Tight bounds in the quadtree complexity theorem and the maximal number of pixels crossed by a curve of given length

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Tight Bounds in the Quadtree Complexity Theorem and the Maximal Number of Pixels Crossed by a Curve of Given Length

Yan Gerard[a,*], Antoine Vacavant[a], Jean-Marie Favreau[a]

[a] *ISIT - UMR 6284 Université d'Auvergne / CNRS - Batiment 3C - Faculé de médecine - 28, place Henri-Dunant - 63001 Clermont-Ferrand, France*

Abstract

The main purpose of this work is to determine the exact maximum number of pixels (a bi-dimensional sequence of unit squares tiling a plane) that a rectifiable curve of given length l can cross. In other words, given $l \in \mathbb{R}$, we provide the value $N(l)$ of the maximal cardinality of the digital cover of a rectifiable curve of length l. The optimal curves are polygonal curves with integer vertices, 0, 1 or 2 vertical or horizontal steps and an arbitrary number of diagonal steps. We also report the properties of the staircase function $N(l)$, which is affinely periodic in the sense that $N(l+\sqrt{2}) = N(l)+3$ and a bound $N(l) \leq 4 + \frac{3}{\sqrt{2}}l$.
Our second aim is to look at the restricted class of closed curves and offer some conjectures on the maximum number $N_{\text{closed}}(l)$ of pixels that a closed curve of length l can cross.
This work finds its application in the quadtree complexity theorem. This well-known result bounds the number of quads with a shape of perimeter p by $16q - 11 + 16p$. However, this linear bound is not tight. From our new upper bound $N_{\text{closed}}(l) \leq N(l) \leq 4 + \frac{3}{\sqrt{2}}l$ we derive a new improved multiresolution complexity theorem: Number(quads) $\leq 16q - 11 + 6\sqrt{2}p$. Lastly, we show that this new bound is tight up to a maximal error of $16(q-1)$.

Keywords:

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*Corresponding author

Email addresses: `yan.gerard@udamail.fr` (Yan Gerard), `antoine.vacavant@udamail.fr` (Antoine Vacavant), `J-Marie.Favreau@udamail.fr` (Jean-Marie Favreau)

quadtree complexity theorem, digital cover cardinality, rectifiable curve, length

1. Introduction

The multiresolution representation of images with quadtrees was introduced in the mid-seventies (1; 2; 3) and began to find wider use in the eighties. The main reason for this success is of course the gain in memory size over bitmaps. This gain is not only an experimental assessment: it has been proved (4) that on increasing the resolution of the image, the size of the quadtree representation is linear, whereas a bitmap is quadratic in the number of minimal cells (respectively quads of minimal size or pixels) along one side of the square image. This means that on increasing the resolution, the quadtrees become more and more efficient compared with raster graphics. This important fact is one of the main reasons for using quadtrees. It was first published in 1978 in (4), where it is called the *tree complexity bound theorem* (see (5)). Currently the linear bound on the number of quads is most often called *quadtree* or *multiresolution complexity theorem* (6). More precisely, the theorem states that the total number of quads (the intermediary nodes and the leaves) needed to represent a polygonal shape is linear in the length of the perimeter of the shape

$$\text{Number(quads)} \leq 16q - 11 + 16p \tag{1}$$

where p is the perimeter of the shape (in comparison, the size of the minimal quads or pixels is 1) and q is the depth of the quadtree (Fig. 1).

This result prompted further research in the eighties. In (7), Dyer sought the average and worst case number of quads needed to represent a black square aligned with the axis. In (8), a bound is given on the number of black quads according to the number of pixels of the border of the black shape, with a proof that the bound is sharp to within a factor of two. In the same year, (9) gave other bounds, still according to the number of pixels of the border. Some of these results were extended to higher dimensions (10), where bounds are given on PR-quadtrees to represent discrete sets of points (11).

We note an important difference between the initial result of Hunter and Steiglitz (Equation 1) and subsequent ones: Equation 1 is a bound according to the perimeter of a real polygon, whereas the others do not consider a real shape, but only its digital cover through the number of pixels of its boundary.

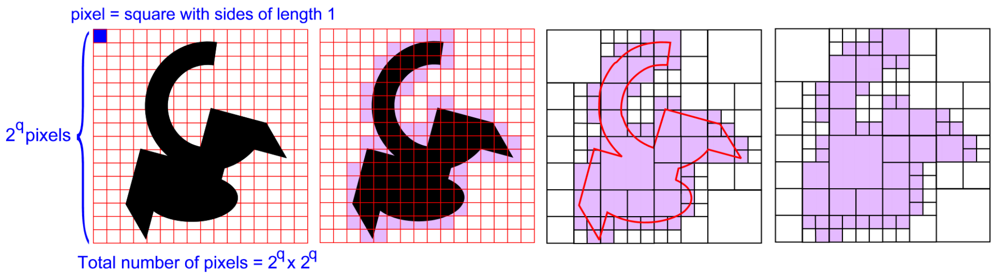

Figure 1: On the left, a black shape is rasterized in a set of colored and white pixels: pixels are colored if and only if their intersection with the shape is non-empty. On the right, instead of describing the shape as a set of colored and white pixels, we use a quadtree structure. The quads can be bigger than single pixels if their color is uniform. It follows that the number of quads needed to represent the shape is much smaller than the total number of pixels. The quadtree complexity theorem states that fewer than $16q - 11 + 16p$ quads are necessary to represent a shape of perimeter p.

The proof of Equation 1 uses the assumption on the continuous perimeter of the shape by stating that (still with pixels of size 1), a polygonal line of length l cannot cross more than $4\lceil l \rceil$ pixels :

$$N(l) \leq 4\lceil l \rceil \tag{2}$$

where $N(l)$ denotes the maximal number of pixels that a rectifiable curve of length l can cross. However, this bound is not tight. The main purpose of the work described here is to compute the exact maximum $N(l)$ and provide the corresponding optimal curves (Fig. 2). It provides for instance a tight inequality:

$$N(l) \leq 4 + 3\frac{l}{\sqrt{2}}. \tag{3}$$

The slope of l in Equation 3 is not 4 as in Equation 2 but $\frac{3}{\sqrt{2}}$. It also improves the main Quadtree Complexity Theorem, namely Equation 1 from the same factor $(\frac{4\sqrt{2}}{3})$.

This paper has six sections. After this introduction, Section 2 is devoted to the definitions and the main result. Its proof is given in Section 3. It is broken down into three steps. In Section 4, we provide an extension of the result to the more restrictive class of closed rectifiable curves. In Section 5, we return to the multiresolution complexity theorem and show that our new bound is tight.

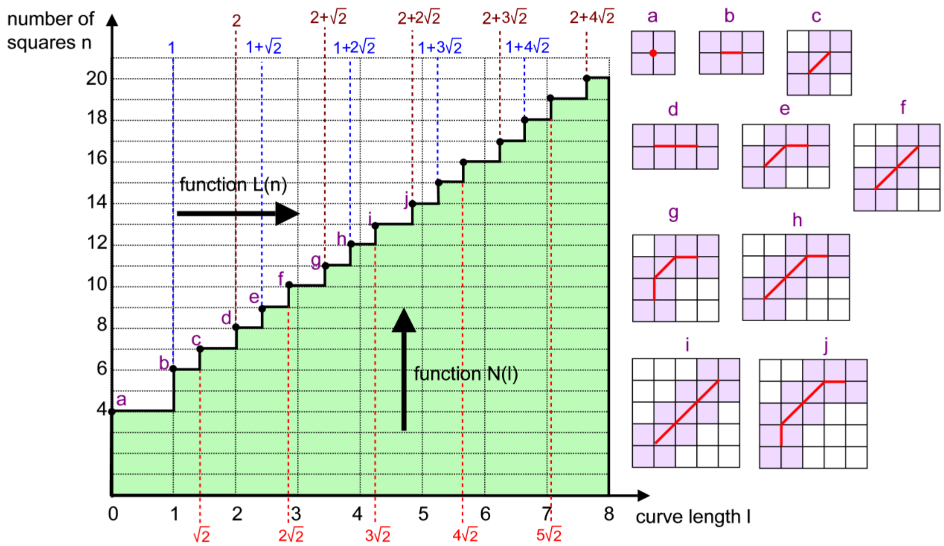

Figure 2: $N(l)$ is the maximal number of pixels of size 1 that a rectifiable curve of length l can cross. $L(n)$ is the minimal length needed to cross n pixels. Some optimal curves are drawn on the right of the graph. They have an arbitrary number of diagonal steps and 0, 1 or 2 horizontal or vertical steps.

2. Maximal number of pixels crossed by a curve of given length

2.1. Definitions and notations

We call *pixels* the closed squares $[i, i+1] \times [j, j+1]$ with $i, j \in \mathbb{Z}$. Let $S \subset \mathbb{R}^2$ be a subset of the plane. The *cover* of S is the set of squares crossing S. We denote it cover(S) so that we have

$$\text{cover}(S) = \{(i, j) \in \mathbb{Z}^2 | \exists (x, y) \in S, x \in [i, i+1] \text{ and } y \in [j, j+1]\}.$$

In the framework of digital geometry, the cover of S is usually called *supercover*. We have removed the prefix *super* for simplicity. The cardinality of the cover of a real set S is simply denoted $|\text{cover}(S)|$. In other words, we will say that *the cardinality of the cover of S has n squares* or alternatively S *cuts, crosses* or *covers* n squares.

