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# A SHORT NOTE ON A PAIR OF MEROMORPHIC FUNCTIONS IN A $p$ -ADIC FIELD, SHARING A FEW SMALL ONES

ALAIN ESCASSUT AND C.C. YANG

ABSTRACT. Let  $f, g$ , be two meromorphic functions on a complete ultrametric algebraically closed field  $\mathbb{K}$  of characteristic 0, or two meromorphic functions in an open disk of  $\mathbb{K}$ , that are not quotients of bounded analytic functions by polynomials. If  $f$  and  $g$  share I.M. 9 small meromorphic functions then  $f = g$ .

If  $f$  and  $g$  have finitely many poles and share I.M. 4 small meromorphic functions, then  $f = g$ .

## 1. Main results

Let  $\mathbb{K}$  be a complete ultrametric algebraically closed field of characteristic 0. Let us fix  $a \in \mathbb{K}$  and let  $R \in ]0, +\infty[$ . We denote by  $d(a, R^-)$  the disk  $\{x \in \mathbb{K} \mid |x - a| < R\}$ .

We denote by  $\mathcal{A}(\mathbb{K})$  the  $\mathbb{K}$ -algebra of entire functions in  $\mathbb{K}$  and by  $\mathcal{M}(\mathbb{K})$  the field of meromorphic functions which is its field of fractions. We denote by  $\mathcal{A}(d(a, R^-))$  the  $\mathbb{K}$ -algebra of analytic functions in  $d(a, R^-)$  i.e. the set of power series converging in the disk  $d(a, R^-)$ . and by  $\mathcal{M}(d(a, R^-))$  the field of meromorphic functions in  $d(a, R^-)$  i.e. the field of fractions of  $\mathcal{A}(d(a, R^-))$ . Moreover, we denote by  $\mathcal{A}_b(d(a, R^-))$  the  $\mathbb{K}$ -algebra of functions  $f \in \mathcal{A}(d(a, R^-))$  that are bounded in  $d(a, R^-)$ , by  $\mathcal{M}_b(d(a, R^-))$  its field of fractions and we put  $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$ .

We define  $N(r, f)$  ([1], chapter 40 or [3], chapter 2) in the same way as for complex meromorphic functions [2]. Let  $f$  be a meromorphic function in all  $\mathbb{K}$  having no zero and no pole at 0. Let  $(a_n)_{n \in \mathbb{N}}$  be the sequence of poles of  $f$ , of respective order  $s_n$ , with  $|a_n| \leq |a_{n+1}|$  and, given  $r > 0$ , (resp.  $r \in ]0, R[$ ), let  $q(r)$  be such that  $|a_{q(r)}| \leq r, |a_{q(r)+1}| > r$ . We then denote by  $N(r, f)$  the counting function of the zeros of  $f$ , counting multiplicity, as usual: for all  $r > 0$ , we put

$$N(r, f) = \sum_{j=0}^{q(r)} s_j (\log |(a_j)| - \log(r)).$$

Moreover, we denote by  $\bar{N}(r, f)$  the count-

ing function of the poles of  $f$ , ignoring multiplicity as  $\sum_{j=0}^{q(r)} (\log |(a_j)| - \log(r))$ .

Next, we define the counting function of zeros of  $f$  as  $Z(r, f) = N(r, \frac{1}{f})$  and we put  $\bar{Z}(r, f) = \bar{N}(r, \frac{1}{f})$ . Finally we put  $T(r, f) = \max(Z(r, f), N(r, f))$ . Then

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$T(r, f)$  is strictly increasing and has most of properties of the characteristic function of a complex function, concerning operations (see [1], chapter 40 and [3], chapter 2.)

A function  $w \in \mathcal{M}(\mathbb{K})$  (resp.  $w \in \mathcal{M}(d(a, R^-))$ ) is called a *small function with respect to  $f$*  if  $\lim_{r \rightarrow +\infty} \frac{T(r, w)}{T(r, f)} = 0$  (resp.  $\lim_{r \rightarrow R^-} \frac{T(r, w)}{T(r, f)} = 0$ ) and we denote by  $\mathcal{M}_f(\mathbb{K})$  (resp.  $\mathcal{M}_f(d(a, R^-))$ ) the set of small functions with respect to  $f$ .

Two functions  $f, g \in \mathcal{M}(\mathbb{K})$  (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ) are said to *share I.M.* a small function  $w \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  (resp.  $w \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ ) if  $f(z) = w(z)$  holds if and only if  $g(z) = w(z)$ .

Here we aim at studying the problem of two meromorphic functions sharing I.M. a few small meromorphic functions in order to show that these two functions are equal. Indeed thanks to Yamanoi's Nevanlinna Second Main Theorem, a similar result is known in complex analysis when two meromorphic functions share 5 small meromorphic functions. But in  $p$ -adic analysis, no Theorem similar to Yamanoi's Theorem is known. Thus, we have studied and used tools or arguments other than Yamanoi's Theorem to derive some similar results as follows.

