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About the decidability of polyhedral separability in the lattice \mathbb{Z}^d

Recognizing digital polyhedra with a prescribed number of faces

Yan Gerard

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Abstract The recognition of primitives in digital geometry is deeply linked with separability problems. This framework leads us to consider the following problem of pattern recognition : Given a finite lattice set $S \subset \mathbb{Z}^d$ and a positive integer n , is it possible to separate S from $\mathbb{Z}^d \setminus S$ by n half-spaces? In other words, does there exist a polyhedron P defined by at most n half-spaces satisfying $P \cap \mathbb{Z}^d = S$? The difficulty comes from the infinite number of constraints generated by all the points of $\mathbb{Z}^d \setminus S$. It makes the decidability of the problem non straightforward since the classical algorithms of polyhedral separability can not be applied in this framework.

We conjecture that the problem is nevertheless decidable and prove it under some assumptions: in arbitrary dimension, if the interior of the convex hull of S contains at least one lattice point or if the dimension d is 2 or if the dimension $d = 3$ and S is not in a specific configuration of lattice width 0 or 1. The proof strategy is to reduce the set of outliers $\mathbb{Z}^d \setminus S$ to its minimal elements according to a partial order “*is in the shadow of*”. These minimal elements are called the *lattice jewels* of S . We prove that under some assumptions, the set S admits only a finite number of lattice jewels. The result about the decidability of the problem is a corollary of this fundamental property.

Keywords Polyhedral separability · Lattice · Convex and Discrete geometry · Polyhedron simplification · Decidability

Mathematics Subject Classification (2000) (2010) 52B20 · 52B10 · 52B11 · 52C05 · 52C07

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1 Introduction

Digital geometry is focused on *lattice sets* (subsets of the lattice \mathbb{Z}^d). Some of these sets are considered as the digital counterparts of the classical figures of the Euclidean geometry. They are called *digital primitives* and can be obtained by the digitization of the usual Euclidean figures [28]. There exists however a class of digital primitives which has been neglected in the past years: the digital triangles, quadrilaterals, pentagons... or more generally, the digital polyhedra. We define here a digital polyhedron as the intersection of a real polyhedron P of \mathbb{R}^d and the lattice \mathbb{Z}^d . The set $P \cap \mathbb{Z}^d$ is sometimes called the *Gauss digitization* of the polyhedron P .

The study of the relationship between lattices and polyhedra is an mathematical field investigated since Hermann Minkowski at the end of the nineteenth century. The geometry of numbers provides many combinatorial results regarding the integer points in *rational polytopes* (a rational polytope is a polytope whose vertices are not necessarily on the integer lattice but have rational coordinates), in *real polytopes* (the vertices may have irrational coordinates) or in *lattice polytopes* (the vertices are integer points). Many of these results, from Pick’s theorem [38] to Ehrhart’s polynomials [19] or Barvinok’s algorithm [6] are focused on the cardinality of the set of integer points in a polyhedron [7]. There exist also many other results dealing with the relations between a rational polytope P and the convex hull P' of its interior integral points, for instance regarding the maximal number of vertices of P' according to the inequalities defining P [5, 14].

Since the sixties, a deep interest on these combinatorial issues comes from computer science and more precisely the field of Integer Linear Programming which

is widely used in Operations Research. In Integer Linear Programming, a rational polyhedron is described by rational linear inequalities, and the problem is to provide some of its interior integer points if any [43]. We can remind to the reader that this class of problems is NP-complete while it can be solved in polynomial time in a fixed dimension [30]. One of the main tools used to prove this result is the notion of *lattice width* which plays also an important role in the following results.

1.1 Problem statement

We start with a classical problem regarding the recognition of digital primitives:

Problem 1 (Digital Recognition: $\text{Reco}(d, \mathcal{F}, S)$)

Input: A dimension d , a family \mathcal{F} of subsets $F \subset \mathbb{R}^d$ and a subset S of \mathbb{Z}^d .

Output: Does there exists a shape F in \mathcal{F} such that $F \cap \mathbb{Z}^d = S$?

The aforementioned problem $[\text{Reco}(d, \mathcal{F}, S)]$ lies in the field of Pattern Recognition. The recognition of digital primitives has been investigated for several families of shapes, such as digital straight segments in dimension 2 [11, 17, 31, 16, 42, 45], pieces of digital planes in dimension 3 or hyperplanes in dimension d [10, 24, 25, 39], circular arcs [13, 35, 47] or generic primitives in higher dimensions [2, 3, 8, 9, 21, 26, 40, 46, 47]. The problem of the recognition of the digital circles in dimension 2 can for instance be denoted $[\text{Reco}(2, \mathcal{F}, S)]$ with a set of annuli as set \mathcal{F} [3].

The purpose of the paper is to investigate the case where the family of shapes \mathcal{F} is a set of polyhedra of \mathbb{R}^d . We denote \mathcal{P} the set of all polyhedra of \mathbb{R}^d and \mathcal{P}_n the set of the polyhedra of \mathbb{R}^d defined by at most n linear inequalities. For any integer n , we have of course the inclusion $\mathcal{P}_n \subset \mathcal{P}_{n+1}$. If we fix the dimension $d = 2$ and the number of faces $n = 3$, the problem $[\text{Reco}(2, \mathcal{P}_3, S)]$ is for instance to provide a triangle having a prescribed intersection $S \subset \mathbb{Z}^2$ with the lattice (Fig.1). The general problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ is one of the open problems mentioned in [4]. There exists some recognition algorithms for more restricted classes of polyhedra in dimension 2 such as squares [22, 34] and rectangles [29]. In any cases, the recognition of a class of digital primitives benefits from the properties of the intersections of the shapes of \mathcal{F} and the lattice. A property of connectivity allows for instance to reduce the problem of recognition to a neighborhood of S . This makes most digital primitives recognition problems much easier than $[\text{Reco}(d, \mathcal{P}_n, S)]$.

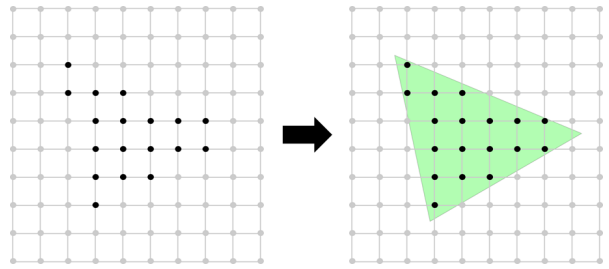


Fig. 1 On the left, an instance of $[\text{Reco}(d, \mathcal{P}_n, S)]$ with $d = 2$ and $n = 3$: a finite lattice set $S \subset \mathbb{Z}^2$ is given and the problem is to find a polyhedron P with at most 3 faces i.e a triangle satisfying $P \cap \mathbb{Z}^2 = S$. A solution is drawn on the right. It is a polyhedron with the prescribed number of faces separating the points of S from the other lattice points.

At last, we notice that the problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ is somehow the converse of Integer Linear Programming. The input is a set of integer points S and the problem is to find a polyhedron with a prescribed number of faces and S as set of interior integral points. Some other classes of polyhedra could be considered, for instance with constraints on their number of vertices or with other complexities (total number of faces of any dimension).

1.2 Polyhedral separability

A solution of an instance $[\text{Reco}(d, \mathcal{P}_n, S)]$ is a polyhedron with at most n faces separating S from the integer points outside S . Polyhedral separability has been intensively investigated in the framework of computational geometry:

Problem 2 (Polyhedral Separability: $\text{PolySep}(d, \mathcal{P}_n, S, T)$)

Input: A dimension d , an integer n and two subsets S and T of \mathbb{R}^d .

Output: Find a polyhedron $P \in \mathcal{P}_n$ with $S \subset P$ and $P \cap T = \emptyset$ (Fig.2).

In other words, we have two sets of points: a set S of inliers and a set T of outliers. The problem is to find a polyhedron $P \in \mathcal{P}_n$, containing S and with an empty intersection with T . The problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ can be reformulated as $[\text{PolySep}(d, \mathcal{P}_n, S, \mathbb{Z}^d \setminus S)]$.

If we fix the number of faces n to 1, then a solution of an instance $[\text{PolySep}(d, \mathcal{P}_1, S, T)]$ is an hyperplane separating S and T . This problem can be expressed as a system of linear constraints (one constraint per point of $S \cup T$) and solved by Linear Programming. It can be solved in pseudo-polynomial time or in a worst-case linear time complexity in the number of points if the dimension d is fixed [32].

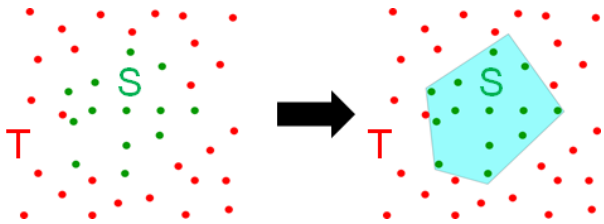


Fig. 2 On the left, an instance of $[\text{PolySep}(d, \mathcal{P}_n, S, T)]$ with $d = 2$ and $n = 5$. We have a set S of inliers and a set T of outliers. A solution, drawn on the right, is a polyhedron P with at most n faces containing S but no point of T .

In dimension $d = 2$, an algorithm with a worst-case complexity in $O((|S| + |T|) \log(|S| + |T|))$ solves the problem $[\text{PolySep}(2, \mathcal{P}_n, S, T)]$ for an arbitrarily large number n of faces in polynomial time [18].

In arbitrary dimension, $[\text{PolySep}(d, \mathcal{P}_n, S, T)]$ is NP-complete, even with a number of faces restricted to $n = 2$ [33].

As far as we know, the worst-case time complexity of the class of problems $[\text{PolySep}(3, \mathcal{P}_n, S, T)]$ in dimension $d = 3$ is an open question. There exists however very close results in the framework of nested polyhedra. In this class of problems introduced by Victor Klee [37], the two sets of points S and T given in the input are replaced by two polyhedra P_{small} and P_{big} [12]. The solution P has to contain P_{small} and be contained by P_{big} . This last condition $P \subset P_{big}$ can be rewritten $P \cap (\mathbb{R}^d \setminus P_{big}) = \emptyset$. It means that the complementary $\mathbb{R}^d \setminus P_{big}$ plays the role of the set of outliers T . It is however no more a finite set as considered in previous results of polyhedral separability. The computation of nested polyhedra can be solved efficiently in dimension $d = 2$ [1], and is NP-complete for larger dimensions $d \geq 3$ [15].