The four points (i, j), $(i+1, j)$, $(i, j+1)$ and $(i+1, j+1)$ are called the corners of the square $[i, i+1] \times [j, j+1]$ and they all belong to it. Some other ways to digitize continuous shapes of $\mathbb{R}^2$ in digital subsets of $\mathbb{Z}^2$ (e.g. with squares $[i, i+1[\times[j, j+1[$) could be introduced here. They might lead to different results. We consider the notion of cover with closed squares because it is the one used in the proofs of the main results, but results with other digitization schemes are very close, as noted in (12).

The purpose of this work was to provide a maximal cardinality for the cover of a curve of given length. We consider rectifiable curves in order to have a general definition of length. Let us denote Γ a curve defined through a continuous function $\Gamma : [0,1] \to \mathbb{R}^2$. Γ is *rectifiable* if for any subdivision of the interval $[0,1]$ in $\cup_{k=0}^{m}[t_k, t_{k+1}]$ with $t_0 = 0$, $t_{m+1} = 1$ and $t_k \leq t_{k+1}$, the length $l_{(t_k)_{0\leq k\leq m}}(\Gamma) = \sum_{k=1}^{m-1} d(\Gamma(t_k), \Gamma(t_{k+1}))$ of the polygonal line joining the points $\Gamma(t_k)$ is bounded. The upper bound of the lengths of polygonal lines inscribed in Γ is, by definition, the length of Γ and is denoted $l(\Gamma)$. We write d for the Euclidean distance d_2 (other distances are mentioned in (12)). The main property of the length we use in what follows is a direct consequence of the definition: the length of polygonal lines with vertices on Γ in the same order as on the curve is less than or equal to the length of Γ. We also use the property that the curve can be subdivided into finitely many consecutive arcs $\Gamma[t_k, t_{k+1}]$, each one of length < 1 (this is easily computed in the particular case where the function Γ is Lipschitz).

We are interested in the maximal cardinality of the cover of a curve of given length or, conversely, the minimal length needed to cover a given number of squares. These functions are defined as follows:

Definition 1. *Given a length $l \in \mathbb{R}$, we denote $N(l)$ the maximal cardinality of the covers of rectifiable curves Γ of length l (or less $\leq l$).*
Given a positive integer $n \in \mathbb{N}$, we denote $L(n)$ the lower bound of the lengths of the rectifiable curves Γ whose cover contains n squares (or more).

The two functions $L(n)$ and $N(l)$ are closely related. The function L is defined over a discrete set. It is a sequence of values $L(n)$ with $n \in \mathbb{N}$. If we consider a length l with $L(n) < l < L(n+1)$, it is sufficient to cross n squares, but not sufficient to cross $n+1$. We then have exactly $N(l) = n$.

If we consider a value l equal to $L(n)$, according to Definition 1 we could have both $N(l) = n$ if the lower bound $L(n)$ is a minimum, or $N(l) = n - 1$ if the lower bound $L(n)$ is not reached. It follows that, except for the lengths $l = L(n)$ where there is an ambiguity, the function $N(l)$ is given directly by the sequence $L(n)$.

2.2. Values of $L(n)$ from $n = 1$ to 6 and consequences

We can easily compute the first values of the sequence $L(n)$.

- The cover of a curve of length 0 contains 1 square if the point of the curve is in the interior of the square. It contains 2 squares if it is in the interior of an edge. It contains 4 squares if is an integer point (i, j) of the lattice $\mathbb{Z}^2$. It is the maximum. It follows that $N(0) = 4$ and $L(1) = L(2) = L(3) = L(4) = 0$.

- Any subset of 5 squares or more contains two squares with one coordinate that differs from 2. The length needed to run from one of these two squares to the other is then at least 1. It follows that $L(5) \geq 1$ and for any $n \geq 5$, $L(n) \geq 1$.

- A curve of length $l = 1$ can cross 6 squares: let us consider a horizontal edge from (i, j) to $(i + 1, j)$ or a vertical one, from (i, j) to $(i, j + 1)$: its cover contains 6 squares. It follows that $N(1) \geq 6$. We then have $L(5) = 1$ and $L(6) = 1$.

2.3. Counting the pixels with multiplicities

We now introduce a different way to count the number of pixels crossed by a curve. In the previous definition with $|\mathrm{cover}(\Gamma)|$, if a curve enters a pixel, leaves it and then re-enters it, the pixel counts only for one. This means that the cardinality of the cover is not the sum of the cardinalities of some segments of the curve. It cannot be computed locally and then summed to obtain the overall cardinality. This process leads to another notion, mostly useful for the proof, that we can introduce now. The idea is that if the curve enters a pixel x times, then we count it with a multiplicity equal to x.

Definition 2. *The* multiplicity cardinality *of a continuous curve* $\Gamma[0, 1]$*, denoted* $\mathrm{mult}(\Gamma)$*, is the maximum of*

$$\sum_{0 \leq k \leq m-1} |\mathrm{cover}(\Gamma(t_k, t_{k+1}))| - \sum_{1 \leq k \leq m-1} |\mathrm{cover}(\{\Gamma(t_k)\}|$$

over the set of all finite subdivisions $(t_k)_{0 \leq k \leq m}$ *of the interval* $[0, 1]$*.*

In other words, we independently count the cardinality of all the pieces of curves of Γ and sum them. We remove the cover of the junction points $\Gamma(t_k)$ because we count them twice in the sum $\sum_{0 \leq k \leq m-1} |\mathrm{cover}(\Gamma(t_k, t_{k+1})|$. In the case of a segment, due to the convexity of the figure and that of the pixels, the multiplicity cardinality is equal to the cardinality of its cover. It leads to the next formula:

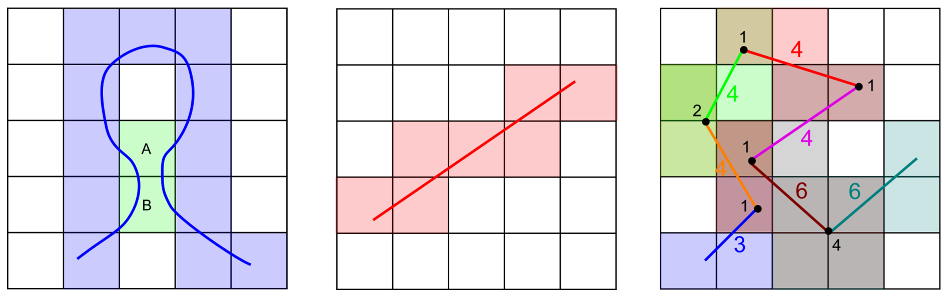

Figure 3: On the left, a curve with a cover of cardinality 14. Its *multiplicity cardinality* is 16 : the pixels A and B count twice because the curve runs out of them and then turns back and enters them again. In the middle, the two notions coincide for segments because due to convexity, a segment cannot leave and re-enter a pixel. On the right, we illustrate Lemma 1, which allows to easily compute the multiplicity cardinality for a polygonal curve by adding the cardinals of the covers of the segments and subtract those of the intermediary vertices. The multiplicity cardinality of the right polygonal curve is $3+4+4+4+4+6+6-1-2-1-1-1-4 = 31-10 = 21$.

Lemma 1. *The multiplicity cardinality* $\mathrm{mult}(\Gamma)$ *of a polygonal curve* Γ *is the sum of the cardinality of the covers of its segments minus the sum of the cardinals of its intermediary vertices.*

Remark that the symmetries according to lines $x = i$, $y = j$, diagonals $x - y = i - j$ or $x + y = i + j$ and rotations obtained by their composition preserve the multiplicity cardinality of a segment. According to lemma 1, they also preserve the multiplicity cardinality of any polygonal curve, but more complex transformations can be applied: we can apply one of these symmetries only to the curve segment after a point (invariant by the transformation) as in Fig. 4. Lemma 1 proves that it does not modify the multiplicity cardinality of the overall curve. This property, which does not hold for the cardinality of the cover, is the main reason for introducing the notion of multiplicity cardinality. It enables us to unfold polygonal curves as done in subsection 3.2.

2.4. Main result

The main purpose of this work is to provide the exact values of functions $N(l)$ of the maximum number of pixels covered by a rectifiable curve of length l and $L(n)$ of the minimal length needed to cover n pixels (the graph is drawn in Fig. 2 with the corresponding optimal curves):

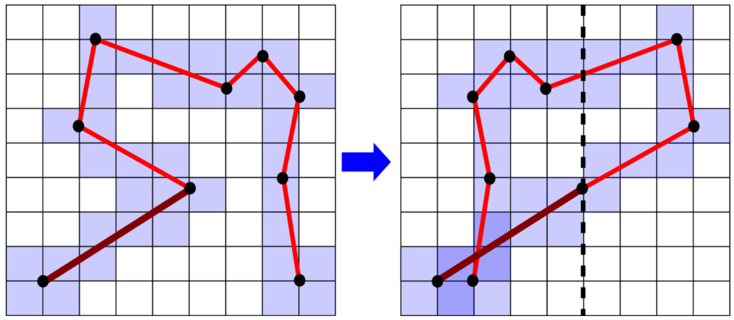

Figure 4: If we apply a symmetry on the segment of a curve before or after a point p on an edge (symmetry according to the support line of this edge), it does not change the multiplicity cardinality of the curve while the cardinality of the cover may change. We illustrate this here with a polygonal curve, as we will use it in the proofs but It is a general property.

