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The identifying code, the locating-dominating, the open locating-dominating and the locating total-dominating problems under some graph operations*

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Abstract

The problems of determining minimum identifying, locating-dominating, open locating-dominating or locating total-dominating codes in a graph G are variations of the classical minimum dominating set problem in G and are all known to be hard for general graphs. A typical line of attack is therefore to determine the cardinality of minimum such codes in special graphs. In this work we study the change of minimum such codes under three operations in graphs: adding a universal vertex, taking the generalized corona of a graph, and taking the square of a graph. We apply these operations to paths and cycles which allows us to provide minimum codes in most of the resulting graph classes.

1 Introduction and preliminaries

Let $G = (V, E)$ be a graph. The open neighborhood of a vertex u is the set $N(u)$ of all vertices of G adjacent to u, and $N[u] = \{u\} \cup N(u)$ is the closed neighborhood of u. A subset $C \subseteq V$ is *dominating* (resp. *total-dominating*) if $N[i] \cap C$ (resp. $N(i) \cap C$) are non-empty sets for all $i \in V$.

In this work we study four problems that have been actively studied during the last decade, see e.g. the bibliography maintained by Lobstein [14].

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A subset $C \subseteq V$ is:

- an *identifying code* (ID) if it is a dominating set and $N[i] \cap C \neq N[j] \cap C$, for $i, j \in V$ [13].
- a *locating-dominating set* (LD) if it is a dominating set and $N(i) \cap C \neq N(j) \cap C$, for $i, j \in V - C$ [19].
- an *open locating-dominating set* (OLD) if it is a total-dominating set and $N(i) \cap C \neq N(j) \cap C$, for $i, j \in V$ [18].
- a *locating total-dominating set* (LTD) if it is a total-dominating set and $N(i) \cap C \neq N(j) \cap C$, for $i, j \in V - C$ [11].

Note that not every graph admits an identifying code, in fact, a graph G admits an identifying code (or G is *identifiable*) if there are no true twins in G, i.e., there is no pair of distinct vertices $i, j \in V$ such that $N[i] = N[j]$, see [13]. Analogously, a graph G without isolated vertices admits a open locating-dominating set if there are no false twins in G, i.e., there is no pair of distinct vertices $i, j \in V$ such that $N(i) = N(j)$, see [18].

Given a graph G, for $X \in \{ID, LD, OLD, LTD\}$, the X-problem on G is the problem of finding an X-set of minimum size of G. The size of such a set is called the X-number of G and it is denoted by $\gamma_X(G)$. From the definitions, the following relations hold for any graph G (admitting an X-set):

$$\gamma_{LD}(G) \leq \gamma_{LTD}(G) \leq \gamma_{OLD}(G), \tag{1}$$

and

$$\gamma_{LD}(G) \leq \gamma_{ID}(G).$$

Note that $\gamma_{ID}(G)$ and $\gamma_{OLD}(G)$ are not comparable as the following examples show:

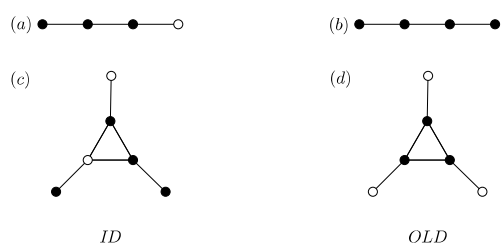

Figure 1: (a, b) $\gamma_{ID}(P_4) = 3 < 4 = \gamma_{OLD}(P_4)$; (c, d) $\gamma_{ID}(G) = 4 > 3 = \gamma_{OLD}(G)$

Determining $\gamma_{ID}(G)$ is in general NP-hard [5] and even remains hard for several graph classes where other in general hard problems are easy to solve, including bipartite graphs [5] and two classes of chordal graphs, namely split graphs and interval graphs [8]. The identifying code problem has been actively studied during the last decade, where typical lines of attack are to determine

minimum identifying codes of special graphs or to provide bounds for their size. Closed formulas for the exact value of $\gamma_{ID}(G)$ have been found so far only for restricted graph families (e.g. for paths and cycles [4], for stars [9], for complete multipartite graphs [1] and some subclasses of split graphs [2]).

