infty(\Omega, M_3(\mathbb{R}))} \leq \alpha\}}(I - \delta \nabla v^n).$$

Since  $\psi_j \in L^2(\Omega, M_3(\mathbb{R}))$  and  $\nabla v^n \xrightarrow{n \rightarrow +\infty} \nabla \bar{v}$  in  $L^4(\Omega, M_3(\mathbb{R}))$  and so in  $L^2(\Omega, M_3(\mathbb{R}))$ ,  
we have that  $-\delta^2 \int_{\Omega} \psi_j : \nabla \bar{v} dx = \lim_{n \rightarrow +\infty} -\delta^2 \int_{\Omega} \psi_j : \nabla v^n dx$ .

By combining all the results, we get that

$$\liminf_{n \rightarrow +\infty} \mathcal{F}(v^n) \geq \mathcal{F}(\bar{v}).$$

669 Also  $\bar{v} \in W_0^{1,\infty}(\Omega, \mathbb{R}^3)$ ,  $\frac{\|\text{Cof}((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1})\|_F^4}{\det((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1})^3} \in L^1(\Omega)$ ,  $\frac{\|(I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1}\|_F^4}{\det((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1})^3} \in L^1(\Omega)$   
670 by finiteness of  $\mathcal{F}(\bar{v})$  inducing the finiteness of  $\int_{\Omega} \det \nabla \varphi_k \tilde{W}_{Op}((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1},$   
671  $\text{Cof}((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1}), \det((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1})) dx$  for all  $k \in \{1, \dots, M\}$ . Since  
672  $\tilde{W}_{Op}((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1}, \text{Cof}((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1}), \det((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1})) = +\infty$  when  
673  $\det((I - \delta \nabla \bar{v})(\nabla \varphi_k)^{-1}) = \frac{\det(I - \delta \nabla \bar{v})(x)}{\det \nabla \varphi_k(x)} \leq 0 \iff \det(I - \delta \nabla \bar{v}) \leq 0$  since  $\det \nabla \varphi_k(x) > 0$   
674 on  $\Omega$ , the set on which it happens must be of null measure otherwise we would have  
675  $\mathcal{F}(\bar{v}) = +\infty$ . So  $\det(I - \delta \nabla \bar{v}) > 0$  almost everywhere in  $\Omega$  and there exists a minimizer  
676 to our initial problem  $\bar{v} \in \hat{\mathcal{W}}_1$ . ■

677 This problem is hard to solve in practice and we follow the same strategy as previously by  
678 adding additional variables and using an alternative optimisation scheme.

SM5. Computation of the derivatives.

$$\begin{aligned}
 679 \quad \frac{\partial \bar{W}_{Op}}{\partial I_b} &= \frac{2a_2 I_b}{\sqrt{III_b^3}} = 6a_2|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 680 \quad \frac{\partial \bar{W}_{Op}}{\partial II_b} &= \frac{2a_1 I_b}{\sqrt{III_b^3}} = 6a_1|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 681 \quad \frac{\partial \bar{W}_{Op}}{\partial III_b} &= \frac{-3a_1 II_b^2}{2\sqrt{III_b^5}} - \frac{3a_2 I_b^2}{2\sqrt{III_b^5}} + \frac{11}{2} a_4 \sqrt{III_b^9} - \frac{a_3}{2III_b \sqrt{III_b}} + \frac{a_3}{2\sqrt{III_b^3}} \\
 682 \quad &- \frac{9a_1 + 9a_2 + a_4}{2\sqrt{III_b}} = -18a_1 - 18a_2 + 5a_4|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 683 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial I_b^2} &= \frac{2a_2}{\sqrt{III_b^3}} = 2a_2|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 684 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial II_b \partial I_b} &= 0, \\
 685 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial I_b} &= -\frac{-3a_2 I_b}{III_b^2 \sqrt{III_b}} = -9a_2|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 686 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial II_b} &= 0, \\
 687 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial II_b^2} &= \frac{2a_1}{\sqrt{III_b^3}} = 2a_1|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 688 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial II_b} &= -\frac{-3a_1 II_b}{III_b^2 \sqrt{III_b}} = -9a_1|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 689 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial III_b} &= \frac{-3a_2 I_b}{III_b^2 \sqrt{III_b}} = -9a_2|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 690 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial II_b \partial III_b} &= \frac{-3a_1 II_b}{III_b^2 \sqrt{III_b}} = -9a_1|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 691 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial III_b^2} &= \frac{15a_1 II_b^2}{4\sqrt{III_b^7}} + \frac{15a_2 I_b^2}{4\sqrt{III_b^7}} + \frac{99a_4}{4} \sqrt{III_b^7} \\
 692 \quad &+ \frac{3a_3}{4\sqrt{III_b^3}} - \frac{a_3}{4III_b \sqrt{III_b}} + \frac{9a_1 + 9a_2 + a_4}{4\sqrt{III_b^3}}, \\
 693 \quad &= 36a_1 + 36a_2 + \frac{a_3}{2} + 25a_4|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 694 \quad \frac{\partial I_b}{\partial b} &= I, \\
 695 \quad \frac{\partial II_b}{\partial b} &= I_b I - b = 2I|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 696 \quad \frac{\partial III_b}{\partial b} &= III_b b^{-1} = I|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 697 \quad \frac{\partial I}{\partial b} &= 0, \\
 698
 \end{aligned}$$

699

$$\begin{aligned}
 700 \quad \frac{\partial b}{\partial b} &= \mathbf{I}, \text{ Identity tensor of order 4 such that } \mathbf{I} : A : A = \text{Tr}(A^T A), \\
 701 \quad \frac{\partial b^{-1}}{\partial b} &= \frac{\mathbf{I} + \bar{\mathbf{I}}}{2} \Big|_{(I_b, II_b, III_b)=(3,3,1)}, \text{ with } \bar{\mathbf{I}} : A : A = \text{Tr}(A^2), \\
 702 \quad \frac{\partial \bar{W}_{Op}}{\partial b} &= \frac{\partial \bar{W}_{Op}}{\partial I_b} \frac{\partial I_b}{\partial b} + \frac{\partial \bar{W}_{Op}}{\partial II_b} \frac{\partial II_b}{\partial b} + \frac{\partial \bar{W}_{Op}}{\partial III_b} \frac{\partial III_b}{\partial b} = (-12a_1 - 6a_1 + 5a_4)I \Big|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 703 \quad \frac{\partial^2 \bar{W}_{Op}}{\partial b^2} &= \frac{\partial \bar{W}_{Op}}{\partial I_b} \frac{\partial I}{\partial b} + I \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b^2} \frac{\partial I_b}{\partial b} + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b \partial I_b} \frac{\partial II_b}{\partial b} + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial I_b} \frac{\partial III_b}{\partial b} \right) \\
 704 \quad &+ \frac{\partial \bar{W}_{Op}}{\partial II_b} \frac{\partial (I_b I - b)}{\partial b} + (I_b I - b) \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial II_b} \frac{\partial I_b}{\partial b} + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b^2} \frac{\partial II_b}{\partial b} + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial II_b} \frac{\partial III_b}{\partial b} \right) \\
 705 \quad &+ \frac{\partial \bar{W}_{Op}}{\partial III_b} \frac{\partial (III_b b^{-1})}{\partial b} + (III_b b^{-1}) \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial III_b} \frac{\partial I_b}{\partial b} + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b \partial III_b} \frac{\partial II_b}{\partial b} + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b^2} \frac{\partial III_b}{\partial b} \right), \\
 706 \quad &= I \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b^2} I + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b \partial I_b} (I_b I - b) + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial I_b} III_b b^{-1} \right) \\
 707 \quad &+ \frac{\partial \bar{W}_{Op}}{\partial II_b} (I \otimes I - \mathbf{I}) + (I_b I - b) \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial II_b} I + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b^2} (I_b I - b) + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial II_b} III_b b^{-1} \right) \\
 708 \quad &+ \frac{\partial \bar{W}_{Op}}{\partial III_b} (b^{-1} \otimes III_b b^{-1} + III_b \frac{\partial b^{-1}}{\partial b}) \\
 709 \quad &+ III_b b^{-1} \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial III_b} I + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b \partial III_b} (I_b I - b) + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b^2} III_b b^{-1} \right), \\
 710 \quad &= \frac{\partial \bar{W}_{Op}}{\partial II_b} (I \otimes I - \mathbf{I}) + \frac{\partial \bar{W}_{Op}}{\partial III_b} III_b (b^{-1} \otimes b^1 + \frac{\partial b^{-1}}{\partial b}) \\
 711 \quad &+ I \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b^2} I + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b \partial I_b} (I_b I - b) + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial I_b} III_b b^{-1} \right) \\
 712 \quad &+ (I_b I - b) \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial II_b} I + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b^2} (I_b I - b) + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial II_b} III_b b^{-1} \right) \\
 713 \quad &+ III_b b^{-1} \otimes \left( \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial III_b} I + \frac{\partial^2 \bar{W}_{Op}}{\partial II_b \partial III_b} (I_b I - b) + \frac{\partial^2 \bar{W}_{Op}}{\partial III_b^2} III_b b^{-1} \right), \\
 714 \quad &= -\left( \frac{\partial \bar{W}_{Op}}{\partial II_b} \mathbf{I} + \frac{1}{2} \frac{\partial \bar{W}_{Op}}{\partial III_b} (\mathbf{I} + \bar{\mathbf{I}}) \right) + I \otimes I \left( \frac{\partial \bar{W}_{Op}}{\partial II_b} + \frac{\partial \bar{W}_{Op}}{\partial III_b} + \frac{\partial^2 \bar{W}_{Op}}{\partial I_b^2} + 4 \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial II_b} \right. \\
 715 \quad &+ 2 \frac{\partial^2 \bar{W}_{Op}}{\partial I_b \partial III_b} + 4 \frac{\partial^2 \bar{W}_{Op}}{\partial II_b^2} + 4 \frac{\partial^2 \bar{W}_{Op}}{\partial III_b \partial II_b} + \left. \frac{\partial^2 \bar{W}_{Op}}{\partial III_b^2} \right) \Big|_{(I_b, II_b, III_b)=(3,3,1)}, \\
 716 \quad &= (15a_1 + 9a_2 - \frac{5}{2}a_4)\mathbf{I} + (9a_1 + 9a_2 - \frac{5a_4}{2})\bar{\mathbf{I}} + I \otimes I (-4a_1 + 20a_2 + 30a_4 + \frac{a_3}{2}). \\
 717
 \end{aligned}$$

