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ULTRAFILTERS AND ULTRAMETRIC BANACH ALGEBRAS OF LIPSCHITZ FUNCTIONS

MONIQUE CHICOURRAT AND ALAIN ESCASSUT

ABSTRACT. The aim of this paper is to examine Banach algebras of bounded Lipschitz functions from an ultrametric space \mathbb{E} to a complete ultrametric field \mathbb{K} . Considering them as a particular case of what we call *C-compatible algebras* we study the interactions between their maximal ideals or their multiplicative spectrum and ultrafilters on \mathbb{E} . We study also their Shilov boundary and topological divisors of zero. Furthermore, we give some conditions on abstract Banach \mathbb{K} -algebras in order to show that they are algebras of Lipschitz functions on an ultrametric space through a kind of Gelfand transform. Actually, given such an algebra A , its elements can be considered as Lipschitz functions from the set of characters on A provided with some distance λ_A . If A is already the Banach algebra of all bounded Lipschitz functions on a closed subset \mathbb{E} of \mathbb{K} , then the two structures are equivalent and we can compare the original distance defined by the absolute value of \mathbb{K} , with λ_A .

1. Introduction and general results in topology

Let \mathbb{K} be an ultrametric complete field and \mathbb{E} be an ultrametric space. It is well known that the set of maximal ideals of a Banach \mathbb{K} -algebra is not sufficient to describe its spectral properties: we have to consider the set of continuous multiplicative semi-norms often called the multiplicative spectrum [1], [6], [10].

First, we generalize results obtained in [8] and [9] to some Banach algebras of uniformly continuous bounded functions which we call *semi-compatible* and *C-compatible* and which are related to the *contiguity relation* yet considered in these papers. Actually, considering such an algebra S , we describe interactions between the contiguity relation defined on ultrafilters on \mathbb{E} and maximal ideals or the multiplicative spectrum of S . We prove that the Shilov boundary of S is the multiplicative spectrum itself. Next we consider the Stone space of some Boolean subring of the clopen sets of \mathbb{E} which turns out to be a compactification of \mathbb{E} homeomorphic to the multiplicative spectrum.

These results particularly apply to the algebra of bounded Lipschitz functions and also to the algebras of bounded differentiable or strictly differentiable functions when \mathbb{E} is a subset of \mathbb{K} . Furthermore we get some particular properties in these algebras for example about their topological divisors of zero.

Finally, we define a kind of Gelfand transform and we give conditions in order to make an ultrametric Banach \mathbb{K} -algebra appear as an algebra of Lipschitz functions on an ultrametric space.

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The main results were stated without any proof in a short paper that was published in the proceedings of a conference [3]. Here we detail them with all the main proofs.

Notations and definitions: Let \mathbb{K} be a field which is complete with respect to an ultrametric absolute value $|\cdot|$ and let \mathbb{E} denote a metric space whose distance δ is ultrametric. Given $a \in \mathbb{E}$ and $r > 0$, we denote by $d_{\mathbb{E}}(a, r^-)$ the open ball $\{x \in \mathbb{E} \mid \delta(a, x) < r\}$ and particularly in \mathbb{K} we denote by $d(a, r^-)$ the open disk $\{x \in \mathbb{K} \mid |x - a| < r\}$. In the same way we denote by $d_{\mathbb{E}}(a, r)$ the closed ball $\{x \in \mathbb{E} \mid \delta(a, x) \leq r\}$ and by $d(a, r)$ the closed disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$. Moreover, in \mathbb{K} , we denote by $C(a, r)$ the set $\{x \in \mathbb{K} \mid |x - a| = r\}$.

We denote by $|\cdot|_{\infty}$ the Archimedean absolute value of \mathbb{R} .

If $F \subset \mathbb{E}$, the function u defined on \mathbb{E} by $u(x) = 1$ if $x \in F$ and $u(x) = 0$ if $x \notin F$, is called *the characteristic function of F* .

Given two subsets A, B of \mathbb{E} , we put $\delta(A, B) = \inf\{\delta(x, y) \mid x \in A, y \in B\}$. Given a subset F of \mathbb{E} such that $F \neq \emptyset$ and $F \neq \mathbb{E}$, we call *codiameter of F* the number $\delta(F, \mathbb{E} \setminus F)$. If $F = \emptyset$ or $F = \mathbb{E}$, we say that its codiameter is infinite. The set F will be said to be *uniformly open* if its codiameter is strictly positive.

We will denote by $\mathbb{G}(\mathbb{E})$ the family of uniformly open subsets of \mathbb{E} . In [8] and [13], dealing with the Banaschewski compactification of \mathbb{E} the authors considered the Boolean ring of clopen sets of \mathbb{E} (with the usual addition Δ and multiplication \cap). In Section 4 we will consider the Boolean ring of uniformly open sets. Actually, we have the following lemmas that are easily checked:

Lemma 1.1. *Given two uniformly open subsets F, G , then $F \cup G, F \cap G$ and $\mathbb{E} \setminus G$ are uniformly open.*

Corollary 1.2. *$\mathbb{G}(\mathbb{E})$ is a Boolean ring with respect to the addition Δ and the multiplication \cap .*

Lemma 1.3. *Given two subsets A and B of \mathbb{E} , there exists a uniformly open subset F such that $A \subset F$ and $B \subset \mathbb{E} \setminus F$ if and only if $\delta(A, B) > 0$.*

Lemma 1.4. *Let f be a uniformly continuous function from \mathbb{E} to \mathbb{K} and let $M > 0$.*

1) *If D is a uniformly open subset of \mathbb{K} , then so is the set $F = \{x \in \mathbb{E} \mid f(x) \in D\}$.*

2) *Given $M > 0$, the sets $E_1 = \{x \in \mathbb{E} \mid |f(x)| \geq M\}$ and $E_2 = \{x \in \mathbb{E} \mid |f(x)| \leq M\}$ are uniformly open.*

Corollary 1.5. *Let f be a uniformly continuous function from \mathbb{E} to \mathbb{K} , let $M > 0$ and let $h > 0$. Then $\{x \in \mathbb{E} \mid \left| |f(x)| - M \right|_{\infty} \leq h\}$ is uniformly open.*

We can easily prove the following:

Lemma 1.6. *Let F be a subset of \mathbb{E} and let u be its characteristic function. The 3 following statements are equivalent:*

1) *F is uniformly open,*

- 2) u is Lipschitz,
 3) u is uniformly continuous.

Notations: Given a normed \mathbb{K} -algebra whose norm is $\| \cdot \|$, we denote by $\| \cdot \|_{sp}$ the spectral semi-norm that is associated and defined as $\|f\|_{sp} = \lim_{n \rightarrow +\infty} \left(\|f^n\| \right)^{\frac{1}{n}}$. We denote by \mathcal{B} the algebra of bounded uniformly continuous functions from \mathbb{E} to \mathbb{K} and by $\| \cdot \|_0$ the norm of uniform convergence on \mathbb{E} defined on any subalgebra of \mathcal{B} .

Lemma 1.7. *Let A be a commutative unital Banach \mathbb{K} -algebra of bounded functions defined on \mathbb{E} . Then $\|f\|_0 \leq \|f\|_{sp} \leq \|f\| \quad \forall f \in A$. Moreover, given $f \in A$ satisfying $\|f\|_{sp} < 1$, then $\lim_{n \rightarrow +\infty} \|f^n\| = 0$.*

Proof. The norm $\| \cdot \|_0$ is power multiplicative and classically it is bounded by the norm $\| \cdot \|$ of A , it is then bounded by $\| \cdot \|_{sp}$. The last claim is immediate. \square

Definition: We will call *semi-compatible algebra* a unital Banach \mathbb{K} -algebra S of uniformly continuous bounded functions f from \mathbb{E} to \mathbb{K} satisfying the two following properties:

- 1) every function $f \in S$ such that $\inf_{x \in E} |f(x)| > 0$ is invertible in S ,
- 2) for every subset $F \subset \mathbb{E}$, the characteristic function of F belongs to S if and only if F is uniformly open,

Moreover, a semi-compatible algebra S will be said to be *C-compatible* if it satisfies

- 3) the spectral semi-norm of S is equal to the norm $\| \cdot \|_0$.

Given a subset X of S , we call *spectral closure of X* , denoted by \tilde{X} , the closure of X with respect to the semi-norm $\| \cdot \|_{sp}$; and X will be said to be *spectrally closed* if $X = \tilde{X}$.

Remark 1.8. \mathcal{B} provided with the norm $\| \cdot \|_0$ is easily seen to be C-compatible.

Throughout the paper, we will denote by S a semi-compatible \mathbb{K} -algebra.

More notations: Let \mathcal{F} be a filter on \mathbb{E} . Given a function f from \mathbb{E} to \mathbb{K} admitting a limit along \mathcal{F} , we will denote by $\lim_{\mathcal{F}} f(x)$ this limit.

Then we will denote by $\mathcal{I}(\mathcal{F}, S)$ the ideal of the $f \in S$ such that $\lim_{\mathcal{F}} f(x) = 0$.

Notice that the unity does not belong to $\mathcal{I}(\mathcal{F}, S)$, so $\mathcal{I}(\mathcal{F}, S) \neq S$.

Given $a \in \mathbb{E}$, we will denote by $\mathcal{I}(a, S)$ the ideal of the $f \in S$ such that $f(a) = 0$ and by $\mathcal{I}'(a, S)$ the ideal of the $f \in S$ such that there exists an open neighborhood L of a such that $f(x) = 0 \quad \forall x \in L$.

We will denote by $Max(S)$ the set of maximal ideals of S and by $Max_{\mathbb{E}}(S)$ the set of maximal ideals of S of the form $\mathcal{I}(a, S)$, $a \in \mathbb{E}$.

Given a set F , we shall denote by $U(F)$ the set of ultrafilters on F .

Definition: Ultrafilters \mathcal{U}, \mathcal{V} on \mathbb{E} are said to be *contiguous* if for every $H \in \mathcal{U}$, $L \in \mathcal{V}$, we have $\delta(H, L) = 0$. We shall denote by (\mathcal{R}) the relation defined on $U(\mathbb{E})$ as $\mathcal{U}(\mathcal{R})\mathcal{V}$ if \mathcal{U} and \mathcal{V} are contiguous.

Remark 1.9. The contiguity relation on ultrafilters on \mathbb{E} is a particular case of the relation on ultrafilters defined by Labib Haddad and in other terms by Pierre Samuel in a uniform space [11], [12]. This relation on a uniform space actually is an equivalence relation.

Lemma 1.10 is classical:

Lemma 1.10. *Let \mathcal{U} be an ultrafilter on \mathbb{E} . Let f be a bounded function from \mathbb{E} to \mathbb{K} . The function $|f|$ from \mathbb{E} to \mathbb{R}_+ defined as $|f|(x) = |f(x)|$ admits a limit along \mathcal{U} . Moreover, if \mathbb{K} is locally compact, then $f(x)$ admits a limit along \mathcal{U} .*

The following lemmas are immediate:

Lemma 1.11. *The spectral closure of an ideal of S is an ideal of S .*

Lemma 1.12. *If a subset Y of S is spectrally closed, it is closed with respect to the norm $\| \cdot \|$ of S .*

Lemma 1.13. *Every maximal ideal \mathcal{M} of S is spectrally closed.*

Proof. By Lemma 1.11 the spectral closure $\widetilde{\mathcal{M}}$ of \mathcal{M} is an ideal. If \mathcal{M} is not spectrally closed, then $\widetilde{\mathcal{M}} = S$, hence there exists $t \in S$ such that $1 - t \in \mathcal{M}$ and $\|t\|_{sp} < 1$. Consequently, $\lim_{n \rightarrow +\infty} \|t^n\| = 0$, therefore the series $(\sum_{n=0}^{\infty} t^n)$ converges and $(\sum_{n=0}^{\infty} t^n)(1 - t) = 1$ and hence the unity belongs to \mathcal{M} , a contradiction. \square

Proposition 1.14 now is easy:

Proposition 1.14. *Given an ultrafilter \mathcal{U} on \mathbb{E} , $\mathcal{I}(\mathcal{U}, S)$ is a prime ideal. Moreover, $\mathcal{I}(\mathcal{U}, S)$ is closed with respect to the norm $\| \cdot \|_0$ and then is spectrally closed.*

Proof. Since \mathcal{U} is an ultrafilter, it is straightforward that $\mathcal{I}(\mathcal{U}, S)$ is prime. Let us now check that $\mathcal{I}(\mathcal{U}, S)$ is closed with respect to the norm $\| \cdot \|_0$. Indeed let g belong to the closure of $\mathcal{I}(\mathcal{U}, S)$ with respect to $\| \cdot \|_0$, let $b = \lim_{\mathcal{U}} |g(x)|$ and suppose $b > 0$. There exists $f \in \mathcal{I}(\mathcal{U}, S)$ such that $\|f - g\|_0 < b$ and then

$$b = \left| \lim_{\mathcal{U}} |f(x)| - \lim_{\mathcal{U}} |g(x)| \right|_{\infty} \leq \lim_{\mathcal{U}} |f(x) - g(x)| \leq \|f - g\|_0 < b,$$

a contradiction showing that $\mathcal{I}(\mathcal{U}, S)$ is closed with respect to the norm $\| \cdot \|_0$. Therefore, since $\| \cdot \|_0 \leq \| \cdot \|_{sp}$, it is closed with respect to the norm $\| \cdot \|_{sp}$. \square

By lemma 1.3, we have the following lemma:

Lemma 1.15. *Let \mathcal{U}, \mathcal{V} be ultrafilters on \mathbb{E} . Then \mathcal{U} and \mathcal{V} are not contiguous if and only if there exists a uniformly open set $H \in \mathcal{U}$ such that $\mathbb{E} \setminus H \in \mathcal{V}$.*

Corollary 1.16. *Let \mathcal{U}, \mathcal{V} be ultrafilters on \mathbb{E} . Then \mathcal{U} and \mathcal{V} are contiguous if and only if they contain the same uniformly open sets.*

Corollary 1.17. *Relation (\mathcal{R}) is an equivalence relation on $U(\mathbb{E})$.*

Theorem 1.18. *Let \mathcal{U}, \mathcal{V} be ultrafilters on \mathbb{E} . Then $\mathcal{I}(\mathcal{U}, S) = \mathcal{I}(\mathcal{V}, S)$ if and only if \mathcal{U} and \mathcal{V} are contiguous.*

Proof. Suppose that \mathcal{U}, \mathcal{V} are not contiguous. By Lemma 1.15, there exists a uniformly open set $H \in \mathcal{U}$ such that $\mathbb{E} \setminus H \in \mathcal{V}$. Then the characteristic function u of H belongs to $\mathcal{I}(\mathcal{V}, S)$ and does not belong to $\mathcal{I}(\mathcal{U}, S)$.

Conversely, suppose that $\mathcal{I}(\mathcal{U}, S) \neq \mathcal{I}(\mathcal{V}, S)$. Without loss of generality, we can assume that there exists $f \in \mathcal{I}(\mathcal{U}, S) \setminus \mathcal{I}(\mathcal{V}, S)$. Then $\lim_{\mathcal{V}} |f(x)|$ is a number $l > 0$.