Theorem 1. *The values of the function $N(l)$, which give the maximal number of squares of the cover of a rectifiable curve of length l, are*

- $N(l) = 3\lfloor \frac{l}{\sqrt{2}} \rfloor + 4$ *and optimal curves have* $\lfloor \frac{l}{\sqrt{2}} \rfloor$ *diagonal steps if* $l \bmod \sqrt{2} < 2 \bmod \sqrt{2}$.
- $N(l) = 3\lfloor \frac{l}{\sqrt{2}} \rfloor + 5$ *and optimal curves have* $\lfloor \frac{l}{\sqrt{2}} \rfloor - 1$ *diagonal steps and* 2 *horizontal or vertical steps if* $2 \bmod \sqrt{2} \leq l \bmod \sqrt{2} < 1$ *and* $l > 1$. *We have* $N(l) = 4$ *if* $2 - \sqrt{2} \leq l < 1$ *(the first step of the staircase is broken).*
- $N(l) = 3\lfloor \frac{l}{\sqrt{2}} \rfloor + 6$ *and optimal curves have* $\lfloor \frac{l}{\sqrt{2}} \rfloor$ *diagonal steps and* 1 *horizontal or vertical step if* $1 \leq l \bmod \sqrt{2}$.

Conversely, the function $L(n)$ is given by:

- $L(1) = L(2) = L(3) = L(4) = 0,\ L(5) = 1.$
- $L(n) = 1 + (\frac{n}{3} - 2)\sqrt{2}$ *if* $n \bmod 3 = 0$ *(and* $n \geq 6$*).*
- $L(n) = (\lfloor \frac{n}{3} \rfloor - 1)\sqrt{2}$ *if* $n \bmod 3 = 1$ *(and* $n \geq 6$*).*
- $L(n) = 2 + (\lfloor \frac{n}{3} \rfloor - 2)\sqrt{2}$ *if* $n \bmod 3 = 2$ *(and* $n \geq 6$*),.*

As the function $N(l)$ has integer values, it is a staircase function. Theorem 1 provides a property of periodicity: For all $l > 1$,

$$N(l + \sqrt{2}) = N(l) + 3. \tag{4}$$

It explains the general behavior of the function: the graph of the function $N(l)$, namely the points of coordinates $(l, N(l))$, are in a strip of slope $\frac{3}{\sqrt{2}}$. It provides the corollary that we have already mentioned partially as Equation 3 :

Corollary 1. *For any length $l \geq 1$, we have*

$$\frac{3}{\sqrt{2}}l + 7 - 3\sqrt{2} < N(l) \leq \frac{3}{\sqrt{2}}l + 4. \tag{5}$$

If we set $l \geq 0$, then the left inequality becomes larger, and we have

$$\frac{3}{\sqrt{2}}l + 4 - \frac{3\sqrt{2}}{2} < N(l) \leq \frac{3}{\sqrt{2}}l + 4.$$

Proof. Due to periodicity (see Equation 4), we have to check the double inequality 5 over the period $[1, 1 + \sqrt{2}]$. In this interval, the graph of the staircase function $N(l)$ has 4 upper vertices: $(l, n) = (1, 6)$, $(\sqrt{2}, 7)$, $(2, 8)$ and $(1 + \sqrt{2}, 9)$ (points b, c, d and e in Fig. 2). Then we can easily check that $n \leq \frac{3}{\sqrt{2}}l + 4$ for these four vertices (we have equality for $c(\sqrt{2}, 7)$). The right inequality follows.
For the left inequality, we consider the lower vertices over the same interval: $(l, n) = (\sqrt{2}, 6)$, $(2, 7)$ and $(1 + \sqrt{2}, 8)$. They both satisfy $\frac{3}{\sqrt{2}}l + 7 - 3\sqrt{2} \leq n$. We have equality for $(l, n) = (2, 7)$. We can then note that the values of $N(l)$ are strictly greater than the ordinates of these three vertices, providing a strict inequality $\frac{3}{\sqrt{2}}l + 7 - 3\sqrt{2} < N(l)$.

For the extended version with $l \geq 0$ instead of $l \geq 1$, we take into account the vertex of coordinates $(1, 4)$ for the lower bound. $\square$

3. Proofs

The proof of the main theorem is rather technical and has to be broken down into several steps. The main difficulty is that the space of rectifiable curves is very large and contains a lot of pathological cases. The first task is to show that we do not need to consider the whole space of rectifiable curves for our purpose. In a first lemma, we show that we can restrict ourselves to the set of polygonal curves. The second step is to go from polygonal curves to polygonal curves with integer vertices. This point is the one that requires the most work, since we have to go beyond the notion of multiplicity cardinality and use an argument of compacity (a more constructive process is suggested in the proof but is not given in detail). Now that we can consider only polygonal curves with integer vertices, the task is almost done, since we are in a discrete space. The final task is that of optimization over the set of integer parameters, which would otherwise have been hard, but which is now quite easy.

3.1. From rectifiable curves to polygonal lines

The first step is to show that we can restrict ourselves to polygonal curves, as can be expressed in the next lemma.

Lemma 2. *Given any length $l \in \mathbb{R}$, $N(l)$ is equal to the maximal cardinality of the covers of polygonal curves of length l (or less $\leq l$).*
Given a positive integer $n \in \mathbb{N}$, $L(n)$ is equal to the lower bound of the lengths of the polygonal curves that cover contains n squares (or more).

Proof. Let us denote temporarily $N_{\text{polylines}}(l)$ the maximal cardinality of the covers of polygonal curves of length l. We want to prove that $N(l) = N_{\text{polylines}}(l)$. If we compare the definition of $N(l)$, the value $N_{\text{polylines}}(l)$ is defined by considering a reduced space of polygonal curves. Remark that $N(l)$ is the maximum for all rectifiable curves so that $N_{\text{polylines}}(l) \leq N(l)$.

Let us now assume that we have a rectifiable curve Γ of length l with a cover of cardinality n. We are going to build a polygonal line P with a shorter or equal length and with the same or a larger cover, and thus a cardinality greater or equal than n. Let us consider a subdivision of the curve Γ in m consecutive arcs $\Gamma[t_k, t_{k+1}]$, each one of length < 1. We have already noted that any subset of 5 squares contains two squares with a difference of coordinates equal to 2. We again use this property of 5 squares. Due to its length strictly smaller than 1, the cover of $\Gamma[t_k, t_{k+1}]$ contains 1 square or 2, 3 or 4 squares sharing a corner. The main idea of the proof is that we can build a polygonal line P_k starting from $\Gamma(t_k)$ "crossing at least the same number of squares as $\Gamma[t_k, t_{k+1}]$" and ending at $\Gamma(t_{k+1})$ with a length shorter than or equal to $\Gamma[t_k, t_{k+1}]$. In other words, we are going to build a polygonal shortcut of the segment of curve $\Gamma[t_k, t_{k+1}]$ with at least the same cover. There are here some easy technical details that can be readily understood from Fig. 5.

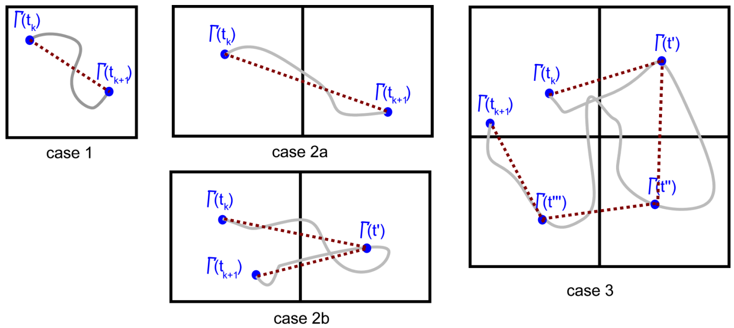

Figure 5: It is easy to show how small segments of curves $\Gamma[t_k, t_{k+1}]$ of length < 1 can be broken down into polygonal lines P_k crossing at least the same squares and with a shorter or equal length.