718

**SM6. Proof of Theorem Theorem 4.2.**

719 *Proof.* Let us denote by  $a$  the mapping defined by :

$$720 \quad a : \begin{cases} H^3(\Omega, \mathbb{R}^2) \times H^3(\Omega, \mathbb{R}^2) \rightarrow \mathbb{R} \\ (u, v) \mapsto \langle \rho_0(u), \rho_0(v) \rangle_{\mathbb{R}^2, N_0} + \frac{\gamma}{2} \langle \rho_1(\nabla u + \nabla u^T), \rho_1(\nabla v + \nabla v^T) \rangle_{M_2(\mathbb{R}), N_1} , \\ 721 \quad + \epsilon(u, v)_{3, \Omega, \mathbb{R}^2} \end{cases}$$

722 with  $(\cdot, \cdot)_{3, \Omega, \mathbb{R}^2}$  denoting the semi-norm in  $H^3(\Omega, \mathbb{R}^2)$ , and by  $L$  the mapping defined by :

$$723 \quad L : \begin{cases} H^3(\Omega, \mathbb{R}^2) \rightarrow \mathbb{R} \\ 724 \quad v \mapsto \langle \rho_0(f_k), \rho_0(v) \rangle_{\mathbb{R}^2, N_0} + \frac{\gamma}{2} \langle \rho_1(\nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k), \rho_1(\nabla v + \nabla v^T) \rangle_{M_2(\mathbb{R}), N_1} . \end{cases}$$

725 Let us notice that  $\forall v \in H^3(\Omega, \mathbb{R}^2)$ ,

$$\begin{aligned} 726 \quad \mathcal{F}_{\epsilon, k}(v) &= \langle \rho_0(v) - \rho_0(f_k) \rangle_{\mathbb{R}^2, N_0}^2 + \frac{\gamma}{2} \langle \rho_1(\nabla v + \nabla v^T) - \rho_1(\nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N_1}^2 \\ 727 \quad &+ \epsilon |v|_{3, \Omega, \mathbb{R}^2}^2, \\ 728 \quad &= \langle \rho_0(v) \rangle_{\mathbb{R}^2, N_0}^2 + \langle \rho_0(f_k) \rangle_{\mathbb{R}^2, N_0}^2 - 2 \langle \rho_0(v), \rho_0(f_k) \rangle_{\mathbb{R}^2, N_0} + \frac{\gamma}{2} \langle \rho_1(\nabla v + \nabla v^T) \rangle_{M_2(\mathbb{R}), N_1}^2 \\ 729 \quad &+ \frac{\gamma}{2} \langle \rho_1(\nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N_1}^2 \\ 730 \quad &- \gamma \langle \rho_1(\nabla v + \nabla v^T), \rho_1(\nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N_1} + \epsilon |v|_{3, \Omega, \mathbb{R}^2}^2, \\ 731 \quad &= a(v, v) - 2L(v) + \langle \rho_0(f_k) \rangle_{\mathbb{R}^2, N_0}^2 + \frac{\gamma}{2} \langle \rho_1(\nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N_1}^2. \end{aligned}$$

733 Also problem (4.2) is equivalent to :

$$734 \quad \begin{cases} \text{Search for } u_{\epsilon, k} \in H^3(\Omega, \mathbb{R}^2) \text{ such that:} \\ 735 \quad \forall v \in H^3(\Omega, \mathbb{R}^2), \forall \mu \in \mathbb{R}, \mathcal{F}_{\epsilon, k}(u_{\epsilon, k}) \leq \mathcal{F}_{\epsilon, k}(u_{\epsilon, k} + \mu v) . \end{cases}$$

736 Using the bilinearity and symmetry of the mapping  $a$ , and the linearity of the mapping  $L$ ,  
737 one has :

$$\begin{aligned} 738 \quad \mathcal{F}_{\epsilon, k}(u_{\epsilon, k} + \mu v) &= a(u_{\epsilon, k}, u_{\epsilon, k}) + 2\mu a(u_{\epsilon, k}, v) + \mu^2 a(v, v) - 2L(u_{\epsilon, k}) - 2\mu L(v) + \langle \rho_0(f_k) \rangle_{\mathbb{R}^2, N_0}^2 \\ 739 \quad &+ \langle \rho_1(\nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N_1}^2, \\ 740 \quad &= \mathcal{F}_{\epsilon, k}(u_{\epsilon, k}) + 2\mu [a(u_{\epsilon, k}, v) - L(v)] + \mu^2 a(v, v), \end{aligned}$$

742 which means, using the previous reformulation that :

$$743 \quad 2\mu [a(u_{\epsilon, k}, v) - L(v)] + \mu^2 a(v, v) \geq 0.$$

745 Assuming now that  $\mu > 0$ , and dividing the previous inequality by  $\mu$ , and finally letting  $\mu$   
746 tend to 0, we get :

$$747 \quad a(u_{\epsilon, k}, v) - L(v) \geq 0.$$

749 Assuming that  $\mu < 0$ , dividing the inequality by  $\mu$  and letting  $\mu$  tend to 0, we finally obtain :

$$750 \quad a(u_{\epsilon, k}, v) - L(v) \leq 0.$$

752 We then deduce that  $a(u_{\epsilon, k}, v) = L(v)$ , for all  $v \in H^3(\Omega, \mathbb{R}^2)$ . The inverse is readily obtained  
753 thanks to the previous relations. ■

754 **SM7. Proof of Lemma Lemma 4.3.**

755 *Proof.* Let us take  $f \in H^3(\Omega, \mathbb{R}^2)$  such that  $\|f\|_{A_0,3,\Omega,\mathbb{R}^2} = 0$ . It implies that  $|f|_{3,\Omega,\mathbb{R}^2} = 0$   
 756 and taking into account the connectedness of  $\Omega$ , it yields  $f \in P^2(\Omega, \mathbb{R}^2)$ . We also have  
 757 that  $\rho_1(\nabla f + \nabla f^T) = 0$  meaning that  $\frac{\partial f_1}{\partial x}(b_i) = 0$ ,  $\frac{\partial f_2}{\partial y}(b_i) = 0$ ,  $\frac{\partial f_1}{\partial y}(b_i) + \frac{\partial f_2}{\partial x}(b_i) = 0$ , for  
 758  $i = 1, \dots, N_1$ . Since  $f \in P^2(\Omega, \mathbb{R}^2)$ , it means that  $\frac{\partial f_1}{\partial x} \in P^1(\Omega)$ , and  $\frac{\partial f_2}{\partial y} \in P^1(\Omega)$ . Since  $A_1$   
 759 contains a  $P^1(\Omega, \mathbb{R}^2)$ -unisolvent subset, it results that  $\frac{\partial f_1}{\partial x} \equiv 0$  and  $\frac{\partial f_2}{\partial y} \equiv 0$ . We therefore  
 760 have that

$$\begin{aligned} 761 \quad & f_1(x, y) = g_1(y), \\ 762 \quad & f_2(x, y) = g_1(x). \end{aligned}$$

764 Since,  $\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} \in P^1(\Omega)$  and  $A_1$  contains a  $P^1(\Omega, \mathbb{R}^2)$  unisolvent subset, we deduce that

$$765 \quad \frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x} \equiv 0 \Leftrightarrow \frac{\partial f_1}{\partial y} = g'_1(y) \equiv -\frac{\partial f_2}{\partial x} = -g'_2(x), \forall (x, y) \in \Omega.$$

767 It thus means that  $f_1(x, y) = ay + b$ ,  $f_2(x, y) = -ax + c$  and are in  $P^1(\Omega)$  with  $f_1(a_i) =$   
 768  $f_2(a_i) = 0$  for all  $i = 1, \dots, N_0$ . Since  $A_0$  contains a  $P^1(\Omega, \mathbb{R}^2)$ -unisolvent subset, we deduce  
 769 that  $f_1 \equiv f_2 \equiv 0$ . It is now clear that  $\|\cdot\|_{A_0,3,\Omega,\mathbb{R}^2}$  is a norm on  $H^3(\Omega, \mathbb{R}^2)$  associated with a  
 770 scalar product. We now prove the equivalence of the norm  $\|\cdot\|_{A_0,3,\Omega,\mathbb{R}^2}$  with the norm  $\|\cdot\|_{3,\Omega,\mathbb{R}^2}$ .

772 First, we have  $\forall f \in H^3(\Omega, \mathbb{R}^2)$ ,  $\forall a_i \in A_0$ ,  $i = 1, \dots, N_0$ ,  $\forall b_j \in A_1$ ,  $j = 1, \dots, N_1$  :

$$\begin{aligned} 773 \quad & \langle f(a_i) \rangle_{\mathbb{R}^2} \leq \|f\|_{C^0(\Omega, \mathbb{R}^2)} \leq c \|f\|_{3,\Omega,\mathbb{R}^2} \text{ thanks to Sobolev's embedding,} \\ 774 \quad & \langle \nabla f(b_j) + \nabla f(b_j)^T \rangle_{M_2(\mathbb{R})} \leq 2 \|f\|_{C^1(\Omega, \mathbb{R}^2)} \leq 2c' \|f\|_{3,\Omega,\mathbb{R}^2} \text{ thanks to Sobolev's embedding,} \\ 775 \quad & \text{so, } \|f\|_{A,3,\Omega,\mathbb{R}^2} \leq (1 + c^2 N + 4c'^2 N)^{\frac{1}{2}} \|f\|_{3,\Omega,\mathbb{R}^2}. \end{aligned}$$

777 Furthermore, let us take  $k = 3 = m$ ,  $p = 2$  in Nečas theorem [?, Chapter 2, section 7.1], and  
 778 let us take  $\rho_0(f)$  and  $\rho_1(\nabla f + \nabla f^T)$  as functionals  $f_i$  (using the same reasoning as previously  
 779 to show the property (7.1 bis)  $\forall v \in P^2(\Omega, \mathbb{R}^2)$ ,  $\sum_{i=1}^l |f_i(v)|^2 = 0 \Leftrightarrow v \equiv 0$ ). Then there exists a  
 780 positive constant  $c_1 > 0$  such that

$$781 \quad c_1 \|f\|_{3,\Omega,\mathbb{R}^2} \leq \left[ \|f\|_{3,\Omega,\mathbb{R}^2}^2 + \sum_{i=1}^{N_0} \langle f(a_i) \rangle_{\mathbb{R}^2}^2 + \sum_{j=1}^{N_1} \langle \nabla f(b_j) + \nabla f(b_j)^T \rangle_{M_2(\mathbb{R})}^2 \right]^{\frac{1}{2}},$$