There exists $L \in \mathcal{V}$ such that $\left| |f(x)| - l \right|_{\infty} < \frac{l}{3} \forall x \in L$ and then $|f(x)| \geq \frac{2l}{3} \forall x \in L$. Therefore, by Lemma 1.4 the set $L' = \{x \in \mathbb{E} \mid |f(x)| \geq \frac{2l}{3}\}$ is a uniformly open set that belongs to \mathcal{V} . But the set $H = \{x \in \mathbb{E} \mid |f(x)| \leq \frac{l}{3}\}$ is a uniformly open set of \mathcal{U} since $\lim_{\mathcal{U}} |f(x)| = 0$ and clearly $H \cap L' = \emptyset$. Consequently, \mathcal{U} and \mathcal{V} are not contiguous. \square

Theorem 1.19 looks like certain Bezout-Corona statements and is obtained through an adaptation of Theorem 5 in [8] :

Theorem 1.19. *Let $f_1, \dots, f_q \in S$ satisfy $\inf_{x \in \mathbb{E}} (\max_{1 \leq j \leq q} |f_j(x)|) > 0$. Then there exists $g_1, \dots, g_q \in S$ such that*

$$\sum_{j=1}^q f_j(x)g_j(x) = 1 \quad \forall x \in \mathbb{E}.$$

Notation: Let $f \in S$ and let $\epsilon > 0$. We set $D(f, \epsilon) = \{x \in \mathbb{E} \mid |f(x)| \leq \epsilon\}$.

Corollary 1.20. *Let I be an ideal of S different from S . The family of sets*

$$\{D(f, \epsilon), f \in I, \epsilon > 0\}$$

generates a filter $\mathcal{F}_{I,S}$ on \mathbb{E} such that $I \subset \mathcal{I}(\mathcal{F}_{I,S}, S)$.

2. Maximal and prime ideals of S

Except Theorems 2.8 and 2.10 and their corollaries, most of results of this section were given in [8] for the algebra of uniformly continuous functions.

Theorem 2.1. *Let \mathcal{M} be a maximal ideal of S . There exists an ultrafilter \mathcal{U} on \mathbb{E} such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$. Moreover, \mathcal{M} is of codimension 1 if and only if every element of S converges along \mathcal{U} . In particular if \mathcal{U} is convergent, then \mathcal{M} is of codimension 1.*

Proof. Indeed, by Corollary 1.20, we can consider the filter $\mathcal{F}_{\mathcal{M},S}$ and we have $\mathcal{M} \subset \mathcal{I}(\mathcal{F}_{\mathcal{M},S}, S)$. Let \mathcal{U} be an ultrafilter thinner than $\mathcal{F}_{\mathcal{M},S}$. So, we have $\mathcal{M} \subset \mathcal{I}(\mathcal{F}_{\mathcal{M},S}, S) \subset \mathcal{I}(\mathcal{U}, S)$. But since \mathcal{M} is a maximal ideal, either $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$, or $\mathcal{I}(\mathcal{U}, S) = S$. But obviously, $\mathcal{I}(\mathcal{U}, S) \neq S$, hence $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$.

Now assume that \mathcal{M} is of codimension 1 and let χ be the \mathbb{K} -algebra homomorphism from S to \mathbb{K} admitting \mathcal{M} for kernel. Let $f \in S$ and let $b = \chi(f)$. Then $f - b$ belongs to the kernel of \mathcal{M} , hence $\lim_{\mathcal{U}} f(x) - b = 0$, hence $\lim_{\mathcal{U}} f(x) = b$ and every element of S converges along \mathcal{U} .

Conversely if every element of S admits a limit along \mathcal{U} then the mapping χ which associates to each $f \in S$ its limit along \mathcal{U} is a \mathbb{K} -algebra homomorphism from S to \mathbb{K} admitting \mathcal{M} for kernel.

In particular if \mathcal{U} converges to a point a then each f in S converges to $f(a)$ along \mathcal{U} . \square

By Lemma 1.10 and Theorem 2.1, the following corollary is immediate:

Corollary 2.2. *Let \mathbb{K} be a locally compact field. Then every maximal ideal of S is of codimension 1.*

Remark 2.3. If \mathbb{K} is locally compact, a maximal ideal of codimension 1 of S is not necessarily of the form $\mathcal{I}(\mathcal{U}, S)$ for some Cauchy ultrafilter \mathcal{U} , as shown in [8].

Notation: We will denote by $Y_{(\mathcal{R})}(\mathbb{E})$ the set of equivalence classes on $U(\mathbb{E})$ with respect to the relation (\mathcal{R}) .

By Theorem 1.18, we can get Corollary 2.4:

Corollary 2.4. *Let \mathcal{M} be a maximal ideal of S . There exists a unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$ for every $\mathcal{U} \in \mathcal{H}$.*

Now, the following Theorem together with Theorem 2.1 characterize all maximal ideals of S .

Theorem 2.5. *Let \mathcal{U} be an ultrafilter on \mathbb{E} . Then $\mathcal{I}(\mathcal{U}, S)$ is a maximal ideal of S .*

Proof. Let $I = \mathcal{I}(\mathcal{U}, S)$ and let \mathcal{M} be a maximal ideal of S containing I . Then by Theorem 2.1 there exists an ultrafilter \mathcal{V} such that $\mathcal{M} = \mathcal{I}(\mathcal{V}, S)$. Suppose now $\mathcal{I}(\mathcal{U}, S) \neq \mathcal{I}(\mathcal{V}, S)$. Then, \mathcal{U} and \mathcal{V} are not contiguous. Consequently, by Lemma 1.15 there exists a uniformly open subset $F \in \mathcal{V}$ that does not belong to \mathcal{U} and hence its characteristic function $u \in S$ belongs to $\mathcal{I}(\mathcal{U}, S)$ but does not belong to $\mathcal{I}(\mathcal{V}, S)$. Thus, u belongs to I but does not belong to \mathcal{M} , a contradiction to the hypothesis. \square

Using Corollary 2.4 and Theorem 2.5, we can prove Corollary 2.6:

Corollary 2.6. *The mapping that associates to each maximal ideal \mathcal{M} of S the class with respect to (\mathcal{R}) of ultrafilters \mathcal{U} , such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$, is a bijection from $Max(S)$ onto $Y_{(\mathcal{R})}(\mathbb{E})$.*

The following Theorem is quite easy:

Theorem 2.7. *Let \mathcal{F} be a Cauchy filter on \mathbb{E} and let $\mathcal{M} = \mathcal{I}(\mathcal{F}, S)$. Then every function $f \in S$ converges to a limit $\theta(f)$ along \mathcal{F} and \mathcal{M} is a maximal ideal of codimension 1.*

Notation: For any subset F of \mathbb{E} , we denote by u_F its characteristic function. Let \mathcal{M} be a maximal ideal of S and let $\mathcal{U} \in U(\mathbb{E})$ be such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$. By Theorem 1.18 and Corollary 1.16 we can define the set $\mathcal{O}_{\mathcal{M}}$ of all uniformly open subsets of \mathbb{E} which belong to \mathcal{U} . We denote by $\mathcal{C}_{\mathcal{M}}$ the set $\{u_{\mathbb{E} \setminus L} \mid L \in \mathcal{O}_{\mathcal{M}}\}$ and by $\mathcal{J}_{\mathcal{M}}$ the set of all functions $f \in S$ which are equal to 0 on some $L \in \mathcal{O}_{\mathcal{M}}$.

Given $a \in \mathbb{E}$, recall that $\mathcal{I}'(a, S)$ is the ideal of the functions $f \in S$ equal to 0 on an open subset of \mathbb{E} containing a .

Theorem 2.8. *Let \mathcal{M} be a maximal ideal of S .*

- 1) $\mathcal{J}_{\mathcal{M}}$ is an ideal of S containing $\mathcal{C}_{\mathcal{M}}$,
- 2) $\mathcal{J}_{\mathcal{M}}$ is the ideal of S generated by $\mathcal{C}_{\mathcal{M}}$ and $\mathcal{J}_{\mathcal{M}} = \{fu \mid f \in S, u \in \mathcal{C}_{\mathcal{M}}\}$,
- 3) If \mathcal{P} is a prime ideal of S contained in \mathcal{M} , then $\mathcal{J}_{\mathcal{M}} \subset \mathcal{P}$.
- 4) if $\mathcal{M} = \mathcal{I}(a, S)$, then $\mathcal{I}'(a, S) = \mathcal{J}_{\mathcal{M}}$.

Proof. 1) Let us check that $\mathcal{J}_{\mathcal{M}}$ is an ideal of S . Let $f, g \in \mathcal{J}_{\mathcal{M}}$. So, there exist $F, G \in \mathcal{O}_{\mathcal{M}}$ such that $f(x) = 0 \forall x \in F$, $g(x) = 0 \forall x \in G$, hence $f(x) - g(x) = 0 \forall x \in F \cap G$. Since $F \cap G$ belongs to $\mathcal{O}_{\mathcal{M}}$, $f - g$ lies in $\mathcal{J}_{\mathcal{M}}$. And obviously, for every $h \in S$, we have $h(x)f(x) = 0 \forall x \in F$, hence fh lies in $\mathcal{J}_{\mathcal{M}}$.

Next, $\mathcal{J}_{\mathcal{M}}$ contains $\mathcal{C}_{\mathcal{M}}$ because given $L \in \mathcal{O}_{\mathcal{M}}$, the set $\mathbb{E} \setminus L$ is uniformly open then $u_{\mathbb{E} \setminus L}$ belongs to S and is equal to 0 on L .

2) Notice that if $f \in S$ and $u \in \mathcal{C}_{\mathcal{M}}$, then by 1) fu belongs to $\mathcal{J}_{\mathcal{M}}$. Conversely, if $f \in \mathcal{J}_{\mathcal{M}}$ and $L \in \mathcal{O}_{\mathcal{M}}$ are such that $f(x)$ is equal to 0 on L , then $u_{\mathbb{E} \setminus L}$ belongs to $\mathcal{C}_{\mathcal{M}}$ and $f = fu_{\mathbb{E} \setminus L}$. This proves that $\mathcal{J}_{\mathcal{M}} = \{fu \mid f \in S, u \in \mathcal{C}_{\mathcal{M}}\}$ and that $\mathcal{J}_{\mathcal{M}}$ is the ideal generated by $\mathcal{C}_{\mathcal{M}}$.

3) It is sufficient to prove that $\mathcal{C}_{\mathcal{M}}$ is included in \mathcal{P} . Indeed, let $\mathcal{U} \in U(\mathbb{E})$ be such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$ and let $L \in \mathcal{O}_{\mathcal{M}}$. Then $L \in \mathcal{U}$ and $u_L \notin \mathcal{M}$. So, $u_L \notin \mathcal{P}$. But $u_L \cdot u_{\mathbb{E} \setminus L} = 0$. Thus $u_{\mathbb{E} \setminus L}$ belongs to \mathcal{P} since \mathcal{P} is prime.

4) Each open neighborhood of a contains a disk that also is a uniformly open neighborhood of a , which ends the proof. \square

Corollary 2.9. *Let \mathcal{U} be an ultrafilter on \mathbb{E} and let \mathcal{P} be a prime ideal included in $\mathcal{I}(\mathcal{U}, S)$. Let $L \in \mathcal{U}$ be uniformly open and let $H = \mathbb{E} \setminus L$. Then the characteristic function u of H belongs to \mathcal{P} .*

Recall that for any normed \mathbb{K} -algebra $(G, \|\cdot\|)$, the closure of an ideal of G is an ideal of G .

Theorem 2.10. *The closure of $\mathcal{J}_{\mathcal{M}}$ with respect to the norm $\|\cdot\|_0$ is equal to \mathcal{M} .*

Proof. Let $f \in \mathcal{M} = \mathcal{I}(\mathcal{U}, S)$. Then for every $\epsilon > 0$ the set $L = D(f, \epsilon)$ belongs to \mathcal{U} and L is uniformly open. Therefore L belongs to $\mathcal{O}_{\mathcal{M}}$ and the characteristic function u of $\mathbb{E} \setminus L$ lies in $\mathcal{C}_{\mathcal{M}}$, so that $fu \in \mathcal{J}_{\mathcal{M}}$. But $f(x) - uf(x) = 0 \forall x \notin L$

and $|f(x) - uf(x)| = |f(x)| \leq \epsilon \forall x \in L$, so $\|f - uf\|_0 \leq \epsilon$ and hence \mathcal{M} is the closure of $\mathcal{J}_{\mathcal{M}}$ with respect to the norm $\|\cdot\|_0$ since, by Proposition 1.14, \mathcal{M} is closed with respect to the norm $\|\cdot\|_0$. \square

Corollary 2.11. *Let \mathcal{P} be a prime ideal contained in \mathcal{M} . Then \mathcal{M} is the closure of \mathcal{P} with respect to $\|\cdot\|_0$.*

Corollary 2.12. *The closure with respect to $\|\cdot\|_0$ of a prime ideal of S is a maximal ideal of S and a prime ideal of S is contained in a unique maximal ideal of S .*

Corollary 2.13. *A prime ideal of S is a maximal ideal if and only if it is closed with respect to $\|\cdot\|_0$.*

If $\mathcal{M} = \mathcal{I}(a, S)$, then $\mathcal{J}_{\mathcal{M}} = \mathcal{I}'(a, S)$. Therefore we have the following Corollary:

Corollary 2.14. *The closure of $\mathcal{I}'(a, S)$ with respect to $\|\cdot\|_0$ is $\mathcal{I}(a, S)$.*

Remark 2.15. $\mathcal{I}(a, S)$ is not necessarily the closure of $\mathcal{I}'(a, S)$ in S with respect to the norm $\|\cdot\|$ of S , see Proposition 7.9 of Section 7.

Corollary 2.16. *If S is C -compatible then:*

- 1) \mathcal{M} is the spectral closure of $\mathcal{J}_{\mathcal{M}}$ and the spectral closure of any prime ideal is contained in \mathcal{M} ;
- 2) A prime ideal is maximal if and only if it is spectrally closed.

3. Multiplicative spectrum

The multiplicative spectrum of a Banach \mathbb{K} -algebra was first introduced by B. Guennebaud [10] and was at the basis of Berkovich's theory [1].

Notations and definitions: Let G be a normed \mathbb{K} -algebra. We denote by $Mult(G, \|\cdot\|)$ the set of continuous multiplicative algebra semi-norms of G provided with the topology of pointwise convergence, which means that a basic neighborhood of some $\psi \in Mult(G, \|\cdot\|)$ is a set of the form $W(\psi, f_1, \dots, f_q, \epsilon)$, with $f_j \in G$ and $\epsilon > 0$, which is the set of $\phi \in Mult(G, \|\cdot\|)$ such that $|\psi(f_j) - \phi(f_j)|_{\infty} \leq \epsilon \forall j = 1, \dots, q$. The topological space $Mult(G, \|\cdot\|)$ is then compact (see [10], or Theorem 6.2 in [6]).