1. The cover of $\Gamma[t_k, t_{k+1}]$ contains only 1 square. In this case, the segment $[\Gamma(t_k), \Gamma(t_{k+1})]$ crosses the same square. We choose it as P_k and notice that by construction, its length $l(P_k) \leq l(\Gamma[t_k, t_{k+1}])$.
2. The cover of $\Gamma[t_k, t_{k+1}]$ contains only 2 squares. These two squares necessarily share an edge (if they share only a corner, then the curve goes through this corner, which makes 4 squares in the cover). Two sub-cases are possible:
   (a) If the cover of the pair $\{\Gamma(t_k), \Gamma(t_{k+1})\}$ already contains the two squares, then the segment $[\Gamma(t_k), \Gamma(t_{k+1})]$ crosses the same squares. We choose it as polygonal line from $\Gamma(t_k)$ to $\Gamma(t_{k+1})$. We have the same cover as the segment of curve of Γ and a shorter or equal length.
   (b) If the cover of the pair $\{\Gamma(t_k), \Gamma(t_{k+1})\}$ does not contain the two squares, there is a value $t' \in [t_k, t_{k+1}]$ with $\Gamma(t')$ in the other square. We choose the two segments $[\Gamma(t_k), \Gamma(t')] \cup [\Gamma(t'), \Gamma(t_{k+1})]$ as polygonal line from $\Gamma(t_k)$ to $\Gamma(t_k)$. We have the same cover as the segment of curve of Γ and again a shorter or equal length.
3. The cover of $\Gamma[t_k, t_{k+1}]$ contains 3 of 4 squares. As in case 2b, if the cover of the pair $\{\Gamma(t_k), \Gamma(t_{k+1})\}$ does not contain all these squares, we merely have to consider points of this curve segment in these unreached squares. We denote them by t' if there is only one remaining square,

by t' and t'' if there are 2 or by t', t'' and t''' if there are 3. We order them so that $t' < t''$ or $t' < t'' < t'''$. Then the polygonal line P_k is defined by:

- $[\Gamma(t_k), \Gamma(t')] \cup [\Gamma(t'), \Gamma(t_{k+1})]$ if there is one remaining square,
- Or $[\Gamma(t_k), \Gamma(t')] \cup [\Gamma(t'), \Gamma(t'')] \cup [\Gamma(t''), \Gamma(t_{k+1})]$ if there are two remaining squares,
- Or $[\Gamma(t_k), \Gamma(t')] \cup [\Gamma(t'), \Gamma(t'')] \cup [\Gamma(t''), \Gamma(t''')] \cup [\Gamma(t'''), \Gamma(t_{k+1})]$ in the case of three remaining squares.

One of the segments may cross a square that was not in the cover in the curve: it increases the cardinality of the cover. In any case, we have built a polygonal line P_k from $\Gamma(t_k)$ to $\Gamma(t_{k+1})$ with a shorter or equal length and a cover that contains at least the same number of squares.

By joining the polygonal lines P_k, we build a polygonal line with a length shorter than or equal to Γ and that crosses at least n squares. If we apply this construction on a polygonal line of length l with a cover of maximal cardinality $N(l)$, then we have a polygonal line with a shorter or equal length and at least a cover of the same cardinality. It follows that $N_{\text{polylines}}(l) \geq N(l)$ and thus equality.

We can also prove that the lower bound $L(n)$ of the lengths necessary to cross n squares is not larger if we consider only polygonal lines, since each time we have a curve of length l crossing n squares, we also have a polygonal line crossing at least the same number of squares with a shorter or equal length. It follows that $L_{\text{polylines}}(n) \leq L(n)$ and thus equality.

$\square$

We notice that the previous proof of (12) has been simplified by considering some segments of curves of length < 1, which obviates detailing the possible pathologies of rectifiable curves (the sequence of the squares that it crosses can have repeats and thus can be infinite).

3.2. From polygonal lines to polygonal lines with integer vertices

We have reduced the problem of the computation of $L(n)$ and $N(l)$ from the space of rectifiable curves to that of polygonal curves. Now we are going to reduce it further by considering only polygonal lines with integer vertices (namely with vertices on the integer lattice).

Lemma 3. *Given any length $l \in \mathbb{R}$, $N(l)$ is equal to the maximal cardinality of the covers of polygonal curves with integer vertices of length l (or less $\leq l$). Given a positive integer $n \in \mathbb{N}$, $L(n)$ is equal to the lower bound of the lengths of the polygonal lines with integer vertices whose cover contains n squares (or more).*

Instead of proving this lemma directly, we are going to prove a weaker version with the notion of multiplicity cardinality introduced in Definition 2 instead of the cardinality of the cover. This notion of multiplicity cardinality enables us to use symmetries as illustrated in Fig. 4. The weak lemma is as follows:

Lemma 4. *Given any length $l \in \mathbb{R}$, $N(l)$ is equal to the maximal multiplicity cardinality* $\mathrm{mult}(\Gamma)$ *of polygonal curves Γ with integer vertices of length l (or less $\leq l$). Given a positive integer $n \in \mathbb{N}$, $L(n)$ is equal to the lower bound of the lengths of the polygonal lines with integer vertices with a multiplicity cardinality* $\mathrm{mult}(\Gamma) \geq n$.

Proof. Let us consider a polygonal curve $\Gamma = \cup_{k=0}^{m-1}[p_k, p_{k+1}]$ (coordinates of p_k are denoted x_k and y_k) with $m+1$ vertices. We consider that the vertices belong to the edges of the integer grid. At least one of their coordinates is an integer . Otherwise we can remove the vertices inside a pixel with a shortcut and remove the part of the curve inside the first and last pixels, keeping only the points on the boundary of the first and last squares. This reduces the length and does not change the cover. Our goal here is to prove the existence of another polygonal curve Γ' with at least the same multiplicity cardinality, a shorter or equal length, and only integer vertices. We prove its existence by induction but first, we cover the particular case where the cover of the curve is just a horizontal or vertical line of squares (since all the values x_k belong to an interval $]i, i+1[$ or all coordinates y_k are in $]j, j+1[$). We then obtain a shorter polygonal line with a larger cover just by considering respectively the horizontal segment $[(\lceil x_0 \rceil, \lceil y_0 \rceil), (\lfloor x_0 \rfloor, \lceil y_0 \rceil)]$ or the vertical one $[(\lceil x_0 \rceil, \lceil y_0 \rceil), (\lceil x_0 \rceil, \lfloor y_0 \rfloor)]$ (see Fig. 6). This proves the result for this particular case.

We initialize our induction by building a polygonal curve Γ' with a shorter or equal length and at least the same multiplicity cardinality as Γ, but now with an initial integer vertex. If the vertex p_0 is already on the integer lattice, then the result holds. If not, as we assumed that all the vertices were on the edges of the integer grid, we can assume without loss of generality (using a

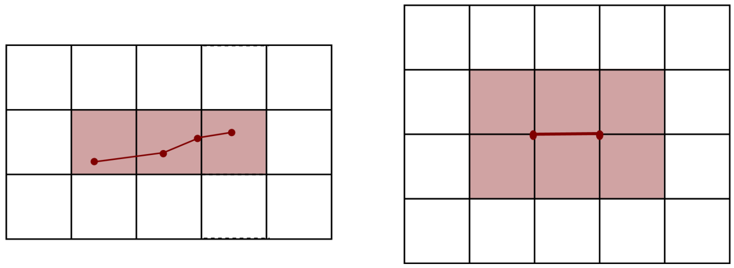

Figure 6: The particular case where the curve covers only a horizontal (or a vertical unit strip) of squares. Horizontal or vertical segments provides a larger cover with a shorter length.

symmetry according to line $x = y$ if necessary) that it is on a vertical edge. It follows that $x_0 = i$ and $j < y_0 < j+1$. Let us now consider the first point p of the polygonal curve with $y = j$ or $y = j+1$ (it exists because otherwise, it is the initial case of unit strip and we have already proved this). We can again assume without loss of generality that $y_0 = j+1$ as drawn for instance in Fig. 7. It follows that the segment from point $(i, j+1) = (x_0, \lceil y_0 \rceil)$ to p is a shortcut with at least the same cover and a shorter or equal length. This lets us build a polygonal curve Γ', with a shorter or equal length, and a larger or equal multiplicity cardinality, now with an integer initial vertex. We can do likewise on the other side of the curve, thus considering a curve Γ' starting and ending on the integer lattice.

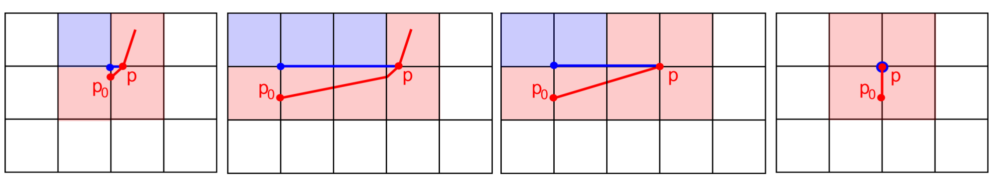

Figure 7: We illustrate how we can build a blue polygonal curve from a red one. The new polygonal line is shorter, has at least the same multiplicity cardinality and now starts from an integer vertex.

For the induction step, we now consider the first non-integer vertex p_k of the curve. The goal is to remove p_k from the curve. In other words, we want a polygonal curve with the same properties (lengths, multiplicity cardinality) but with one fewer non-integer vertex than the previous curve. This is shown in the next and last part of the proof. Before giving the details, we consider the consequence of this construction. By induction on the number of non-integer vertices, this yields a polygonal curve with a length shorter than or equal to Γ, at least the same multiplicity cardinality and only integer vertices, thus proving the result. Consequently, our only task now is to prove that we can remove p_k from the vertices of the polygonal curve without introducing any other new non-integer vertex.