783 which concludes the proof. ■

784 **SM8. Proof of Theorem Theorem 4.4.**

785 *Proof.* The mapping  $L$  is a linear and continuous form on  $H^3(\Omega, \mathbb{R}^2)$ . Indeed,

$$\begin{aligned}
 786 \quad |L(v)| &= \left| \sum_{i=1}^{N_0} \langle f_k(a_i), v(a_i) \rangle_{\mathbb{R}^2} + \sum_{j=1}^{N_1} \frac{\gamma}{2} \langle \nabla f_k(b_j) + \nabla f_k(b_j)^T + \nabla f_k(b_j)^T \nabla f_k(b_j), \nabla v(b_j) \right. \\
 787 \quad &\quad \left. + \nabla v(b_j)^T \rangle_{M_2(\mathbb{R})} \right|, \\
 788 \quad &\leq \sum_{i=1}^{N_0} |\langle f_k(a_i), v(a_i) \rangle_{\mathbb{R}^2}| + \frac{\gamma}{2} \sum_{j=1}^{N_1} |\langle \nabla f_k(b_j) + \nabla f_k(b_j)^T + \nabla f_k(b_j)^T \nabla f_k(b_j), \nabla v(b_j) \\
 789 \quad &\quad + \nabla v(b_j)^T \rangle_{M_2(\mathbb{R})}|, \\
 790 \quad &\leq \sum_{i=1}^{N_0} |\langle f_k(a_i) \rangle_{\mathbb{R}^2}| |\langle v(a_i) \rangle_{\mathbb{R}^2}| + \frac{\gamma}{2} \sum_{j=1}^{N_1} |\langle \nabla f_k(b_j) + \nabla f_k(b_j)^T + \nabla f_k(b_j)^T \nabla f_k(b_j) \rangle_{M_2(\mathbb{R})}| \\
 791 \quad &\quad |\langle \nabla v(b_j) + \nabla v(b_j)^T \rangle_{M_2(\mathbb{R})}|, \\
 792 \quad &\leq \sum_{i=1}^{N_0} |\langle f_k(a_i) \rangle_{\mathbb{R}^2}| \|v\|_{C^0(\Omega, \mathbb{R}^2)} + \gamma \sum_{j=1}^{N_1} |\langle \nabla f_k(b_j) + \nabla f_k(b_j)^T + \nabla f_k(b_j)^T \nabla f_k(b_j) \rangle_{M_2(\mathbb{R})}| \\
 793 \quad &\quad \|v\|_{C^1(\Omega, \mathbb{R}^2)}, \\
 794 \quad &\leq (cN_0 \|f_k\|_{C^0(\Omega, \mathbb{R}^2)} + c'N_1 \gamma \max_{i=1, \dots, N_1} |\langle \nabla f_k(b_i) + \nabla f_k(b_i)^T + \nabla f_k(b_i)^T \nabla f_k(b_i) \rangle_{M_2(\mathbb{R})}) \\
 795 \quad &\quad \|v\|_{3, \Omega, \mathbb{R}^2}.
 \end{aligned}$$

797 Moreover, the mapping  $a$  is a symmetric, bilinear form, continuous on  $H^3(\Omega, \mathbb{R}^2) \times H^3(\Omega, \mathbb{R}^2)$ .  
 798 While symmetry and bilinearity are obvious, the continuity of  $a$  can be obtained by using the  
 799 equivalence of norms established in the previous lemma :  $\forall (u, v) \in H^3(\Omega, \mathbb{R}^2) \times H^3(\Omega, \mathbb{R}^2)$ ,

$$\begin{aligned}
 800 \quad |a(u, v)| &= \left| \sum_{i=1}^{N_0} \langle u(a_i), v(a_i) \rangle_{\mathbb{R}^2} + \frac{\gamma}{2} \sum_{j=1}^{N_1} \langle \nabla u(b_j) + \nabla u(b_j)^T, \nabla v(b_j) + \nabla v(b_j)^T \rangle_{M_2(\mathbb{R})} + \epsilon(u, v)_{3, \Omega, \mathbb{R}^2}, \right. \\
 801 \quad &\leq \max(1, \epsilon, \frac{\gamma}{2}) \left[ \sum_{i=1}^{N_0} |\langle u(a_i), v(a_i) \rangle_{\mathbb{R}^2}| + \sum_{j=1}^{N_1} |\langle \nabla u(b_j) + \nabla u(b_j)^T, \nabla v(b_j) + \nabla v(b_j)^T \rangle_{M_2(\mathbb{R})}| \right. \\
 802 \quad &\quad \left. + |(u, v)_{3, \Omega, \mathbb{R}^2}| \right], \\
 803 \quad &\leq 4 \max(1, \epsilon, \frac{\gamma}{2}) \left[ \sum_{i=1}^{N_0} \langle u(a_i) \rangle_{\mathbb{R}^2} \langle v(a_i) \rangle_{\mathbb{R}^2} + \sum_{j=1}^{N_1} \langle \nabla u(b_j) \rangle_{M_2(\mathbb{R})} \langle \nabla v(b_j) \rangle_{M_2(\mathbb{R})} + |u|_{3, \Omega, \mathbb{R}^2} |v|_{3, \Omega, \mathbb{R}^2}. \right. \\
 804 \quad &\quad \left. \right]
 \end{aligned}$$

805 This last inequality proves that  $a$  is continuous on  $H^3(\Omega, \mathbb{R}^2) \times H^3(\Omega, \mathbb{R}^2)$ .

806 To finish with, we prove that  $a$  is  $H^3(\Omega, \mathbb{R}^3)$ -elliptic. Let  $v \in H^3(\Omega, \mathbb{R}^2)$ ,

$$\begin{aligned}
 807 \quad a(v, v) &= \langle \rho_0(u) \rangle_{\mathbb{R}^2, N_0}^2 + \frac{\gamma}{2} \langle \rho_1(\nabla v + \nabla v^T) \rangle_{M_2(\mathbb{R}), N_1}^2 + \epsilon |v|_{3, \Omega, \mathbb{R}^2}^2, \\
 808 \quad &\geq \min(1, \epsilon, \frac{\gamma}{2}) \|v\|_{A, 3, \Omega, \mathbb{R}^2}^2. \\
 809 \quad &
 \end{aligned}$$



810 Using again the equivalence of norms established above, we deduce that  $a$  is  $H^3(\Omega, \mathbb{R}^2)$ -elliptic.  
 811 The Lax-Milgram theorem enables us to conclude that the variational problem (4.3) has a  
 812 unique solution denoted by  $u_{\epsilon,k}$ . ■

813 **SM9. Proof of Lemma Lemma 4.5.**

814 *Proof.* The Sobolev's embedding gives that

$$815 \quad \exists C_1 > 0, \forall d \in D, \forall f \in H^3(\Omega, \mathbb{R}^2), \|f\|_{A_0^d, 3, \Omega, \mathbb{R}^2} \leq C_1 \|f\|_{3, \Omega, \mathbb{R}^2},$$

817 by using the same arguments as previously, with  $C_1$  independent of  $d$  and depending only  
 818 on  $\mathcal{N}$ . Let us now find a constant  $C_2$  independent of  $d$  such that the inequality  $\|f\|_{3, \Omega, \mathbb{R}^2} \leq$   
 819  $C_2 \|f\|_{A_0^d, 3, \Omega, \mathbb{R}^2}$  holds. For  $f \in H^3(\Omega, \mathbb{R}^2)$ , we have

$$\begin{aligned} 820 \quad \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \langle f(b_{0j}) \rangle_{\mathbb{R}^2}^2 &= \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \langle f(b_{0j}) - f(a_{0j}^d) + f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2, \\ 821 \quad &= \frac{1}{2} \sum_{i=1}^{\mathcal{N}} \langle f(b_{0j}) - f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 + \langle f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 + 2 \langle f(b_{0j}) - f(a_{0j}^d), f(a_{0j}^d) \rangle_{\mathbb{R}^2}, \\ 822 \quad &\leq \sum_{i=1}^{\mathcal{N}} \langle f(b_{0j}) - f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 + \langle f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2. \\ 823 \end{aligned}$$

824 The open subset  $\Omega$  having a Lipschitz continuous boundary, the space  $H^3(\Omega, \mathbb{R}^2)$  verifies the  
 825 Sobolev's Hölder embedding theorem namely  $\exists \lambda \in ]0, 1]$ ,  $H^3(\Omega, \mathbb{R}^2) \hookrightarrow C^{0, \lambda}(\bar{\Omega}, \mathbb{R}^2)$ . Thus  
 826  $f \in C^{0, \lambda}(\bar{\Omega}, \mathbb{R}^2)$ , and  $\exists C > 0, \forall j = 1, \dots, \mathcal{N}, \forall d \in D$ ,

$$\begin{aligned} 827 \quad \langle f(b_{0j}) - f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 &\leq \|f\|_{C^{0, \lambda}(\bar{\Omega}, \mathbb{R}^2)}^2 \langle b_{0j} - a_{0j}^d \rangle_{\mathbb{R}^2}^{2\lambda}, \\ 828 \quad &\leq C^2 \|f\|_{H^3(\Omega, \mathbb{R}^2)}^2 \langle b_{0j} - a_{0j}^d \rangle_{\mathbb{R}^2}^{2\lambda} \end{aligned}$$

830 Besides, from the hypotheses, it comes that  $\forall j = 1, \dots, \mathcal{N}$

$$831 \quad \forall \beta_j > 0, \exists \eta_{\beta_j} > 0, \forall d \in D, (d \leq \eta_{\beta_j} \Rightarrow \langle a_{0j}^d - b_{0j} \rangle_{\mathbb{R}^2} \leq \beta_{\eta_j}).$$

833 Then  $\forall j = 1, \dots, \mathcal{N}$ ,

$$834 \quad \forall \beta_j > 0, \exists \eta_{\beta_j}, \forall d \in D, (d \leq \eta_{\beta_j} \Rightarrow \langle f(b_{0j}) - f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 \leq C^2 \|f\|_{H^3(\Omega, \mathbb{R}^2)}^2).$$

836 Let  $\beta > 0$  and let us take  $\beta_j = \beta$  for all  $j = 1, \dots, \mathcal{N}$ , and  $\eta = \min\{\eta_{\beta_1}, \dots, \eta_{\beta_{\mathcal{N}}}\}$ , then

$$837 \quad d \leq \eta \Rightarrow \sum_{j=1}^{\mathcal{N}} \langle f(b_{0j}) - f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 \leq C^2 \beta^{2\lambda} \mathcal{N} \|f\|_{H^3(\Omega, \mathbb{R}^2)}^2.$$

839 This implies that  $\forall \beta > 0, \exists \eta > 0, \forall d \in D, \forall f \in H^3(\Omega, \mathbb{R}^2)$ ,

$$840 \quad d \leq \eta \Rightarrow \sum_{j=1}^{\mathcal{N}} \langle f(b_{0j}) - f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 \leq C^2 \beta^{2\lambda} \mathcal{N} \|f\|_{H^3(\Omega, \mathbb{R}^2)}^2.$$