Given $\phi \in Mult(G, \|\cdot\|)$, we call *kernel of ϕ* the set of the $x \in S$ such that $\phi(x) = 0$ and we denote it by $Ker(\phi)$. It is a prime closed ideal of G with respect to the norm $\|\cdot\|$ [6].

We denote by $Mult_m(G, \|\cdot\|)$ the set of continuous multiplicative semi-norms of G whose kernel is a maximal ideal and by $Mult_1(G, \|\cdot\|)$ the set of continuous multiplicative semi-norms of G whose kernel is a maximal ideal of codimension 1. Particularly, considering the algebra S , we denote by $Mult_{\mathbb{E}}(S, \|\cdot\|)$ the set of continuous multiplicative semi-norms of S whose kernel is a maximal ideal of the form $\mathcal{I}(a, S)$, $a \in \mathbb{E}$. We denote by $\Upsilon(G)$ the set of \mathbb{K} -algebra homomorphisms from G to \mathbb{K} .

Theorem 3.1 is classical [5]:

Theorem 3.1. *Let G be a unital commutative ultrametric Banach \mathbb{K} -algebra. For each $f \in G$, $\|f\|_{sp} = \sup\{\phi(f) \mid \phi \in \text{Mult}(G, \|\cdot\|)\}$. For every $\chi \in \Upsilon(G)$, we have $|\chi(f)| \leq \|f\|_{sp} \forall f \in G$.*

More notations: Let us recall that in S , we have $\|\cdot\|_0 \leq \|\cdot\|_{sp}$ and that if S is C -compatible, then $\|\cdot\|_0 = \|\cdot\|_{sp}$.

For any ultrafilter $\mathcal{U} \in U(\mathbb{E})$ and any $f \in S$, $|f(x)|$ has a limit along \mathcal{U} since f is bounded (see lemma 1.10.). Then we denote by $\varphi_{\mathcal{U}}$ the mapping from S to \mathbb{R} defined by $\varphi_{\mathcal{U}}(f) = \lim_{\mathcal{U}} |f(x)|$. Given $a \in \mathbb{E}$ we denote by φ_a the mapping from S to \mathbb{R} defined by $\varphi_a(f) = |f(a)|$. These maps belong to $\text{Mult}(S, \|\cdot\|)$ since $\|\cdot\|_0 \leq \|\cdot\|_{sp} \leq \|\cdot\|$. Particularly, the elements of $\text{Mult}_{\mathbb{E}}(S, \|\cdot\|)$ are the φ_a , $a \in \mathbb{E}$.

Proposition 3.2. *Let $a \in \mathbb{E}$. Then $\mathcal{I}(a, S)$ is a maximal ideal of S of codimension 1 and φ_a belongs to $\text{Mult}_1(S, \|\cdot\|)$.*

Moreover, for every algebra homomorphism χ from S to \mathbb{K} , its kernel is a maximal ideal of codimension 1 of the form $\mathcal{I}(\mathcal{U}, S)$ with $\mathcal{U} \in U(\mathbb{E})$ and χ is defined as $\chi(f) = \lim_{\mathcal{U}} f(x)$, while $\varphi_{\mathcal{U}}(f) = |\chi(f)|$.

Proof. The first part is clear. The second comes from the proof of theorem 2.1: actually we proved that if \mathcal{M} is of codimension 1 then every $f \in S$ has a limit along \mathcal{U} and $\lim_{\mathcal{U}} f(x) = \chi(f)$. \square

The following Proposition 3.3 is immediate:

Proposition 3.3. *Let \mathcal{U} be an ultrafilter on \mathbb{E} . Then $\varphi_{\mathcal{U}}$ belongs to the closure of $\text{Mult}_{\mathbb{E}}(S, \|\cdot\|)$.*

Remark 3.4. According to Remark 2.3, if \mathbb{K} is locally compact we have $\text{Max}_1(S) \neq \text{Max}_{\mathbb{E}}(S)$ and hence $\text{Mult}_1(S, \|\cdot\|) \neq \text{Mult}_{\mathbb{E}}(S, \|\cdot\|)$ because given a maximal ideal $\mathcal{I}(\mathcal{U}, S)$ which does not belong to $\text{Max}_{\mathbb{E}}$, we can define $\varphi_{\mathcal{U}} \in \text{Mult}(S, \|\cdot\|)$ which does not belong to $\text{Mult}_{\mathbb{E}}(S, \|\cdot\|)$.

Given $\phi \in \text{Mult}(S, \|\cdot\|)$, it is well known that $\text{Ker}(\phi)$ is a prime closed ideal, with respect to the norm $\|\cdot\|$ of S . Actually we have the following proposition:

Proposition 3.5. *For each $\phi \in \text{Mult}(S, \|\cdot\|)$, $\text{Ker}(\phi)$ is a prime spectrally closed ideal.*

Proof. Let $\phi \in \text{Mult}(S, \|\cdot\|)$ and let f belong to the spectral closure of $\text{Ker}(\phi)$. There exists a sequence $(f_n)_{n \in \mathbb{N}}$ of $\text{Ker}(\phi)$ such that $\lim_{n \rightarrow \infty} \|f_n - f\|_{sp} = 0$. By Theorem 3.1, since $\phi(g) \leq \|g\|_{sp} \forall g \in S$, we have $\lim_{n \rightarrow \infty} \phi(f_n - f) = 0$. But $\phi(f_n) = 0 \forall n \in \mathbb{N}$, hence $\phi(f_n - f) = \phi(f)$ and hence $\phi(f) = 0$. Therefore, f belongs to $\text{Ker}(\phi)$, which means that $\text{Ker}(\phi) = \widetilde{\text{Ker}(\phi)}$. \square

By Corollary 2.16, we have the following corollary:

Corollary 3.6. *If S is C -compatible, then $\text{Mult}(S, \|\cdot\|) = \text{Mult}_m(S, \|\cdot\|)$.*

The following Theorem is classical (Theorem 6.15 in [6]).

Theorem 3.7. *Let G be a commutative unital ultrametric Banach \mathbb{K} -algebra. For every maximal ideal \mathcal{M} of G , there exists $\phi \in \text{Mult}_m(S, \|\cdot\|)$ such that $\mathcal{M} = \text{Ker}(\phi)$.*

Definition: Recall that a unital commutative Banach \mathbb{K} -algebra is said to be *multibjective* if every maximal ideal is the kernel of only one continuous multiplicative semi-norm.

Remark 3.8. There exist some rare cases of ultrametric Banach algebras that are not multibjective [5].

Theorem 3.9. *Suppose S is C-compatible. Then S is multibjective. Precisely if $\psi \in \text{Mult}(S, \|\cdot\|)$ and $\text{Ker}(\psi) = \mathcal{M}$ then $\psi = \varphi_{\mathcal{U}}$ for every ultrafilter \mathcal{U} such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$.*

Proof. Let $\psi \in \text{Mult}_m(S, \|\cdot\|)$, let $\mathcal{M} = \text{Ker}(\psi)$ and \mathcal{U} be an ultrafilter such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, S)$.

Let $f \in S$. Notice that if $f \in \mathcal{M}$ then $\psi(f) = \varphi_{\mathcal{U}}(f) = 0$. Now we assume that $f \notin \mathcal{M}$. So $\psi(f)$ and $\varphi_{\mathcal{U}}(f)$ are both strictly positive. We prove that they are equal.

First let $\epsilon > 0$ and consider the set $L = \{x \in \mathbb{E} : |f(x)| \leq \varphi_{\mathcal{U}}(f) + \epsilon\}$. This set belongs to \mathcal{U} and by Lemma 1.4, it is uniformly open. Therefore its characteristic function u lies in S . We have $\varphi_{\mathcal{U}}(u) = 1$. Consequently, we can derive that $\psi(u) = 1$ because u is idempotent and does not belong to \mathcal{M} . Therefore $\psi(uf) = \psi(f)$ and $\varphi_{\mathcal{U}}(uf) = \varphi_{\mathcal{U}}(f)$. By Theorem 3.1, we have $\psi(f) = \psi(uf) \leq \|uf\|_{sp} = \|uf\|_0$ because S is C-compatible. But by definition of L we have: $\|uf\|_0 \leq \varphi_{\mathcal{U}}(f) + \epsilon$. Therefore, $\psi(f) \leq \varphi_{\mathcal{U}}(f) + \epsilon$. This holds for every $\epsilon > 0$. Consequently we may conclude that $\psi(f) \leq \varphi_{\mathcal{U}}(f)$ for every $f \in S$.

We prove now the inverse inequality. We have $\varphi_{\mathcal{U}}(f) > 0$, so consider the set $W = \{x \in \mathbb{E} : |f(x)| \geq \frac{\varphi_{\mathcal{U}}(f)}{2}\}$. Again this is a uniformly open set which belongs to \mathcal{U} . Let w be the characteristic function of W and put $g = wf + (1 - w)$. We have $\varphi_{\mathcal{U}}(w) = 1$ and $\varphi_{\mathcal{U}}(1 - w) = 0$ so $w \notin \mathcal{M}$ and $1 - w \in \mathcal{M}$. Since $\mathcal{M} = \text{Ker}(\psi)$ we then have $\psi(1 - w) = 0$ and $\psi(w) = 1$ because w is idempotent. Finally $\psi(g) = \psi(f)$ and $\varphi_{\mathcal{U}}(g) = \varphi_{\mathcal{U}}(f)$.

On the other hand, we can check that $|g(x)| \geq \min(1, \frac{\varphi_{\mathcal{U}}(f)}{2})$ for all $x \in \mathbb{E}$, hence g is invertible in S . Putting $h = \frac{1}{g}$, using the first inequality proved above, we have

$$\psi(f) = \psi(g) = \frac{1}{\psi(h)} \geq \frac{1}{\varphi_{\mathcal{U}}(h)} = \varphi_{\mathcal{U}}(g) = \varphi_{\mathcal{U}}(f).$$

That concludes the proof. □

Remark 3.10. It follows from the preceding theorem that for a C-compatible S , two ultrafilters on \mathbb{E} that are not contiguous define two distinct continuous multiplicative semi-norms on S , this particularly applies to the algebra \mathcal{B} of all uniformly continuous functions [8].

Corollary 3.11. *Suppose S is C-compatible. For every $\phi \in \text{Mult}(S, \|\cdot\|)$ there exists a unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in S, \forall \mathcal{U} \in \mathcal{H}$.*

Moreover, the mapping Ψ that associates to each $\phi \in \text{Mult}(S, \|\cdot\|)$ the unique $\mathcal{H} \in Y_{\mathcal{R}}(\mathbb{E})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)| \forall f \in S, \forall \mathcal{U} \in \mathcal{H}$, is a bijection from $\text{Mult}(S, \|\cdot\|)$ onto $Y_{(\mathcal{R})}(\mathbb{E})$.

Assuming that S is C-compatible, since by Theorem 3.9 each element $\phi \in \text{Mult}(S, \|\cdot\|)$ is of the form $\varphi_{\mathcal{U}}$, Corollary 3.12 is obvious:

Corollary 3.12. *If S is C-compatible, then $\text{Mult}_{\mathbb{E}}(S, \|\cdot\|)$ is dense in $\text{Mult}(S, \|\cdot\|)$.*

The following theorem below is given as Theorem 14 in [8] concerning the algebra \mathcal{B} of uniformly continuous functions and we can generalize it through a similar proof:

Theorem 3.13. *The topological space \mathbb{E} , provided with its distance δ , is homeomorphic to $\text{Mult}_{\mathbb{E}}(S, \|\cdot\|)$ provided with the restricted topology from that of $\text{Mult}(S, \|\cdot\|)$.*

Corollary 3.14. *$\text{Mult}(S, \|\cdot\|)$ is a compactification of the topological space \mathbb{E} .*

Theorem 3.15. *Let $\phi = \varphi_{\mathcal{U}} \in \text{Mult}_m(S, \|\cdot\|)$, with \mathcal{U} an ultrafilter on \mathbb{E} , let Γ be the field $\frac{S}{\text{Ker}(\phi)}$ and let θ be the canonical surjection from S onto Γ . Then, the mapping defined on Γ by $|\theta(f)| = \phi(f) \forall f \in S$, is the quotient norm $\|\cdot\|'$ of $\|\cdot\|_0$ defined on Γ and is an absolute value on Γ . Moreover, if $\text{Ker}(\phi)$ is of codimension 1, then this absolute value is the one defined on \mathbb{K} and coincides with the quotient norm of the norm $\|\cdot\|$ of S .*

Proof. Let $\mathcal{M} = \text{Ker}(\varphi_{\mathcal{U}})$. Let $t \in \Gamma$ and let $f \in S$ be such that $\theta(f) = t$. So, $\|t\|' \geq \lim_{\mathcal{U}} |f(s)|$. Conversely, take $\epsilon > 0$ and let $V = \{x \in \mathbb{E} : |f(x)| \leq \lim_{\mathcal{U}} |f(s)| + \epsilon\}$. By Lemma 1.4, the set V is uniformly open and belongs to \mathcal{U} . The characteristic function u of $\mathbb{E} \setminus V$ belongs to \mathcal{M} and so does uf . But by construction, $(f - uf)(x) = 0 \forall x \in \mathbb{E} \setminus V$ and $(f - uf)(x) = f(x) \forall x \in V$. Consequently, $\|f - uf\|_0 \leq \lim_{\mathcal{U}} |f(s)| + \epsilon$ and therefore $\|t\|' \leq \|f - uf\|_0 \leq \lim_{\mathcal{U}} |f(s)| + \epsilon$. This finishes proving the equality $\|\theta(f)\|' = \lim_{\mathcal{U}} |f(s)|$ and hence the mapping defined by $|\theta(f)| = \phi(f), f \in S$ is the quotient norm $\|\cdot\|'$ of $\|\cdot\|_0$. Then it is multiplicative, hence it is an absolute value on Γ .

Now, suppose that \mathcal{M} is of codimension 1. Then Γ is isomorphic to \mathbb{K} and its absolute value $\|\cdot\|'$ is continuous with respect to the topology of \mathbb{K} , hence it is equal to the absolute value of \mathbb{K} . Finally consider the quotient norm $\|\cdot\|_q$ of the norm $\|\cdot\|$ of S : that quotient norm of course bounds the quotient norm $\|\cdot\|'$ which is the absolute value of \mathbb{K} . If $f \in S$ and $b = \theta(f)$, we have $f - b \in \mathcal{M}$ and $\|\theta(f)\|_q \leq \|b\| = |b| = |\theta(f)| = \|\theta(f)\|'$, which ends the proof. \square

Corollary 3.16. *Suppose that S is C-compatible. Let $\phi \in \text{Mult}(S, \|\cdot\|)$, let Γ be the field $\frac{S}{\text{Ker}(\phi)}$ and let θ be the canonical surjection from S onto Γ . Then, the mapping defined on Γ by $|\theta(f)| = \phi(f), \forall f \in S$ is the quotient norm $\|\cdot\|'$ of $\|\cdot\|_0$ on Γ and is an absolute value on Γ . Moreover, if $\text{Ker}(\phi)$ is of codimension*

1, then this absolute value is the one defined on \mathbb{K} and coincides with the quotient norm of the norm $\| \cdot \|$ of S .