1.3 On decidability

Whatever the worst-case time complexity of polyhedral separability, it can be easily proved that any instance $[\text{PolySep}(d, \mathcal{P}_n, S, T)]$ is decidable for finite subsets S and T . A brute-force approach is to consider all partitions of T in n subsets $T = \cup_{i=1}^n T_i$ and for each set T_i use a Linear Programming algorithm solving $[\text{PolySep}(d, \mathcal{P}_{n=1}, S, T_i)]$. This approach does not work with an infinite set T of outliers. Unfortunately, by rewriting the problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ as an instance of polyhedral separability $[\text{PolySep}(d, \mathcal{P}_n, S, \mathbb{Z}^d \setminus S)]$, the set of outliers $T = \mathbb{Z}^d \setminus S$ is not finite. It makes the problem intractable for the usual algorithms of polyhedral separability. The infinite number of outliers does not allow to guarantee that any instance $[\text{Reco}(d, \mathcal{P}_n, S)]$ is

decidable. Unlike to a remark of [4], the decidability of the problem in dimension $d = 2$ is only granted by MinDSS algorithms [20] in some usual cases without acute angles but not in particular cases, especially if the set S is non connected. Even for triangles in dimension 2, the decidability of $[\text{Reco}(2, \mathcal{P}_3, S)]$ is not obvious (Fig.3).

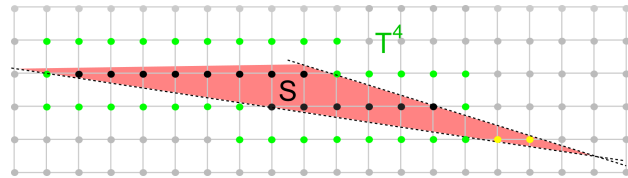


Fig. 3 A finite lattice set $S \subset \mathbb{Z}^2$ (in black) and the set $T^4 \subset \mathbb{Z}^2 \setminus S$ of its outer 4-neighbors. There exists triangles containing S and no 4-neighbor of S : the red triangle is a solution of $[\text{PolySep}(d, \mathcal{P}_{n=1}, S, T^4)]$ but it is not a solution of $[\text{Reco}(d, \mathcal{P}_n, S)]$ because it contains lattice points outside S and its boundary (in yellow).

Although the decidability of most problems of computational geometry is usually rather easy to obtain, the infinite cardinality of the lattice makes the one of $[\text{Reco}(d, \mathcal{P}_n, S)]$ non trivial. Each integer point outside S is by itself a constraint. It makes the problem much harder than it seems at first glance.

1.4 Overview of the conjectures and results

We conjecture that the recognition of digital polyhedra with a given number of faces is decidable:

Conjecture 1 (Main Conjecture)

The problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ is decidable for any finite subset $S \subset \mathbb{Z}^d$.

We prove the conjecture under some complementary assumptions: the problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ is decidable if

- the interior of the convex hull of S contains at least one lattice point, or
- $d = 2$, or
- $d = 3$ with sets S of lattice width $\text{width}(S)$ equal to 1 avoiding the specific configurations enumerated in Theorem 3, or
- $d = 3$ with sets S having a lattice width $\text{width}(S)$ larger than or equal to 2.

It means that the decidability of $[\text{Reco}(d, \mathcal{P}_n, S)]$ remains an open problem if $d > 3$ with sets S such that the interior of the convex hull does not contain any lattice point or in dimension 3, for the specific configurations of Theorem 3. These results come from properties of particular points called the *lattice jewels* of S .

They are the minimal elements of a partial order relation “*is in the shadow of*”. We prove that except in the cases enumerated in Theorem 3, the set S has only a finite number of such points. With this result, it is easy to provide an algorithm for $[\text{Reco}(d, \mathcal{P}_n, S)]$ with the guarantee that it always finishes.

In Sec. 2, we present the main conjectures and results. In Sec. 3, we introduce the *shadow*, the *jewels* and the *crown*. In Sec. 4, we provide theorems about the number of lattice jewels while the last section is devoted to the proofs of decidability.

2 Conjectures and results

2.1 Conjectures

Conjecture 1 is the main conjecture of the paper. It claims the decidability of $[\text{Reco}(d, \mathcal{P}_n, S)]$ for any finite subset $S \subset \mathbb{Z}^d$ and any integer n .

We introduce a second conjecture regarding the existence of rational solutions for any instance $[\text{Reco}(d, \mathcal{P}_n, S)]$. We denote $\mathcal{P}^{\mathbb{Q}}$ the set of rational polyhedra and $\mathcal{P}_n^{\mathbb{Q}}$ the set of rational polyhedra defined by at most n rational linear inequalities.

Conjecture 2 (Rational solutions)

Given a finite set $S \subset \mathbb{Z}^d$ of cardinality at least 2, if the instance of Digital Recognition $[\text{Reco}(d, \mathcal{P}_n, S)]$ has a solution P , then it admits a solution $P' \in \mathcal{P}_n^{\mathbb{Q}}$ ($[\text{Reco}(d, \mathcal{P}_n^{\mathbb{Q}}, S)]$ has also a solution).

In other words, Conjecture 2 means that if there exists a solution, there exists a rational one. The conjecture excludes the singletons because it would be false by including them. If we consider for instance the set $S = \{(0, 0)\}$, the instance $[\text{Reco}(2, \mathcal{P}_2, S)]$ has a solution with the real polyhedron P defined by the linear inequalities $x + \sqrt{2}y \leq 0$ and $x + \sqrt{2}y \geq 0$ (P is the straight line $x + \sqrt{2}y = 0$ of irrational slope) while it admits no rational solution (in $\mathcal{P}_2^{\mathbb{Q}}$). The strategy to prove such a result is usually to obtain a property that the set of the solutions is open. It allows to find a rational solution in the neighborhood of a real one, but we don't have such a topological property here. It makes Conjecture 2 difficult to prove.

2.2 Results

The main result of the paper is the following theorem which proves Conjecture 1 under some complementary assumptions.

Theorem 1 *The problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ is decidable if*

- $d = 2$
- $d = 3$ and $\text{width}(S) \geq 2$,
- $d = 3$ and $\text{width}_u(S) = 1$ and the property that the two planar sections $S \cap \{x \in \mathbb{Z}^3 \mid u \cdot x = h\}$ of S in the consecutive planes $u \cdot x = h$ and $u \cdot x = h + 1$ with $u \in \mathbb{Z}^3$ and $h \in \mathbb{Z}$ covering S are of dimension 2 (Fig.4),
- or in arbitrary dimension, if the interior of the convex hull of S contains at least one lattice point.

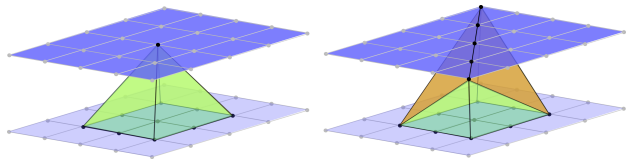


Fig. 4 For some finite lattice sets $S \subset \mathbb{Z}^3$ in dimension $d = 3$, the decidability of $[\text{Reco}(3, \mathcal{P}_n, S)]$ is not given by Theorem 1. It is for instance the case of the two “pyramids” above. Their lattice width is 1 (they are covered by two consecutive horizontal planes) but the planar section of S in the upper plane is reduced to a point (on the left) or to a segment (on the right). These particular sets have an infinite set of *lattice jewels* (see Fig.14 and Fig.15). It does not allow to obtain their decidability as in other cases of Theorem 1.

This result is presented in [23] for dimension $d = 2$ and sets S with a convex hull containing an integer point in their interior. It was not given in dimension $d = 3$. The proof of Theorem 1 is given in Sec. 5. It is based on the fundamental result that under some assumptions, the set S has only a finite number of *lattice jewels*. These theorems are proved in Sec. 4.2 in dimension d and in Sec. 4.3 for a complete characterization in dimensions 2 and 3. The question of the decidability of $[\text{Reco}(d, \mathcal{P}_n, S)]$ remains open even in dimension $d = 3$, for the sets S which are not full-dimensional or with a lattice width $\text{width}_u(S) = 1$ and a degenerated section in one of the two consecutive planes of normal direction u crossing S (Fig.4). For readers which are not interested in decidability but in this specific notion of jewels, theorems 2 and 3 are the main contribution of the paper.

3 The shadow, the jewels and the crown

We have already noticed that the class of problems $[\text{Reco}(d, \mathcal{P}_n, S)]$ can be written $[\text{PolySep}(d, \mathcal{P}_n, S, T)]$ with $T = \mathbb{Z}^d \setminus S$ namely the lattice points which are not in S . The difficulty to provide an algorithm comes from the infinite cardinality of $\mathbb{Z}^d \setminus S$. The goal of this section is to prove that we can reduce the outliers to a set of points that we call the *lattice jewels* of S . Then

the decidability of $[\text{Reco}(d, \mathcal{P}_n, S)]$ can be obtained in the case where there are only finitely many lattice jewels. The investigation of this property requires first to introduce the notion of *shadow*.

3.1 The shadow of a point

We place us in a general framework where S is a subset of \mathbb{R}^d . Its convex hull is denoted $\text{conv}(S)$. We define the *shadow* of a point $x \in \mathbb{R}^d$ relatively to S as the set of points which are hidden by x :

Definition 1 Given $S \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$, the **shadow** of x relatively to S is the set of points $y \in \mathbb{R}^d$ whose convex hull with S contains x (Fig.5):

$$\text{shadow}_S(x) = \{y \in \mathbb{R}^d \mid x \in \text{conv}(S \cup \{y\})\}. \quad (1)$$

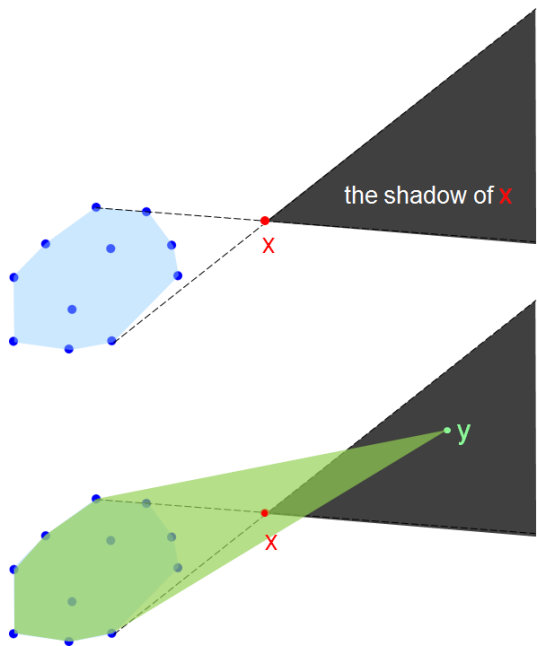


Fig. 5 Above, the shadow of a point x according to S . Below, according to the definition, a point y is in the shadow of x if x is in the convex hull $\text{conv}(S \cup \{y\})$ (in green).