Let us denote p_{k-1} the vertex before p_k in the sequence it is on the integer lattice, but we will not use this property and p_{k+1} its consecutive vertex. We recall here that all vertices have at least one integer coordinate, which means they are on the edges of the pixels. There are three cases to consider as illustrated in Fig. 8.

- a) The curve re-enters the pixel before p_k. In other words, p_{k-1} and p_{k+1} are on the same side toward the support line of the edge.
- b) The curve follows the edge of p_k.
- c) The curve crosses the edge of p_k. In other words, p_{k-1} and p_{k+1} are on the opposite sides toward the support line of the edge.

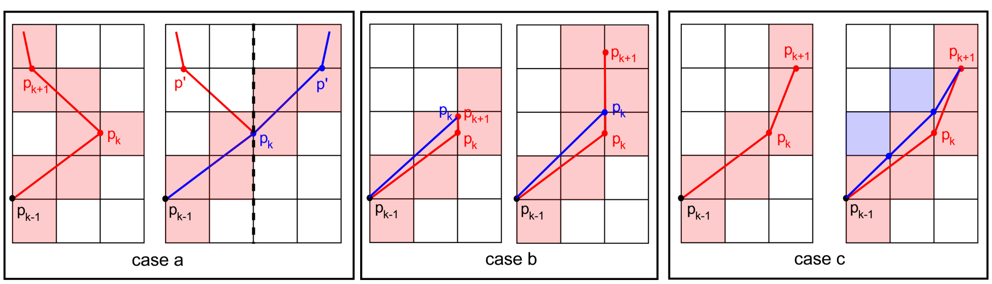

Figure 8: Case a): if the curve does not cross the edge at vertex p_k, then we operate a symmetry to avoid it and fall into case c). Case b): if the curve follows the edge, before or after p_k, we can easily provide a shortcut with the same cover by removing p_k or with an integer vertex. Case c): If the three vertices p_{k-1}, p_k and p_{k+1} are not aligned, we can build a new polygonal curve with the same end points p_{k-1} and p_{k+1}, a shorter or equal length, and at least the same multiplicity cardinality, as in Minimum Length Polygon algorithms (13). These transformations enable us to remove the first non-integer vertices by induction.

In case a), we operate a symmetry of the part of the curve after the point p_k according to the integer line containing p_k, as in Fig. 4. As noted in subsection 2.3, this preserves the multiplicity cardinality. It also preserves the length of the curve. This lets us fall into case c), which we consider shortly. In case b), there are two sub-cases. First, we can remove the vertex p_k and go directly from the initial integer vertex p_{k-1} to the next vertex p_{k+1} if p_k is on the same edge, or we move p_k until the next integer point on the edge, as illustrated in Fig. 8. In case c), if the three points p_{k-1}, p_k and p_{k+1} are aligned, then we remove point p_k from the list of the vertices since there is no angle at that point. It provides our new polygonal curve Γ' with a number of non-integer vertices reduced from 1, as required. The main task we now have to perform is on the sub-case where the three consecutive vertices p_{k-1}, p_k and p_{k+1} are not aligned. There are two options for the proof: First, construct our new curve Γ' with one fewer non-integer vertex than in Γ or second, simply prove its existence.

- The construction itself is easier to understand: we simply reduce the length of the polygonal line from p_{k-1} to p_{k+1} with a shortcut by moving the point p_k along the edge until we obtain a minimal length. This optimization step is similar to the construction of the Minimum Length Polygon (14; 15; 16; 13) whose vertices are lattice points. The difference from this classical framework of digital geometry is that we are not on a simple contour of the shape. This makes no real difference. We move p_k along its edge and reduce the sum of the length $p_{k-1}p_k + p_kp_{k+1}$. The constraint is to preserve the local cover of each segment. It follows that each segment is moving as a chord between the lattice of integer points and by moving p_k on its edge, the chord can come into contact with lattice points. In this case, a finite number of new integer vertices is introduced. The minimal length polygonal line going from p_{k-1} to p_{k+1} and with the same cover (for each segment) as the initial polygonal line $[p_{k-1}p_k] \cup [p_kp_{k+1}]$ has only integer vertices. We do not provide the details of this algorithm. This first reasoning is more an intuitive argument than a real robust proof.

- Second, we provide a more robust proof of the existence of a segment of a polygonal curve Γ' from p_{k-1} to p_{k+1} with a length shorter than or equal to the two segments, with at least the same multiplicity cardinality and with only integer vertices between p_{k-1} and p_{k+1}. It is a result in two steps: first we prove the existence of a polygonal curve with a minimal length in a compact space of polygonal curves, and then we prove that such a curve cannot have non-integer vertices between p_{k-1} and p_{k+1}. For the existence, we consider the sequence of closed edges (the vertices are considered inside each edge) crossed by the two segments $[p_{k-1}p_k] \cup [p_k p_{k+1}]$ (except the ones of p_{k-1} and p_k) as in Fig. 9. The Cartesian product of the set of edges is one-to-one with $[0,1]^q$ (q is the number of edges) and is a compact set for any distance associated with a norm. If we now consider the set of polygonal curves with one vertex on each edge, it is one-to-one with the Cartesian product of the set of edges and thus with $[0,1]^q$. The length of such a polygonal curve is a continuous function over $[0,1]^q$. Since it is positive, it has a minimum. Indeed, there exists at least one polygonal curve with its vertices on the edges and with a minimal length. This minimal polygonal curve has at least the same multiplicity cardinality as the two segments.

The second step is simply to note that the polygonal curves of minimal length from p_{k-1} to p_{k+1} crossing the same sequence of edges cannot have an angle at a non-integer vertex:. It is straightforward because we have previously avoided all the problematic cases where the polygonal curve is just touching an edge and re-entering the same pixel (case a)), as illustrated Fig. 10. It implies that the curves with minimal length going through the same edges as $[p_{k-1}p_k] \cup [p_k p_{k+1}]$ have only intermediary integer vertices.

It follows that in all cases, we can provide a new polygonal line with a shorter or equal length, with at least the same multiplicity cardinality and where the first non-integer vertex has been removed.

By induction over the number of non-integer vertices, we can build a polygonal curve with at least the same properties as the initial polygonal line Γ (shorter or equal length, at least the same multiplicity cardinality), but now with only integer vertices. This proves the weak lemma. $\square$

The "strong" lemma 3 is an easy corollary of Lemma 4 since we can perform symmetries on polygonal lines without changing the multiplicity cardinality. Lemma 4 lets us consider a polygonal line Γ of length l, with only integer vertices and with a multiplicity cardinality equal to $N(l)$. We can easily build a sequence of horizontal and vertical symmetries as in Fig. 4 in order to unfold Γ and make it monotonic. This does not change the multiplicity cardinality. The new polygonal curve Γ' obtained by unfolding Γ has the same length l and the same multiplicity cardinality as Γ. As this

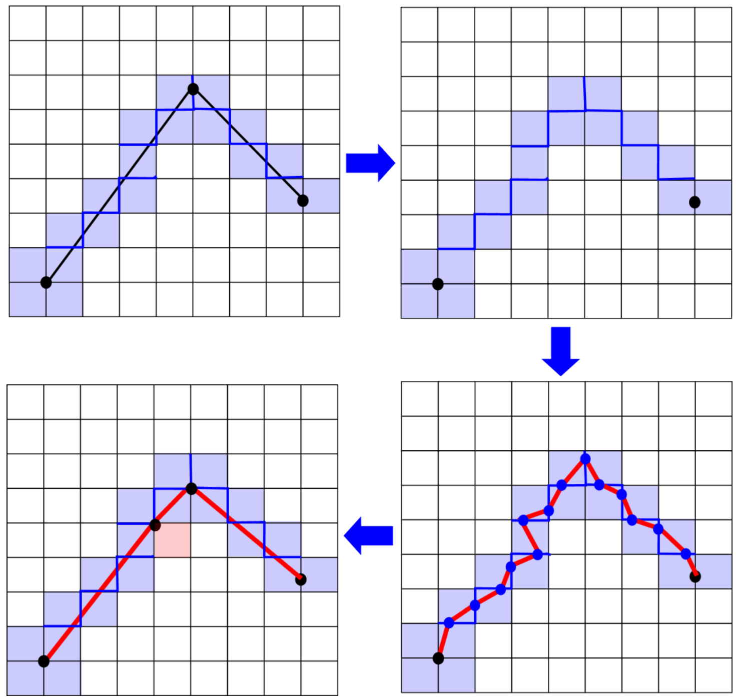

Figure 9: Starting from two segments of a polynomial curve in case c), we build the sequence of consecutive edges crossed by the curve. We then consider the space of polygonal curves with a sequence of vertices on previous edges. This space is compact and the function *length* is continuous and positive. It follows that there exist polygonal curves in this space with a minimal length. The next step is to show that they can only have integer vertices between their endpoints.

unfolded curve is monotonic, it crosses the pixels of its cover only once. The cardinality of the cover of Γ' is then equal to its multiplicity cardinality, itself equal to the multiplicity cardinality of Γ (it is $N(l)$). We have thus built a polygonal line Γ' of length l and with only integer vertices and crossing $N(l)$ pixels. This proves Lemma 3.