Remark 3.17. It is not clear whether an algebra S admits a prime closed ideal \mathcal{P} (with respect to the norm $\| \cdot \|$) which is not a maximal ideal. If it admits such a prime closed ideal, then it is not the kernel of a continuous multiplicative semi-norm. In such a case, the quotient algebra by \mathcal{P} has no continuous absolute value extending that of \mathbb{K} , although it has no divisors of zero. Such a situation can happen in certain Banach algebras [2].

Definition and notation: Given a \mathbb{K} -normed algebra G , we call *Shilov boundary of G* a closed subset F of $Mult(G, \| \cdot \|)$ that is minimum with respect to inclusion, such that, for every $x \in G$, there exists $\phi \in F$ such that $\phi(x) = \|x\|_{sp}$.

Let us recall the following Theorem given in [7]:

Theorem 3.18. *Every normed \mathbb{K} -algebra admits a Shilov boundary.*

Notation: Given a normed \mathbb{K} -algebra G , we denote by $Shil(G)$ the Shilov boundary of G .

Lemma 3.19. *Let us fix $a \in \mathbb{E}$. For every $r > 0$, let $Z(a, r)$ be the set of $\varphi_{\mathcal{U}}$, $\mathcal{U} \in U(\mathbb{E})$, such that $d_{\mathbb{E}}(a, r)$ belongs to \mathcal{U} . The family $\{Z(a, r) \mid r \in]0, 1[\}$ makes a basis of the filter of neighborhoods of φ_a .*

Proof. Let $W(\varphi_a, f_1, \dots, f_q, \epsilon)$ be a neighborhood of φ_a in $Mult(S, \| \cdot \|)$. There exists $r > 0$ such that, whenever $\delta(a, x) \leq r$ we have $|f_j(x) - f_j(a)| \leq \epsilon \forall j = 1, \dots, q$ and therefore, clearly, $|\varphi_{\mathcal{U}}(f_j) - \varphi_a(f_j)|_{\infty} \leq \epsilon \forall j = 1, \dots, q$ for every \mathcal{U} containing $d_{\mathbb{E}}(a, r)$. Thus $Z(a, r)$ is included in $W(\varphi_a, f_1, \dots, f_q, \epsilon)$.

Conversely, consider a set $Z(a, r)$ with $r \in]0, 1[$, let u be the characteristic function of $d_{\mathbb{E}}(a, r)$ and consider $W(\varphi_a, u, r)$. Given $\psi = \varphi_{\mathcal{U}} \in W(\varphi_a, u, r)$, we have $|\psi(u) - \varphi_a(u)|_{\infty} \leq r$. But $|\psi(u) - \varphi_a(u)|_{\infty} = |\psi(u) - 1|_{\infty} = |\lim_{\mathcal{U}} |u(x)| - 1|_{\infty}$. If $d_{\mathbb{E}}(a, r)$ belongs to \mathcal{U} , then $\lim_{\mathcal{U}} |u(x)| = 1$ and therefore $|\lim_{\mathcal{U}} |u(x)| - 1|_{\infty} = 0$. But if $d_{\mathbb{E}}(a, r)$ does not belong to \mathcal{U} , then $\lim_{\mathcal{U}} |u(x)| = 0$ and therefore $|\lim_{\mathcal{U}} |u(x)| - 1|_{\infty} = 1$. Consequently, since $r < 1$, $W(\varphi_a, u, r)$ is included in $Z(a, r)$, which finishes proving that the family of $Z(a, r)$, $r \in]0, 1[$ is a basis of the filter of neighborhoods of φ_a . \square

Theorem 3.20. *Suppose S is C -compatible. The Shilov boundary of S is equal to $Mult(S, \| \cdot \|)$.*

Proof. We will show that for every $a \in \mathbb{E}$, φ_a belongs to $Shil(S)$. So, let us fix $a \in \mathbb{E}$ and suppose that φ_a does not belong to $Shil(S)$. Since $Shil(S)$ is a closed subset of $Mult(S, \| \cdot \|)$, there exists a neighborhood of φ_a that contains no element of $Shil(S)$. Therefore, by the preceding lemma, there exists $s > 0$ such that $Z(a, s)$ contains no element of $Shil(S)$. Now, let $D = d_{\mathbb{E}}(a, s)$ and let u be the characteristic function of D . Since any $\phi \in Mult(S, \| \cdot \|)$ satisfies either $\phi(u) = 1$ or $\phi(u) = 0$, there exists $\theta \in Shil(S)$ be such that $\theta(u) = \|u\|_{sp} = 1$. Then, θ is of the form $\varphi_{\mathcal{U}}$, with $\mathcal{U} \in U(\mathbb{E})$ and \mathcal{U} does not contain D . But since

$u(x) = 0 \forall x \in \mathbb{E} \setminus D$, we have $\theta(u) = 0$, a contradiction. Consequently, for every $a \in \mathbb{E}$, φ_a belongs to $Shil(S)$ which is a closed subset of $Mult(S, \|\cdot\|)$ and since, by Corollary 3.12, $Mult_{\mathbb{E}}(S, \|\cdot\|)$ is dense in $Mult(S, \|\cdot\|)$, $Shil(S)$ is equal to $Mult(S, \|\cdot\|)$. \square

We finally give the following theorem. It will be useful in section 8. It was proved in [10] and is also stated with a different proof in [1].

Theorem 3.21. *Let G be a unital commutative ultrametric Banach \mathbb{K} -algebra and let X be a clopen subset of $Mult(G, \|\cdot\|)$. Then there exists an idempotent u in G such that $\phi(u) = 1 \forall \phi \in X$, $\phi(u) = 0 \forall \phi \notin X$.*

4. The Stone space of $\mathbb{G}(\mathbb{E})$.

It was proved in [8] that for the algebra \mathcal{A} of continuous bounded functions from \mathbb{E} to \mathbb{K} , the Banaschewski compactification of \mathbb{E} is homeomorphic to $Mult(\mathcal{A}, \|\cdot\|_0)$. Here we get some similar version for C-compatible algebras.

We have defined the Boolean ring $\mathbb{G}(\mathbb{E})$ of uniformly open subsets of \mathbb{E} provided with the operations Δ for the addition and \cap for the multiplication. Let $\Sigma(\mathbb{E})$ be the set of non-zero ring homomorphisms from $\mathbb{G}(\mathbb{E})$ onto the field \mathbb{F}_2 provided with the topology of pointwise convergence. This is the Stone space of the Boolean ring $\mathbb{G}(\mathbb{E})$, it is a compact space (see for example [13] for further details).

For every $\mathcal{U} \in U(\mathbb{E})$, we denote by $\zeta_{\mathcal{U}}$ the ring homomorphism from $\mathbb{G}(\mathbb{E})$ onto \mathbb{F}_2 defined by $\zeta_{\mathcal{U}}(O) = 1$ for every $O \in \mathbb{G}(\mathbb{E})$ that belongs to \mathcal{U} and $\zeta_{\mathcal{U}}(O) = 0$ for every $O \in \mathbb{G}(\mathbb{E})$ that does not belong to \mathcal{U} .

Particularly, given $a \in \mathbb{E}$, we denote by ζ_a the ring homomorphism from $\mathbb{G}(\mathbb{E})$ onto \mathbb{F}_2 defined by $\zeta_a(O) = 1$ for every $O \in \mathbb{G}(\mathbb{E})$ that contains a and $\zeta_a(O) = 0$ for every $O \in \mathbb{G}(\mathbb{E})$ that does not contain a .

Throughout this section, we suppose that S is a C-compatible algebra.

Remark 4.1. Let $\Sigma'(\mathbb{E})$ be the set of ζ_a , $a \in \mathbb{E}$. The mapping that associates ζ_a to any $a \in \mathbb{E}$ defines a surjective mapping from \mathbb{E} onto $\Sigma'(\mathbb{E})$. That mapping is also injective because given $a, b \in \mathbb{E}$, there exists a uniformly open subset F such that $a \in F$ and $b \notin F$.

By Corollary 3.11, we have a bijection Ψ from $Mult(S, \|\cdot\|)$ onto $Y_{(\mathcal{R})}(\mathbb{E})$ associating to each $\phi \in Mult(S, \|\cdot\|)$ the unique $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ such that $\phi(f) = \lim_{\mathcal{U}} |f(x)|$, $\mathcal{U} \in \mathcal{H}$, $f \in S$, i.e. $\phi = \varphi_{\mathcal{U}}$ for every $\mathcal{U} \in \mathcal{H}$.

On the other hand, let us take some $\mathcal{H} \in Y_{(\mathcal{R})}(\mathbb{E})$ and ultrafilters \mathcal{U}, \mathcal{V} in \mathcal{H} . Since \mathcal{U}, \mathcal{V} have the same uniformly open subsets of \mathbb{E} , we have $\zeta_{\mathcal{U}} = \zeta_{\mathcal{V}}$ and hence we can define a mapping Ξ from $Y_{\mathcal{R}}(\mathbb{E})$ into $\Sigma(\mathbb{E})$ which associates to each $\mathcal{H} \in Y_{\mathcal{R}}(\mathbb{E})$ the $\zeta_{\mathcal{U}}$ such that $\mathcal{U} \in \mathcal{H}$.

Lemma 4.2. Ξ is a bijection from $Y_{(\mathcal{R})}(\mathbb{E})$ onto $\Sigma(\mathbb{E})$.

Proof. Indeed, Ξ is clearly injective by Corollary 1.16.

Now, let us check that Ξ is surjective. Let $\theta \in \Sigma(\mathbb{I}\mathbb{E})$. Since θ is a ring homomorphism for the Boolean operations, the family of uniformly open sets X satisfying $\theta(X) = 1$ generates a filter \mathcal{F} . Let $\mathcal{U} \in U(\mathbb{I}\mathbb{E})$ be thinner than \mathcal{F} and let \mathcal{H} be the class of \mathcal{U} with respect to (\mathcal{R}) . We will check that $\theta = \Xi(\mathcal{H}) = \zeta_{\mathcal{U}}$. Let O be a uniformly open subset that belongs to \mathcal{U} . Then $\mathbb{I}\mathbb{E} \setminus O$ does not belong to \mathcal{U} and therefore it does not belong to \mathcal{F} , so $\theta(\mathbb{I}\mathbb{E} \setminus O) = 0$, consequently $\theta(O) = 1$. And now, let O be a uniformly open subset that does not belong to \mathcal{U} . Then O does not belong to \mathcal{F} , hence $\theta(O) = 0$, which ends the proof. \square

We put $\Phi = \Xi \circ \Psi$ and hence Φ is a bijection from $Mult(S, \|\cdot\|)$ onto $\Sigma(\mathbb{I}\mathbb{E})$. Notice that for every ultrafilter \mathcal{U} , $\Psi(\varphi_{\mathcal{U}})$ is the class \mathcal{H} of \mathcal{U} with respect to (\mathcal{R}) and $\Xi(\mathcal{H}) = \zeta_{\mathcal{U}}$ so $\Phi(\varphi_{\mathcal{U}}) = \zeta_{\mathcal{U}}$.

Theorem 4.3. *Φ is a homeomorphism once $\Sigma(\mathbb{I}\mathbb{E})$ and $Mult(S, \|\cdot\|)$ are provided with topologies of pointwise convergence.*

Proof. Recall that for any $\mathcal{U} \in U(\mathbb{I}\mathbb{E})$, a neighborhoods basis of $\varphi_{\mathcal{U}}$ in $Mult(S, \|\cdot\|)$ is given by the family of sets of the form $W(\varphi_{\mathcal{U}}, f_1, \dots, f_q, \epsilon)$ with $f_1, \dots, f_q \in S$, $\epsilon > 0$ and

$$W(\varphi_{\mathcal{U}}, f_1, \dots, f_q, \epsilon) = \{\varphi_{\mathcal{V}} \mid \left| \lim_{\mathcal{U}} |f_j(x)| - \lim_{\mathcal{V}} |f_j(x)| \right|_{\infty} \leq \epsilon, j = 1, \dots, q\}.$$

On the other hand, for any $\mathcal{U} \in U(\mathbb{I}\mathbb{E})$, a neighborhood basis for $\zeta_{\mathcal{U}}$ in $\Sigma(\mathbb{I}\mathbb{E})$ is given by the family of sets $V(\zeta_{\mathcal{U}}, O_1, \dots, O_q)$ where O_1, \dots, O_q belong to $\mathbb{G}(E)$ and

$$V(\zeta_{\mathcal{U}}, O_1, \dots, O_q) = \{\zeta_{\mathcal{V}} \mid \zeta_{\mathcal{U}}(O_j) = \zeta_{\mathcal{V}}(O_j), j = 1, \dots, q\}.$$

Notice also that if F belongs to $\mathbb{G}(\mathbb{I}\mathbb{E})$ and if u is its characteristic function, then for any $\mathcal{U} \in U(\mathbb{I}\mathbb{E})$, we have $\zeta_{\mathcal{U}}(F) = 1$ if and only if $F \in \mathcal{U}$, i.e. if and only if $\lim_{\mathcal{U}} |u(x)| = 1$. Otherwise, both $\zeta_{\mathcal{U}}(F)$ and $\lim_{\mathcal{U}} |u(x)|$ are equal to 0. Therefore, the relation

$$\left| \lim_{\mathcal{U}} |u(x)| - \lim_{\mathcal{V}} |u(x)| \right|_{\infty} \leq \frac{1}{2}$$

holds if and only if $\zeta_{\mathcal{U}}(F) = \zeta_{\mathcal{V}}(F)$. Recall that for every $\mathcal{U} \in U(\mathbb{I}\mathbb{E})$ we have $\Phi(\varphi_{\mathcal{U}}) = \zeta_{\mathcal{U}}$.

We will show that Φ is continuous. Consider $O_1, \dots, O_q \in \mathbb{G}(\mathbb{I}\mathbb{E})$, $\mathcal{U} \in U(\mathbb{I}\mathbb{E})$ and the neighborhood $V(\zeta_{\mathcal{U}}, O_1, \dots, O_q)$ of $\zeta_{\mathcal{U}}$. From the preceding remark, $\zeta_{\mathcal{V}}$ belongs to $V(\zeta_{\mathcal{U}}, O_1, \dots, O_q)$ if and only if for every $j = 1, \dots, q$, $\zeta_{\mathcal{U}}(O_j) = \zeta_{\mathcal{V}}(O_j)$, i.e. if for every $j = 1, \dots, q$,

$$\left| \lim_{\mathcal{U}} |u_j(s)| - \lim_{\mathcal{V}} |u_j(x)| \right|_{\infty} \leq \frac{1}{2}$$

i.e. if $\varphi_{\mathcal{V}}$ belongs to $W(\varphi_{\mathcal{U}}, u_1, \dots, u_q, \frac{1}{2})$. Consequently, this proves that Φ is continuous.