If x is in the convex hull of S , its shadow is the whole space \mathbb{R}^d . The definition is interesting only if the point x does not belong to the convex hull of S . We provide another characterization of the shadow of x :

Lemma 1 Let S be a subset of \mathbb{R}^d and x, y two points of \mathbb{R}^d . The point y belongs to the shadow of x if and only if they are equal or if the half-line starting from x in the direction of $x - y$ crosses the convex hull of S (Fig.6).

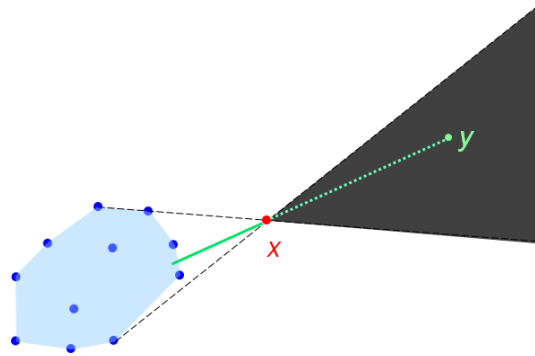


Fig. 6 A point y is in the own shadow (in grey) of x relatively to S if and only if the half-line (in green) starting from x in the direction of $x - y$ crosses the convex hull of S (in blue).

Proof In the case where $x = y$, by definition a point is in its shadow. If x and y are different and the half-line starting from x in the direction $x - y$ crosses the convex hull of S , then x is in the convex hull of the union $S \cup \{y\}$. It proves that y is in the shadow of x .

For the converse, we consider two cases: if x is in the convex hull of S , the shadow of x is the whole space \mathbb{R}^d and the proposition is trivial. Otherwise, we use Caratheodory's theorem. By definition, y is in the shadow of x if the convex hull of $\{y\} \cup S$ contains x . It follows from Caratheodory's theorem that x is the barycenter of a finite subset of $\{y\} \cup S$: there exists $d+1$ reals $\lambda_k \in [0, 1]$ and $d+1$ points $x_k \in S$ with a sum $\sum_{k=0}^d \lambda_k = 1$ and satisfying $x = \sum_{k=0}^d \lambda_k x_k$. Since x is not in the convex hull of S , y is one of the x_k , let us say x_0 . It leads to $x - \lambda_0 y = \sum_{k=1}^d \lambda_k x_k$. After reorganization, we have $x + \frac{\lambda_0}{\sum_{k=1}^d \lambda_k} (x - y) = \frac{1}{\sum_{k=1}^d \lambda_k} \sum_{k=1}^d \lambda_k x_k$. The expression $x + \frac{\lambda_0}{\sum_{k=1}^d \lambda_k} (x - y)$ determines a point of the half-line issued from x in the direction of $x - y$ while $\frac{1}{\sum_{k=1}^d \lambda_k} \sum_{k=1}^d \lambda_k x_k$ is a point of the convex hull of S . It proves the second direction of the lemma. \square

The shadow of a point has many properties. If S is finite, the main one is to be a polyhedral cone with x as vertex. It can be seen geometrically: we have the polyhedral cone containing all the half-lines issued from x and crossing a point of the convex hull of S . According to Lemma 1, the shadow of x is its symmetric with respect to x (Fig.7)

3.2 The binary relation "is in the shadow of"

We consider now the binary relation "is in the shadow of". We denote it with a triangle \triangleright_S or \triangleleft_S indexed by the set S : y is in the shadow of x is denoted $y \triangleright_S x$ or equiv-

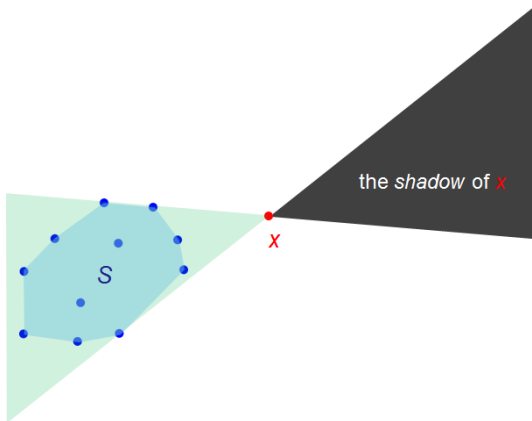


Fig. 7 The blue polyhedral cone is the union of the half-lines issued from x and crossing the convex hull of S . The shadow of x is its symmetric with respect to x .

alently $x \triangleleft_S y$. The relation \triangleleft_S has interesting properties that we can summarize in the following proposition:

Property 1 The binary relation “is in the shadow of” noticed \triangleright_S is a partial order on the complementary $\mathbb{R}^d \setminus \text{conv}(S)$ of the convex hull of S .

Proof The relation is reflexive: by definition x is in the shadow of x .

The transitivity means that if $x \triangleleft_S y$ and $y \triangleleft_S z$, then we have $x \triangleleft_S z$. We use Lemma 1 to prove it. We denote $p_x(y)$ a point of the convex hull of S crossed by the half-line issued from x in the direction of $x - y$. We denote $p_y(z)$ a point of the convex hull of S crossed by the half-line issued from y in the direction of $y - z$ and our goal is to prove the existence of a similar point $p_x(z)$ of the convex hull of S crossed by the half-line issued from x in the direction of $x - z$. This result is an exercise of planar geometry illustrated Fig.8 (we let to the reader).

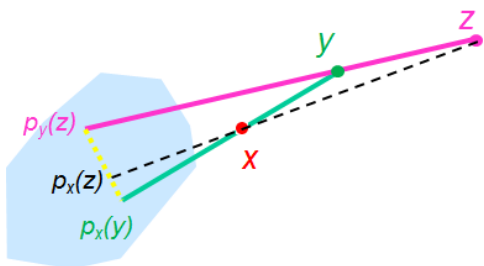


Fig. 8 The transitivity of the partial order relation \triangleleft_S comes from the possibility to build a point $p_x(z)$ in the intersection of the convex hull of S and the half-line issued from x in the direction of $x - z$. The existence of this point can be obtained from the existence of similar points $p_x(y)$ and $p_y(z)$.

The antisymmetry means that $x \triangleleft_S y$ and $y \triangleleft_S x$ implies $x = y$. It is not true if x and y are in the convex hull of S . If the two points are different, according to Lemma 1, it follows from $x \triangleleft_S y$ and $y \triangleleft_S x$ that both half-lines issued from x in the direction of $x - y$ and issued from y in the direction $y - x$ contain a point of the convex hull of S . It follows that the line xy contains a point of the convex hull of S on both sides of the segment $[x, y]$. We obtain that x and y are both in the convex hull of S , which is a contradiction if we consider points outside from it. \square

The transitivity of the partial order \triangleleft_S means exactly that if a point y is in the shadow of x namely $x \triangleleft_S y$, then the shadow of y is included in the shadow of x .

3.3 The shadow of a set of points

The notion of shadow can be easily extended to sets of points. The shadow of a set T is defined as the union of the shadows of its elements:

$$\text{shadow}_S(T) = \bigcup_{x \in T} \text{shadow}_x(T).$$

It follows from the transitivity of the partial order \triangleleft_S that the shadow of T is fully determined by its minimal elements according to \triangleleft_S . We call these minimal elements the *jewels* of T (Fig.9):

Definition 2 Let S and T be two subsets of \mathbb{R}^d . A point $x \in T$ is a **jewel** of T if there exist no point $x' \in T$ satisfying $x' \triangleleft_S x$. We denote $\text{jewels}_S(T)$ their set.

The jewels of T are the counterparts of the vertices of the convex hull of S on the side of T . By construction, we can reduce the problem of polyhedral separability of two sets S and T to the polyhedral separability of S and the set of the jewels of T :

Lemma 2 Let S and T be two subsets of \mathbb{R}^d . The instance of polyhedral separability $[\text{PolySep}(d, \mathcal{P}_n, S, T)]$ is equivalent to the reduced instance $[\text{PolySep}(d, \mathcal{P}_n, S, \text{jewels}_S(T))]$ where the set of outliers T is reduced to the jewels $\text{jewels}_S(T)$ of T .

Proof Let P be a polyhedron solution of $[\text{PolySep}(d, \mathcal{P}_n, S, T)]$. As the set $\text{jewels}_S(T)$ is a subset of T , P is also a solution of $[\text{PolySep}(d, \mathcal{P}_n, S, \text{jewels}_S(T))]$.

In order to prove the converse, we assume that P is a solution of $[\text{PolySep}(d, \mathcal{P}_n, S, \text{jewels}_S(T))]$ and not a solution of the complete instance $[\text{PolySep}(d, \mathcal{P}_n, S, T)]$. It follows that P contains a point x of T which is not a

jewel of T . Then there exists a jewel $x' \in T$ with $x' \triangleleft_S x$. By definition of the shadow, the jewel x' is in the convex hull of $S \cup \{x\}$. As $S \cup \{x\}$ is included in the polyhedron P , it follows from the convexity of P that the jewel x' is also in P . It is in contradiction with the assumption that P is a solution of $[\text{PolySep}(d, \mathcal{P}_n, S, \text{jewels}_S(T))]$. \square

Lemma 2 is not only true for polyhedra. As it can be noticed by reading the proof, it holds for any family of convex shapes \mathcal{F} : Any instance of separability of inliers and outliers by convex shapes is equivalent to a reduced instance where the outliers are reduced to the jewels $\text{jewels}_S(T)$. This approach has been used for instance in the framework of digital circles recognition in [13] where the set of the outliers is reduced to its subset of jewels.