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842 This implies that  $\forall \beta > 0, \exists \eta > 0, \forall d \in D, \forall f \in H^3(\Omega, \mathbb{R}^2)$ ,

$$843 \quad d \leq \eta \Rightarrow \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \langle f(b_{0j}) \rangle_{\mathbb{R}^2}^2 + |f|_{3,\Omega,\mathbb{R}^2}^2 - C^2 \beta^{2\lambda} \mathcal{N} \|f\|_{H^3(\Omega,\mathbb{R}^2)}^2 \leq \sum_{i=1}^{\mathcal{N}} \langle f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 + |f|_{3,\Omega,\mathbb{R}^2}^2,$$

$$844 \quad d \leq \eta \Rightarrow \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \langle f(b_{0j}) \rangle_{\mathbb{R}^2}^2 + \sum_{i=1}^{N_1} \langle \nabla f(b_i) + \nabla f(b_i)^T \rangle_{M_2(\mathbb{R})}^2 |f|_{3,\Omega,\mathbb{R}^2}^2 - C^2 \beta^{2\lambda} \mathcal{N} \|f\|_{H^3(\Omega,\mathbb{R}^2)}^2$$

$$845 \quad \leq \sum_{i=1}^{\mathcal{N}} \langle f(a_{0j}^d) \rangle_{\mathbb{R}^2}^2 + \sum_{i=1}^{N_1} \langle \nabla f(b_i) + \nabla f(b_i)^T \rangle_{M_2(\mathbb{R})}^2 + |f|_{3,\Omega,\mathbb{R}^2}^2.$$

847 As previously shown, the mapping  $f \in H^3(\Omega, \mathbb{R}^2) \mapsto (\frac{1}{2} \sum_{j=1}^{\mathcal{N}} \langle f(b_{0j}) \rangle_{\mathbb{R}^2}^2 + \sum_{i=1}^{N_1} \langle \nabla f(b_i) + \nabla f(b_i)^T$

848  $\rangle_{M_2(\mathbb{R})}^2 + |f|_{3,\Omega,\mathbb{R}^2}^2)^{\frac{1}{2}}$  is a norm on  $H^3(\Omega, \mathbb{R}^2)$  equivalent to the norm  $\|\cdot\|_{3,\Omega,\mathbb{R}^2}$ , so  $\forall \beta > 0, \exists \eta >$   
849  $0, \forall d \in D, \forall f \in H^3(\Omega, \mathbb{R}^2)$ ,

$$850 \quad d \leq \eta \Rightarrow (C'' - C^2 \beta^{2\lambda} \mathcal{N}) \|f\|_{H^3(\Omega,\mathbb{R}^2)}^2 \leq \|f\|_{A_0^d,3,\Omega,\mathbb{R}^2}^2.$$

852 By choosing  $\beta$  adequately, the norm equivalence is obtained. ■

### 853 SM10. Proof of Theorem Theorem 4.6.

854 *Proof.* The proof is divided into 3 steps that we detail hereafter :

855 • **First step** : We start by proving that the sequence  $(u_\epsilon^d)_{d \in D \cap ]0,\eta]}$  is bounded in  
856  $H^3(\Omega, \mathbb{R}^2)$  for fixed  $N_1$ , and  $\epsilon$ . In the minimisation problem (4.4), let us take  $v = f_k$ .  
857 Then we have

$$858 \quad \langle \rho^d(u_\epsilon^d - f_k) \rangle_{\mathbb{R}^2, N(d)}^2 + \frac{\gamma}{2} \langle \rho_1(\nabla u_\epsilon^d + (\nabla u_\epsilon^d)^T - \nabla f_k - \nabla f_k^T - \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N_1}^2 + \epsilon |u_\epsilon^d|_{3,\Omega,\mathbb{R}^2}^2$$

$$859 \quad \leq \epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + \frac{\gamma}{2} \sum_{i=1}^{N_1} \langle \nabla f_k(b_i)^T \nabla f_k(b_i) \rangle_{M_2(\mathbb{R})}^2,$$

$$860 \quad \leq \epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 2\gamma \sum_{i=1}^{N_1} \langle \nabla f_k(b_i) \rangle_{M_2(\mathbb{R})}^2,$$

$$861 \quad \leq \epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 2\gamma N_1 \|f_k\|_{C^1(\bar{\Omega}, \mathbb{R}^2)}^2, \quad \blacksquare$$

863 from which we deduce that

$$864 \quad \begin{cases} |u_\epsilon^d|_{3,\Omega,\mathbb{R}^2}^2 \leq |f_k|_{3,\Omega,\mathbb{R}^2}^2 + \frac{2\gamma}{\epsilon} \|f_k\|_{C^1(\bar{\Omega}, \mathbb{R}^2)}^2 N_1, \\ \langle \rho^d(u_\epsilon^d - f_k) \rangle_{\mathbb{R}^2, N}^2 \leq \epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 2\gamma N_1 \|f_k\|_{C^1(\bar{\Omega}, \mathbb{R}^2)}^2, \\ \langle \rho_1(\nabla u_\epsilon^d + (\nabla u_\epsilon^d)^T - \nabla f_k - \nabla f_k^T - \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N_1}^2 \leq \frac{2\epsilon}{\gamma} |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 4N_1 \|f_k\|_{C^1(\bar{\Omega}, \mathbb{R}^2)}^2. \end{cases}$$

866 As  $A_0^d \subset A^d$  with  $A_0^d$  containing  $\mathcal{N}$  elements forming a  $P^1$ -unisolvent set. So one has

$$867 \quad \sum_{a \in A_0^d} \langle u_\epsilon^d(a) - f_k(a) \rangle_{\mathbb{R}^2}^2 \leq \sum_{a \in A^d} \langle u_\epsilon^d(a) - f_k(a) \rangle_{\mathbb{R}^2}^2 \leq \epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 2\gamma N_1 \|f_k\|_{C^1(\bar{\Omega}, \mathbb{R}^2)}^2.$$

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Moreover

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$$\sum_{a \in A_0^d} \langle u_\epsilon^d(a) \rangle_{\mathbb{R}^2}^2 = \sum_{a \in A_0^d} \langle u_\epsilon^d(a) - f_k(a) + f_k(a) \rangle_{\mathbb{R}^2}^2,$$

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$$\leq 2 \sum_{a \in A_0^d} \langle u_\epsilon^d(a) - f_k(a) \rangle_{\mathbb{R}^2}^2 + \langle f_k(a) \rangle_{\mathbb{R}^2}^2,$$

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$$\leq 2\epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 4N_1\gamma \|f_k\|_{C^1(\bar{\Omega},\mathbb{R}^2)}^2 + 2\mathcal{N} \|f_k\|_{C^0(\bar{\Omega},\mathbb{R}^2)}^2,$$

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$$\sum_{a \in A_1} \langle \nabla u_\epsilon^d(a) + (\nabla u_\epsilon^d(a))^T \rangle_{M_2(\mathbb{R})}^2 = \sum_{a \in A_1} \langle \nabla u_\epsilon^d(a) + \nabla u_\epsilon^d(a)^T - \nabla f_k(a) - \nabla f_k(a)^T$$

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$$- \nabla f_k(a)^T \nabla f_k(a) + \nabla f_k(a) + \nabla f_k(a)^T + \nabla f_k(a)^T \nabla f_k(a) \rangle_{M_2(\mathbb{R})}^2,$$

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$$\leq 2 \sum_{a \in A_1} \langle \nabla u_\epsilon^d(a) + \nabla u_\epsilon^d(a)^T - \nabla f_k(a) - \nabla f_k(a)^T - \nabla f_k(a)^T \nabla f_k(a) \rangle_{M_2(\mathbb{R})}^2$$

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$$+ \langle \nabla f_k(a) + \nabla f_k(a)^T + \nabla f_k(a)^T \nabla f_k(a) \rangle_{M_2(\mathbb{R})}^2,$$

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$$\leq \frac{2\epsilon}{\gamma} |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 8N_1 \|f_k\|_{C^1(\bar{\Omega},\mathbb{R}^2)}^2 + 32N_1 \|f_k\|_{C^1(\bar{\Omega},\mathbb{R}^2)}^2.$$

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Finally, using the equivalence of norm from the previous lemmas, and the previous inequalities, we obtain that

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$$\exists \nu > 0, \forall d \in D, d \leq \eta \Rightarrow \|u_\epsilon^d\|_{H^3(\Omega,\mathbb{R}^2)}^2 \leq \nu.$$

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The sequence  $(u_\epsilon^d)_{d \in D \cap ]0,\eta]}$  is bounded in  $H^3(\Omega, \mathbb{R}^2)$  independently of  $d$  so one can extract a subsequence  $(u_\epsilon^{d_l})_{l \in \mathbb{N}}$  with  $\lim_{l \rightarrow +\infty} d_l = 0$  (since 0 is an accumulation point of  $D$ ) that weakly converges to an element of  $H^3(\Omega, \mathbb{R}^2)$  denoted by  $f_k^* : u_\epsilon^{d_l} \rightharpoonup f_k^*$  in  $H^3(\Omega, \mathbb{R}^2)$ .

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- **Second step** : In the second step, we prove that  $f_k^* = f_k$ . Let us assume that  $f_k^* \neq f_k$  that is, there exists a non-empty open set  $w$  included in  $\Omega$  and a positive real  $\alpha$  such that

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$$\forall x \in w, \langle f_k(x) - f_k^*(x) \rangle_{\mathbb{R}^2} > \alpha.$$

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Let us now set  $\xi = 1 + E \left[ \frac{\epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 2\gamma N_1 \|f_k\|_{C^1(\Omega,\mathbb{R}^2)}^2}{\alpha^2} \right]$ , where  $E[\cdot]$  denotes the integer part of the argument. Let  $B_0 = \{p_{01}, p_{02}, \dots, p_{0\xi}\}$  be a subset of  $\xi$  distinct points from  $w$ . One has :

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$$\forall i = 1, \dots, \xi, \exists (p_{0i}^d)_{d \in D}, \forall d \in D, p_{0i}^d \in A^d \text{ and } p_{0i}^d = \lim_{d \rightarrow 0} p_{0i}^d.$$

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For any  $d \in D$ , let  $B_0^d$  be the set  $\{p_{01}^d, \dots, p_{0\xi}^d\}$ . As previously proved and taking into account that  $B_0^d \subset A^d$ , we have

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$$\sum_{i=1}^{\xi} u_\epsilon^{d_l} (p_{0i}^{d_l} - f_k(p_{0i}^{d_l}))_{\mathbb{R}^2}^2 \leq \epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 2\gamma N_1 \|f_k\|_{C^1(\bar{\Omega},\mathbb{R}^2)}^2.$$

903 Besides,  $\forall i = 1, \dots, \xi$ ,

$$\begin{aligned} 904 \quad \langle u_\epsilon^{d_l}(p_{0i}^{d_l}) - f_k^*(p_{0i}) \rangle_{\mathbb{R}^2} &= \langle u_\epsilon^{d_l}(p_{0i}^{d_l}) - u_\epsilon^{d_l}(p_{0i}) + u_\epsilon^{d_l}(p_{0i}) - f_k^*(p_{0i}) \rangle_{\mathbb{R}^2}, \\ 905 \quad &\leq \langle u_\epsilon^{d_l}(p_{0i}^{d_l}) - u_\epsilon^{d_l}(p_{0i}) \rangle_{\mathbb{R}^2} + \langle u_\epsilon^{d_l}(p_{0i}) - f_k^*(p_{0i}) \rangle_{\mathbb{R}^2} \end{aligned}$$