3.3. Computation of $N(l)$ and $L(n)$

For our purpose of counting the maximal number of pixels crossed by a curve (without multiplicity), we have reduced the space of the rectifiable curves to the discrete set of (monotonic) polygonal curves with integer vertices. These polygonal curves $\Gamma = \cup_{k=0}^{m-1}[p_k, p_{k+1}]$ with $m+1$ vertices from p_0 to p_m can be described by the sequence of vectors $(a_k, b_k)_{0\leq k\leq m-1} = p_{k+1} - p_k$

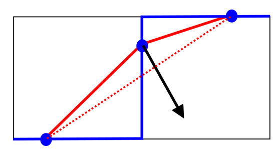

Figure 10: The polygonal curves of minimal length going through a sequence of edges with each time, previous and next edges on both sides of the middle edge, cannot have an angle at a vertex unless it is on the boundary of the edge. Hence the vertices of the curve can only be integer points.

going from p_k to p_{k+1}. Owing to monotonicity, we can assume that a_k and b_k are positive, and we can assume with no loss of generality that the integers a_k and b_k are co-prime (otherwise, we introduce intermediary vertices). It also follows from monotonicity that the cardinality of the cover of Γ is

$$|\mathrm{cover}(\Gamma)| = \sum_{k=0}^{m} |\mathrm{cover}([p_k, p_{k+1}])| - 4m + 4.$$

The cardinality $|\mathrm{cover}([p_k, p_{k+1}])|$ of the cover of each segment depends only on its vector (a_k, b_k). As a_k and b_k are co-prime, there is no integer point on the segment $[p_k, p_{k+1}]$. We can express this value. The starting point covers 4 pixels. A new pixel arises in the cover each time the curve crosses a horizontal or vertical line. It provides $a_k + b_k$ new pixels and a last pixel has not been counted at the end point. It follows that $|\mathrm{cover}([p_k, p_{k+1}])| = a_k + b_k + 5$. We then have

$$|\mathrm{cover}(\Gamma)| = \sum_{k=0}^{m} a_k + b_k + m + 4.$$

Let us now consider the case $0 < b_k < a_k$. The segment covers $5 + a_k + b_k$ pixels. Let us compare it with a segment of vector $(a_k, 0)$. This segment made of a_k unit horizontal segments has a cover of cardinality $4 + 2a_k$. This cardinality $4 + 2a_k$ verifies $4 + 2a_k > 4 + a_k + b_k$. We can rewrite it $4 + 2a_k \geq 5 + a_k + b_k = |\mathrm{cover}([p_k, p_{k+1}])|$ and with a length a_k instead of $\sqrt{a_k^2 + b_k^2}$. It follows that the segments with $0 < b_k < a_k$ are not optimal: there exist shorter segments covering at least the same number of pixels. Likewise, if $0 < a_k < b_k$. There remain only three cases that can provide curves of minimal length with a cover of given cardinality by setting $(a, b) \in \{(1, 0), (0, 1), (1, 1)\}$. It follows that a polygonal curve with integer vertices and a segment other than $(1, 0)$, $(1, 1)$, $(0, 1)$ is not of minimal length among all the curves with a cover having the same cardinality. Thus, the minimal value $L(n)$ is obtained with curves having only horizontal, vertical and diagonal segments.

Let us denote c its number of horizontal or vertical steps of length 1 and d its number of diagonal steps of length $\sqrt{2}$. The length of the curve is $\mathcal{L}(c,d) = c+d\sqrt{2}$. The cardinality of the cover is $\mathcal{N}(c,d) = 4+2c+3d$. Given a minimal cardinality n, which values of c and d provide the shortest curves? This question is a problem of minimization of the linear function $\mathcal{L}(c,d) = c+d\sqrt{2}$ over the subset $(c,d) \in \mathbb{N}^2$ verifying $\mathcal{N}(c,d) = 4+2c+3d \geq n$ (see Fig. 11). Conversely, given a maximal length l, the value $N(l)$ is the maximum of the function $\mathcal{N}(c,d) = 4+2c+3d$ subject to $\mathcal{L}(c,d) = c+d\sqrt{2} \leq l$. We formulate this in the next lemma:

Lemma 5. *We have*

$$N(l) = \max_{(c,d)\in\mathbb{N}^2, c+d\sqrt{2}\leq l} 4+2c+3d$$

and

$$L(n) = \min_{(c,d)\in\mathbb{N}^2, 4+2c+3d\geq n} c+d\sqrt{2}.$$

We can already note that if $c \geq 3$, then the value $\mathcal{N}(c-3,d+2) = \mathcal{N}(c,d)$ with a shorter length $\mathcal{L}(c-3,d+2) = \mathcal{L}(c,d) - 3 + 2\sqrt{2} < \mathcal{L}(c,d)$. This proves directly that given n, the values $(c,d) \in \mathbb{N}^2$ verifying $\mathcal{N}(c,d) \geq n$ with a minimal length $\mathcal{L}(c,d)$ are necessarily obtained with $c < 3$. Optimal curves can then only have $c = 0$, $c = 1$ or $c = 2$ horizontal or vertical steps.

By ordering the values of the set $\mathcal{N}(\{0,1,2\},\mathbb{N})$ and $\mathcal{L}(\{0,1,2\},\mathbb{N})$, we prove the main theorem 1.

4. Extensions to closed curves

The main result of our work concerns the maximal cardinality of a curve of given length, and we provide the optimal curves. In the framework of the quadtree complexity theorem, the curves are closed. It appears that the maximal cardinality of a closed curve of length l is probably less than or equal to the value $N(l)$ obtained by considering the whole set of rectifiable curves.

Definition 3. *Given a length $l \in \mathbb{R}$, $N_{\text{closed}}(l)$ is the maximal cardinality of the covers of rectifiable closed curves Γ of length l (or less $\leq l$).*

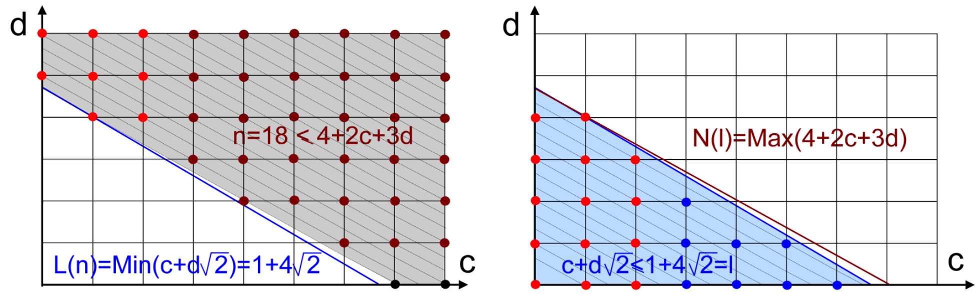

Figure 11: On the left, $L(n)$ is the minimum of $\mathcal{L}(c,d) = c + d\sqrt{2} \geq n$ under constraint $\mathcal{N}(c,d) = 4 + 2c + 3d \geq n$. On the right, $N(l)$ is the maximum of $\mathcal{L}(c,d) = 4 + 2c + 3d$ subject to $\mathcal{L}(c,d) = c + d\sqrt{2} \geq l$. Due to the respective slopes of the extremal lines $-\frac{\sqrt{2}}{2}$ and $-\frac{2}{3}$, the optimal points necessarily have a coordinate c equal to 0, 1 or 2 (the red points).

Given a positive integer $n \in \mathbb{N}$, we denote $L_{\text{closed}}(n)$ the lower bound of the lengths of the rectifiable closed curves Γ whose cover contains n squares (or more).

Let us start with a few remarks. First, we have $N_{\text{closed}}(l) \leq N(l)$ and $L_{\text{closed}}(n) \geq L(n)$. With horizontal curves from $(0,0)$ to $(\frac{l}{2}, 0)$ and back, we have $N_{\text{closed}}(l) \geq 4 + 2\lfloor \frac{l}{2} \rfloor$.
Second, for the multiplicity cardinality, we have $N_{\text{closed}}(0) = 4$ and $L_{\text{closed}}(5) = L_{\text{closed}}(6) = 2$, since any set of 5 pixels has two squares with a coordinate that differs from 2 (a horizontal or vertical step of length 1 provides a cover with 6 pixels). Then $N(2) = 6$. It is easy to provide lower bounds for $N(l)$, since it merely requires considering a curve and computing the cardinality of its cover. It is less trivial to provide upper bounds or more precisely a better upper bound than $N(l)$. Until now, we have not succeeded in doing so. We nevertheless conjectured the values of $N_{\text{closed}}(l)$ for l going from 0 to $2 + 4\sqrt{2}$. The conjectured graph is drawn in Fig. 12 with corresponding closed curves that we conjectured as optimal. In any case, the value of the function $N_{\text{closed}}(l)$ is at least that given in the graph and at most $N(l)$.