In order to complete the medieval glossary, we introduce the *crown* of S relatively to T . It is the complementary of the convex hull of S and the shadow of T :

Definition 3 Let S and T be two subsets of \mathbb{R}^d . The **crown** of S with respect to T is the set of points $x \in \mathbb{R}^d$ such that the convex hull of the union of S with $\{x\}$ has an empty intersection with T (Fig.9 and 10):

$$\text{crown}_T(S) = \{x \in \mathbb{R}^d \mid T \cap \text{conv}(S \cup \{x\}) = \emptyset\}.$$

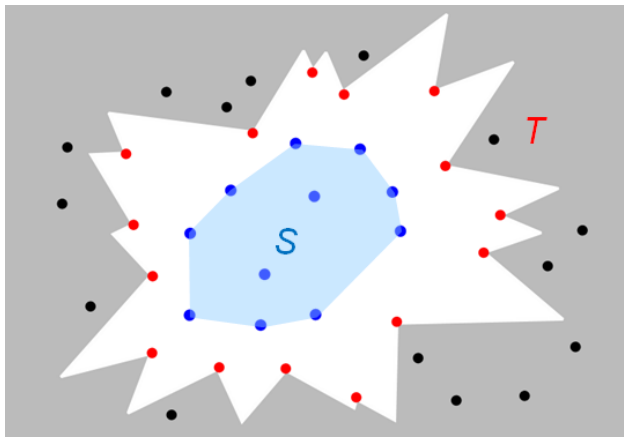


Fig. 9 The partition of the space \mathbb{R}^d in the convex hull of S (blue), the crown (white) and the shadow of T (grey). The set T contains the red and black points. The red ones are the minimal elements of T according to the partial order \triangleleft_S namely the jewels of T while the black points are the non minimal elements of T . It can be noticed on the picture that the shadow and the crown are fully determined by S and the jewels of T . The jewels play the same role than the vertices of the convex hull of S on the side of the outliers.

3.4 The lattice jewels

In the framework of our problem of decidability, we are interested in the polyhedral separability of a lattice set $S \subset \mathbb{Z}^d$ from the integral points outside S . The set T of the outliers is $\mathbb{Z}^d \setminus S$. Its jewels are called the *lattice jewels* of T .

Definition 4 Let S be a subset of \mathbb{R}^d satisfying a property of digital convexity. The convex hull $\text{conv}(S)$ of S does not contain any other lattice points than the ones already in S : $\text{conv}(S) \cap \mathbb{Z}^d = S \cap \mathbb{Z}^d$. The **lattice jewels** of S are the jewels of the complementary $\mathbb{Z}^d \setminus S$ relatively to S (Fig.10). We denote jewels_S the set $\text{jewels}_S(\mathbb{Z}^d \setminus S)$.

The condition $\text{conv}(S) \cap \mathbb{Z}^d = S \cap \mathbb{Z}^d$ of Definition 4 is necessary to guarantee that the relation \triangleleft_S is a partial order on $\mathbb{Z}^d \setminus S$. Otherwise, for any lattice point x in the complementary of $x \in \mathbb{Z}^d \setminus S$, the lattice points y of $(\text{conv}(S) \cap \mathbb{Z}^d) \setminus S$ satisfy $y \triangleleft_S x$. There can not be any other jewels than the lattice points of $(\text{conv}(S) \cap \mathbb{Z}^d) \setminus S$. If we consider a lattice set S , the condition $\text{conv}(S) \cap \mathbb{Z}^d = S \cap \mathbb{Z}^d$ is usually called *digital convexity*: the set S is said digitally convex. It's a necessary condition to speak about its lattice jewels.

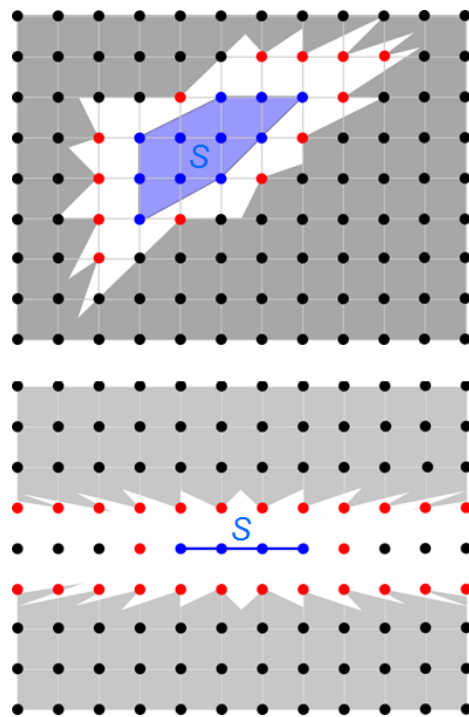


Fig. 10 Two lattice sets S (deep blue), their convex hulls (blue), their crown (white), their lattice jewels (red points) and for each, the shadow (grey) of $\mathbb{Z}^2 \setminus S$.

As corollary of Lemma 2, we can reduce any instance $[\text{Reco}(d, \mathcal{P}_n, S)]$ to polyhedral separability with a restricted set of outliers:

Lemma 3 *Let S be a subset of \mathbb{R}^d . Any instance of problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ is equivalent to the instance of polyhedral separability $[\text{PolySep}(d, \mathcal{P}_n, S, \text{jewels}_S)]$.*

This reduction becomes interesting if the set of the lattice jewels is finite since the polyhedral separability becomes decidable. It leads to the question of the number of the lattice jewels of a lattice set S . It can be seen in Fig.10 that the set of lattice jewels of an aligned point's set has an infinite cardinality while it can be finite for other examples.

4 About the number of lattice jewels

The results about the existence of a finite or infinite number of lattice jewels are decomposed in two theorems. A sufficient condition is given in arbitrary dimension in Theorem 2. A complete characterization is given in dimensions $d = 2$ and 3 in Theorem 3.

4.1 Affine dimension and lattice width

We provide a list of cases where it can be guaranteed that the set jewels_S of the lattice jewels of S remains finite. It requires to introduce first the notions of *dimension* and *lattice width*.

Definition 5 The affine **dimension** of $S \subset \mathbb{R}^d$ is the dimension of the minimal affine space containing S . It is denoted $\text{dim}(S)$.

The **width** of S in a direction $u \in \mathbb{R}^d$ is

$$\text{width}_u(S) = \max\{u \cdot x \mid x \in S\} - \min\{u \cdot x \mid x \in S\}. \quad (2)$$

The **lattice width** of S is the minimal of the width by choosing the direction u in all integral directions (Fig.11):

$$\text{width}(S) = \min\{\text{width}_u(S) \mid u \in \mathbb{Z}^d, u \neq 0\}. \quad (3)$$

Geometrically, the lattice width of S provides the minimal number of consecutive diophantine hyperplanes required to cover S . This notion has been used in the framework of integer programming to prove that Integer Linear Programming can be solved in polynomial time in a fixed dimensions [30]. It is especially used for characterizing empty lattice polytopes [44,27].

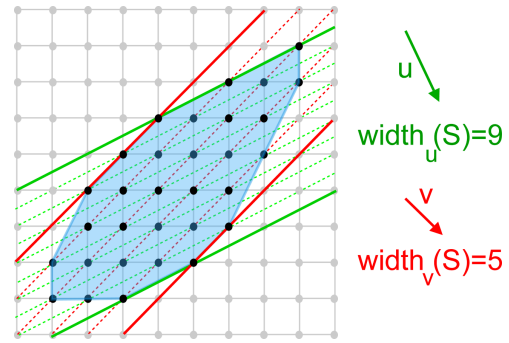


Fig. 11 A lattice set S and its lattice width in the two directions $u = (1, -2)$ and $v = (1, -1)$. Its lattice width $\text{width}(S)$ is the minimum achieved in all directions. For this set, it is obtained in the diagonal direction: we have $\text{width}(S) = 5$

4.2 In arbitrary dimension

In a first theorem (Theorem 2), we provide a condition in arbitrary dimension. This condition is sufficient to guarantee a finite number of lattice jewels but it is not necessary (cf Sec. 4.3).

Theorem 2 *Let S be a finite digitally convex subset of \mathbb{Z}^d . If the interior of the convex hull of S contains an integer point, the set jewels_S of the lattice jewels is finite.*

Proof We assume that the interior of the convex hull of S contains an integer point denoted s and that the set of the lattice jewels jewels_S of $\mathbb{Z}^d \setminus S$ is infinite. Our goal is to obtain a contradiction.

As the set jewels_S is not finite, let $x_k \in \text{jewels}_S$ be a sequence of pairwise distinct jewels that we assume different from the origin ($\forall k \in \mathbb{N}, x_k \neq (0)_{1 \leq i \leq d}$). We consider the direction $y_k = \frac{x_k}{\|x_k\|_2}$ which belongs to the hypersphere S^d centered at the origin and of radius 1. As S^d is compact, the sequence y_k admits a convergent subsequence y'_k with a non null limit z : $\lim_{k \rightarrow +\infty} y'_k = z$. It means that the corresponding subsequence of points $x'_k \in \text{jewels}_S$ is going in direction z (A).

We intend to prove now that there exists an integer point in the interior of $(\text{conv}(S) \oplus \mathbb{R}^+ z) \setminus \text{conv}(S)$ (B1) where the operator \oplus denotes Minkowski addition and $\mathbb{R}^+ z$ is the half-line starting from the origin in the direction z . This part of the proof is illustrated in Fig.12.