Also determining $\gamma_{LD}(G)$ is in general NP-hard [5] and even remains hard for bipartite graphs [5]. This result is extended to planar bipartite unit disk graphs in [15]. Closed formulas for the exact value of $\gamma_{LD}(G)$ have been found so far for restricted graph families as e.g. paths [19], cycles [4], stars, complete multipartite graphs and thin suns [3].

Determining $\gamma_{OLD}(G)$ is in general NP-hard [18] and remains NP-hard for perfect elimination bipartite graphs and APX-complete for chordal graphs with maximum degree 4 [16]. Closed formulas for the exact value of $\gamma_{OLD}(G)$ have been found so far only for restricted graph families such as cliques and paths [18].

Concerning the LTD-problem we observe that it is as hard as the OLD-problem by just using the same arguments as in [18]. Bounds for the LTD-number of trees are given in [11, 12]. In addition, the LTD-number in special families of graphs, including cubic graphs and grid graphs, is investigated in [12].

To apply polyhedral methods, a reformulation as set covering problem is in order. For a 0/1-matrix M with n columns, the set covering polyhedron is $Q^*(M) = conv\left\{x \in \mathbb{Z}_+^n : Mx \geq \mathbf{1}\right\}$ and $Q(M) = \left\{x \in \mathbb{R}_+^n : Mx \geq \mathbf{1}\right\}$ is its linear relaxation. By [3] and [2] such constraint systems $M_X x \geq \mathbf{1}$ with $X \in \{ID, LD\}$, respectively, are

$$M_{ID}(G) = \begin{pmatrix} N[G] \\ \triangle_1[i,j] \\ \triangle_2[i,j] \end{pmatrix} \qquad M_{LD}(G) = \begin{pmatrix} N[G] \\ \triangle_1(i,j) \\ \triangle_2[i,j] \end{pmatrix}$$

where every row in matrix $N[G]$ (resp. $N(G)$) is the characteristic vector of a closed (resp. open) neighborhood of a vertex in G and $\triangle_k(i,j)$ (resp. $\triangle_k[i,j]$) is the characteristic vector of a symmetric difference of open (resp. closed) neighborhoods of vertices at distance k. It is not hard to verify that, if $X \in \{OLD, LTD\}$, we have:

$$M_{OLD}(G) = \begin{pmatrix} N(G) \\ \triangle_1(i,j) \\ \triangle_2(i,j) \end{pmatrix} \qquad M_{LTD}(G) = \begin{pmatrix} N(G) \\ \triangle_1(i,j) \\ \triangle_2[i,j] \end{pmatrix}$$

Observe that, when considering these problems as set covering problems, we can delete from $M_X(G)$ the redundant (duplicated or dominated) rows.

The work is organized as follows: in Section 2, given a graph G, we study the change of $\gamma_X(G)$ with $X \in \{ID, LD, OLD, LTD\}$ under the addition of a universal vertex to G. Then we apply these results to calculate $\gamma_X(G)$, with $X \in \{ID, LD, OLD, LTD\}$ when G is a fan or a wheel. In Section 3 we use the polyhedral approach to find $\gamma_X(G)$, with $X \in \{ID, LD, OLD, LTD\}$ when G

is the generalized corona of a graph. Finally, in Section 4, we study the same numbers when G is the square of a path or cycle.

2 Graphs obtained from adding a universal vertex

Let $G = (V, E)$ be a connected graph and $0 \notin V$. We define the graph obtained by *adding a universal vertex* $G' = (V', E')$ as the graph such that $V' = V \cup \{0\}$ and $E' = E \cup \{0i, i \in V\}$.