Let us now consider rotated squares oriented according to the angle $\frac{\pi}{4}$. Their vertices are $(a, 0)$, $(0, a)$, $(-a, 0)$ and $(0, -a)$ with $a \in \mathbb{N}$. This curve has a length $l = 4a\sqrt{2}$ and crosses $12a$ pixels (see Fig. 13).

It follows that $N_{\text{closed}}(4a\sqrt{2}) \geq 12a$ while $N(4a\sqrt{2}) = 4 + 3 \cdot 4a = 4 + 12a$. The difference between these two values is constant and equal to 4 for any

a. It follows that for any length l, we have

$$N_{\text{closed}}(l) \geq N_{\text{closed}}(4 \left\lfloor \frac{l}{4\sqrt{2}} \right\rfloor \sqrt{2}) \geq 4 + 12 \left\lfloor \frac{l}{4\sqrt{2}} \right\rfloor .$$

As we have $\left\lfloor \frac{l}{4\sqrt{2}} \right\rfloor > \frac{l}{4\sqrt{2}} - 1$, we obtain

$$N_{\text{closed}}(l) > 4 + 12\frac{l}{4\sqrt{2}} - 12 = -8 + \frac{3}{\sqrt{2}}l.$$

Lemma 6. *$N_{\text{closed}}(l)$ is strictly bigger than $-8 + \frac{3}{\sqrt{2}}l$.*

We notice that due to corollary 1, we have $N(l) \leq 4 + \frac{3}{\sqrt{2}}l$. This leads to a difference between $N(l)$ and $N_{\text{closed}}(l)$ which verifies $N(l) - N_{\text{closed}}(l) < 12$ and thus

$$N(l) - N_{\text{closed}}(l) \leq 11.$$

In other words, the difference between the maximal cardinality of the cover for general and closed curves is bounded by a constant. We nevertheless believe that 11 is not a tight bound. By looking at the conjectured graph drawn in Fig. 12, we conjecture the following inequalities, both stricter than $N(l) - 11 \leq N_{\text{closed}}(l) \leq N(l)$:

Conjecture 1. *For any $l \in \mathbb{R}$, $N(l) - 5 \leq N_{\text{closed}}(l)$.*

Conjecture 2. *For $l \geq 3\sqrt{2}$, $N(l) - 5 \leq N_{\text{closed}}(l) \leq N(l) - 4$.*

Conjecture 2 seems true as the difference between the multiplicity cardinality and the cardinality of the cover of a closed curve is 4 for many polygonal curves with an integer initial vertex. With corollary 1, conjecture 2 provides the inequality:

$$\forall l \geq 3\sqrt{2},\ N_{\text{closed}}(l) \leq \frac{3}{\sqrt{2}}l. \tag{6}$$

5. New Bound in the Quadtree Complexity Theorem

5.1. New Bound

As we noted in our introduction, the best-known consequence of the main result is the Quadtree Complexity Theorem (4) expressed by equation 1

$$\text{Number(quads)} \leq 16q - 11 + 16p$$

where p is the perimeter of the polygonal shape, and Number(quads) the total number of quads necessary to represent the shape with a chosen depth q. More precisely, in order to make the comparison easier with the reference paper (4), we adopt its formalism: we consider a quadtree where each quad of level k is of size 2^{q-k+1} and the root (level $k = 1$) is of size 2^q. The most important statement of the quadtree complexity theorem is the linearity of the number of quads in p. We will now improve this linear bound by providing the best possible factor of p and work as much as possible on the constant, according to the results that we stated or conjectured.

Theorem 2. *For a shape having a rectifiable contour of length p, the number of quads necessary to represent it with a depth q verifies*

$$\text{Number(quads)} \leq 16q - 11 + 6\sqrt{2}p.$$

Proof. We use the same computation with the upper bound $N_{\text{closed}}(l) \leq N(l) \leq 4 + \frac{3}{\sqrt{2}}l$ of corollary 1 instead of equation 2: $N(l) \leq 4\lceil l \rceil$ (in (6), they even use the worst bound $N(l) \leq 6\lceil l \rceil$). The proof of the quadtree complexity theorem in (4) gives $\text{Number(quads)} = 1 + 4 + \sum_{k=2}^{q} 4B(k) \leq 1+4+\sum_{k=2}^{q} 4N(\frac{p}{2^{q-k+1}})$ where $B(k)$ is the number of quads of level k crossed by the curve. The inequality $N(l) \leq 4\lceil l \rceil \leq 4 + 4l$ provides the bound $\text{Number(quads)} \leq 5+4\sum_{k=2}^{q} 4+4\frac{p}{2^{q-k+1}}$, but with our better bound, we now have $\text{Number(quads)} \leq 5+4\sum_{k=2}^{q} 4+\frac{3}{\sqrt{2}}\frac{p}{2^{q-k+1}}$. It leads to $\text{Number(quads)} \leq 16q-11+\frac{12}{\sqrt{2}}p\sum_{k=2}^{q}\frac{1}{2^{q-k+1}}$. We obtain at least $\text{Number(quads)} \leq 16q-11+6\sqrt{2}p$. □

Conjecture 3. *If we accept conjecture 2, then we have the following propositions: if $p \geq 3 \cdots 2^{q-\frac{1}{2}}$, then we have*

$$\text{Number(quads)} \leq 5 + 6\sqrt{2}p.$$

If $6\sqrt{2} \leq p < 3 \cdot 2^{q-\frac{1}{2}}$, then

$$\text{Number(quads)} \leq 5 + 6\sqrt{2}p + 16q - 16\left\lfloor \log_2(\frac{p}{3}) + \frac{1}{2} \right\rfloor.$$

Proof. The proof is exactly the same as in Theorem 2, with only one difference: we use the equation $N_{\text{closed}}(l) \leq \frac{3}{\sqrt{2}}l$ instead of $N_{\text{closed}}(l) \leq 4 + \frac{3}{\sqrt{2}}l$ if $l \geq 3\sqrt{2}$. In the first case, all the lengths $l_k = \frac{p}{2^{q-k+1}}$ with indices from $k = 2$ to q verify it. In the second case, we introduce the index $k_p = q + \lceil \frac{1}{2} - \log_2(\frac{p}{3}) \rceil$. For $k \le k_p$, we have $l_k = \frac{p}{2^{q-k+1}} < 3\sqrt{2}$. We then have the constant 4 in the upper bound of $N(l_k)$ (we cannot apply the conjectured value). For $k > k_p$, we do not have to add 4. It provides Number(quads) $\le 5 + 4\sum_{k=2}^{k_p} 4 + \frac{3}{\sqrt{2}}\frac{p}{2^{q-k+1}} + 4\sum_{k=k_p+1}^{q} \frac{3}{\sqrt{2}}\frac{p}{2^{q-k+1}}$. We then have Number(quads) $\le 5+4\sum_{k=2}^{k_p} 4+4\sum_{k=2}^{q} \frac{3}{\sqrt{2}}\frac{p}{2^{q-k+1}} = 5+4(k_p-1)+6\sqrt{2}p$. We then merely have to change $4\sum_{k=2}^{k_p} 4$ in $16(k_p-1) = 16(q+\lceil \frac{1}{2}-\log_2(\frac{p}{3})\rceil - 1) = 16(q+\lceil -\frac{1}{2}-\log_2(\frac{p}{3})\rceil)$ which can be rewritten $16q-16\lfloor \frac{1}{2}+\log_2(\frac{p}{3})\rfloor$. $\square$

5.2. The new bounds are tight

We obtained two bounds: that of theorem 2 Number(quads) $\le 16q - 11 + 6\sqrt{2}p$ or Number(quads) $\le 5 + 6\sqrt{2}p$ if we assume $p \ge 3.2^{q-\frac{1}{2}}$ and more importantly, if we assume conjecture 2. How do we know whether these bounds are tight? To answer this question, we provide a sequence of curves with a number of quads close to the upper bounds.

Let us again consider some curves coming from a rotated square (see Fig. 14) in a quadtree $[0, 2^q]^2$. The four vertices of the square are $(2^{q-2}, 2^{q-1})$, $(2^{q-1}, 2^{q-2})$, $(3.2^{q-2}, 2^{q-1})$ and $(2^{q-1}, 3.2^{q-2})$. Its perimeter is $p = 2^{q+1}\sqrt{2}$. According to theorem 2, for this rotated square, we have Number(quads) $\le 16q - 11 + 6\sqrt{2}p$. This can be rewritten

$$\text{Number(quads)} \le 16q - 11 + 12 \cdot 2^q.$$

According to conjecture 3, since the perimeter is large enough ($p = 2^q\sqrt{2} \ge 3 \cdot 2^{q-\frac{1}{2}}$), we should have

$$\text{Number(quads)} \le 5 + 12.2^q.$$

Let us now compare these two bounds with the exact number of quads of the quadtree representation of the rotated square. We have 1 quad of level $k = 1$: the square $[0, 2^q]^2$ itself, 4 quads of level $k = 2$, 16 quads of level $k = 3$, and exactly $B(k) = 48.2^{k-3}$ quads of level $k \ge 3$. By summing these values, we obtain Number(quads) $= 1 + 4 + 16 + \sum_{k=3}^{q} B(k)$. With $21 + 48(\sum_{k'=0}^{q-3} 2^{k'}) = 21 + 48(2^{q-2} - 1)$, it follows that

$$\text{Number(quads)} = -27 + 12.2^q.$$

The difference between the proven bound $16q - 11 + 12.2^q$ and the exact value $-27+12.2^q$ is $16(q-1)$. The difference between the conjectured bound $5 + 12.2^q$ and the exact value $-27 + 12.2^q$ is 32.