907 Since  $u_\epsilon^{d_l} \in H^3(\Omega, \mathbb{R}^2) \hookrightarrow C^{0,\lambda}(\bar{\Omega}, \mathbb{R}^2)$ , there exists  $C_3 > 0$  such that

$$908 \quad \langle u_\epsilon^{d_l}(p_{0i}^{d_l}) - u_\epsilon^{d_l}(p_{0i}) \rangle_{\mathbb{R}^2} \leq C_3 \langle p_{0i}^{d_l} - p_{0i} \rangle_{\mathbb{R}^2}^\lambda.$$

910 But  $\lim_{l \rightarrow +\infty} d_l = 0$ , and  $p_{0i} = \lim_{l \rightarrow +\infty} p_{0i}^{d_l}$ , we thus deduce that

$$911 \quad \lim_{l \rightarrow +\infty} \langle u_\epsilon^{d_l}(p_{0i}^{d_l}) - u_\epsilon^{d_l}(p_{0i}) \rangle_{\mathbb{R}^2} = 0.$$

913 The Rellich-Kondrachov compact embedding theorem gives that  $H^3(\Omega, \mathbb{R}^2)$   
914  $\hookrightarrow C^{0,\lambda}(\bar{\Omega}, \mathbb{R}^2)$ . Thus,  $(u_\epsilon^{d_l})_{l \in \mathbb{N}}$  uniformly converges to  $f_k^*$  and then

$$915 \quad \lim_{l \rightarrow +\infty} \langle u_\epsilon^{d_l}(p_{0i}) - f_k^*(p_{0i}) \rangle_{\mathbb{R}^2} = 0.$$

917 Therefore, we can conclude that

$$918 \quad \lim_{l \rightarrow +\infty} \langle u_\epsilon^{d_l}(p_{0i}^{d_l}) - f_k^*(p_{0i}) \rangle_{\mathbb{R}^2} = 0, \text{ that is } \lim_{l \rightarrow +\infty} u_\epsilon^{d_l}(p_{0i}^{d_l}) = f_k^*(p_{0i}).$$

920 Letting  $l$  tend to infinity in the first inequality, we get

$$921 \quad \sum_{l=1}^{\xi} \langle f_k^*(p_{0i}) - f_k(p_{0i}) \rangle_{\mathbb{R}^2}^2 \leq \epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 2\gamma N_1 \|f_k\|_{C^1(\bar{\Omega},\mathbb{R}^2)}^2,$$

923 and so

$$924 \quad \xi \alpha^2 \leq \epsilon |f_k|_{3,\Omega,\mathbb{R}^2}^2 + 2\gamma N_1 \|f_k\|_{C^1(\bar{\Omega},\mathbb{R}^2)}^2,$$

926 which is in contradiction with the choice of  $\xi$ . Then  $f_k^* = f_k$ .

927 • **Third step** : In the last step, we come back to the initial displacements in  $W^{1,\infty}(\Omega, \mathbb{R}^2)$ .  
928 We have

$$\begin{aligned} 929 \quad \lim_{k \rightarrow +\infty, l \rightarrow +\infty} \|u_\epsilon^{d_l} - u\|_{1,\infty} &\leq \lim_{k \rightarrow +\infty} ( \lim_{l \rightarrow +\infty} \|u_\epsilon^{d_l} - f_k\|_{1,\infty} + \|f_k - u\|_{1,\infty} ), \\ 930 \quad &\leq 0, \end{aligned} \quad \blacksquare$$

932 since  $\|f_k - u\|_{1,\infty} \xrightarrow{k \rightarrow +\infty} 0$  by construction and for all  $k \in \mathbb{N}$ ,  $\lim_{l \rightarrow +\infty} \|u_\epsilon^{d_l} - f_k\|_{1,\infty}$  as  
933  $H^3(\Omega, \mathbb{R}^2) \hookrightarrow W^{1,\infty}(\Omega, \mathbb{R}^2)$ .

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935 **SM11. Alternative convergence analysis for the third method.** Let us now consider the  
 936 following problem :

(SM11.1)

$$937 \left\{ \begin{array}{l} \text{Search for } u_\epsilon^d \in H_0^3(\Omega) \text{ such that } \forall t \in H^3(\Omega, \mathbb{R}^2), \\ \langle \rho_0(u_\epsilon^d - f_k) \rangle_{\mathbb{R}^2, N_0}^2 + \frac{\gamma}{2} \langle \rho^d(\nabla u_\epsilon^d + (\nabla u_\epsilon^d)^T - \nabla f_k - \nabla f_k^T - \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N}^2 + \epsilon |u_\epsilon^d|_{3, \Omega, \mathbb{R}^2}^2 \\ 938 \leq \langle \rho_0(v - f_k) \rangle_{\mathbb{R}^2, N_0}^2 + \frac{\gamma}{2} \langle \rho^d(\nabla v + \nabla v^T - \nabla f_k - \nabla f_k^T - \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R}), N}^2 + \epsilon |v|_{3, \Omega, \mathbb{R}^2}^2 \end{array} \right.$$

939 In this case, the norm  $\|\cdot\|_{A^d, 3, \Omega, \mathbb{R}^2}$  is defined by

$$940 \quad \|f\|_{A^d, 3, \Omega, \mathbb{R}^2} = [\langle \rho_0(f_k) \rangle_{\mathbb{R}^2, N_0}^2 + \langle \rho^d(\nabla f + \nabla f^T) \rangle_{M_2(\mathbb{R}), N}^2 + |f|_{3, \Omega, \mathbb{R}^2}^2]^{\frac{1}{2}}.$$

942 As shown in the previous lemma, this norm  $\|\cdot\|_{A^d, 3, \Omega, \mathbb{R}^2}$  is equivalent to the norm  $\|\cdot\|_{3, \Omega, \mathbb{R}^2}$  on  
 943  $H^3(\Omega, \mathbb{R}^2)$ . We first derive the following lemma.

944 **Lemma SM11.1.** *Let  $B_1 = \{b_{01}, \dots, b_{0N}\}$  be a fixed  $P^1$ -unisolvent subset of  $\bar{\Omega}$ . By hypoth-*  
 945 *esis,  $0 \in \bar{D}$  and  $\limsup_{d \rightarrow 0} \delta(x, A^d) = 0$  holds so*

$$946 \quad \forall j = 1, \dots, N, \exists (a_{0j}^d)_{d \in D}, \forall d \in D, a_{0j}^d \in A^d \text{ and } b_{0j} = \lim_{d \rightarrow 0} a_{0j}^d.$$

948 For any  $d \in D$ , let  $A_0^d$  be the set  $\{a_{0j}^d, \dots, a_{0N}^d\}$  and let  $\|\cdot\|_{A_0^d, 3, \Omega, \mathbb{R}^2}$  be the sum defined by  
 949  $\|f\|_{A_0^d, 3, \Omega, \mathbb{R}^2}$ ,

$$950 \quad \|f\|_{A_0^d, 3, \Omega, \mathbb{R}^2} = \left[ \sum_{j=1}^{N_0} \langle f(a_j) \rangle_{\mathbb{R}^2}^2 + \sum_{j=1}^N \langle \nabla f(a_{0j}^d) + \nabla f(a_{0j}^d)^T \rangle_{M_2(\mathbb{R})}^2 \right]^{\frac{1}{2}}.$$

952 Then, there exists  $\eta > 0$  such that for any  $d \leq \eta$ , the norm  $\|\cdot\|_{A_0^d, 3, \Omega, \mathbb{R}^2}$  is uniformly equivalent  
 953 over  $D \cap ]0, \eta]$  to the norm  $\|\cdot\|_{3, \Omega, \mathbb{R}^2}$  in  $H^3(\Omega, \mathbb{R}^2)$ .

954 *Proof.* The Sobolev's embedding gives that

$$955 \quad \exists c_1 > 0, \forall d \in D, \forall f \in H^3(\Omega, \mathbb{R}^2), \|f\|_{A_0^d, 3, \Omega, \mathbb{R}^2} \leq C_1 \|f\|_{3, \Omega, \mathbb{R}^2},$$

957 by using similar arguments as previously. The constant  $C_1$  is independent of  $d$  and only  
 958 depends on  $N_0$  and  $N$ .

959 Let us now find a constant  $C_2$  independent of  $d$  such that the inequality

$$960 \quad \|f\|_{3, \Omega, \mathbb{R}^2} \leq C_2 \|f\|_{A_0^d, 3, \Omega, \mathbb{R}^2},$$

962 holds. For  $f \in H^3(\Omega, \mathbb{R}^2)$ ,

$$963 \quad \frac{1}{2} \sum_{j=1}^N \langle \nabla f(b_{0j}) + \nabla f(b_{0j})^T \rangle_{\mathbb{R}^2}^2 = \frac{1}{2} \sum_{j=1}^N \langle \nabla f(b_{0j}) + \nabla f(b_{0j})^T - \nabla f(a_{0j}^d) - \nabla f(a_{0j}^d)^T \\ 964 + \nabla f(a_{0j}^d) + \nabla f(a_{0j}^d)^T \rangle_{M_2(\mathbb{R})}^2, \\ 965 \leq \sum_{j=1}^N \langle \nabla f(b_{0j}) + \nabla f(b_{0j})^T - \nabla f(a_{0j}^d) - \nabla f(a_{0j}^d)^T \rangle_{M_2(\mathbb{R})}^2 + \langle \nabla f(a_{0j}^d) + \nabla f(a_{0j}^d)^T \rangle_{M_2(\mathbb{R})}^2. \\ 966$$

967 The open subset  $\Omega$  having a Lipschitz-continuous boundary and the space  $H^3(\Omega, \mathbb{R}^2)$  verifies  
968 the Sobolev's Hölder embedding theorem, namely

$$969 \quad \exists \lambda \in ]0, 1], H^3(\Omega, \mathbb{R}^2) \hookrightarrow \mathcal{C}^{1,\lambda}(\bar{\Omega}, \mathbb{R}^2).$$

971 Thus  $f \in \mathcal{C}^{1,\lambda}(\Omega, \mathbb{R}^2)$  and  $\exists C > 0, \forall j = 1, \dots, \mathcal{N}, \forall d \in D,$

$$972 \quad \langle \nabla f(b_{0j}) + \nabla f(b_{0j})^T - \nabla f(a_{0j}^d) - \nabla f(a_{0j}^d)^T \rangle_{M_2(\mathbb{R})}^2 \leq 2 \|f\|_{\mathcal{C}^{1,\lambda}(\Omega, \mathbb{R}^2)}^2 \langle b_{0j} - a_{0j}^d \rangle_{\mathbb{R}^2}^{2\lambda},$$

$$973 \quad \leq 2C^2 \|f\|_{3,\Omega, \mathbb{R}^2}^2 \langle b_{0j} - a_{0j}^d \rangle_{\mathbb{R}^2}^{2\lambda}.$$