We will now prove that Φ^{-1} is also continuous. Consider $f_1, \dots, f_q \in S$, $\epsilon > 0$, $\mathcal{U} \in U(\mathbb{I}\mathbb{E})$ and the neighborhood $W(\varphi_{\mathcal{U}}, f_1, \dots, f_q, \epsilon)$ which is obviously $\bigcap_{j=1}^q W(\varphi_{\mathcal{U}}, f_j, \epsilon)$. Let us fix $i \in \{1, \dots, q\}$. Put $a_i = \lim_{\mathcal{U}} |f_i(x)|$ and $O_i = \{x \in$

$\mathbb{E} \mid \left| |f_i(x)| - a_i \right|_\infty \leq \frac{\epsilon}{2}$. By Lemma 1.4 O_i is uniformly open and of course it belongs to \mathcal{U} . Thus we have $V(\zeta_{\mathcal{U}}, O_i) = \{\zeta_{\mathcal{V}} \mid O_i \in \mathcal{V}\}$. Now let $\mathcal{V} \in U(\mathbb{E})$ be such that $O_i \in \mathcal{V}$ and put

$O'_i = \{x \in \mathbb{E} \mid \left| |f_i(x)| - \lim_{\mathcal{V}} |f_i(x)| \right|_\infty \leq \frac{\epsilon}{2}\}$. Then O'_i is uniformly open also and it belongs to \mathcal{V} . Therefore, $O_i \cap O'_i$ belongs to \mathcal{V} . Take $x \in O_i \cap O'_i$. We have $\left| |f_i(x)| - a_i \right|_\infty \leq \frac{\epsilon}{2}$ and $\left| |f_i(x)| - \lim_{\mathcal{V}} |f_i(x)| \right|_\infty \leq \frac{\epsilon}{2}$, so $\left| \lim_{\mathcal{V}} |f_i(x)| - a_i \right|_\infty \leq \epsilon$ and hence $\varphi_{\mathcal{V}}$ belongs to $W(\varphi_{\mathcal{U}}, f_i, \epsilon)$. This holds for every $i = 1, \dots, q$. Therefore

we can conclude that if $\zeta_{\mathcal{V}}$ belongs to $V(\zeta_{\mathcal{U}}, O_1, \dots, O_q)$, which is $\bigcap_{i=1}^q V(\zeta_{\mathcal{U}}, O_i)$, then

$\varphi_{\mathcal{V}}$ belongs to $\bigcap_{i=1}^q W(\varphi_{\mathcal{U}}, f_i, \epsilon)$ which is $W(\varphi_{\mathcal{U}}, f_1, \dots, f_q, \epsilon)$. This finishes proving

that Φ^{-1} is continuous too, and hence it is a homeomorphism. \square

Corollary 4.4. *The space $\Sigma(\mathbb{E})$ is a compactification of \mathbb{E} which is equivalent to the compactification $Mult(S, \|\cdot\|)$.*

Remark 4.5. For a C-compatible algebra S , the compactification $\Sigma(\mathbb{E})$ coincides with the Guennebaud-Berkovich multiplicative spectrum of S . This is not the Banaschewski compactification, which corresponds to the Stone space associated to the Boolean ring of clopen sets of \mathbb{E} .

5. About the completion of \mathbb{E}

Notations: We denote by $\widehat{\mathbb{E}}$ the completion of \mathbb{E} and by $\widehat{\delta}$ the continuation of δ to \mathbb{E} . We then identify \mathbb{E} with a dense subset of $\widehat{\mathbb{E}}$. The following theorem is well known:

Theorem 5.1. *Every uniformly continuous function f from \mathbb{E} to \mathbb{K} has a unique extension to a uniformly continuous function \widehat{f} from $\widehat{\mathbb{E}}$ to \mathbb{K} and we have $\|f\|_0 = \|\widehat{f}\|_0$.*

Notations: We denote by \widehat{S} the set of functions \widehat{f} , $f \in S$. Given $f \in S$, we put $\|\widehat{f}\| = \|f\|$.

We have the following proposition:

Proposition 5.2. *The normed \mathbb{K} -algebra $(\widehat{S}, \|\cdot\|)$ is isomorphic to $(S, \|\cdot\|)$ and it is semi-compatible with respect to $\widehat{\mathbb{E}}$. Moreover, if S is C-compatible, so is \widehat{S} .*

Proof. Obviously $(\widehat{S}, \|\cdot\|)$ is isomorphic to $(S, \|\cdot\|)$ and therefore it is a Banach \mathbb{K} -algebra. Now we prove that it is semi-compatible.

1) Take $f \in S$ such that $\inf_{\widehat{\mathbb{E}}} \{|\widehat{f}(x)| \mid x \in \widehat{\mathbb{E}}\} > 0$. Then $\inf_{\mathbb{E}} \{|f(x)| \mid x \in \mathbb{E}\} > 0$ and hence f is invertible in S . Now, if $g \in S$ and $fg = 1$, then $\widehat{f}\widehat{g} = 1$ hence \widehat{f} is invertible in \widehat{S} .

2) Let \widehat{F} be a subset of $\widehat{\mathbb{E}}$ and let u be its characteristic function. Obviously, if $u \in \widehat{S}$, then, using Lemma 1.6, \widehat{F} is uniformly open in $\widehat{\mathbb{E}}$ since u is uniformly continuous.

Assume now that \widehat{F} is uniformly open in $\widehat{\mathbb{E}}$. Put $F = \widehat{F} \cap \mathbb{E}$. If $\widehat{F} = \emptyset$, then $F = \emptyset$ and $u = 0$. Next, if $\widehat{F} \neq \emptyset$, then $F \neq \emptyset$ because \widehat{F} is open and \mathbb{E} is dense in $\widehat{\mathbb{E}}$. So, we have $\mathbb{E} \setminus F = (\widehat{\mathbb{E}} \setminus \widehat{F}) \cap \mathbb{E}$ and $\delta(F, \mathbb{E} \setminus F) \geq \delta(\widehat{F}, \widehat{\mathbb{E}} \setminus \widehat{F}) > 0$. Therefore, F is uniformly open in \mathbb{E} , hence its characteristic function u' lies in S . But u' is the restriction of u to \mathbb{E} and u is uniformly continuous. Consequently, by theorem 5.1, $u = \widehat{u'}$ and hence u belongs to \widehat{S} .

3) Suppose now that S is C-compatible. Clearly, the spectral norm on \widehat{S} is induced by that of S and hence it is $\| \cdot \|_0$. Therefore \widehat{S} is C-compatible. \square

Remark 5.3. Every result obtained in sections 2 and 3 also holds for the algebra $(\widehat{S}, \| \cdot \|)$. Particularly, we have Corollary 5.4.

Corollary 5.4. *The mapping which associates to every $\widehat{\mathcal{M}} \in \text{Max}(\widehat{S})$ the equivalence class $\mathcal{H} \in Y_{(\mathcal{R})}(\widehat{\mathbb{E}})$ such that $\mathcal{M} = \mathcal{I}(\mathcal{U}, \widehat{S})$ for all $\mathcal{U} \in \mathcal{H}$, is a bijection from $\text{Max}(\widehat{S})$ onto $Y_{(\mathcal{R})}(\widehat{\mathbb{E}})$.*

On another hand, the algebras $(S, \| \cdot \|)$ and $(\widehat{S}, \| \cdot \|)$ are isometric and so are $(S, \| \cdot \|_0)$ and $(\widehat{S}, \| \cdot \|_0)$. Therefore, the mapping from $\text{Max}(S)$ to $\text{Max}(\widehat{S})$ which associates to each maximal ideal \mathcal{M} of S the ideal $\widehat{\mathcal{M}} = \{\widehat{f} \mid f \in \mathcal{M}\}$ is bijective and the ideals $\widehat{\mathcal{M}}$ and \mathcal{M} are isometric. Furthermore, since $(S, \| \cdot \|)$ is multibjective, so is $(\widehat{S}, \| \cdot \|)$ and $\text{Mult}(\widehat{S}, \| \cdot \|)$ can be identified with $\text{Mult}(S, \| \cdot \|)$.

Notice that any ultrafilter \mathcal{U} on \mathbb{E} generates an ultrafilter $\widehat{\mathcal{U}}$ on $\widehat{\mathbb{E}}$ and for any $f \in S$, we have $\lim_{\mathcal{U}} |f(x)| = \lim_{\widehat{\mathcal{U}}} |\widehat{f}(x)|$. Thus, for each ultrafilter \mathcal{U} on \mathbb{E} , the ideal $\mathcal{I}(\mathcal{U}, S)$ corresponds to the ideal $\mathcal{I}(\widehat{\mathcal{U}}, \widehat{S})$ of \widehat{S} . Then, using results of Section 2 concerning S , we have the following Theorem:

Theorem 5.5. *For each maximal ideal $\widehat{\mathcal{M}}$ of \widehat{S} :*

- 1) *There exists an ultrafilter \mathcal{U} on \mathbb{E} such that $\widehat{\mathcal{M}} = \mathcal{I}(\widehat{\mathcal{U}}, \widehat{S})$,*
- 2) *There exists a unique equivalence class \mathcal{H} of $Y_{(\mathcal{R})}(\mathbb{E})$ such that \mathcal{U} belongs to \mathcal{H} if and only if $\widehat{\mathcal{M}} = \mathcal{I}(\widehat{\mathcal{U}}, \widehat{S})$.*

By Corollary 5.4 and Theorem 5.5, we get this Corollary 5.6:

Corollary 5.6. *The mapping from $Y_{(\mathcal{R})}(\mathbb{E})$ to $Y_{(\mathcal{R})}(\widehat{\mathbb{E}})$ that associates to the equivalence class of any ultrafilter \mathcal{U} in $Y_{(\mathcal{R})}(\mathbb{E})$ the equivalence class of $\widehat{\mathcal{U}}$ in $Y_{(\mathcal{R})}(\widehat{\mathbb{E}})$ is bijective. In particular every ultrafilter of $\widehat{\mathbb{E}}$ is contiguous in $\widehat{\mathbb{E}}$ to $\widehat{\mathcal{U}}$ for some ultrafilter \mathcal{U} of \mathbb{E} .*

6. Algebras \mathcal{B} , \mathcal{L} , \mathcal{D} , \mathcal{E}

As recalled in Section 1, we denote by \mathcal{B} the Banach \mathbb{K} -algebra of bounded uniformly continuous functions from \mathbb{E} to \mathbb{K} . Next, we denote by \mathcal{L} the set of

bounded Lipschitz functions from \mathbb{E} to \mathbb{K} . Whenever \mathbb{E} is a subset of \mathbb{K} , we denote by \mathcal{D} the subset of \mathcal{L} of differentiable functions in \mathbb{E} and by \mathcal{E} the subset of \mathcal{L} of functions such that for every $a \in \mathbb{E}$, $\frac{f(x) - f(y)}{x - y}$ has a limit when x and y tend to a separately. Following [9] the functions of \mathcal{E} are called *strictly differentiable*.

Given $f \in \mathcal{L}$, we put $\|f\|_1 = \sup_{\substack{x, y \in \mathbb{E} \\ x \neq y}} \frac{|f(x) - f(y)|}{\delta(x, y)}$ and $\|f\| = \max(\|f\|_0, \|f\|_1)$.

In particular, if $f \in \mathcal{D}$, then $\|f\|_1 = \sup_{\substack{x, y \in \mathbb{E} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|}$.

Remark 6.1. If $\mathbb{E} \subset \mathbb{K}$, then $\mathcal{E} \subset \mathcal{D} \subset \mathcal{L}$.

As noticed in Section 1, $(\mathcal{B}, \|\cdot\|_0)$ is a semi-compatible algebra. In [9], it was proved that the algebra here denoted by \mathcal{E} is a Banach \mathbb{K} -algebra with respect to the norm $\|\cdot\|$.

Theorem 6.2 presents no difficulty:

Theorem 6.2. \mathcal{L} , \mathcal{D} and \mathcal{E} are normed \mathbb{K} -algebras with respect to the norm $\|\cdot\|$.

Now we can prove that \mathcal{L} , \mathcal{D} and \mathcal{E} also are Banach algebras. First we will prove that \mathcal{L} is a Banach \mathbb{K} -algebra and then we will show that \mathcal{D} and \mathcal{E} are closed in \mathcal{L} when $\mathbb{E} \subset \mathbb{K}$.

Theorem 6.3. \mathcal{L} , \mathcal{D} and \mathcal{E} are Banach \mathbb{K} -algebras with respect to the norm $\|\cdot\|$.

Proof. Let $(f_n)_{n \in \mathbb{N}}$ be a Cauchy sequence of \mathcal{L} . Take $\epsilon > 0$ and let $N(\epsilon) \in \mathbb{N}$ be such that $\|f_n - f_m\| \leq \epsilon \forall m, n \geq N(\epsilon)$. Since $\|f_n - f_m\|_0 \leq \epsilon \forall m, n \geq N(\epsilon)$, the sequence $(f_n)_{n \in \mathbb{N}}$ converges with respect to the norm $\|\cdot\|_0$ to a function g such that $\|f_n - g\|_0 \leq \epsilon \forall n \geq N(\epsilon)$. On the other hand, since the sequence $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence for the norm $\|\cdot\|_1$, then for all $x, y \in \mathbb{E}$, such that $x \neq y$, we have

$$\frac{|f_n(x) - f_m(x) - (f_n(y) - f_m(y))|}{\delta(x, y)} \leq \epsilon \forall m, n \geq N(\epsilon)$$

and therefore, fixing n and passing to the limit on m , for all $x, y \in \mathbb{E}$, such that $x \neq y$ we get

$$\frac{|f_n(x) - g(x) - (f_n(y) - g(y))|}{\delta(x, y)} \leq \epsilon \forall n \geq N(\epsilon).$$

This is true for all $x, y \in \mathbb{E}$, $x \neq y$ and shows that $f_n - g$ belongs to \mathcal{L} . Consequently, g also belongs to \mathcal{L} . Particularly we notice that $\|g - f_n\|_1 \leq \epsilon$, hence $\|g - f_n\| \leq \epsilon$. Thus the sequence $(f_n)_{n \in \mathbb{N}}$ does converge to g in \mathcal{L} .