**Theorem 1:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ) and share I.M.  $q$  small functions  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, q$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ ) ( $j = 1, \dots, q$ ). Then*

$$\sum_{j=1}^q \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

**Theorem 2:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ) and share I.M. 9 distinct small functions (other than the constant  $\infty$ )  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, 9$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ ) ( $j = 1, \dots, 9$ ). Then  $f = g$ .*

**Theorem 3:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{M}_u(d(a, R^-))$ ) have finitely many poles and share I.M. 4 distinct small functions (other than the constant  $\infty$ )  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, 4$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ ) ( $j = 1, \dots, 4$ ). Then  $f = g$ .*

**Corollary:** *Let  $f, g \in \mathcal{A}(\mathbb{K})$  be transcendental (resp.  $f, g \in \mathcal{A}_u(d(a, R^-))$ ) share I.M. 4 distinct small functions (other than the constant  $\infty$ )  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, 4$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ ) ( $j = 1, \dots, 4$ ). Then  $f = g$ .*

**Remark:** The results known in complex analysis suggest that the number 9 obtained in Theorem 2 might be improved, concerning  $p$ -adic meromorphic functions. On the contrary, concerning analytic functions or meromorphic functions with finitely many poles, the number 4 obtained in Theorem 3 seems the best possible.

## 2. The proofs

In order to prove the theorems, we need to recall the following three lemmas. Lemma 1 comes from Theorem 40.8 in [1].

**Lemma 1:** *Let  $f, g \in \mathcal{M}(\mathbb{K})$  (resp. let  $f, g \in \mathcal{M}(d(a, R^-))$ ). Then  $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$ . Moreover, if  $f$  and  $g$  have finitely many poles then  $T(r, f + g) \leq \max(T(r, f), T(r, g)) + O(\log(r))$ .*

The following Lemma 2 is Lemma 40.10 in [1] (see also Proposition 2.5 in [3]):

**Lemma 2:** *Let  $f \in \mathcal{M}(\mathbb{K})$ . Then  $T(r, f) \leq O(\log(r))$  in  $]0, +\infty[$  if and only if  $f$  belongs to  $\mathbb{K}(x)$ . Let  $f \in \mathcal{M}(d(a, R^-))$ . Either  $f \in \mathcal{M}_b(d(a, R^-))$  and then  $T(r, f)$  is bounded in  $]0, R[$  or  $f \in \mathcal{M}_u(d(a, R^-))$  and then  $\lim_{r \rightarrow R^-} T(r, f) = +\infty$ .*

The following Lemma 3 comes from Theorem 43.10 in [1] (see also Theorem 2.21 in [3]).

**Lemma 3:** *Let  $f \in \mathcal{M}(\mathbb{K})$  (resp. let  $f \in \mathcal{M}_u(d(a, R^-))$ ) and let  $w_1, w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$  (resp. let  $w_1, w_2, w_3 \in \mathcal{M}_f(d(a, R^-))$ ). Then*

$$T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + \bar{Z}(r, f - w_3) + o(T(r, f)).$$

The following Lemma 4 comes from Corollary 43.12 in [1].

**Lemma 4:** *Let  $f \in \mathcal{M}(\mathbb{K})$  (resp. let  $f \in \mathcal{M}_u(d(a, R^-))$ ) have finitely many poles and let  $w_1, w_2 \in \mathcal{M}_f(\mathbb{K})$  (resp. let  $w_1, w_2 \in \mathcal{M}_f(d(a, R^-))$ ). Then*

$$T(r, f) \leq \bar{Z}(r, f - w_1) + \bar{Z}(r, f - w_2) + o(T(r, f)).$$

**Proof of Theorem 1:** Suppose that  $f$  and  $g$  belong to  $\mathcal{M}(\mathbb{K})$ , are not identical and share I.M.  $q$  small functions  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, q$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, \dots, q$ )).

Let  $b$  be a zero of  $f - w_i$  for a certain index  $i$ . Then it is also a zero of  $g - w_i$ .