975 Besides, we also have from the assumptions that for all  $j = 1, \dots, \mathcal{N},$

$$976 \quad \forall \beta_j > 0, \exists \eta_{\beta_j} > 0, \forall d \in D, (d \leq \eta_{\beta_j} \Rightarrow \langle a_{0j}^d - b_{0j} \rangle_{\mathbb{R}^2} \leq \beta_j).$$

978 Then  $\forall j = 1, \dots, \mathcal{N}$

$$979 \quad \forall \beta_j > 0, \exists \eta_{\beta_j} > 0, \forall d \in D, (d \leq \eta_{\beta_j} \Rightarrow \langle \nabla f(b_{0j}) + \nabla f(b_{0j})^T - \nabla f(a_{0j}^d) - \nabla f(a_{0j}^d)^T \rangle_{M_2(\mathbb{R})}^2$$

$$980 \quad \leq 2C^2 \|f\|_{3,\Omega, \mathbb{R}^2}^2 \beta_j^{2\lambda}.$$

982 Let  $\delta > 0$  and let us take  $\beta_j = \beta, \forall j = 1 \dots, \mathcal{N}, \eta = \min\{\eta_{\beta_1}, \dots, \eta_{\beta_{\mathcal{N}}}\},$  then

$$983 \quad \forall d \in D, (d \leq \eta \Rightarrow \sum_{j=1}^{\mathcal{N}} \langle \nabla f(b_{0j}) + \nabla f(b_{0j})^T - \nabla f(a_{0j}^d) - \nabla f(a_{0j}^d)^T \rangle_{M_2(\mathbb{R})}^2 \leq \mathcal{N} C^2 \beta^{2\lambda} \|f\|_{3,\Omega, \mathbb{R}^2}^2.$$

985 Finally,

$$986 \quad \forall \beta > 0, \exists \eta > 0, \forall d \in D, \forall f \in H^3(\Omega, \mathbb{R}^2),$$

$$987 \quad d \leq \eta \Rightarrow \sum_{j=1}^{\mathcal{N}} \langle \nabla f(b_{0j}) + \nabla f(b_{0j})^T - \nabla f(a_{0j}^d) - \nabla f(a_{0j}^d)^T \rangle_{M_2(\mathbb{R})}^2 \leq C^2 \beta^{2\lambda} \mathcal{N} \|f\|_{3,\Omega, \mathbb{R}^2}^2.$$

989 This implies that

$$990 \quad \forall \beta > 0, \exists \eta > 0, \forall d \in D, \forall f \in H^3(\Omega, \mathbb{R}^2), d \leq \eta$$

$$991 \quad \Rightarrow \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \langle \nabla f(b_{0j}) + \nabla f(b_{0j}^T) \rangle_{M_2(\mathbb{R})}^2 + \sum_{j=1}^{N_0} \langle f(a_i) \rangle_{\mathbb{R}^2}^2 + |f|_{3,\Omega, \mathbb{R}^2}^2 - C^2 \beta^{2\lambda} \mathcal{N} \|f\|_{3,\Omega, \mathbb{R}^2}^2 \leq \|f\|_{A_0^d, 3, \Omega, \mathbb{R}^2}^2.$$

993 As previously shown, the mapping  $f \in H^3(\Omega, \mathbb{R}^2) \mapsto \left[ \sum_{j=1}^{N_0} \langle f(a_i) \rangle_{\mathbb{R}^2}^2 + \frac{1}{2} \sum_{j=1}^{\mathcal{N}} \langle \nabla f(b_{0j}) + \nabla f(b_{0j})^T \rangle_{M_2(\mathbb{R})}^2 \right. \\ 994 \quad \left. + |f|_{3,\Omega, \mathbb{R}^2}^2 \right]^{\frac{1}{2}}$  is a norm on  $H^3(\Omega, \mathbb{R}^2)$  equivalent to the norm  $\|\cdot\|_{3,\Omega, \mathbb{R}^2}$  so

$$995 \quad \forall \beta > 0, \exists \eta > 0, \forall d \in D, \forall f \in H^3(\Omega, \mathbb{R}^2),$$

$$996 \quad d \leq \eta \Rightarrow (C'^2 - C^2 \beta^{2\lambda} \mathcal{N}) \|f\|_{3,\Omega, \mathbb{R}^2}^2 \leq \|f\|_{3,\Omega, \mathbb{R}^2}^2.$$

998 By choosing  $\beta$  adequately, the norm equivalence is obtained. ■

999 Before considering a convergence result, let us first introduce the following problem

1000 (SM11.2) 
$$\inf_{u \in H^3(\Omega, \mathbb{R}^2)} |u|_{3, \Omega, \mathbb{R}^2},$$

1001 s.t.  $\nabla u + \nabla u^T = \nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k$  everywhere on  $\Omega$ ,

1002  $\rho_0(u) = \rho_0(f_k),$

1004 and prove the existence of minimisers.

1005 **Theorem SM11.2 (Existence of minimisers).** *Let us assume that there exists  $u_0 \in H^3(\Omega, \mathbb{R}^2)$*   
 1006 *such that*

1007 
$$\frac{\partial u_{0,1}}{\partial x} = \frac{\partial f_{k,1}}{\partial x} + \left(\frac{\partial f_{k,1}}{\partial x}\right)^2 + \left(\frac{\partial f_{k,2}}{\partial x}\right)^2,$$

1008 
$$\frac{\partial u_{0,2}}{\partial y} = \frac{\partial f_{k,2}}{\partial y} + \left(\frac{\partial f_{k,2}}{\partial y}\right)^2 + \left(\frac{\partial f_{k,1}}{\partial y}\right)^2,$$

1009 
$$\frac{\partial u_{0,1}}{\partial y} + \frac{\partial u_{0,2}}{\partial x} = \frac{\partial f_{k,1}}{\partial y} + \frac{\partial f_{k,2}}{\partial x} + \frac{\partial f_{k,1}}{\partial x} \frac{\partial f_{k,1}}{\partial y} + \frac{\partial f_{k,2}}{\partial x} \frac{\partial f_{k,2}}{\partial y},$$

1010

1011 *everywhere on  $\Omega$  and for all  $i = 1, \dots, N_0$ ,  $u_0(a_i) = f_k(a_i)$ . (This shouldn't be too restrictive*  
 1012 *if  $N_0$  is small enough). Then there is at least one minimiser of the problem (SM11.2).*

1013 *Proof.* Let  $(u_j)$  be a minimising sequence of the problem such that  $\forall j \in \mathbb{N}$ ,

1014 
$$u_j \in H^3(\Omega, \mathbb{R}^2),$$

1015 
$$\nabla u_j + \nabla u_j^T = \nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k, \text{ everywhere on } \Omega,$$

1016 
$$\rho(u_j) = \rho(f_k).$$

1018 For  $j$  large enough, we have the following coercivity inequality

1019  $|u_j|_{3, \Omega, \mathbb{R}^2} \leq |u_0|_{3, \Omega, \mathbb{R}^2}$ , and so

1020  $\langle \rho(\nabla u_j + \nabla u_j^T) \rangle_{M_2(\mathbb{R})}^2 + \langle \rho(u_j) \rangle_{\mathbb{R}^2}^2 + |u_j|_{3, \Omega, \mathbb{R}^2}^2 \leq \langle \rho(\nabla u_0 + \nabla u_0^T) \rangle_{M_2(\mathbb{R})}^2 + \langle \rho(u_0) \rangle_{\mathbb{R}^2}^2 + |u_0|_{3, \Omega, \mathbb{R}^2}^2. \blacksquare$

1022 Thanks to the previous lemma that gives an equivalence of norms, we get that  $(u_j)$  is uniformly  
 1023 bounded according to  $j$  in  $H^3(\Omega, \mathbb{R}^2)$ . We thus can extract a sub-sequence still denoted by  
 1024  $(u_j)$  such that

1025 
$$u_j \xrightarrow{j \rightarrow +\infty} \bar{u} \text{ in } H^3(\Omega, \mathbb{R}^2),$$

1026

1027 with  $\bar{u} \in H^3(\Omega, \mathbb{R}^2)$ . Since  $H^3(\Omega, \mathbb{R}^2) \hookrightarrow_c \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ , we deduce that  $u_k \rightarrow \bar{u}$  in  $\mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ , and  
 1028 therefore

1029 
$$u_j \xrightarrow{j \rightarrow +\infty} \bar{u} \text{ everywhere in } \Omega,$$

1030 
$$\nabla u_j \xrightarrow{j \rightarrow +\infty} \nabla \bar{u} \text{ everywhere in } \Omega,$$

1031 
$$\nabla u_j^T \xrightarrow{j \rightarrow +\infty} \nabla \bar{u}^T \text{ everywhere in } \Omega.$$

1032

1033 Hence we have that

$$1034 \quad \rho(\bar{u}) = \lim_{j \rightarrow +\infty} \rho(u_j) = \lim_{j \rightarrow +\infty} \rho(f_k) = \rho(f_k),$$

$$1035 \quad \nabla \bar{u}(x) + \nabla \bar{u}^T(x) = \lim_{j \rightarrow +\infty} \nabla u_j(x) + \nabla u_j(x)^T = \lim_{j \rightarrow +\infty} \nabla f_k(x) + \nabla f_k^T(x) + \nabla f_k^T(x) \nabla f_k(x)$$

$$1036 \quad = \nabla f_k(x) + \nabla f_k^T(x) + \nabla f_k^T(x) \nabla f_k(x), \text{ for every } x \in \Omega.$$

1038 Finally, by the semi-continuity of  $|\cdot|_{3,\Omega,\mathbb{R}^2}$ , we get that

$$1039 \quad |\bar{u}|_{3,\Omega,\mathbb{R}^2} \leq \liminf_{j \rightarrow +\infty} |u_j|_{3,\Omega,\mathbb{R}^2},$$

1041 and  $\bar{u}$  is a minimiser of the problem (SM11.2). ■

1042 We now establish the following convergence result

1043 **Theorem SM11.3 (Convergence).** *Let  $\hat{f}_k$  be a minimiser of problem (SM11.2). For any*  
 1044  *$d \in D$ , and  $\epsilon \in ]0, \epsilon_0]$ , we denote by  $u_{\epsilon,k}^d$  the unique solution of problem (SM11.1). Then under*  
 1045 *the above assumptions, there exists a subsequence  $(u_{\epsilon_l,k}^{d_l})$  with  $\lim_{l \rightarrow +\infty} d_l = \lim_{l \rightarrow +\infty} \epsilon_l = 0$ , such that*

$$1046 \quad u_{\epsilon_l,k}^{d_l} \xrightarrow{l \rightarrow +\infty} \hat{f}_k$$

1048 *in  $H^3(\Omega, \mathbb{R}^2)$  with  $\lim_{k \rightarrow +\infty} \lim_{l \rightarrow +\infty} \|\nabla u_{\epsilon_l,k}^{d_l} + (\nabla u_{\epsilon_l,k}^{d_l})^T - \nabla u - \nabla u^T - \nabla u^T \nabla u\|_{L^\infty(\Omega, M_2(\mathbb{R}))} = 0$ .*

1049 *Proof.* The proof is divided into four steps that we detail hereafter.