Suppose now that $\mathbb{E} \subset \mathbb{K}$ and let us show that \mathcal{D} is closed in \mathcal{L} . Take a sequence $(f_n)_{n \in \mathbb{N}}$ converging to a limit $f \in \mathcal{L}$ and let us show that f belongs to \mathcal{D} . As noticed above, since the sequence $(f_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_1$, the sequence $(f'_n)_{n \in \mathbb{N}}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_0$. Let h be its limit for this norm. This limit is then bounded in

\mathbb{E} . We will show that f is differentiable and that $f' = h$. Fix $a \in \mathbb{E}$ and $\epsilon > 0$. For all $x \in \mathbb{E}$ and for every $n \in \mathbb{N}$, we have

$$\begin{aligned} & \left| \frac{f(x) - f(a)}{x - a} - h(a) \right| \\ &= \left| \frac{f(x) - f(a) - (f_n(x) - f_n(a))}{x - a} + \frac{f_n(x) - f_n(a)}{x - a} - f'_n(a) + f'_n(a) - h(a) \right| \\ &\leq \max \left(\left| \frac{f(x) - f(a) - (f_n(x) - f_n(a))}{x - a} \right|, \left| \frac{f_n(x) - f_n(a)}{x - a} - f'_n(a) \right|, |f'_n(a) - h(a)| \right) \\ &\leq \max \left(\|f - f_n\|, \left| \frac{f_n(x) - f_n(a)}{x - a} - f'_n(a) \right|, \|f'_n - h\|_0 \right). \end{aligned}$$

We can fix $N \in \mathbb{N}$ such that $\|f - f_N\| \leq \epsilon$ and $\|f'_N - h\|_0 \leq \epsilon$. Then there exists $r > 0$ such that, for all $x \in d(a, r)$, we have

$$\left| \frac{f_N(x) - f_N(a)}{x - a} - f'_N(a) \right| \leq \epsilon$$

and thus, we have

$$\left| \frac{f(x) - f(a)}{x - a} - h(a) \right| \leq \epsilon.$$

This proves that $f'(a) = h(a)$. Therefore f is differentiable and $f' = h$.

A similar proof shows that \mathcal{E} also is closed in \mathcal{D} . \square

Remark 6.4. Concerning \mathcal{D} , one can ask why we do not consider a norm of the form $\|f\|_d = \max(\|f\|_0, \|f'\|_0)$ instead of the above norm $\|\cdot\|$ where $\|\cdot\|_1$ is defined with the help of the Lipschitz inequality. Indeed, $\|\cdot\|_d$ is a norm of \mathbb{K} -algebra. But the problem is that the algebra \mathcal{D} is not complete with respect to that norm, in the general case. The example given in [9] (Remark 2) shows that we can't obtain a Banach algebra in that way because a sequence that converges with respect to that norm may have a limit which is not differentiable at certain points.

Remark 6.5. Now, suppose that every non empty circle $C(0, r)$ has at least two classes and consider a function f differentiable in \mathbb{E} and a a point of $\widehat{\mathbb{E}} \setminus \mathbb{E}$. Then in general, \widehat{f} is not differentiable at a , as the following example shows. Let \mathbb{E} be the set $\{x \in \mathbb{K} \mid 0 < |x| \leq 1\}$ and let $(a_n)_{n \in \mathbb{N}}$ be a sequence in \mathbb{E} such that

$$|a_n| < |a_{n-1}|, \quad \lim_{n \rightarrow +\infty} |a_n| = 0.$$

For each $n \in \mathbb{N}$, put $r_n = |a_n|$. Let g be the function defined on \mathbb{E} by $g(x) = a_n \forall x \in d(a_n, r_n^-)$ and

$$g(x) = 0 \quad \forall x \in \mathbb{E} \setminus \left(\bigcup_{n=1}^{\infty} d(a_n, r_n^-) \right).$$

We can check that g is differentiable and Lipschitz in \mathbb{E} . But $\widehat{g}(0) = 0$ and \widehat{g} is not differentiable at 0. Indeed, let $(b_n)_{n \in \mathbb{N}}$ be a sequence of \mathbb{E} such that $|b_n| = |b_n - a_n| = r_n \forall n \in \mathbb{N}$. Now,

$$\frac{g(a_n) - \widehat{g}(0)}{a_n - 0} = 1 \quad \forall n \in \mathbb{N}$$

whereas

$$\frac{g(b_n) - \widehat{g}(0)}{b_n - 0} = 0 \quad \forall n \in \mathbb{N},$$

which shows that $\frac{\widehat{g}(x) - \widehat{g}(0)}{x}$ has no limit at 0. Therefore g is not differentiable in $\widehat{\mathbb{E}}$. In the same way, we can show that \widehat{g} is strictly differentiable in \mathbb{E} but not in $\widehat{\mathbb{E}}$.

Theorem 6.6. *An element of $\mathcal{L}, \mathcal{D}, \mathcal{E}$ is invertible if and only if $\inf\{|f(x)| \mid x \in \mathbb{E}\} > 0$.*

Proof. Suppose that $\inf\{|f(x)| \mid x \in \mathbb{E}\} > 0$ and put $g(x) = \frac{1}{f(x)}$. Let us first show that f belongs to \mathcal{L} . Indeed, let $m = \inf\{|f(x)| \mid x \in \mathbb{E}\}$. Then

$$\frac{|g(x) - g(y)|}{\delta(x, y)} = \frac{|f(y) - f(x)|}{|f(x)f(y)|\delta(x, y)} \leq \frac{|f(y) - f(x)|}{m^2\delta(x, y)}$$

which proves that g belongs to \mathcal{L} . Similarly, if $f \in \mathcal{D}$ (resp. $f \in \mathcal{E}$), then g belongs to \mathcal{D} (resp. to \mathcal{E}). \square

Theorem 6.7. *In each algebra $\mathcal{L}, \mathcal{D}, \mathcal{E}$, the spectral norm $\|\cdot\|_{sp}$ is $\|\cdot\|_0$.*

Proof. Take $f \in \mathcal{L}$ and $n \in \mathbb{N}$. Without loss of generality, we can suppose that $\|f\|_0 \geq 1$. We have

$$\|f^n\| = \max\left(\|f^n\|_0, \sup_{x, y \in E, x \neq y} \frac{|(f(x))^n - (f(y))^n|}{\delta(x; y)}\right).$$

We notice that $|(f(x))^n - (f(y))^n| \leq |f(x) - f(y)|(\|f\|_0)^{n-1}$ and hence

$$\sup_{x, y \in E, x \neq y} \frac{|(f(x))^n - (f(y))^n|}{\delta(x; y)} \leq (\|f\|_0)^{n-1} \sup_{x, y \in E, x \neq y} \frac{|f(x) - f(y)|}{\delta(x; y)} = (\|f\|_0)^{n-1} \|f\|_1.$$

Consequently, we get

$$\|f\|_0 \leq \|f\|_{sp} \leq \sqrt[n]{\|f\|_1} (\|f\|_0)^{\frac{n-1}{n}}.$$

Then when n tends to $+\infty$, we get $\|f\|_{sp} = \|f\|_0 \forall f \in \mathcal{L}$. This is then true in \mathcal{D} and \mathcal{E} too. \square

Theorem 6.8. *The \mathbb{K} -algebras $\mathcal{L}, \mathcal{D}, \mathcal{E}$ are C -compatible algebras.*

Proof. Indeed, by Theorems 6.3, 6.6 and 6.7, we just have to check that a subset F of \mathbb{E} is uniformly open if and only if its characteristic function belongs to \mathcal{L} (resp. to \mathcal{D} , resp. to \mathcal{E}), which is immediate by Lemma 1.6. \square

7. Particular properties of the algebras $\mathcal{B}, \mathcal{L}, \mathcal{D}, \mathcal{E}$.

A first specific property of the algebras $\mathcal{B}, \mathcal{L}, \mathcal{D}, \mathcal{E}$ concerns maximal ideals of finite codimension.

Notation: For convenience, we will denote here by T one of the algebras $\mathcal{B}, \mathcal{L}, \mathcal{D}, \mathcal{E}$ and by T^* the \mathbb{L} -algebra of bounded uniformly continuous functions (resp. Lipschitz functions, resp. differentiable functions, resp. strictly differentiable functions) from \mathbb{E} to \mathbb{L} .

Lemmas 7.1 and 7.2 are basic results in algebra:

Lemma 7.1. *Let \mathbb{L} be a finite algebraic extension of \mathbb{K} of the form $\mathbb{L} = \mathbb{K}[a]$ of degree d provided with the absolute value which extends that of \mathbb{K} . Let $f \in T^*$.*

Then f is of the form $\sum_{j=0}^{d-1} a^j f_j$, with $f_j \in T$ for $j = 0, \dots, d-1$. So, T^ is isomorphic to $T \otimes \mathbb{L}$.*

Lemma 7.2. *Let \mathbb{L} be a finite algebraic extension of \mathbb{K} provided with the absolute value which extends that of \mathbb{K} . Suppose there exists a morphism of \mathbb{K} -algebra, χ , from T onto \mathbb{L} . Then χ has a continuation to a surjective morphism of \mathbb{L} -algebra χ^* from T^* to \mathbb{L} .*

Proof. Let $d = [\mathbb{L} : \mathbb{K}]$. Suppose first that \mathbb{L} is of the form $\mathbb{K}[a]$. Then by

Lemma 7.1, any f in T^* is of the form $\sum_{j=0}^{d-1} a^j f_j$, where the f_j are functions from

\mathbb{E} to \mathbb{K} for $j = 0, \dots, d-1$. We then set $\chi^*(f) = \sum_{n=0}^{d-1} a_n \chi(f_j)$. The conclusion is

then straightforward.

Consider now the general case. We can obviously write \mathbb{L} in the form $\mathbb{K}[b_1, \dots, b_q]$. Writing \mathbb{L}_j for the extension $\mathbb{K}[b_1, \dots, b_j]$ we have $\mathbb{L}_j = \mathbb{L}_{j-1}[b_j]$. By induction on j , using the just proved preceding result we get that for each $j = 1, \dots, q$, χ has a continuation to a surjective morphism of \mathbb{L}_j -algebra, χ_j^* , from $(T \otimes \mathbb{L}_j)$ onto \mathbb{L}_j . Taking $j = q$ ends the proof since $T \otimes \mathbb{L}_q = T^*$. \square

We can now state the following theorem whose proof is similar to that of Theorem 3.7 in [9] but here concerns the algebras $\mathcal{L}, \mathcal{D}, \mathcal{E}$. Actually, that result may be generalized to all semi-compatible algebras, provided that \mathbb{K} is a perfect field [4], a condition that we avoid here.

Theorem 7.3. *Every maximal ideal \mathcal{M} of T of finite codimension is of codimension 1.*

Proof. Let \mathbb{L} be the field $\frac{T}{\mathcal{M}}$ and T^* be defined as in the preceding theorem. Then T^* is a C-compatible \mathbb{L} -algebra. Now, let χ be the quotient morphism from T over \mathbb{L} whose kernel is \mathcal{M} . By Lemma 7.2 χ admits an extension to a morphism χ^* from T^* to \mathbb{L} . Since T^* is semi-compatible and since the kernel

of χ^* is a maximal ideal \mathcal{M}^* of T^* , there exists an ultrafilter \mathcal{U} on \mathbb{E} such that $\mathcal{M}^* = \mathcal{I}(\mathcal{U}, T^*)$.

Let $g \in T$ and let $b = \chi(g) \in \mathbb{L}$. Then we have $\chi^*(g - b) = 0$, hence $g - b$ belongs to \mathcal{M}^* , therefore $\lim_{\mathcal{U}} g(x) - b = 0$ i.e. $\lim_{\mathcal{U}} g(x) = b$. But since $g \in T$, $g(x)$ belongs to \mathbb{K} for all $x \in \mathbb{E}$. Therefore, since \mathbb{K} is complete, b belongs to \mathbb{K} . But by definition χ is a surjection from T onto \mathbb{L} , hence every value b of \mathbb{L} is the image of some $g \in T$ and hence it lies in \mathbb{K} , therefore $\mathbb{L} = \mathbb{K}$. \square

Remark 7.4. In [5], it is shown that in the algebra of bounded analytic functions in the open unit disk of a complete ultrametric algebraically closed field, any maximal ideal which is not defined by a point of the open unit disk is of infinite codimension. Here, we may ask whether the same holds. In the general case no answer is obvious. We can only answer a particular case.

Theorem 7.5. *Suppose $\mathbb{E} \subset \mathbb{K}$ and let $\mathcal{M} = \mathcal{I}(\mathcal{U}, T)$ be a maximal ideal of T where \mathcal{U} is an ultrafilter on \mathbb{E} . If \mathcal{U} is a Cauchy filter, then \mathcal{M} is of codimension 1. Else, \mathcal{M} is of infinite codimension.*

Proof. Suppose first that \mathcal{U} is a Cauchy ultrafilter. By Theorem 2.7, \mathcal{M} is of codimension 1. Now, suppose that \mathcal{U} is not a Cauchy filter and consider the identity map g defined on \mathbb{E} . Then g has no limit on \mathcal{U} , therefore by Theorem 2.1 again, \mathcal{M} is not of codimension 1. But then by Theorem 7.3, \mathcal{M} is not of finite codimension. \square

Notation: Given a commutative unital Banach \mathbb{K} -algebra S , we denote by $Max_1(S)$ the set of maximal ideals of S of codimension 1 and by $Max_{\mathbb{E}}(S)$ the set of maximal ideal of S of the form $\mathcal{I}(a, S)$, $a \in \mathbb{E}$.

Corollary 7.6. *Suppose \mathbb{E} is a closed subset of \mathbb{K} . Then $Max_1(T) = Max_{\mathbb{E}}(T)$.*

Remark 7.7. Consider $\phi \in Mult(\mathcal{L}, \|\cdot\|)$, let $\mathcal{M} = Ker(\phi)$ and let θ be the canonical surjection from \mathcal{L} onto $\frac{\mathcal{L}}{\mathcal{M}}$. By Theorem 3.15, the quotient norm of the quotient field $\frac{\mathcal{L}}{\mathcal{M}}$ is just the quotient norm of the uniform convergence norm $\|\cdot\|_0$ and is equal to the absolute value of \mathbb{K} . In the case of a maximal ideal of infinite codimension, we can't apply Banach's Theorem and there is no reason to think that the quotient norm is equivalent to the absolute value defined as $|\theta(f)| = \lim_{\mathcal{U}} |f(x)|$.

Definition: Given a \mathbb{K} -normed algebra A whose norm is $\|\cdot\|$, we call *topological divisor of zero* an element $f \in A$ such that there exists a sequence $(u_n)_{n \in \mathbb{N}}$ of elements of A such that $\inf_{n \in \mathbb{N}} \|u_n\| > 0$ and $\lim_{n \rightarrow +\infty} fu_n = 0$.

Theorem 7.8. *Suppose that \mathbb{E} has no isolated points. Then an element of an algebra \mathcal{L} is a topological divisor of zero if and only if it is not invertible.*

Proof. It is obvious that an invertible element of \mathcal{L} is not a topological divisor of zero. Now, consider an element $f \in \mathcal{L}$ that is not invertible. By Theorem 6.6, we

have $\inf_{x \in \mathbb{E}} |f(x)| = 0$. Therefore, there exists a sequence of disks $(d_{\mathbb{E}}(a_n, r_n))_{n \in \mathbb{N}}$ with $\lim_{n \rightarrow \infty} r_n = 0$, such that $|f(x)| \leq \frac{1}{n}$, $\forall x \in d_{\mathbb{E}}(a_n, r_n)$, $\forall n \in \mathbb{N}^*$.