If the direction z is rational (the line $\mathbb{R}z$ contains integer points), then by translation, the half-line $s + \mathbb{R}^+ z$ contains infinitely many integer points. The ones which are not in the convex hull of S are in the interior of $(\text{conv}(S) \oplus \mathbb{R}^+ z) \setminus \text{conv}(S)$. Let's take such an integer point and call it x : this step is denoted (B1). We have $x = s + \lambda z$ and we notice that by definition of s , there

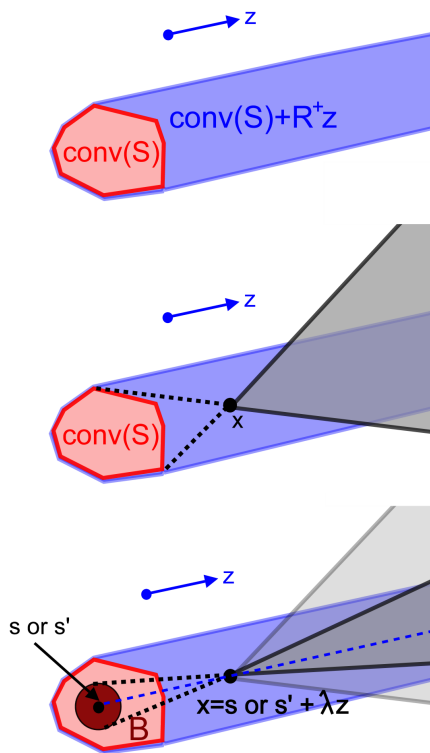


Fig. 12 Above, the set S , its convex hull $\text{conv}(S)$ (in red) and the polyhedron $\text{conv}(S) \oplus \mathbb{R}^+z$ (in blue). In the middle, an integer point x in the interior of $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$. Its shadow (in grey) covers a polyhedron cone which the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ except a bounded part. Below, for the proof, we introduce a ball in the convex hull of S and an integer point on the half-line from its center in direction z . By construction, the shadow $\text{shadow}_B(x)$ of x with respect to the ball B contradicts the existence of jewels tending in direction z .

is a ball B centered at s with a strictly positive radius r included in the convex hull of S (B2).

If the direction z is not rational, we use a classical result of simultaneous rational approximation. As s is in the interior of the convex hull of S , there exists a ball $B_r(s)$ centered at s of radius $r > 0$ included in the interior of the convex hull of S . Then we consider the Minkowski sum $B_r(s) + \mathbb{R}^+z$. By construction, this half-cylinder is in the interior of $\text{conv}(S) \oplus \mathbb{R}^+z$. Simultaneous rational approximation, for instance with Dirichlet's approximation theorem, allows to prove that $B_r(s) + \mathbb{R}^+z$ contains infinitely many integer points. As they can not be all in the convex hull of S , it proves (B1): we have an integer point x in the interior of $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$. We can also notice that x can be written $s' + \lambda z$ with a ball B centered at s' contained by the convex hull of S (B2).

With (A), (B1) and (B2), the last step is to obtain the contradiction: the shadow $\text{shadow}_S(x)$ of the integer point x is a polyhedral cone which contradicts

the existence of jewels x'_k going to infinite in direction z (Fig.12). The sequence of the integer points x'_k is necessarily going to infinite since there is only a finite number of lattice points at a given distance of the origin. It follows that limit of $\lim_{k \rightarrow +\infty} \frac{x'_k - x}{\|x'_k - x\|} = \lim_{k \rightarrow +\infty} \frac{x'_k}{\|x'_k\|} = z$. As the ball B centered at s (rational case) or s' (irrational case) is in $\text{conv}(S)$, the shadow $\text{shadow}_S(x)$ of x contains the shadow $\text{shadow}_B(x)$. Since the points x'_k are jewels, they don't belong to the shadow of any other integer point and thus, they can not belong to $\text{shadow}_S(x)$. It provides a contradiction with the limit $\lim_{k \rightarrow +\infty} \frac{x'_k - x}{\|x'_k - x\|} = z$. \square

4.3 Complete characterization in dimensions $d = 2$ and 3

Theorem 2 provides a sufficient condition in arbitrary dimension to prove that a set S has only a finite number of lattice jewels. This condition is not necessary. The square $S = \{(0, 0), (1, 0), (1, 1), (0, 1)\}$ has, for instance, a finite number of lattice jewels without having any integer point in the interior of its convex hull (Fig.13).

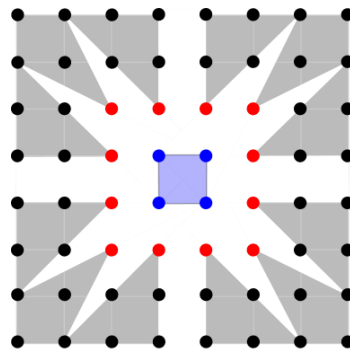


Fig. 13 The lattice set S (in blue) does not satisfy the conditions of Theorem 2: it has no integer point in the interior of its convex hull but it has only a finite number of lattice jewels (in red). It shows that the condition of Theorem 2 is sufficient but not necessary.

We improve Theorem 2 in dimension $d = 2$ and 3 by providing a complete characterization of the finite lattice sets having a finite or infinite number of lattice jewels:

Theorem 3 *A finite digitally convex set $S \subset \mathbb{Z}^2$ has infinitely many lattice jewels if and only if its affine dimension is 0 or 1 (a set of aligned points).*

A finite digitally convex set $S \subset \mathbb{Z}^3$ has infinitely many lattice jewels if and only if

- *its affine dimension is $\dim(S) \leq 2$,*

- or if there exists a direction $u \in \mathbb{Z}^3$ with $\text{width}_u(S) = 1$ and such that one of the two non empty planar sections $S_{u,h} = \{x \in S | x \cdot u = h\}$ of S with u as normal direction has an affine dimension equal to 0 or 1 (Fig.4 and 14).

Proof We prove first that the configurations mentioned in Theorem 3 have infinitely many lattice jewels. If a lattice set S is not full dimensional, it is contained by an hyperplane $u \cdot x = h$ with $u \in \mathbb{Z}^d$ having coprime coordinates and $h \in \mathbb{Z}$. In this case, any lattice point of the neighboring diophantine hyperplanes $u \cdot x = h - 1$ or $u \cdot x = h + 1$ is a lattice jewel.

For the last case, we have $d = 3$ and assume that there exists $u \in \mathbb{Z}^3$ (with coprime coordinates) satisfying $\text{width}_u(S) = 1$. We add the condition that one of the two non empty planar sections of S namely $S_{u,h} = \{x \in S | u \cdot x = h\}$ is of affine dimension 0 or 1. The set $S_{u,h}$ is a singleton (Fig.14) or a set of aligned points (Fig.15). In both cases, we consider a line L (with a rational direction) containing $S_{u,h}$. The sub-lattice of the integer points in the plane $u \cdot x = h$ can be decomposed in diophantine lines parallel to L . It can be easily checked that the lattice points in the two diophantine lines parallel to L and surrounding L in the plane $u \cdot x = h$ are lattice jewels (even if it is in dimension 2 and not 3, the figure Fig. can help to see it).

It proves that all the configurations enumerated in Theorem 3 have an infinite number of lattice jewels.

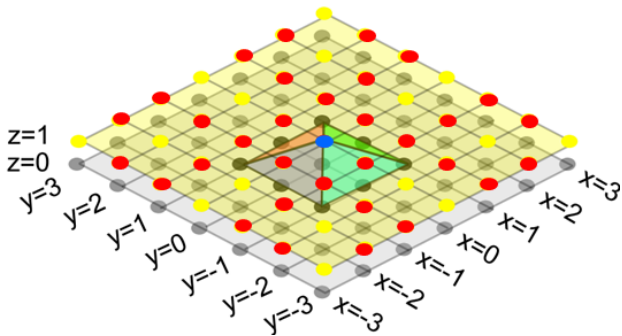


Fig. 14 The set S already drawn Fig.4 is a pyramid with its basis in the plane $z = 0$ and its vertex of coordinate $(0, 0, 1)$. Its lattice width is $\text{width}(S) = 1$. The set S has infinitely many lattice jewels (in red): any lattice point of coordinates $(x, y, 1)$ with coprime x and y is a lattice jewel.

In order to prove the converse, we use the same principle than the proof of Theorem 2 without giving again all its technical details. We assume that S is not in one of the configurations mentioned in Theorem 3 and that it admits an infinite number of lattice jewels. Then there exists a sub-sequence of lattice jewels tending to

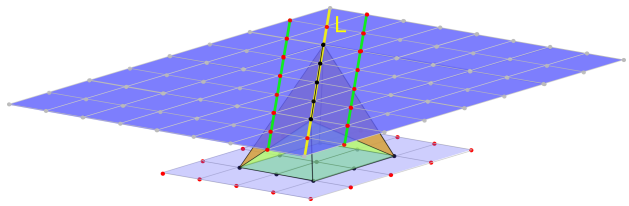


Fig. 15 The set S already drawn Fig.4 has a lattice width $\text{width}(S) = 1$ and a degenerated section $S_{u,h}$ in its upper plane $u \cdot x = h$. The section $S_{u,h}$ is contained by the line L (in yellow). All the integer points in the neighboring diophantine lines (in green) of its degenerated section are lattice jewels (in red).

infinite in a direction z , as in part (A) of the proof of Theorem 2. Our goal consists in obtaining a contradiction.

As S does not satisfy the conditions of Theorem 3, it is of full dimension and it is necessarily in one of the three following configurations:

- $d = 2$ and $\dim(S) = 2$. We refer to this case as (i).
- $d = 3$ and $\text{width}(S) \geq 2$ (ii).
- $d = 3$ and $\text{width}(S) = 1$ and for all direction u such that $\text{width}_u(S) = 1$, the two consecutive planar sections of S with u as normal direction, are of dimension 2 (iii). This last condition excludes the examples shown in Fig.4 or in Fig.14.

Case (i) The lattice set $S \subset \mathbb{Z}^2$ is fully dimensional in \mathbb{R}^2 . The polyhedron $\text{conv}(S) \oplus \mathbb{R}^+ z$ is bounded by two half-lines $y \oplus \mathbb{R}^+ z$ and $y' \oplus \mathbb{R}^+ z$ (Fig.16). If the direction z is rational, then there are integer points x and x' on each half-line. Their shadows $\text{shadow}_S(x)$ and $\text{shadow}_S(x')$ cover two cones around the polyhedron $\text{conv}(S) \oplus \mathbb{R}^+ z$. As the jewels are not in the shadows of any other integer points, the only possibility for the sequence of jewels x'_k to go in direction z is to be in the strip $\text{conv}(S) \oplus \mathbb{R}^+ z$. It follows that we have an integer point x'_k in the interior of $\text{conv}(S) \oplus \mathbb{R}^+ z$ and outside $\text{conv}(S)$. If the direction z is irrational, the existence of an integer point in the interior of $(\text{conv}(S) \oplus \mathbb{R}^+ z) \setminus \text{conv}(S)$ is a consequence of the Dirichlet approximation theorem. As in the proof of Theorem 2, it leads to a contradiction.