Remark 1 *Let $G = (V, E)$ be a graph and $0 \notin V$.*

1. *G' has true twins iff G has true twins or a universal vertex (i.e. a vertex i such that $N[i] = V$).*
2. *G' has false twins iff G also has.*

Theorem 2 *Let $X \in \{ID, LD, OLD, LTD\}$ and $G = (V, E)$ be a connected graph admitting an X-set. Then*

$$\gamma_X(G) \leq \gamma_X(G') \leq \gamma_X(G) + 1.$$

Moreover, $\gamma_X(G') = \gamma_X(G)$ if and only if there is a minimum X-set C such that for all $i \in V$, $C \nsubseteq N[i]$ when $X \in \{ID, LD\}$ or $C \nsubseteq N(i)$ when $X \in \{OLD, LTD\}$.

Proof Let C' be an X-set of minimum size of G', i.e., $\gamma_X(G') = |C'|$.

If $0 \notin C'$ then C' is an X-set of G and $\gamma_X(G) \leq |C'| = \gamma_X(G')$.

If $0 \in C'$, as $0 \in N[i]$ for all $i \in V$, 0 does not identify the vertices in V. On the other hand, as C' is minimal, then there exists $j \in V$ such that $N[j] \cap C' = \{0\}$. Moreover j is the only vertex with this property since C' is an X-set of G'. Since G is connected, there exists $k \in N(j) \cap V$ then we define $C = C' - \{0\} \cup \{k\}$. It holds that C is an X-set of G such that $|C| = |C'|$. Therefore, $\gamma_X(G) \leq |C| = |C'| = \gamma_X(G')$

Assume that for every X-set of minimum size of G, C, there is $i_C \in V$ such that $C \subseteq N[i_C]$. Clearly C is not an X-set of G'. But if $X = ID$ then there is $j \notin N[i_C]$ and $C \cup \{j\}$ is an ID-set of G'. If $X \in \{LD, OLD, LTD\}$ then $C \cup \{0\}$ is an X-set of G'. In any case $\gamma_X(G') \leq \gamma_X(G) + 1$.

Now, assume that there is an X-set C of minimum size of G such that for all $i \in V$, $C \nsubseteq N[i]$ when $X \in \{ID, LD\}$ or $C \nsubseteq N(i)$ when $X \in \{OLD, LTD\}$. We will prove that C is an X-set of G'.

As $N(0) \cap C = C \neq \emptyset$ then C total-dominates or dominates the vertices in V'.

On the other hand, we suppose that $N[0] \cap C = N[i] \cap C$ (or $N(0) \cap C = N(i) \cap C$) for some $i \in V$ if C is an X-set with $X \in \{ID, LD\}$ (or $X \in \{OLD, LTD\}$) then $C \subseteq N[i] \cap C$ (resp. $C \subseteq N(i) \cap C$) and this contradicts the assumption

on C. Then, $N[0] \cap C \neq N[i] \cap C$ ($N(0) \cap C \neq N(i) \cap C$) for all $i \in V$, i.e., C is an X-set of G' with $X \in \{ID, LD, OLD, LTD\}$.

Therefore, $\gamma_X(G') \leq |C| = \gamma_X(G)$.

Finally, assume that for every minimum X-set C of G there exists $i \in V$ such that $C \subseteq N[i]$ if $X \in \{ID, LD\}$ ($C \subseteq N(i)$ if $X \in \{OLD, LTD\}$). Let D be a minimum X-code of G' such that $|D| = \gamma_X(G)$. It is clear that $0 \in D$. Hence there is a unique $k \in V$ such that $N[k] \cap D = \{0\}$. Let $j \in N(k)$ and let $D' = D - \{0\} \cup \{j\}$. It is easy to check that D' is an X-set of G' of cardinality $\gamma_X(G)$ not containing vertex 0. Hence D' is an X-set of G of cardinality $\gamma_X(G)$, then from assumption there exists $h \in V$ such that $D' \subseteq N[h]$, but this contradicts the fact that D' is an X-set of G' since vertices 0 and h are not separated. $\square$

Let F_n (resp. W_n) denote the *fan* (resp. *wheel*) of $n+1$ vertices, i.e., F_n (resp. W_n) is the graph obtained by adding a universal vertex to the path P_n (resp. cycle C_n).