We have already noted that the proven bound $16q-11+12.2^q$ was larger than expected due to the gap between $N_{\mathrm{closed}}(l)$ and $N(l)$. However, the error on this particular class of curves remains linear in q. On the other hand, if we consider the bound $16q-11+16p$ of the state of the art from (4), this bound overestimates the number of quads from $(16\sqrt{2}-12)2^q+16(q-1)$. This class of curves shows that the overestimation of the new proven bound $(16q-11+12.2^q)$ is at worst linear in q with a factor 16 (the error could be even less with other curves). In other words, we have shown that the difference between the bound $\mathrm{Number(quads)} \leq 16q-11+6\sqrt{2}p$ of theorem 2 and the real maximum of the number of quads for a figure of perimeter $p=2^q\sqrt{2}$ is less than or equal to $16(q-1)$. If we consider the conjectured bound $\mathrm{Number(quads)} \leq 5+6\sqrt{2}p$, for $p=2^q\sqrt{2}$, then the difference from the real maximum is less than or equal to 32. On the other hand, the overestimation on the number of quads provided by the state of the art upper bound $16q-11+16p$ is exponential. This proves that the two bounds provided in Theorem 2 and Conjecture 3 are much tighter.

6. Conclusion

The main result of this paper lies in the field of digital geometry. It is a result of *complexity*, which provides a relation between the length of a rectifiable curve and the cardinality of its cover. We have provided the exact maximum number $N(l)$ of pixels that a curve of length l can cross. As far as we know, no such results have ever been published before (12), which is very surprising given its fundamental aspect. The values of $N(l)$ are easy to obtain: the difficulty is proving the optimality of the polygonal lines with integer vertices, 0, 1 or 2 horizontal and vertical steps, and an arbitrary number of diagonal steps. Once we have the proof of the expression of the function $N(l)$, it is quite straightforward to derive the properties of the function, as for instance the inequality $N(l) \leq 4+\frac{3}{\sqrt{2}}l$.
In a second step, we considered only closed curves. Except for the multiplicity cardinality of the function $N_{\mathrm{closed}}(l)$ with $l \in [0,2]$ and the straightforward inequality $N_{\mathrm{closed}}(l) \leq N(l)$, we give only conjectures.
All this work finds its application in the quadtree complexity theorem (4). The theorem states that the number of quads needed to represent a shape of perimeter p is less than $16q-11+16p$. This bound is proved by using the inequality $N_{\mathrm{closed}}(l) \leq N(l) \leq 4\lceil l \rceil \leq 4l+4$ which is not tight. By using a tighter bound on $N_{\mathrm{closed}}(l)$, we obtain a tighter bound $\mathrm{Number(quads)} \leq$

$16q - 11 + 6\sqrt{2}p$. We then proved that this bound is tight at worst up to $16(q-1)$. Of course, our conjectured bound on $N_{\text{closed}}(l)$ provides an even better result: the conjectured bound in quadtree complexity theorem becomes tight at worst up to a constant.

Much work obviously remains, at least in four directions:

- The first task is to prove the conjectures on closed curves that provide the value of $N_{\text{closed}}(l)$.
- The second challenge is to extend the results to curves in $\mathbb{R}^d$ with dimension 3, 4 or more.
- Then, we can also consider the length of curves associated with other norms $||.||_k$. Some investigations in this direction have been made in (12), but this topic is far from closed.
- The last direction is to increase the dimension of the manifold. What can we say about the number of voxels crossed by a surface with a given characteristic ? The first task is to determine a convenient characteristic, because it seems the notion of area does not work.

Generally, this framework is of interest because through these questions of maximal cardinality, we obtain deep relations between continuous objects and their discrete counterparts.

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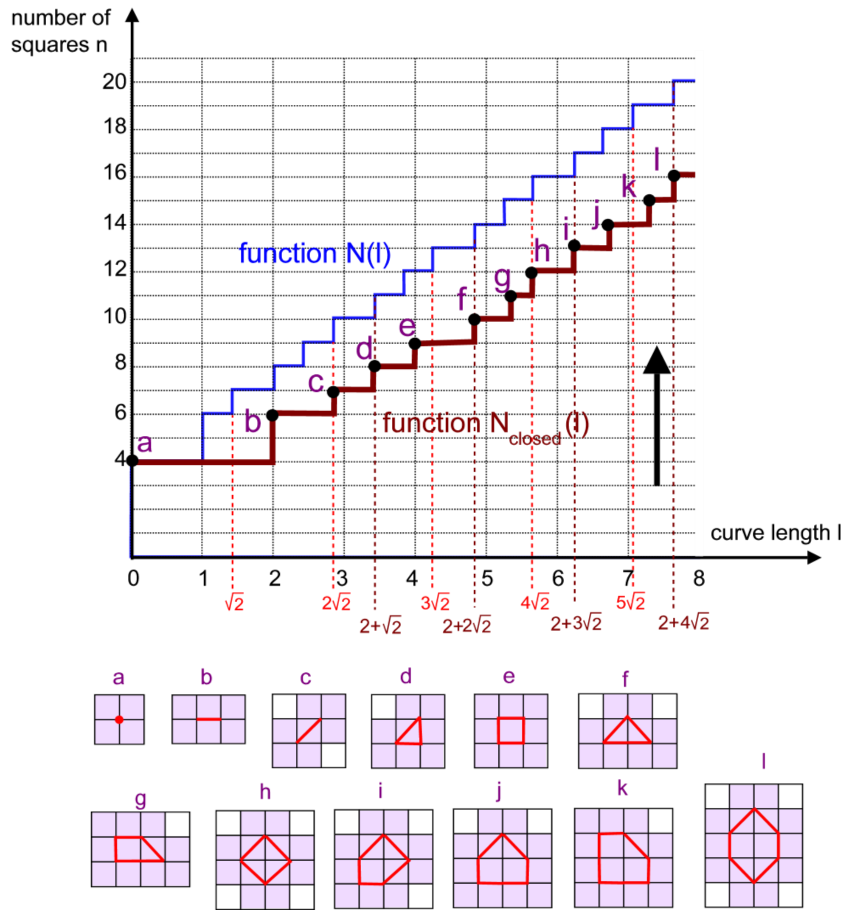

Figure 12: Our conjecture for the values of $N_{\text{closed}}(l)$ with a length l going from 0 to $2+4\sqrt{2}$. We have yet to prove that these values are optimal, but we can assert that the real values of $N_{\text{closed}}(l)$ are at least equal to these values.

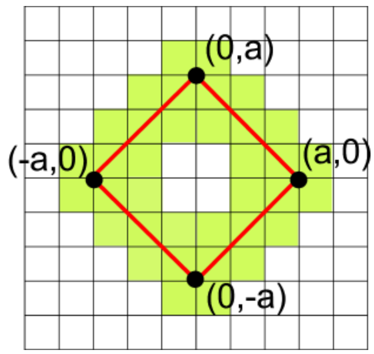

Figure 13: A rotated squared curve of length $l = 4a\sqrt{2}$ crosses $12a$ pixels. It follows that $N(4a\sqrt{2}) \geq 12a$.

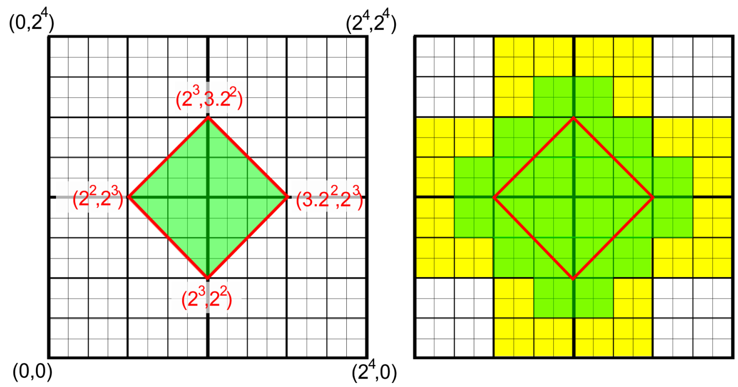

Figure 14: We consider a rotated square in a quadtree. We count the number of quads necessary to represent it. We have 1 root of size 2^4, 4 quads of size 2^3, all of them cut the curve making $4 \cdot 4 = 16$ quads of size 2^2. We have 12 quads of size 2^2 which cut the curve and which are broken down: this makes $4.12 = 48$ quads of size 2^1. We have 24 quads of size 2^1 that touch the curve making $4 \cdot 24 = 96$ quads of size 1. The sum is $1 + 4 + 16 + 48 + 96 = 165$ quads, which is also the value given by the formula $\text{Number(quads)} = -27 + 12.2^q$ for $q = 4$.