Since the points a_n are not isolated, for every $n \in \mathbb{N}$ we can fix $b_n \in d_{\mathbb{E}}(a_n, r_n)$ such that $b_n \neq a_n$.

For each $n \in \mathbb{N}^*$, let $t_n = \delta(a_n, b_n)$ and h_n be the characteristic function of $d_{\mathbb{E}}(a_n, t_n^-)$. Notice that $0 < t_n \leq r_n$ so $\lim_{n \rightarrow \infty} t_n = 0$. Now h_n belongs to \mathcal{L} and clearly satisfies

$$(1) \quad \frac{|h_n(x) - h_n(y)|}{\delta(x, y)} \leq \frac{1}{t_n} \quad \forall x, y \in \mathbb{E}, x \neq y.$$

Moreover, we notice that $\frac{|h_n(a_n) - h_n(b_n)|}{\delta(a_n, b_n)} = \frac{1}{t_n}$ hence

$$(2) \quad \|h_n\|_1 = \frac{1}{t_n} \quad \forall n \in \mathbb{N}^*.$$

Let $l \in \mathbb{K}$ be such that $|l| \in]0, 1[$. Since the valuation on \mathbb{K} is not trivial, for each $n \in \mathbb{N}$, we can find an element $\tau_n \in \mathbb{K}$ such that $|l|t_n \leq |\tau_n| \leq t_n$. We put $w_n = \tau_n h_n$ for all $n \in \mathbb{N}$. Then clearly we have

$$(3) \quad \|w_n\|_0 = |\tau_n| \|h_n\|_0 = |\tau_n| \leq t_n$$

and by (2), we have

$$(4) \quad |l| \leq \|w_n\|_1 = \frac{|\tau_n|}{t_n} \leq 1$$

Hence

$$(5) \quad |l| \leq \|w_n\| \leq \max(1, |\tau_n|) \quad \forall n \in \mathbb{N}^*.$$

Consider now the sequence $(fw_n)_{n \in \mathbb{N}^*}$. By (3), we have $\|fw_n\|_0 \leq t_n \|f\|_0$, hence

$$(6) \quad \lim_{n \rightarrow \infty} \|fw_n\|_0 = 0.$$

Furthermore for all $x, y \in \mathbb{E}$, we have

$$\frac{|f(x)w_n(x) - f(y)w_n(y)|}{\delta(x, y)} \leq \max\left(|f(x)| \cdot \frac{|w_n(x) - w_n(y)|}{\delta(x, y)}, |w_n(y)| \cdot \frac{|f(x) - f(y)|}{\delta(x, y)}\right)$$

and by (3) it is easily seen that

$$(7) \quad |w_n(y)| \cdot \frac{|f(x) - f(y)|}{\delta(x, y)} \leq t_n \|f\|_1.$$

On the other hand, if $x \in d_{\mathbb{E}}(a_n, r_n)$, we have : $|f(x)| \leq \frac{1}{n}$, hence by (4),

$$(8) \quad |f(x)| \cdot \frac{|w_n(x) - w_n(y)|}{\delta(x, y)} \leq \frac{1}{n}$$

and by (7) and (8), we obtain

$$(9) \quad \frac{|f(x)w_n(x) - f(y)w_n(y)|}{\delta(x, y)} \leq \max\left(\frac{1}{n}, t_n \|f\|_1\right).$$

Similarly, since x and y play the same role, if y belongs to $d_{\mathbb{E}}(a_n, r_n)$, we obtain the same inequality.

Suppose now that neither x nor y belongs to $d_{\mathbb{E}}(a_n, r_n)$. Then $w_n(x) = w_n(y) = 0$, therefore

$$(10) \quad \frac{|f(x)w_n(x) - f(y)w_n(y)|}{\delta(x, y)} \leq \max\left(\frac{1}{n}, t_n \|f\|_1\right).$$

Consequently, by (9) and (10) we have proved that $\|fw_n\|_1 \leq \max\left(\frac{1}{n}, t_n \|f\|_1\right)$. Hence $\lim_{n \rightarrow \infty} \|fw_n\|_1 = 0$ and by (6) $\lim_{n \rightarrow \infty} \|fw_n\| = 0$ which, together with (5), ends the proof. \square

Finally, we can prove this last result: given $a \in \mathbb{E}$, $\mathcal{I}(a, \mathcal{L})$ is not necessarily the closure of $\mathcal{I}'(a, \mathcal{L})$, while by Corollary 2.14, it is its spectral closure.

Proposition 7.9. *Suppose that the set \mathbb{E} is included in an ultrametric field \mathbb{F} and contains a disk $d(0, R)$ of the field \mathbb{F} . There exists $f \in \mathcal{I}(0, \mathcal{L})$ that does not belong to the closure of $\mathcal{I}'(0, \mathcal{L})$ with respect to the norm $\| \cdot \|$ of \mathcal{L} .*

Proof. Let $\omega \in \mathbb{F}$ be such that $0 < |\omega| < 1$. For every $n \in \mathbb{N}$, set $r_n = |\omega|^{-n}$, let $a_n \in C(0, r_n)$, let $F_n = d(a_n, r_n^-)$ and let $L = \bigcup_{n=1}^{\infty} F_n$. Let f be the function defined in \mathbb{E} as $f(x) = 0 \forall x \in \mathbb{E} \setminus L$ and $f(x) = a_n \forall x \in F_n$, $n \in \mathbb{N}$.

We notice that f belongs to \mathcal{L} . Indeed, let $x, y \in \mathbb{E}$ with $x \neq y$. If $f(x) \neq f(y)$, then at least one of the points x and y belongs to L . Suppose that $y \in L$.

Suppose first that $x \notin L$. Then $f(x) = 0$ and y belongs to some disk $d(a_n, r_n^-)$ and hence $|f(y)| = |a_n| = r_n$, whereas $|x - y| \geq r_n$, therefore $\left| \frac{f(x) - f(y)}{x - y} \right| \leq 1$.

Suppose now that x and y belong to L . Say, x belongs to $d(a_m, r_m^-)$ and y belongs to $d(a_n, r_n^-)$ with $m < n$ since $f(x) \neq f(y)$. Then $|f(x)| = r_m < r_n = |f(y)|$, hence $|f(x) - f(y)| = |f(y)| = r_n$ and $|x - y| = |y| = r_n$ therefore $\left| \frac{f(x) - f(y)}{x - y} \right| \leq 1$. Thus we have checked that $\left| \frac{f(x) - f(y)}{x - y} \right| \leq 1 \forall x, y \in \mathbb{E}, x \neq y$.

That finishes proving that f belongs to \mathcal{L} .

Now, by construction, we can see that f belongs to $\mathcal{I}(0, \mathcal{L})$. However, we will check that f does not belong to the closure of $\mathcal{I}'(0, \mathcal{L})$. Let $h \in \mathcal{I}'(0, \mathcal{L})$. There exists a disk $d(0, r_q)$ such that $h(x) = 0 \forall x \in d(0, r_q)$. Consequently, $f(x) - h(x) = f(x) \forall x \in d(0, r_q)$. But we notice that $f(x) = 0 \forall x \in C(0, r_q) \setminus F_q$. So, when x belongs to F_q and y belongs to $C(0, r_q) \setminus F_q$, we have $\left| \frac{f(x) - f(y)}{x - y} \right| = 1$, therefore $\|f - h\| \geq \|f - h\|_1 \geq 1$. This proves that $\mathcal{I}(0, \mathcal{L})$ is not the closure of $\mathcal{I}'(0, \mathcal{L})$ with respect to the norm $\| \cdot \|$. \square

8. A kind of Gelfand transform

A Gelfand transform is not easy to find on ultrametric Banach algebras, due to the existence of maximal ideals of infinite codimension. However, here we can obtain a kind of Gelfand transform under certain hypotheses on the multiplicative

spectrum in order to find again an algebra of bounded Lipschitz functions on some ultrametric space.

Notations: Let $(A, \| \cdot \|)$ be a commutative unital Banach \mathbb{K} -algebra which is not a field. Let $\Upsilon(A)$ be the set of algebra homomorphisms from A onto \mathbb{K} and let λ_A be the mapping from $A \times A$ to \mathbb{R}_+ defined by $\lambda_A(\chi, \zeta) = \sup\{|\chi(f) - \zeta(f)| \mid \|f\| \leq 1\}$.

Given $\chi \in \Upsilon(A)$, we denote by $|\chi|$ the element of $Mult(A, \| \cdot \|)$ defined as $|\chi|(f) = |\chi(f)|$, $f \in A$. Given $D \subset \Upsilon(A)$, we put $|D| = \{|\chi|, \chi \in D\}$.

The following Lemma is easily checked:

Lemma 8.1. λ_A is an ultrametric distance on $\Upsilon(A)$ such that $\lambda_A(\chi, \xi) \leq 1 \forall \chi, \xi \in A$.

Definition: Let $(A, \| \cdot \|)$ be a unital commutative ultrametric Banach \mathbb{K} -algebra. The algebra $(A, \| \cdot \|)$ will be said to be *L-based* if it satisfies the following:

a) $Mult_1(A, \| \cdot \|)$ is dense in $Mult(A, \| \cdot \|)$,

b) the spectral semi-norm $\| \cdot \|_{sp}$ is a norm,

c) For every uniformly open subset D of $\Upsilon(A)$ with respect to λ_A , the closures of $|D|$ and $|\Upsilon(A) \setminus D|$ are disjoint open subsets of $Mult(A, \| \cdot \|)$.

Theorem 8.2. Let $(A, \| \cdot \|)$ satisfy properties a) and b) above. Then the algebra A is algebraically isomorphic to an algebra \mathcal{C} of bounded Lipschitz functions from the ultrametric space $\mathbb{E} = (\Upsilon(A), \lambda_A)$ to \mathbb{K} . Identifying A with \mathcal{C} , the following are true.

i) The spectral norm $\| \cdot \|_{sp}$ of A is equal to the uniform convergence norm $\| \cdot \|_0$ on \mathbb{E} .

ii) Every $f \in A$ such that $\inf\{|\chi(f)| : \chi \in \Upsilon(A)\} > 0$ is invertible in A .

iii) There exists a constant $c \geq 1$ such that the Lipschitz semi-norm defined as $\|f\|_1 = \sup \left(\left\{ \frac{|f(x) - f(y)|}{\lambda_A(x, y)} \mid x, y \in \mathbb{E}, x \neq y \right\} \right)$ satisfies $\|f\|_1 \leq c\|f\|$ for all $f \in A$ and the topology defined by $\| \cdot \|$ on A is stronger than the topology induced by the norm $\| \cdot \|_{\mathcal{L}}$ of the Banach \mathbb{K} -algebra \mathcal{L} of all bounded Lipschitz functions from \mathbb{E} to \mathbb{K} , where $\|f\|_{\mathcal{L}} = \max(\|f\|_0, \|f\|_1)$, $f \in \mathcal{L}$.

Proof. We first show that A is isomorphic to a sub- \mathbb{K} -algebra of the algebra of bounded functions from \mathbb{E} to \mathbb{K} . For each $f \in A$ and $\chi \in \mathbb{E}$, we put $f^\circ(\chi) = \chi(f)$ and then we define a bounded function f° from \mathbb{E} to \mathbb{K} . Let us check that this mapping Θ associating to each $f \in A$ the function f° is injective. Indeed, Θ is obviously an algebra homomorphism whose kernel is the intersection \mathcal{J} of all maximal ideals of codimension 1. But thanks to Properties a), b) and to Theorem 3.1, we can check that $\mathcal{J} = (0)$. Consequently Θ is injective and hence A is isomorphic to a subalgebra \mathcal{C} of the algebra of bounded functions from \mathbb{E} to \mathbb{K} . Hencefore we will identify an element f of A with the function it defines on \mathbb{E} .

Let us now show that every $g \in A$ is Lipschitz, with respect to the distance λ_A . Let $\chi, \zeta \in \mathbb{E}$ and let $g \in A$ be such that $\|g\| \leq 1$. Then we have

$$|\chi(g) - \zeta(g)| \leq \sup\{|\chi(f) - \zeta(f)|, \|f\| \leq 1\} = \lambda_A(\chi, \zeta).$$

Now in general, take $g \in A$ and $\nu \in \mathbb{K}$ such that $|\nu| \geq \|g\|$. Let $h = \frac{g}{\nu}$. Then $|\chi(h) - \zeta(h)| \leq \lambda_A(\chi, \zeta)$ hence $|\chi(g) - \zeta(g)| \leq |\nu| \lambda_A(\chi, \zeta)$, therefore g is Lipschitz.

Consequently, A can be identified with a Banach \mathbb{K} -algebra of bounded Lipschitz functions from the ultrametric space \mathbb{E} to \mathbb{K} .

We will now show that the statements i), ii) and iii) are true. Thanks to Property a) and Theorem 3.1, it is immediately seen that the spectral norm $\|\cdot\|_{sp}$ of A is the uniform convergence norm $\|\cdot\|_0$ on \mathbb{E} ; hence i) is true.

Let us now show that whenever $|\chi(f)| \geq m > 0$ for all $\chi \in \Upsilon(A)$, then f is invertible in A . Indeed, suppose that f is not invertible. Then there exists a maximal ideal \mathcal{M} that contains f . By theorem 3.7, there exists $\phi \in Mult(A, \|\cdot\|)$ such that $\mathcal{M} = Ker(\phi)$ and then $\phi(f) = 0$. Given $r > 0$, let us denote again by $W(\phi, f, r)$ the neighborhood of $\phi : \{\psi \in Mult(A, \|\cdot\|) \mid |\psi(f) - \phi(f)|_\infty \leq r\}$. Now, let us recall that $\Upsilon(A)$ is the set of characters of A , hence $Mult_1(A, \|\cdot\|)$ is just the set of $|\chi|$, $\chi \in \Upsilon(A)$. Thus, since $Mult_1(A, \|\cdot\|)$ is dense in $Mult(A, \|\cdot\|)$, there exists a sequence χ_n of $\Upsilon(A)$ such that for every $n \in \mathbb{N}$, $|\chi_n|$ belongs to the neighborhood $W(\phi, f, \frac{1}{n})$ and hence, $\phi(f) = \lim_{n \rightarrow +\infty} |\chi_n(f)|$, i.e. $\phi(f) = \lim_{n \rightarrow +\infty} \chi_n(f) = 0$, a contradiction because $|\chi(f)| \geq m \forall \chi \in \Upsilon(A)$. Consequently, f is invertible in A i.e. ii) holds.

Finally let us prove iii). Let $f \in A$. Notice that if $\|f\| \leq 1$ then for every $x, y \in \mathbb{E}$ with $x \neq y$ we have, by definition of λ_A , $\frac{|f(x) - f(y)|}{\lambda_A(x, y)} \leq 1$ and hence $\|f\|_1 \leq 1$.