For $X \in \{ID, LD, OLD, LTD\}$, $\gamma_X(P_n)$ has already been calculated, see the following table.

X	$\gamma_X(P_n)$
ID	$\lceil \frac{n+1}{2} \rceil$ [4]
LD	$\lceil \frac{2n}{5} \rceil$ [4]
OLD	$4k+r$ for $n = 6k+r$, $r \in \{0,1,2,3,4\}$, $4k+4$ for $n = 6k+5$ [18]
LTD	$\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ [11]

Table 1: $\gamma_X(P_n)$

Now, in the case of cycles, $\gamma_{ID}(C_n)$, $\gamma_{LD}(C_n)$ and $\gamma_{LTD}(C_n)$ are known. The value of $\gamma_{OLD}(C_n)$ is provided in the following result.

Lemma 3 *For $n \geq 3$, $\gamma_{OLD}(C_n) = \lceil \frac{2n}{3} \rceil$.*

Proof (Sketch) In [18] it is proved that $\gamma_{OLD}(C_n) \geq \lceil \frac{2n}{3} \rceil$. The bound is tight as we can show, if $n = 3k + r$ with $r = 0, 1, 2$, the sets

- $\{3i-2, 3i-1 : 0 \leq i \leq k\}$ if $r = 0$,
- $\{3i-2, 3i-1 : 0 \leq i \leq k\} \cup \{n-1\}$ if $r = 1$,
- $\{3i-2, 3i-1 : 0 \leq i \leq k\} \cup \{n-1, n-2\}$ if $r = 2$,

are OLD-sets of C_n of cardinality $\lceil \frac{2n}{3} \rceil$. $\square$

We summarize the results for cycles in the following table:

For $n \geq 4$ and $X \in \{ID, LD, OLD, LTD\}$, from Theorem 2 we have $\gamma_X(P_n) \leq \gamma_X(F_n)$ and $\gamma_X(C_n) \leq \gamma_X(W_n)$. If $n = 4$, X-sets of minimum size (the black vertices) are depicted in the figure below.

For W_4 and W_5, X-sets of minimum size are shown in Fig. 3 and 4.

As a consequence of Theorem 2, we obtain:

X	$\gamma_X(C_n)$
ID	$\gamma_{ID}(C_n) = \frac{n}{2}$ if n is even and $\frac{n+1}{2} + 1$ if $n \geq 7$ is odd [10]
LD	$\lceil \frac{2n}{5} \rceil$ [4]
OLD	$\lceil \frac{2n}{3} \rceil$ Lemma 3
LTD	$\lfloor \frac{n}{2} \rfloor + \lfloor \frac{n}{4} \rfloor - \lceil \frac{n}{4} \rceil$ [6]

Table 2: $\gamma_X(C_n)$

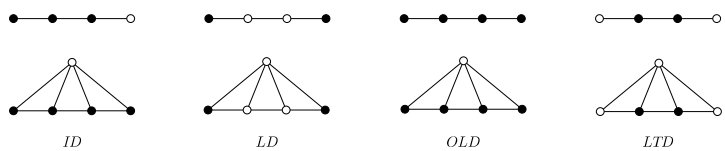

Figure 2: Minimum X-sets for P_4 and F_4

Corollary 4 *For $X \in \{ID, LD, OLD, LTD\}$ we have $\gamma_X(F_n) = \gamma_X(P_n)$ and $\gamma_X(W_n) = \gamma_X(C_n), n \geq 6$.*

Proof If $n \geq 5$, at least one vertex in each of the sets $\{1, 2\}$ and $\{n-1, n\}$ must belong to an X-set of P_n. Then, from Theorem 2, $\gamma_X(F_n) = \gamma_X(P_n)$.