Suppose that  $b$  is counted several times in the sum  $\sum_{j=1}^q \bar{Z}(r, f - w_j)$ , which means

that it is a zero of another function  $f - w_h$  for a certain index  $h \neq i$ . Then we have  $w_i(b) = w_h(b)$  and hence  $b$  is a zero of the function  $w_i - w_h$  which belongs to  $\mathcal{M}_f(\mathbb{K})$ . Now, put  $\tilde{Z}(r, f - w_1) = \bar{Z}(r, f - w_1)$  and for each  $j > 1$ , let  $\tilde{Z}(r, f - w_j)$  be the counting function of zeros of  $f - w_j$  in the disk  $d(a, r^-)$  ignoring multiplicity and avoiding the zeros already counted as zeros of  $f - w_h$  for

some  $h < j$ . Consider now the sum  $\sum_{j=1}^q \tilde{Z}(r, f - w_j)$ . Since the functions  $w_i - w_j$  belong to  $\mathcal{M}_f(\mathbb{K})$ , clearly, we have

$$\sum_{j=1}^q \bar{Z}(r, f - w_j) = \sum_{j=1}^q \tilde{Z}(r, f - w_j) = o(T(r, f))$$

It's clear, from the assumption, that  $f(x) - w_j(x) = 0$  implies  $g(x) - w_j(x) = 0$  and hence  $f - g = 0$ . It follows that

$$\sum_{j=1}^q \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g).$$

Consequently,

$$\sum_{j=1}^q \overline{Z}(r, f - w_j) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g)).$$

Now, if  $f$  and  $g$  belong to  $\mathcal{M}(d(a, R^-))$ , the proof is exactly the same.

**Proof of Theorems 2 and 3:** Consider first the hypothese of Theorem 2. Suppose that  $f$  and  $g$  are not identical and share 9 small functions  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, 9$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, \dots, 9$ )). Let us now put  $M(r) = \max(T(r, f), T(r, g))$ . By Lemma 3, we can write

$$\begin{aligned} T(r, f) &\leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{Z}(r, f - w_3) + o(T(r, f)), \\ T(r, f) &\leq \overline{Z}(r, f - w_4) + \overline{Z}(r, f - w_5) + \overline{Z}(r, f - w_6) + o(T(r, f)) \end{aligned}$$

and

$$T(r, f) \leq \overline{Z}(r, f - w_7) + \overline{Z}(r, f - w_8) + \overline{Z}(r, f - w_9) + o(T(r, f))$$

hence by Theorem 1,

$$3T(r, f) \leq \sum_{j=1}^9 T(r, f - w_j) + o(T(r, f)) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g))$$

and similarly,

$$3T(r, g) \leq \sum_{j=1}^9 T(r, g - w_j) + o(T(r, g)) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g))$$

therefore

$$(1) \quad 3M(r) \leq T(r, f - g) + o(M(r)).$$

But by Lemma 1,  $T(r, f - g) \leq T(r, f) + T(r, g) + O(1) \leq 2M(r) + O(1)$ , therefore by (1) we obtain  $3M(r) \leq 2M(r) + o(M(r))$ .

Now, if  $f$  and  $g$  are transcendental meromorphic functions, this is absurd by Lemma 2 and proves that  $f$  and  $g$  are identical. And if  $f$  and  $g$  belong to  $\mathcal{M}_u(d(a, R^-))$ , then by Lemma 4 this is absurd because  $r$  is then bounded. That finishes the proof of Theorem 2.

Consider now the hypothese of Theorem 3 and suppose that  $f$  and  $g$  are meromorphic transcendental functions with finitely many poles, (resp. belong to  $\mathcal{M}_u(d(a, R^-))$  with finitely many poles), and share 4 small functions  $w_j \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  ( $j = 1, \dots, 4$ ) (resp.  $w_j \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  ( $j = 1, \dots, 4$ )). By Lemma 4, we can write

$$T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + o(T(r, f)),$$

and

$$T(r, f) \leq \overline{Z}(r, f - w_3) + \overline{Z}(r, f - w_4) + o(T(r, f))$$

hence by Theorem 1,

$$2T(r, f) \leq \sum_{j=1}^4 T(r, f - w_j) + o(T(r, f)) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g))$$

and similarly,

$$2T(r, g) \leq \sum_{j=1}^4 T(r, g - w_j) + o(T(r, g)) \leq \overline{Z}(r, f - g) + o(T(r, f)) + o(T(r, g))$$

therefore

$$(2) \quad 2M(r) \leq T(r, f - g) + o(M(r)).$$

But since  $f$  and  $g$  have finitely many poles, by Lemma 1 we have  $T(r, f - g) \leq M(r) + O(1)$ . Therefore by (2) we obtain  $2M(r) \leq M(r) + o(M(r))$ , which is absurd in the same way as for Theorem 2 and ends the proof of Theorem 3.

#### REFERENCES

1. **Escassut, A.** *Value Distribution in  $p$ -adic Analysis*. World Scientific Publishing Co. Pte. Ltd. Singapore, (2015).
2. **Hayman, W.K.** *Picard values of meromorphic functions and their derivatives*, Annals of Mathematics. Vol. 70, no. 1, pp. 9-42 (1959).
3. **Hu P.C. and Yang, C.C.** *Meromorphic Functions over non-Archimedean Fields*, Kluwer Academic Publishers (2000).
4. **Yamanoi, K** *The second main theorem for small functions and related problems*, Acta Mathematica 192, p. 225-294 (2004).

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