1050 • **First step** : We start by proving that the sequence  $(u_{\epsilon,k}^d)_{d \in D \cap ]0, \eta], \epsilon \in ]0, \epsilon_0]}$  is bounded  
 1051 in  $H^3(\Omega, \mathbb{R}^2)$ . In the minimisation problem (SM11.1), let us take  $v = \hat{f}_k$ . Then we  
 1052 have

$$1053 \quad \langle \rho(u_{\epsilon,k}^d - f_k) \rangle_{\mathbb{R}^2, N_0}^2 + \frac{\gamma}{2} \langle \rho^d(\nabla u_{\epsilon,k}^d + (\nabla u_{\epsilon,k}^d)^T - \nabla f_k - \nabla f_k^T - \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R})}^2 + \epsilon |u_{\epsilon,k}^d|_{3,\Omega,\mathbb{R}}^2$$

$$1054 \quad \leq \epsilon |\hat{f}_k|_{3,\Omega,\mathbb{R}}^2, \quad \blacksquare$$

1056 from which we deduce that

$$1057 \quad \begin{cases} |u_{\epsilon,k}^d|_{3,\Omega,\mathbb{R}^2} \leq |\hat{f}_k|_{3,\Omega,\mathbb{R}^2}, \\ \frac{\gamma}{2} \langle \rho^d(\nabla u_{\epsilon,k}^d + (\nabla u_{\epsilon,k}^d)^T - \nabla f_k - \nabla f_k^T - \nabla f_k^T \nabla f_k) \rangle_{M_2(\mathbb{R})}^2 \leq \epsilon |\hat{f}_k|_{3,\Omega,\mathbb{R}^2}^2, \\ \langle \rho_0(u_{\epsilon,k}^d - f_k) \rangle_{\mathbb{R}^2}^2 \leq \epsilon |\hat{f}_k|_{3,\Omega,\mathbb{R}}^2 \leq \epsilon_0 |\hat{f}_k|_{3,\Omega,\mathbb{R}}^2. \end{cases}$$

1059 As  $A_0^d \subset A^d$ , one has

$$1060 \quad \sum_{a \in A_0^d} \langle \nabla u_{\epsilon,k}^d(a) + \nabla u_{\epsilon,k}^d(a)^T - \nabla \hat{f}_k(a) - \nabla \hat{f}_k(a)^T \rangle_{M_2(\mathbb{R})}^2 \leq \epsilon_0 |\hat{f}_k|_{3,\Omega,\mathbb{R}^2}^2.$$

1062 Moreover

$$1063 \quad \sum_{a \in A_0^d} \langle \nabla u_{\epsilon,k}^d(a) + \nabla u_{\epsilon,k}^d(a)^T \rangle_{M_2(\mathbb{R})}^2 \leq 2 \sum_{a \in A_0^d} \langle \nabla u_{\epsilon,k}^d(a) + \nabla u_{\epsilon,k}^d(a)^T - \nabla \hat{f}_k(a) - \nabla \hat{f}_k(a)^T \rangle_{M_2(\mathbb{R})}^2$$

$$1064 \quad + 2 \sum_{a \in A_0^d} \langle \nabla \hat{f}_k(a) + \nabla \hat{f}_k(a)^T \rangle_{M_2(\mathbb{R})}^2,$$

$$1065 \quad \leq 2\epsilon_0 |\hat{f}_k|_{3,\Omega,\mathbb{R}^2}^2 + 4\mathcal{N} \|\hat{f}_k\|_{C^1(\Omega, \mathbb{R}^2)}^2. \quad \blacksquare$$



1067 Finally, using the equivalence of norm previously established, and the inequalities  
1068 above, we obtain that

$$1069 \quad \exists \nu > 0, \forall d \in D, \forall \epsilon \in ]0, \epsilon_0], d \leq \eta \Rightarrow \|u_k^d\|_{3, \Omega, \mathbb{R}^2} \leq \nu.$$

1071 The sequence  $(u_{\epsilon, k}^d)_{d \in D \cap ]0, \eta], \epsilon \in ]0, \epsilon_0]}$  is bounded in  $H^3(\Omega, \mathbb{R}^2)$  so one can extract a sub-  
1072 sequence  $(u_{\epsilon_l, k}^{d_l})$  with  $\lim_{l \rightarrow +\infty} d_l = \lim_{l \rightarrow +\infty} \epsilon_l = 0$  that weakly converges to  $f_k^*$  in  $H^3(\Omega, \mathbb{R}^2)$ .

1073 • **Second step:** We now prove that  $f_k^* = \hat{f}_k$ . Let us assume that  $\nabla f_k^* + (\nabla f_k^*)^T \neq$   
1074  $\nabla \hat{f}_k + \nabla \hat{f}_k^T$ , that is, there exists a non-empty open set  $\omega$  included in  $\Omega$  and a positive  
1075 real  $\alpha$  such that

$$1076 \quad \forall x \in \omega, \langle \nabla f_k^*(x) + (\nabla f_k^*)^T(x) - \nabla \hat{f}_k(x) - \nabla \hat{f}_k^T(x) \rangle_{M_2(\mathbb{R})}^2 > \alpha.$$

1078 Let us set  $\xi = E \left[ \frac{\epsilon |\hat{f}_k|_{3, \Omega, \mathbb{R}^2}^2}{\alpha^2} \right]$ , where  $E[\cdot]$  denotes the integer part of the argument. Let  
1079  $B_0 = \{p_{01}, \dots, p_{0\xi}\}$  be a subset of  $\xi$  distinct points from  $\omega$ . One has

$$1080 \quad \forall i = 1, \dots, \xi, \exists (p_{0i}^d)_{d \in D}, (\forall d \in D, p_{0i} \in A) \text{ and } p_{0i} = \lim_{d \rightarrow 0} p_{0i}^d.$$

1082 For any  $d \in D$ , let  $B_0^d$  be the set  $\{p_{01}^d, \dots, p_{0\xi}^d\}$ . As previously proved and taking into  
1083 account that  $B_0^d \subset A^d$ , we have

$$1084 \quad \sum_{i=1}^{\xi} \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l})^T - \nabla \hat{f}_k(p_{0i}^{d_l}) - \nabla \hat{f}_k(p_{0i}^{d_l})^T \rangle_{M_2(\mathbb{R})}^2 \leq \epsilon |\hat{f}_k|_{3, \Omega, \mathbb{R}^2}^2.$$

1086 Besides, for all  $i = 1, \dots, \xi$ ,

$$\begin{aligned} 1087 & \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) - \nabla f_k^*(p_{0i}) - \nabla f_k^*(p_{0i})^T \rangle_{M_2(\mathbb{R})} \\ 1088 & = \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) - \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}) - \nabla u_{\epsilon_l, k}^{d_l}(p_{0i})^T + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i})^T \\ 1089 & - \nabla f_k^*(p_{0i}) - \nabla f_k^*(p_{0i})^T \rangle_{M_2(\mathbb{R})}, \\ 1090 & \leq \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) - \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}) - \nabla u_{\epsilon_l, k}^{d_l}(p_{0i})^T \rangle_{M_2(\mathbb{R})} + \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}) \\ 1091 & + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i})^T - \nabla f_k^*(p_{0i}) - \nabla f_k^*(p_{0i})^T \rangle_{M_2(\mathbb{R})}. \end{aligned}$$

1093 But  $u_{\epsilon_l, k}^{d_l} \in H^3(\Omega, \mathbb{R}^2) \xrightarrow{c} \mathcal{C}^{1, \lambda}(\Omega, \mathbb{R}^2)$  for  $\lambda \in [0, 1]$  and so there exists  $C_3 > 0$  such  
1094 that

$$1095 \quad \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) - \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}) - \nabla u_{\epsilon_l, k}^{d_l}(p_{0i})^T \rangle_{M_2(\mathbb{R})} \leq 2C_3 \langle p_{0i}^{d_l} - p_{0i} \rangle_{\mathbb{R}^2}^\lambda.$$

1097 But  $\lim_{l \rightarrow +\infty} d_l = 0$  and  $p_{0i} = \lim_{l \rightarrow +\infty} p_{0i}^{d_l}$ , and we deduce that

$$1098 \quad \lim_{l \rightarrow +\infty} \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) - \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}) - \nabla u_{\epsilon_l, k}^{d_l}(p_{0i})^T \rangle_{M_2(\mathbb{R})} = 0.$$

1099

1100 The Rellisch-Kondrachov compact embedding theorem gives that  $H^3(\Omega, \mathbb{R}^2)$   
1101  $\hookrightarrow_c \mathcal{C}^1(\Omega, \mathbb{R}^2)$ , and thus

$$1102 \lim_{l \rightarrow +\infty} \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i})^T \nabla f_k^*(p_{0i}) - \nabla f_k^*(p_{0i})^T \rangle_{M_2(\mathbb{R})} = 0.$$

1104 Therefore, we can conclude that

$$1105 \lim_{l \rightarrow +\infty} \langle \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) - \nabla f_k^*(p_{0i}) - \nabla f_k^*(p_{0i})^T \rangle_{M_2(\mathbb{R})} = 0,$$

1107 and so  $\lim_{l \rightarrow +\infty} \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) + \nabla u_{\epsilon_l, k}^{d_l}(p_{0i}^{d_l}) = \nabla f_k^*(p_{0i}) + \nabla f_k^*(p_{0i})^T$ . Letting  $l$  tend to  
1108 infinity, it comes

$$1109 \sum_{i=1}^{\xi} \langle \nabla f_k^*(p_{0i}) + \nabla f_k^*(p_{0i}) - \nabla \hat{f}_k(p_{0i}) - \nabla \hat{f}_k(p_{0i})^T \rangle_{M_2(\mathbb{R})}^2 \leq \epsilon |\hat{f}_k|_{3, \Omega, \mathbb{R}^2}^2,$$

1111 and so  $\alpha^2 \xi \leq \epsilon |\hat{f}_k|_{3, \Omega, \mathbb{R}^2}^2$  which is in contradiction with the definition of  $\xi$ . Con-  
1112 sequently,  $\nabla f_k^* + (\nabla f_k^*)^T = \nabla \hat{f}_k + \nabla \hat{f}_k^T = \nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k$  everywhere on  
1113  $\Omega$ . Also, since  $H^3(\Omega, \mathbb{R}^2) \hookrightarrow_c \mathcal{C}^0(\bar{\Omega}, \mathbb{R}^2)$ , we have that  $\lim_{l \rightarrow +\infty} \langle \rho_0(u_{\epsilon_l, k}^{d_l} - f_k) \rangle_{\mathbb{R}^2, N_0}^2 =$   
1114  $\langle \rho_0(f_k^* - f_k) \rangle_{\mathbb{R}^2, N_0}^2$ , and by letting  $l$  tend to infinity in the last inequality we get that

$$1115 \langle \rho_0(f_k^* - f_k) \rangle_{\mathbb{R}^2, N_0}^2 \leq \lim_{l \rightarrow +\infty} \epsilon_l |\hat{f}_k|_{3, \Omega, \mathbb{R}^2}^2 = 0.$$