Now, as $n \geq 6$, it is immediate to observe that no minimum X-set is contained in $N[i]$ ($N(i)$) when $X \in \{ID, LD\}$ (when $X \in \{OLD, LTD\}$), then again from Theorem 2, $\gamma_X(W_n) = \gamma_X(C_n)$. $\square$

Observe that by combining the results in Corollary 4 and Tables 1 and 2, we compute the exact value of a minimum X-set of a fan or a wheel for $X \in \{ID, LD, OLD, LTD\}$.

3 Generalized corona of a graph

Let $G = (V, E)$ be a graph and $\mathbf{k} \in \mathbb{Z}_+^{|V|}$. The $\mathbf{k}$-corona of G, denoted by $G^{\mathbf{k}}$ is the graph obtained by adding k_i pendant vertices to each $i \in V$.

As the pendant vertices are false twins if $k_i \geq 2$ for some $i \in V$, the graph $G^{\mathbf{k}}$ does not admit an OLD-set. But in the case $k_i = 1$ for all $i \in V$ the only OLD-set of $G^{\mathbf{k}}$ is V. We now study the remaining problems.

Theorem 5 *Let $G^{\mathbf{k}}$ be the $\mathbf{k}$-corona of a graph G where $\mathbf{k} \in \mathbb{Z}_+^{|V|}$ is a vector with $\mathbf{k} = (k_1, k_2, \ldots, k_{|V|})$, $k_i \geq 2$ for all $i \in \{1, 2, \ldots, |V|\}$. Then, $\gamma_{ID}(G^{\mathbf{k}}) = \gamma_{LD}(G^{\mathbf{k}}) = \gamma_{LTD}(G^{\mathbf{k}}) = k_1 + \cdots + k_{|V|}$.*

Proof Let $V = \{v_1, \ldots, v_{|V|}\}$ and for each $v_i \in V$, $P_i = \{p_i^1, \ldots, p_i^{k_i}\}$ the set of the pendent vertices of v_i with $k_i \geq 3$ for all $i \in \{1, \ldots, |V|\}$.

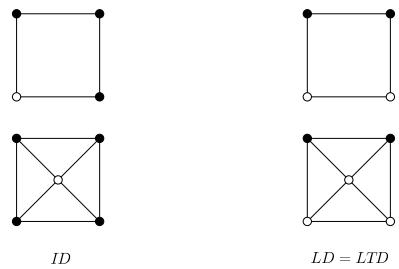

Figure 3: Minimum X-sets for C_4 and W_4

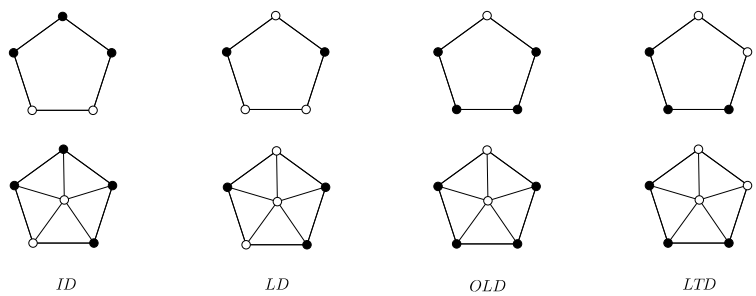

Figure 4: Minimum X-sets for C_5 and W_5

Clearly, $N[p_i^j] = \{p_i^j, v_i\}$ and $\triangle_2[p_i^j, p_i^k] = \{p_i^j, p_i^k\}$. Hence, the rows corresponding to the sets $N[v_i] = P_i \cup \{v_i\} \cup N_G(v_i)$, $\triangle_1[v_i, v_j] = P_i \cup P_j \cup (N_G(v_i) \triangle N_G(v_j))$, $\triangle_1[v_i, p_i^j] = (P_i - \{p_i^j\}) \cup N_G(v_i)$, $\triangle_2[v_i, p_j^k] = P_i \cup (N_G[v_i] - \{v_j\}) \cup \{p_j^k\}$ and $\triangle_2[v_i, v_j] \supseteq P_i \cup \{v_i\} \cup P_j \cup \{v_j\}$ are redundant. Thus $M_{ID}(G^{\mathbf{k}})$ exactly contains all 2-element subsets of V_i for each $i \in \{1, 2, \ldots, |V|\}$ and then