Before starting with the cases (ii) and (iii), we recall some algebraic properties of the projection of the lattice \mathbb{Z}^d in a non null direction $z \in \mathbb{R}^d$. We denote $(z_i)_{1 \leq i \leq d}$ the coordinates of z . The projection can be considered in any hyperplane which is not parallel with z , for instance the plane of equation $z \cdot x = 0$ but it does not matter. We introduce the \mathbb{Q} linear space E_z generated by the coordinates z_i : $E_z = \{q \cdot z | q \in \mathbb{Q}^d\}$. The

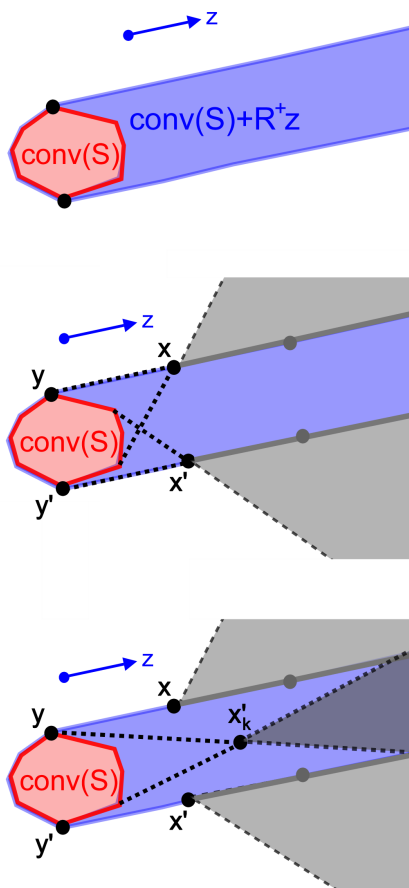


Fig. 16 Case (i) with a rational z : Above, the set S , its convex hull $\text{conv}(S)$ (in red) and the polyhedron $\text{conv}(S) \oplus \mathbb{R}^+z$ (in blue). In the middle, two integer points x and x' on the boundary of $\text{conv}(S) \oplus \mathbb{R}^+z$ and their shadows. Below, the only way to have integer points going in the direction z is that they belong to the strip between $y \oplus \mathbb{R}z$ and $y' \oplus \mathbb{R}z$ but the shadow $\text{shadow}_S(x_k)$ of x_k in the interior of the strip covers its extremity and leads to a contradiction.

dimension of E_z can take all the integral values from 1 to d according to the rational relations between the real coordinates z_i .

If the dimension d is 2, given $z \in \mathbb{R}^2$, the rational dimension $\dim_{\mathbb{Q}}(E_z)$ of E_z can be 1 or 2. If it is 1 it means that there is a rational linear relation between z_1 and z_2 . In other words, z is a rational direction and the projection of the lattice \mathbb{Z}^2 in the direction z is lattice of dimension 1 in the line of projection. Otherwise, the dimension $\dim_{\mathbb{Q}}(E_z)$ is 2 and the projection of the lattice \mathbb{Z}^2 in the direction z is dense on the line of projection.

If the case of \mathbb{Z}^3 , there are three cases to investigate (Fig.17). If the dimension $\dim_{\mathbb{Q}}(E_z)$ is 3, there is no rational relation between the three coordinates of z . As in the two-dimensional case, the projection of \mathbb{Z}^3 is dense in the plane $z \cdot x = 0$. If the dimension $\dim_{\mathbb{Q}}(E_z)$ is 1, there are two rational relations between the three

coordinates of z which makes the z direction rational. Then the projection of the lattice \mathbb{Z}^3 in the z direction is a lattice of dimension 2. An hybrid case occurs if the dimension $\dim_{\mathbb{Q}}(E_z)$ is 2. There exists one rational linear relation between z_1, z_2 and z_3 . By multiplying it by the least common multiple of the denominators, we can denote it $\alpha \cdot z = 0$ with $\alpha \in \mathbb{Z}^3$. This relation provides a partition of \mathbb{Z}^3 in consecutive planes of equations $\alpha \cdot x = h$ with $h \in \mathbb{Z}$. Since α is normal to z , the projection in the z direction preserves the scalar product $\alpha \cdot x$ with the consequence that the projections of the lattice points still satisfy $\alpha \cdot x \in \mathbb{Z}$. The projected points of the lattice \mathbb{Z}^3 belong to the equidistant parallel lines $\alpha \cdot x = h$ in the plane $z \cdot x = 0$. The projection of the lattice is not dense in the plane $z \cdot x = 0$ but it is dense on these straight lines.

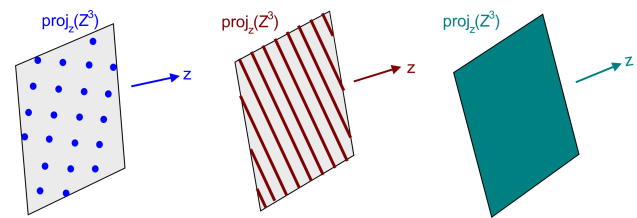


Fig. 17 On the left, the projection of \mathbb{Z}^3 in the z direction if the rational dimension of $E_z = \{q \cdot z | q \in \mathbb{Q}^d\}$ is 1. It means that the z direction is rational and the projection of the lattice is a lattice of dimension 2. In the middle, the rational dimension of $E_z = \{q \cdot z | q \in \mathbb{Q}^d\}$ is 2. We have a rational linear relation between the coordinates of z . The projection of the lattice \mathbb{Z}^3 in the z direction is dense on equidistant parallel lines. On the right, if $\dim_{\mathbb{Q}}(E_z) = 3$, there exists no rational linear relation between the coordinates of z . The projection of the lattice \mathbb{Z}^3 is dense in the plane of projection.

The general strategy to prove the result in the cases (ii) and (iii) remains roughly the same than previously: we prove that there exists a lattice point in the interior of the set $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$. Sometimes, we obtain however more directly that there can not be jewels tending in the direction z . It contradicts the existence of infinitely many lattice jewels.

Although the cases (ii) and (iii) are different, we do not consider them separately. In dimension $d = 3$, the existence of an integer point in the set $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ can be investigated by considering the projections of the lattice \mathbb{Z}^3 and of the convex hull of S according to the direction z in a cutting plane. Six different configurations denoted from A) to F) can occur. They depend on the rational dimension $\dim_{\mathbb{Q}}(E_z)$ and on the direction of z with respect to S (Fig.18).

- The configuration A) occurs with a rational dimension $\dim_{\mathbb{Q}}(E_z)$ equal to 1. The projection of \mathbb{Z}^3 in

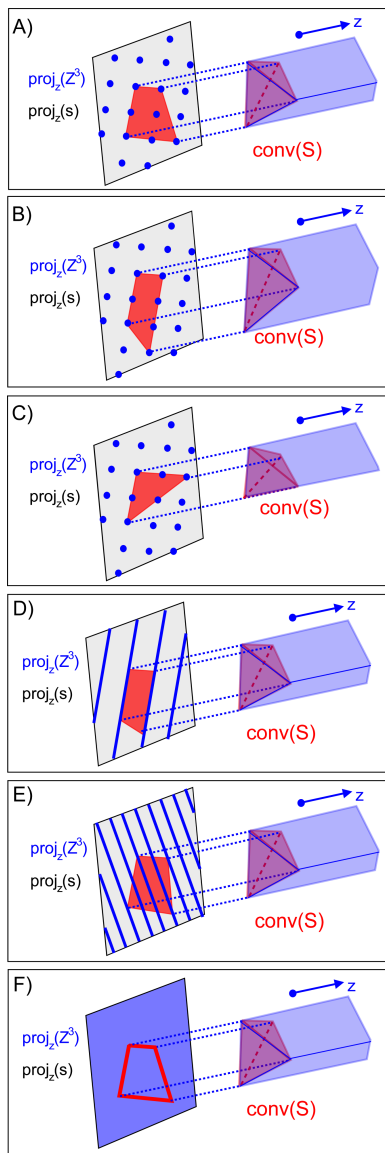


Fig. 18 The figure shows the different possible configurations of the projection of \mathbb{Z}^3 and the projection of the convex hull of S in the direction z . The projection of \mathbb{Z}^3 in the direction z on a cutting plane may have different structures. In A), B) and C), the rational dimension $\dim_{\mathbb{Q}}(E_z)$ of E_z is 1 and the projection of \mathbb{Z}^3 is a lattice. The dimension $\dim_{\mathbb{Q}}(E_z)$ is 2 in D) and E): the projection of \mathbb{Z}^3 is dense on equidistant parallel lines. The dimension $\dim_{\mathbb{Q}}(E_z)$ is 3 in F) with a dense projection of \mathbb{Z}^3 in the cutting plane. The goal is to determine the cases where a projection of a point of \mathbb{Z}^3 (a blue point) in the interior of the projection of the convex hull of S (the red polygon) because they lead to obtain a contradiction. It allows to refute the existence of infinitely many lattice jewels. There are three cases where it is not true: there is no blue point in the interior of the polygon in B), C) and D). B) and D) are excluded from (ii) due to their lattice width $\text{width}(S)$ equal to at most 1. They are investigated in the case (iii). The case C) falls in (ii) and requires a specific proof illustrated in Fig.21 with a similar construction than in Fig.16.

the direction z is a lattice. We place us in the cutting plane. It contains the two-dimensional lattice obtained by projection of \mathbb{Z}^3 . The lattice width of the projection of S in this planar lattice is necessarily at least the lattice width of S . The configuration A) corresponds to the case where the interior of the projection of the convex hull $\text{conv}(S)$ of S contains at least one point of the projection of \mathbb{Z}^3 (Fig.18 A)). Then we apply the same construction than in the proof of Theorem 2 to prove the existence of an integer point in $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$. The shadow of this integer point covers the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ except a compact part. It contradicts the existence of a sequence of lattice jewels of S tending to infinite in the direction z (Fig.12).