Suppose first that the valuation of \mathbb{K} is dense. Take $\epsilon > 0$ and $\nu \in \mathbb{K}$ such that $\|f\| \leq |\nu| \leq \|f\| + \epsilon$. Then $\left\| \frac{f}{\nu} \right\| \leq 1$ and hence $\left\| \frac{f}{\nu} \right\|_1 \leq 1$, i.e. $\|f\|_1 \leq |\nu|$ and $\|f\|_1 \leq \|f\| + \epsilon$. This holds for all $\epsilon > 0$ and hence we have $\|f\|_1 \leq \|f\|$.

Suppose now that \mathbb{K} has a discrete valuation. Let $\mu = \sup\{|x| \mid x \in \mathbb{K}, |x| < 1\}$ and take $\nu \in \mathbb{K}$ such that $|\nu| = \mu$.

If $\|f\| = 1$ then $\|f\|_1 \leq \|f\|$. If $\|f\| < 1$ then we can find $n \in \mathbb{N}$ such that $\mu^{n+1} \leq \|f\| \leq \mu^n$. Hence putting $g = \frac{f}{\nu^n}$, we have $\mu \leq \|g\| \leq 1$ and hence we have $\|g\|_1 \leq 1$. Therefore, $\frac{\|g\|_1}{\|g\|} \leq \frac{\|g\|_1}{\mu} \leq \frac{1}{\mu}$. But $\frac{\|g\|_1}{\|g\|} = \frac{\|f\|_1}{\|f\|}$ hence $\frac{\|f\|_1}{\|f\|} \leq \frac{1}{\mu}$ which finishes proving $\frac{\|f\|_1}{\|f\|} \leq c$, with $c = \frac{1}{\mu} \geq 1$. Consequently, we have $\|f\|_1 \leq c\|f\|$ for all $f \in A$ such that $\|f\| \leq 1$.

If $\|f\| > 1$ then there exists $n \in \mathbb{N}$ such that $\|f\| \leq \frac{1}{\mu^{n+1}}$. Putting $h = \nu^{n+1}f$ we have $\|h\| \leq 1$ hence $\|h\|_1 \leq c\|h\|$ which gives again $\|f\|_1 \leq c\|f\|$. Finally $\|f\|_1 \leq c\|f\|$ for every $f \in A$.

On the other hand, by Lemma 1.7, we have $\|f\|_{sp} \leq \|f\| \forall f \in A$, hence $\|f\| \geq \max(\|f\|_{sp}, \frac{1}{c}\|f\|_1) \geq \frac{1}{c} \max(\|f\|_0, \|f\|_1)$ for all $f \in A$, which proves that the norm $\|\cdot\|$ of A is at least as strong as the norm of the Banach \mathbb{K} -algebra of all bounded Lipschitz functions on \mathbb{E} . \square

Theorem 8.3. *As in the preceding theorem, let $(A, \|\cdot\|)$ satisfy properties a) and b). Identifying again A with \mathcal{C} , the three statements below are equivalent:*

- $\alpha)$ $(A, \|\cdot\|)$ is L -based i.e. Property c) is satisfied;
- $\beta)$ $(A, \|\cdot\|)$ is C -compatible;
- $\gamma)$ $(A, \|\cdot\|)$ satisfies the following Property c'): for every uniformly open subset D of \mathbb{E} , there exists $g_D \in A$ such that $\inf_{\chi \in D} |\chi(g_D)| > \sup_{\chi \notin D} |\chi(g_D)|$.

Proof. Notice that by the preceding theorem, A is C -compatible if and only if A satisfies the property 2) of C -compatible algebras.

Assume that $(A, \|\cdot\|)$ is L -based. Let D be a uniformly open subset of \mathbb{E} . By Property c), the closure F of $|D|$ in $Mult(A, \|\cdot\|)$ is a closed open set and so is the closure G of $|\Upsilon(A) \setminus D|$ and the two closures are disjoint. On the other hand, by Property a), we have $F \cup G = Mult(A, \|\cdot\|)$. Therefore, F and G are two disjoint open closed subsets making a partition of $Mult(A, \|\cdot\|)$. Consequently, by Theorem 3.21, there exists an idempotent $u \in A$ such that $\phi(u) = 1 \forall \phi \in F$ and $\phi(u) = 0 \forall \phi \notin F$. Particularly we have $|\chi|(u) = 1 \forall \chi \in D$ and $|\chi|(u) = 0 \forall \chi \in \Upsilon(A) \setminus D$ and then, since u is idempotent, $\chi(u) = 1 \forall \chi \in D$ and $\chi(u) = 0 \forall \chi \in \Upsilon(A) \setminus D$ i.e. $u(\chi) = 1 \forall \chi \in D$ and $u(\chi) = 0 \forall \chi \notin D$. Consequently, u is the characteristic function of D and it is in A . Hence $(A, \|\cdot\|)$ is a C -compatible algebra and hence α implies β .

Assuming now that A is C -compatible, for every uniformly open subset D of \mathbb{E} , the characteristic function u of D belongs to A and clearly, we have:

$$\inf_{\chi \in D} |\chi(g_D)| = 1 > \sup_{\chi \notin D} |\chi(g_D)| = 0. \quad \text{Thus, Property c')} \text{ is satisfied and hence } \beta$$

implies γ .

Finally assume that property c') is satisfied and let us prove that c) is satisfied. Let D be a uniformly open subset of \mathbb{E} and let $g_D \in A$ be such that

$$\inf_{\chi \in D} |\chi(g_D)| > \sup_{\chi \notin D} |\chi(g_D)|.$$

Let $t = \inf_{\chi \in D} |\chi(g_D)|$ and $s = \sup_{\chi \notin D} |\chi(g_D)|$. The mapping $[\phi \in Mult(A, \|\cdot\|) \mapsto$

$\phi(g_D) \in \mathbb{R}]$ defines a continuous function since $Mult(A, \|\cdot\|)$ is provided with the pointwise convergence. So the sets $L_1 = \{\phi \in Mult(A, \|\cdot\|), \phi(g_D) \geq t\}$ and $L_2 = \{\phi \in Mult(A, \|\cdot\|), \phi(g_D) \leq s\}$ are disjoint closed subsets of $Mult(A, \|\cdot\|)$ and we have $|D| \subset L_1$ and $|\Upsilon(A) \setminus D| \subset L_2$. So L_1 contains the closure of $|D|$ and L_2 contains the closure of $|\Upsilon(A) \setminus D|$. But the closures of $|D|$ and $|\Upsilon(A) \setminus D|$

recover $Mult(A, \|\cdot\|)$ since $Mult_1(A, \|\cdot\|)$ is dense in $Mult(A, \|\cdot\|)$, so we get that the closures of $|D|$ and $|\Upsilon(A) \setminus D|$ make a partition of $Mult(A, \|\cdot\|)$ and finally are disjoint open sets. \square

By Theorems 8.3, 3.9, 3.20 and Corollary 3.6 we obtain the following corollary:

Corollary 8.4. *Let A be a L -based algebra. Then $Mult(A, \|\cdot\|) = Mult_m(A, \|\cdot\|)$. Moreover, A is multibjective. Further, $Shil(A) = Mult(A, \|\cdot\|)$.*

Theorem 8.5. *Let A be a L -based algebra. Then $\Upsilon(A)$ is complete with respect to the distance λ_A .*

Proof. Recall that by Theorems 8.2 and 8.3, A is C -compatible and the elements of A are Lipschitz functions, with respect to λ_A on $\mathbb{E} = \Upsilon(A)$ and particularly they are uniformly continuous functions. Let $(\chi_n)_{n \in \mathbb{N}}$ be a Cauchy sequence of $(\Upsilon(A), \lambda_A)$. Then the Fréchet filter \mathcal{F} generated by this sequence is a Cauchy filter on $\Upsilon(A)$ and by Theorem 2.7, the ideal $\mathcal{M} = \mathcal{I}(\mathcal{F}, A)$ is a maximal ideal of codimension 1 and hence it defines an element θ of $\Upsilon(A)$ satisfying $\theta(f) = \lim_{\mathcal{F}} f(x) = \lim_{n \rightarrow +\infty} f(\chi_n) = \lim_{n \rightarrow +\infty} \chi_n(f)$, for every $f \in A$. Let us check that θ is the limit of the Cauchy sequence $(\chi_n)_{n \in \mathbb{N}}$.

Let us fix $\epsilon > 0$ and let $N \in \mathbb{N}$ be such that for all integers n, m greater than N , $\lambda_A(\chi_n, \chi_m) \leq \epsilon$. Then for every $f \in A$ such that $\|f\| \leq 1$, we have $|\chi_n(f) - \chi_m(f)| \leq \epsilon$ and thus $|\chi_n(f) - \theta(f)| \leq \epsilon$. Hence $\lambda_A(\chi_n, \theta) \leq \epsilon$, which ends the proof. \square

Notation: In the following theorems we denote by T one of the \mathbb{K} -algebras $\mathcal{L}, \mathcal{D}, \mathcal{E}$.

Theorem 8.6. *Suppose \mathbb{E} is a closed subset of \mathbb{K} . Let Z be the mapping from \mathbb{E} into $\Upsilon(T)$ that associates to each point $a \in \mathbb{E}$ the element of $\Upsilon(T)$ whose kernel is $\mathcal{I}(a, T)$. Then Z is a bijection from \mathbb{E} onto $\Upsilon(T)$. Moreover, we have $|b - a| \geq \lambda_T(a, b) \forall a, b \in \mathbb{E}$.*

Proof. Z obviously is an injection from \mathbb{E} into $\Upsilon(T)$. Now, let $\chi \in \Upsilon(T)$ and let $\mathcal{M} = Ker(\chi)$. By Theorem 2.1, there exists an ultrafilter \mathcal{U} on \mathbb{E} such that $Ker(\chi) = \mathcal{I}(\mathcal{U}, T)$. Since \mathcal{M} is of codimension 1 and since \mathbb{E} is closed, by Theorem 7.5, \mathcal{U} converges in \mathbb{E} to a point $c \in \mathbb{E}$. Consequently, Z is surjective.

Now we will show that $|b - a| \geq \lambda_T(a, b) \forall a, b \in \mathbb{E}$. Let us take $a, b \in \mathbb{E}$, with $a \neq b$ and consider $\lambda_T(a, b) = \sup\{|f(a) - f(b)|, \|f\| \leq 1\}$. Recall that, when $\mathbb{E} \subset \mathbb{K}$, in algebras T , we have defined the Lipschitz semi-norm $\|f\|_1$ as $\|f\|_1 = \sup\{\frac{|f(x) - f(y)|}{|x - y|}, x \neq y\}$. For every $f \in T$ such that $\|f\| \leq 1$ we have $\|f\|_1 \leq 1$ and thus $|f(a) - f(b)| \leq |a - b|$. Therefore $|b - a| \geq \lambda_T(a, b)$. \square

Corollary 8.7. *Suppose \mathbb{E} is a closed subset of \mathbb{K} . Then every uniformly open subset of $\Upsilon(T)$ with respect to λ_T is a uniformly open subset of \mathbb{E} with respect to the absolute value of \mathbb{K} .*

Theorem 8.8. *Suppose \mathbb{E} is a closed subset of \mathbb{K} . Then T is a L -based algebra.*

Proof. By Theorem 7.5, $Mult_1(T, \|\cdot\|) = Mult_{\mathbb{E}}(T, \|\cdot\|)$. By Theorem 6.8, T is \mathbb{C} -compatible, hence by Corollary 3.12 $Mult_1(T, \|\cdot\|)$ is dense in $Mult(T, \|\cdot\|)$. Next $\|\cdot\|_0$ is a norm equal to $\|\cdot\|_{sp}$. So, Properties a) and b) are satisfied.

Consider now a uniformly open subset D of \mathbb{E} with respect to λ_T . Identifying \mathbb{E} with $\Upsilon(T)$, by Corollary 8.7 D is also uniformly open with respect to the absolute value of \mathbb{K} . Consequently, the characteristic function u of D belongs to T and we have $\inf_{x \in D} |u(x)| = 1 > 0 = \sup_{x \notin D} |u(x)|$. So we get property c') of theorem 8.3, which ends the proof. \square

Notation: Let $(A, \|\cdot\|)$ be a L -based algebra. We will denote by \tilde{A} the algebra of all bounded Lipschitz functions from the space $\mathbb{E} = (\Upsilon(A), \lambda_A)$ to \mathbb{K} and we denote by $\|\cdot\|_{\tilde{}}$ the norm $\|\cdot\|_{\tilde{}} = \max(\|\cdot\|_0, \|\cdot\|_1)$ where $\|f\|_1 = \sup\{\frac{|f(x) - f(y)|}{\lambda_A(x, y)} \mid x, y \in \mathbb{E} \ x \neq y\}$ for every $f \in \tilde{A}$.

Theorem 8.9. *Let \mathbb{E} be a closed subset of \mathbb{K} and A be the L -based algebra of all bounded Lipschitz functions from \mathbb{E} to \mathbb{K} provided with the norm $\|\cdot\| = \max(\|\cdot\|_0, \|\cdot\|_1)$. Then the algebras A and \tilde{A} are identical and the norms $\|\cdot\|$ and $\|\cdot\|_{\tilde{}}$ are equivalent. In particular, a bounded function f is Lipschitz with respect to the distance $|\cdot|$ of \mathbb{E} if and only if it is Lipschitz with respect to λ_A . Next, we have $|x - y| \geq \lambda_A(x, y) \ \forall x, y \in \mathbb{E}$ and if \mathbb{E} is bounded, then there exists a constant $M \geq 1$ such that $|x - y| \leq M\lambda_A(x, y) \ \forall x, y \in \mathbb{E}$ and the distance λ_A is equivalent to the distance $|\cdot|$ of \mathbb{E} .*

Proof. Identifying \mathbb{E} with $\Upsilon(A)$, by Theorem 8.6 we have $|b - a| \geq \lambda_A(a, b)$ for all a and b in \mathbb{E} and consequently, if $f \in \tilde{A}$ then $f \in A$ and $\|f\|_1 \leq \|f\|_{\tilde{}}$ for all $f \in A$. Furthermore, by Theorem 8.2 (and its proof), we also have $A \subset \tilde{A}$ and there exists a constant $c \geq 1$ such that $\|f\|_{\tilde{}} \leq c\|f\|$ for all $f \in A$.

We conclude that the algebras A and \tilde{A} are equal and for every $f \in A$ we finally have : $\|f\| \leq \|f\|_{\tilde{}} \leq c\|f\|$, which proves that these norms are equivalent.

Moreover, if \mathbb{E} is bounded with respect to the distance defined by the absolute value of \mathbb{K} , then the identity map lies in A and hence is Lipschitz with respect to λ_A , therefore we have a constant $M > 0$ such that $M\lambda_A(x, y) \geq |x - y|, \ \forall x, y \in \mathbb{E}$. \square

Remark 8.10. By Lemma 8.1, we have $\lambda_A(x, y) \leq 1 \ \forall x, y \in A$. Therefore, if \mathbb{E} is not bounded with respect to the absolute value of \mathbb{K} , there exists no $M > 0$ such that $M\lambda_A(x, y) \geq |x - y| \ \forall x, y \in A$.

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