1117 Finally, by weak lower semicontinuity of the seminorm in  $H^3(\Omega, \mathbb{R}^2)$ , we get that

$$1118 |\hat{f}_k|_{3, \Omega, \mathbb{R}^2} \leq \liminf_{l \rightarrow +\infty} |u_{\epsilon_l, k}^{d_l}|_{3, \Omega, \mathbb{R}^2} \leq |\hat{f}_k|_{3, \Omega, \mathbb{R}^2}.$$

1120 Thus  $f_k^*$  is a minimiser of problem (SM11.2). Without loss of generality, we say  
1121  $f_k^* = \hat{f}_k$ .

1122 • **Third step:** We then have

$$1123 \begin{aligned} & \|\nabla u_{\epsilon_l, k}^{d_l} + (\nabla u_{\epsilon_l, k}^{d_l})^T - \nabla u - \nabla u^T - \nabla u^T \nabla u\|_{L^\infty(\Omega, M_2(\mathbb{R}))} \\ 1124 & \leq \|\nabla u_{\epsilon_l, k}^{d_l} + (\nabla u_{\epsilon_l, k}^{d_l})^T - \nabla \hat{f}_k - \nabla \hat{f}_k^T\|_{L^\infty(\Omega, M_2(\mathbb{R}))} + \|\nabla \hat{f}_k + \nabla \hat{f}_k^T - \nabla u - \nabla u^T \\ 1125 & - \nabla u^T \nabla u\|_{L^\infty(\Omega, M_2(\mathbb{R}))}, \\ 1126 & \leq \|\nabla u_{\epsilon_l, k}^{d_l} + (\nabla u_{\epsilon_l, k}^{d_l})^T - \nabla \hat{f}_k - \nabla \hat{f}_k^T\|_{L^\infty(\Omega, M_2(\mathbb{R}))} + \|\nabla f_k + \nabla f_k^T + \nabla f_k^T \nabla f_k - \nabla u \\ 1127 & - \nabla u^T - \nabla u^T \nabla u\|_{L^\infty(\Omega, M_2(\mathbb{R}))}, \end{aligned}$$

1129 with  $\lim_{k \rightarrow +\infty} \lim_{l \rightarrow +\infty} \|\nabla u_{\epsilon_l, k}^{d_l} + (\nabla u_{\epsilon_l, k}^{d_l})^T - \nabla \hat{f}_k - \nabla \hat{f}_k^T\|_{L^\infty(\Omega, M_2(\mathbb{R}))} = 0$  from what precedes  
1130 and the Sobolev embedding  $H^3(\Omega, \mathbb{R}^2) \hookrightarrow_c W^{1, \infty}(\Omega, \mathbb{R}^2)$ , and  $\lim_{k \rightarrow +\infty} \lim_{l \rightarrow +\infty} \|\nabla f_k + \nabla f_k^T +$   
1131  $\nabla f_k^T \nabla f_k - \nabla u - \nabla u^T - \nabla u^T \nabla u\|_{L^\infty(\Omega, M_2(\mathbb{R}))} = 0$ , by construction of the sequence  $f_k$ .  
1132 We thus have  $\lim_{k \rightarrow +\infty} \lim_{l \rightarrow +\infty} \|\nabla u_{\epsilon_l, k}^{d_l} + (\nabla u_{\epsilon_l, k}^{d_l})^T - \nabla u - \nabla u^T - \nabla u^T \nabla u\|_{L^\infty(\Omega, M_2(\mathbb{R}))} = 0$ .

1133 • **Fourth step** : We aim at proving that the sequence  $(u_{\epsilon_l, k}^{d_l})$  strongly converges to  $\hat{f}_k$   
 1134 in  $H^3(\Omega, \mathbb{R}^2)$ . The Rellich-Kondrachov compact embedding theorem gives that

$$1135 \quad \forall r, r' \in \mathbb{R}, r > r', H^r(\Omega, \mathbb{R}^2) \xhookrightarrow{c} H^{r'}(\Omega, \mathbb{R}^2). \\ 1136$$

1137 In our case, it means that the sequence  $(u_{\epsilon_l, k}^{d_l})$  that weakly converges to  $\hat{f}_k$  in  $H^3(\Omega, \mathbb{R}^2)$ ,  
 1138 strongly converges to  $\hat{f}_k$  in  $H^2(\Omega, \mathbb{R}^2)$ . We thus just need to prove that

$$1139 \quad \lim_{l \rightarrow +\infty} |u_{\epsilon_l, k}^{d_l} - \hat{f}_k|_{3, \Omega, \mathbb{R}^2} = 0. \\ 1140$$

1141 One has

$$1142 \quad |u_{\epsilon_l, k}^{d_l} - \hat{f}_k|_{3, \Omega, \mathbb{R}^2} = |u_{\epsilon_l, k}^{d_l}|_{3, \Omega, \mathbb{R}^2} + |\hat{f}_k|_{3, \Omega, \mathbb{R}^2} - 2(u_{\epsilon_l, k}^{d_l}, \hat{f}_k)_{3, \Omega, \mathbb{R}^2}, \\ 1143 \quad \leq 2|\hat{f}_k|_{3, \Omega, \mathbb{R}^2} - 2(u_{\epsilon_l, k}^{d_l}, \hat{f}_k)_{3, \Omega, \mathbb{R}^2},$$

1145 so  $\lim_{l \rightarrow +\infty} |u_{\epsilon_l, k}^{d_l} - \hat{f}_k|_{3, \Omega, \mathbb{R}^2} = 0$ , and finally  $\lim_{l \rightarrow +\infty} \|u_{\epsilon_l, k}^{d_l} - \hat{f}_k\|_{3, \Omega, \mathbb{R}^2} = 0$  which concludes  
 1146 the proof. ■

1147 **SM12. Numerical Resolution of the Third Approach.** We now turn to the discretisation  
 1148 of the variational problem associated with [Theorem 4.4](#) in which  $u$  is a substitute for  $f_k$ . To  
 1149 do so, we use standard notations of the finite element theory similar to those in [\[?, ?\]](#). Let  $\mathcal{H}$   
 1150 be an open bounded subset of  $]0, +\infty[$  admitting 0 as accumulation point. Let us recall that  
 1151 the elements of class  $\mathcal{C}^{k'}$  can be used for the computation of discrete  $D^m$ -splines (in our case,  
 1152  $m = 3$ ) with  $m \leq k' + 1$ . As a consequence,  $(k', m) = (2, 3)$  is a suitable combination. For  
 1153 all  $n \in \mathbb{N}$  and for all subsets  $E$  of  $\mathbb{R}^2$ ,  $Q_l(E)$  denotes the space of the restrictions of  $E$  of the  
 1154 polynomial functions over  $\mathbb{R}^2$  of degree  $\leq l$  with respect to each variable.  $\forall h \in \mathcal{H}$ , let  $(V_h)^2$   
 1155 be the subspace of  $H^3(\Omega, \mathbb{R}^2)$  of finite dimension with  $(V_h)^2 \hookrightarrow \mathcal{C}^1(\bar{\Omega}, \mathbb{R}^2)$ . The reference finite  
 1156 element is the Bogner-Fox-Schmit  $\mathcal{C}^2$  rectangle denoted by  $(K, P_K, \Sigma_K)$ .

1157 Let  $(v^q)_{q=1,2}$  be the components of  $v \in H^3(\Omega, \mathbb{R}^2)$  and  $w_i = \begin{pmatrix} w_{i,11} & w_{i,12} \\ w_{i,21} = w_{i,12} & w_{i,22} \end{pmatrix} :=$   
 1158  $\nabla u(a_i) + \nabla u(a_i)^T + \nabla u(a_i)^T \nabla u(a_i)$ ,  $\forall i \in \{1, \dots, N\}$ . Let also  $(x_i^q)_{q=1,2}$  be the compo-  
 1159 nents of  $u(a_i)$ ,  $\forall i \in \{1, \dots, N\}$ . Now let  $M_h$  be the dimension of  $V_h$  and  $\{P_j^h\}_{j=1, \dots, M_h}$  be  
 1160 basis functions. If we denote by  $u_\epsilon^h$  the solution of the variational problem associated with  
 1161 [Theorem 4.4](#) and approximated in  $(V_h)^2$ , we can thus decompose  $u_\epsilon^h = (u_\epsilon^{h,q})_{q=1,2}$  into:

$$1162 \quad \forall q = 1, 2, \exists (\alpha_j^q)_{j=1, \dots, M_h} \in \mathbb{R}, u_\epsilon^{h,q} = \sum_{j=1}^{M_h} \alpha_j^q P_j^h. \\ 1163$$

1164 Denoting by  $A^h = \left( \frac{\partial P_j^h}{\partial x}(a_i) \right)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M_h}}$ ,  $B^h = \left( \frac{\partial P_j^h}{\partial y}(a_i) \right)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M_h}} \in (M_{N \times M_h}(\mathbb{R}))^2$ ,  $C^h =$   
 1165  $\left( P_j^h(a_i) \right)_{\substack{1 \leq i \leq N \\ 1 \leq j \leq M_h}} \in M_{N \times M_h}(\mathbb{R})$  and  $R^h = \left( (P_j^h, P_i^h)_{3, \Omega, \mathbb{R}} \right)_{1 \leq i, j \leq M_h}$  and taking successively

1166 in the variational problem  $v = \begin{pmatrix} P_l^h \\ 0 \end{pmatrix}$ ,  $l = 1, \dots, M_h$  and then  $v = \begin{pmatrix} 0 \\ P_l^h \end{pmatrix}$ ,  $l = 1, \dots, M_h$ , the  
1167 problem amounts to solving the following linear system

$$1168 \quad \begin{array}{c} M_h \\ M_h \end{array} \left( \begin{array}{c|c} M_h & M_h \\ \hline 2\gamma(A^h)^T A^h + \gamma(B^h)^T B^h & \gamma(B^h)^T A^h \\ + (C^h)^T C^h + \epsilon R^h & \\ \hline \gamma(A^h)^T B^h & \gamma(A^h)^T A^h + 2\gamma(B^h)^T B^h \\ + (C^h)^T C^h + \epsilon R^h & \end{array} \right) \begin{pmatrix} \alpha^1 \\ \alpha^2 \end{pmatrix} = \begin{pmatrix} \gamma(A^h)^T w_{11} + \gamma(B^h)^T w_{12} \\ + (C^h)^T x^1 \\ \hline \gamma(A^h)^T w_{12} + \gamma(B^h)^T w_{22} \\ + (C^h)^T x^2 \end{pmatrix}.$$

1169  
1170