- In the configuration B) of Fig.18, the lattice width of S in a direction $u \in \mathbb{Z}^3$ normal to z is at most 1. It falls in the case (iii). Three different sub-cases can arise (Fig.19). B2) and B3) are the ones where we have obtained an infinite number of lattice jewels. They are excluded from the case (iii) since the lattice width in a direction $u \in \mathbb{Z}^d$ is 1 and one of the two consecutive planar sections of S in the planes $u \cdot x = h$ and $u \cdot x = h + 1$ is degenerated.

It remains only to focus on B1). We introduce four lattice points a, b, c and d on the boundary of the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ (Fig.20) such that the convex hull of their projections in the cutting plane is a fundamental domain of the projected lattice of \mathbb{Z}^3 . The shadows of these four points define a cross between two pairs of consecutive parallel planes. There is no integer point in between. It follows that we can not have a sequence of integer points tending in z direction. It is in contradiction with the existence of infinitely many lattice jewels.

- The configuration C) of Fig.18 is the default configuration if we are not in A) or B) with a rational dimension $\dim_{\mathbb{Q}}(E_z)$ equal to 1. Avoiding A) and B) means that the sub-lattice width of the projection $\text{proj}_z(S)$ of S in the plane of projection is at least 2 -it avoids B- and its convex hull (in red) does not contain any sub-lattice point (in blue) - it avoids A. In dimension 2, such a configuration is isomorphic to a unique subset of \mathbb{Z}^2 : any lattice sets $H \in \mathbb{Z}^2$ with $\text{width}(H) > 1$ and with no lattice point in the interior of its convex hull is isomorphic to the triangle of vertices $\{(0, 0), (2, 0), (0, 2)\}$ [36, 41]. It explains the specific configuration which arises in C). In this case, the strategy is close to the one of B1). There exist three integer points $a, b,$ and c in the interior of the faces of the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ (Fig.21). Their three shadows are polyhedral cones which surround

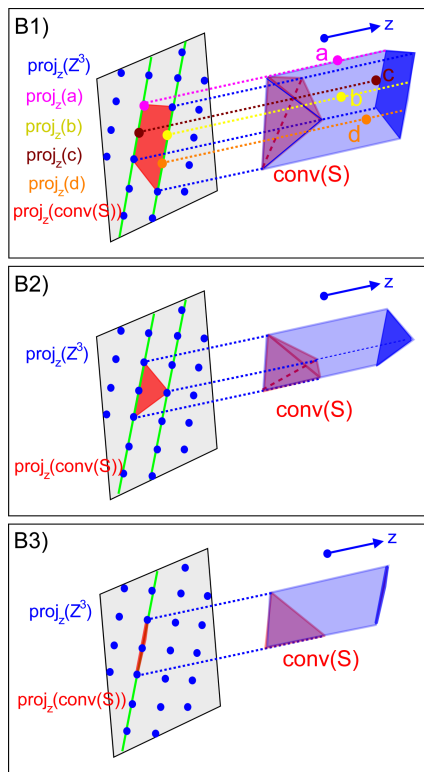


Fig. 19 Three different configurations can arise in the case B) of Fig.18. In these three cases, the projection of S in the cutting plane has a lattice width equal to 1. It follows that the lattice width of S in the direction $u \in \mathbb{Z}^3$ normal to z and to the green lines is at most 1. In case B2), one of the two planar sections $S_{u,h} = \{x \in S | x \cdot u = h\}$ containing the points of S is of affine dimension 0 or 1. In case B3), there is only one non empty planar section. These two configurations can be excluded because they don't satisfy the conditions of (iii) (notice that they provide infinitely many lattice jewels). It remains to prove that B1) provides only a finite number of lattice jewels.

the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$. There is no integer point in between. It contradicts the existence of a sequence of lattice jewels tending in the direction z .

- In the configuration D) of Fig.18), the rational dimension $\dim_{\mathbb{Q}}(E_z)$ is 2. The projection of \mathbb{Z}^3 in the direction z is dense on equidistant parallel lines. Three sub-configurations are possible (Fig.22).

As for B), the sub-configurations D2) and D3) of Fig.22 are excluded. It remains only the case D1) where the two planar sections of S with the normal direction of $u \in \mathbb{Z}^3$ (with $\text{width}_u(S) \leq 1$) are not degenerated. Then there exists two lattice points a and b in the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ having their projections in the interior of the edges of the projection of the convex hull of S (Fig.23). The shadows of these two points define a strip between two consecutive parallel planes. There is no

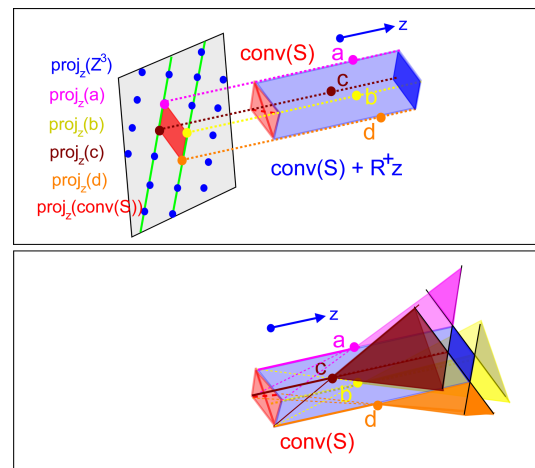


Fig. 20 In the case B1) of Fig.19, we have four integer points a, b, c and d in the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ (in blue). Below, the space between their four shadows (in four colors) defines a cross which does not contain any integer point. It contradicts the existence of a sequence of lattice jewels tending in the direction z .

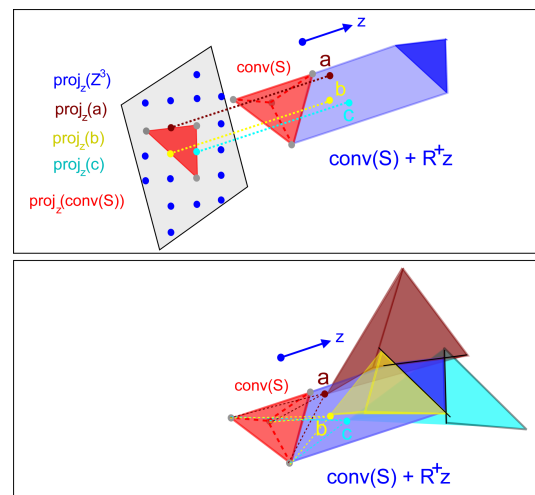


Fig. 21 In the case C) of Fig.18, we have three integer points a, b and c on the faces of $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ while the interior of $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ does not contain any integer point. Below, the shadow of the three lattice points a, b and c are polyhedral cones which surround the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$. It contradicts the existence of jewels tending to infinite in the direction z .

integer point in between. It follows that we can not have a sequence of integer points tending in the direction z . It is in contradiction with the existence of infinitely many lattice jewels.

- In the configurations E) and F) of Fig.18), the existence of integer points in the interior of the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$ is straightforward. It leads to a contradiction. We obtain the same contradiction than previously.

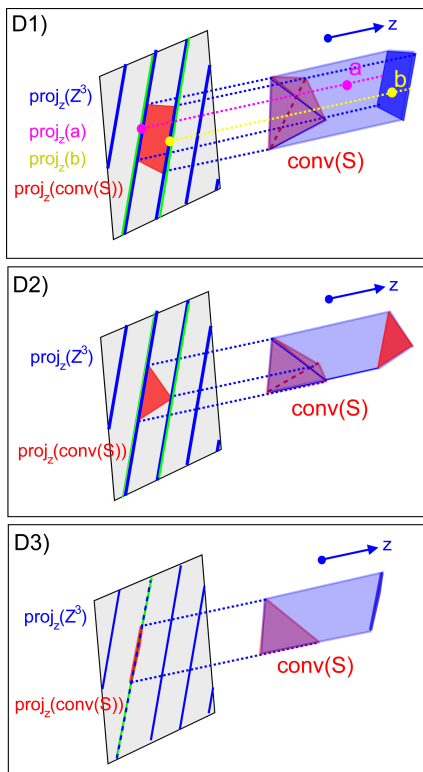


Fig. 22 As for the case B), three different configurations can arise in the case D) of Fig.18. D2) and D3) are excluded because one of the planar sections of S in a direction u with $\text{width}_u(S) \leq 1$ is degenerated. We focus on D1) and prove that the number of lattice jewels is finite.

In all the cases and sub-cases, the existence of infinitely many lattice jewels leads to a contradiction. We conclude that there exists only a finite number of lattice jewels in the case (ii).

We have proved that in all the configurations allowed by the assumptions (i), (ii) or (iii), the hypothesis that there exists infinitely many lattice jewels leads to a contradiction. It proves that the only sets S having a infinite number of lattice jewels are the ones enumerated in Theorem 3. \square

5 Proof of Theorem 1

We prove here the decidability of $[\text{Reco}(d, \mathcal{P}_n, S)]$ stated in Theorem 1. The assumptions are

- $d = 2$, or
- $d = 3$ and the set S has a finite set of lattice jewels (see Theorem 3 for a characterization), or
- in arbitrary dimension, the interior of the convex hull of S contains at least one lattice point.

We decompose the proof in two cases:

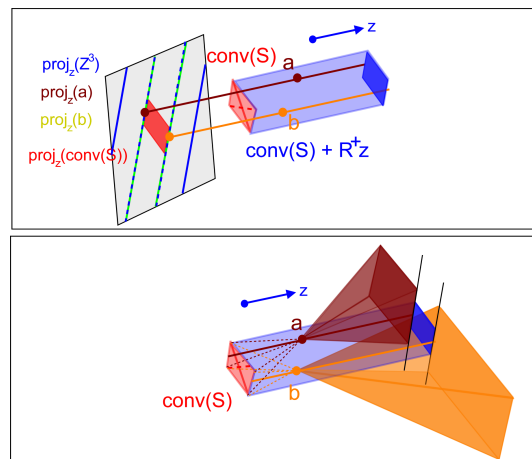


Fig. 23 In the case D1) of Fig.22, we have two integer points a and b in the half-cylinder $(\text{conv}(S) \oplus \mathbb{R}^+z) \setminus \text{conv}(S)$. On the right, the space between their two shadows (brown and orange) defines a strip which does not contain any integer point. It contradicts the existence of a sequence of lattice jewels tending in the direction z .

- If S has a finite number of lattice jewels, the decidability of the instance $[\text{Reco}(d, \mathcal{P}_n, S)]$ is proved in Lemma 5. According to Theorems 2 and 3, it concerns the lattice sets S in dimension $d = 2$ and an affine dimension $\dim(S) = 2$, or the lattice sets S in dimension $d = 3$ which are not in the specific configurations of Theorem 3 or the lattice sets S with an integer point in the interior of their convex hull.
- If the affine dimension of $S \subset \mathbb{Z}^2$ is 0 or 1, the decidability is established in Lemma 6. This specific case is the only one allowed by the assumptions of Theorem 1 but with an infinite number of lattice jewels.

5.1 Sets with a finite number of lattice jewels.

We focus now on the case where the set S admits only a finite number of lattice jewels. According to Lemma 3, we can reduce $[\text{Reco}(d, \mathcal{P}_n, S)]$ to the decidable instance of $[\text{PolySep}(d, \mathcal{P}_n, S, \text{jewels}_S)]$ (decidable since jewels_S is finite). It requires to compute first the set of lattice jewels jewels_S but even if such a strategy is interesting in terms of computational complexity, this computation is not necessary to prove the decidability. We choose another strategy: we can solve a sequence of instances $[\text{PolySep}(d, \mathcal{P}_n, S, T^k)]$ where the set of outliers T^k is a growing subset of integer points outside S .

By solving one of these instances of polyhedral separability $[\text{PolySep}(d, \mathcal{P}_n, S, T^k)]$ where T^k is a set of lattice points outside S , three cases can occur:

- the instance $[\text{PolySep}(d, \mathcal{P}_n, S, T^k)]$ has no solution. This result can be extended to $[\text{PolySep}(d, \mathcal{P}_n, S, \mathbb{Z}^d \setminus S)]$. It follows that $[\text{Reco}(d, \mathcal{P}_n, S)]$ has no solution.
- the instance $[\text{PolySep}(d, \mathcal{P}_n, S, T^k)]$ has a solution P but P might contain some lattice points outside S and T^k . In this case, the solution P is not a valid solution of $[\text{Reco}(d, \mathcal{P}_n, S)]$. We can not conclude and we have to proceed to further computations: we increase T^k .
- the instance $[\text{PolySep}(d, \mathcal{P}_n, S, T^k)]$ has a solution P and this solution does not contain any other lattice point than the ones of S . Then P is a valid solution of $[\text{Reco}(d, \mathcal{P}_n, S)]$.

We notice now that the second case, where we can not conclude, can not arise if the set of outliers T^k contains all the lattice jewels. It is a consequence of the following lemma:

Lemma 4 *Let P be a convex set containing S . If P contains at least one lattice point $x \in \mathbb{Z}^d \setminus S$ outside S , then it contains a lattice jewel of S : $P \cap \text{jewel}_S \neq \emptyset$.*

Proof If x is not itself a lattice jewel, then there exists a lattice jewel $x' \in \text{jewel}_S$ with $x' \triangleleft_S x$. It means that x' is in the convex hull of $S \cup \{x\}$ itself included in the convex set P . It proves that the lattice jewel x' is in P . \square

Lemma 4 states that the second case, where we can not conclude, can not arise if the lattice jewels of S are all in the set of outliers T^k : $\text{jewel}_S \subset T^k$. This remark leads to a simple strategy. We can take for instance

$$T_k = \{x \in \mathbb{Z}^d \mid \|x\| \leq k, x \notin \text{conv}(S)\}. \quad (4)$$

As the set jewel_S is finite, there exists a finite integer K with $\text{jewel}_S \subset T^K$. The algorithm 1 is to solve the sequence of instances of polyhedral separability $[\text{PolySep}(d, \mathcal{P}_n, S, T^k)]$ until finding a conclusion.

Algorithm 1 Algorithm for $[\text{Reco}(d, \mathcal{P}_n, S)]$ with a finite jewels

```

1:  $k \leftarrow 1$ 
2:  $T^k \leftarrow \{x \in \mathbb{Z}^d \mid \|x\| \leq k, x \notin S\}$ 
3: loop
4:    $P \leftarrow \text{PolySep}(d, \mathcal{P}_n, S, T^k)$ 
5:   if ( $P = \text{null}$ ) then
6:     return null
7:   if ( $P \cap \mathbb{Z}^d = S$ ) then
8:     return P
9:    $k \leftarrow k + 1$ 
10:   $T^k \leftarrow \{x \in \mathbb{Z}^d \mid \|x\| \leq k, x \notin S\}$ 

```

The only case where the answer of $[\text{PolySep}(d, \mathcal{P}_n, S, T^k)]$ does not allow to conclude on $[\text{Reco}(d, \mathcal{P}_n, S)]$

is the one where its solution P is not valid since P contains lattice points outside S . For $k = K$, as all the lattice jewels are outliers ($\text{jewel}_S \subset T^k$) and due to Lemma 4, this case can not arise. Then, the algorithm 1 terminates at the worst for $k = K$. It proves the following lemma:

Lemma 5 *If the set of lattice jewels jewel_S of S is finite, then the algorithm 1 solves the instance $[\text{Reco}(d, \mathcal{P}_n, S)]$ in a finite time.*

5.2 Sets $S \subset \mathbb{Z}^2$ of dimension 0 or 1

In dimension $d = 2$, according to Theorem 3, the only case where we have an infinite set of lattice jewels is the one of the degenerated sets: S contains only one point (its dimension is 0) or it is a set of aligned points (its dimension is 1). We can easily notice that, in these two cases, the set of lattice jewels of S is infinite but it is of little importance since the answer of $[\text{Reco}(d, \mathcal{P}_n, S)]$ is straightforward.

Lemma 6 *If the set $S \subset \mathbb{Z}^2$ contains only one point, then the instance $[\text{Reco}(2, \mathcal{P}_n, S)]$ admits solutions for $n \geq 2$.*

If the set $S \subset \mathbb{Z}^2$ is of dimension 1 and convex, then the instance of problem $[\text{Reco}(2, \mathcal{P}_n, S)]$ admits solutions if and only if $n \geq 3$.

If the set S is not convex, then the instance $[\text{Reco}(2, \mathcal{P}_n, S)]$ has no solution.

Proof If the set $S \subset \mathbb{Z}^2$ contains only one point denoted s , a solution of the instance $[\text{Reco}(2, \mathcal{P}_n, S)]$ is given by two half-planes $n.x \leq n.s$ and $n.x \geq n.s$ with an irrational normal vector n (for instance $(1, \sqrt{2})$).

If the set $S \subset \mathbb{Z}^2$ is of dimension 1, there is of course no solution if it is not convex (but this trivial case has to be mentioned). If the set $S \subset \mathbb{Z}^2$ is of dimension 1 and convex, there is no solution with only $n = 2$ half-planes, because they should be of parallel bounding lines and there is either 0 lattice point, or 1 lattice point or infinitely many lattice points in strips between parallel lines. With $n = 3$, many triangles containing the set of aligned points provide solutions (Fig.24).

6 Conclusion

Is the problem $[\text{Reco}(d, \mathcal{P}_n, S)]$ decidable for any finite lattice set S ? Some algorithms for solving this exact problem would be of interest in the framework of pattern recognition even if it is much easier to tackle

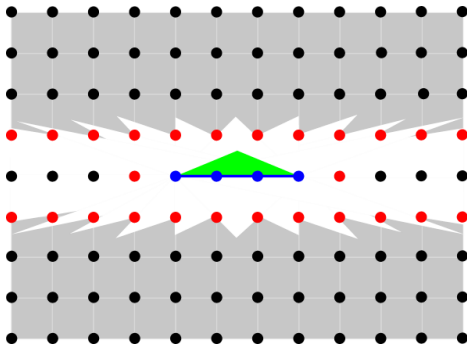


Fig. 24 In the case of aligned points of \mathbb{Z}^2 , a triangle solution namely a solution of $[\text{Reco}(2, \mathcal{P}_3, S)]$ can be easily built.

an approximate version where the set S is only separated from its neighbors. More than its practical interest, the resolution of the exact theoretical issue can attract attention since much of the problems of discrete and computational geometry can be easily proved to be decidable. It is not true with $[\text{Reco}(d, \mathcal{P}_n, S)]$ due to the infinite number of constraints generated by the points of $\mathbb{Z}^2 \setminus S$. Conjecture 1 claims that the problem is decidable. We have proved it under some assumptions given in Theorem 1. This result provides a better understanding of convex primitives recognition problems in the framework of digital geometry. The proof is based on the introduction of a partial order “is in the shadow of” and of the minimal elements of $\mathbb{Z}^d \setminus S$: the lattice jewels of S . These particular lattice points have interesting properties. The most useful in the framework of polyhedral separability is the possibility to reduce the infinite set of outliers $\mathbb{Z}^2 \setminus S$ to the lattice jewels of S . It makes $[\text{Reco}(d, \mathcal{P}_n, S)]$ decidable when the number of lattice jewels is finite. The conditions stated in Theorem 1 leave a lot of unsolved questions:

- The main conjecture itself: is $[\text{Reco}(d, \mathcal{P}_n, S)]$ decidable for any finite lattice set S ?

Or some more restricted questions:

- is $[\text{Reco}(3, \mathcal{P}_n, S)]$ decidable in the dimension $d = 3$? This question can be focused on the lattice sets with a lattice width equal to 1 and with a degenerated non empty planar section as in Fig.14 or 4 (otherwise the finite number of lattice jewels provides the decidability).

- What happens in higher dimensions ? Regarding Theorem 2, is it possible to provide a complete characterization of the sets S with a finite number of lattice jewels in arbitrary dimension?

- Is it always possible to find rational solutions if there exists a real one (it is clearly true for finite lattice sets S having a finite number of lattice jewels), as claimed by Conjecture 2?

- For sets $S \subset \mathbb{Z}^d$ which are not full dimensional, is it possible to expand a solution in their support sub-

lattice?

All these questions show that the decidability of $[\text{Reco}(d, \mathcal{P}_n, S)]$ is the starting point of many open perspectives. More than a conclusion to the problem, the complexity of the numerous cases investigated in the proofs and the remaining open problems are invitations to search further, with more generic tools... or more imagination.

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