$$M_{ID}(G^{\mathbf{k}}) = \begin{pmatrix} R^2_{k_1+1} & 0 & \cdots & 0 \\ 0 & R^2_{k_2+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R^2_{|V|+1} \end{pmatrix},$$

where R_j^2 denotes the matrix whose rows are all the vectors in $\{0,1\}^j$ with exactly two 1's. It is known that the covering number of R_j^2 is $j-1$ (see [17]). Hence, $\gamma_{ID}(G^{\mathbf{k}}) = k_1 + \cdots + k_{|V|}$.

Now, observe that, $N[p_i^j] = \{p_i^j, v_i\}$ and $\triangle_2[p_i^j, p_i^k] = \{p_i^j, p_i^k\}$. Hence, the rows corresponding to the sets $N[v_i] = P_i \cup \{v_i\} \cup N_G(v_i)$, $\triangle_1(v_i, v_j) = P_i \cup \{v_i\} \cup P_j \cup \{v_j\}$, $\triangle_1(v_i, p_i^j) \supseteq \{p_i^j, v_i\}$, $\triangle_2[v_i, p_j^k] \supseteq P_i \cup \{v_i\}$ and $\triangle_2[v_i, v_j] \supseteq P_i \cup \{v_i\}$ are redundant. So, $M_{LD}(G^{\mathbf{k}}) = M_{ID}(G^{\mathbf{k}})$ and hence $\gamma_{ID}(G^{\mathbf{k}}) = \gamma_{LD}(G^{\mathbf{k}})$.

Finally, to study $\gamma_{LTD}(G^{\mathbf{k}})$ observe that the symmetric differences are anal-

ogous to the LD-problem, then they are all dominated except from $\triangle_2[p_i^j, p_i^k] = \{p_i^j, p_i^k\}$ and $N(v_i) = P_i \cup N_G(v_i)$ is dominated by $\triangle_2[p_i^j, p_i^k]$ too. On the other hand, $N(p_i^j) = \{v_i\}$ are not dominated for $i \in \{1, \ldots, |V|\}$. Thus,

$$M_{LTD}(G^{\mathbf{k}}) = \begin{pmatrix} I_n & 0 & \cdots & 0 \\ 0 & R^2_{k_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R^2_{|V|} \end{pmatrix}.$$

Therefore, we obtain that $\gamma_{ID}(G^{\mathbf{k}}) = \gamma_{LD}(G^{\mathbf{k}}) = \gamma_{LTD}(G^{\mathbf{k}}) = k_1 + \cdots + k_{|V|}$. $\square$

4 Square of paths and cycles

The square of a graph $G = (V, E)$ is the graph $G^2 = (V, E')$ where $E' = E \cup \{ij : dist(i, j) = 2\}$. In this section we will analyze the X-sets, for $X \in \{ID, LD, OLD, LTD\}$ in the case $G = P_n$ and $G = C_n$.

Firstly, it is easy to check that $\gamma_{ID}(P_5^2) = \gamma_{ID}(P_6^2) = 4$ and $\gamma_{ID}(P_7^2) = 5$.

Theorem 6 *For P_n^2 with $n \geq 8$ we have that $\gamma_{ID}(P_n^2) = \lceil \frac{n+1}{2} \rceil$.*

Proof (Sketch) We know that $\gamma_{ID}(P_n^2) \geq \lceil \frac{n+1}{2} \rceil$ for every $n \geq 5$ [4]. We can show that the set $\{5\} \cup \{2i : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor\}$ is an identifying code of P_n^2 and has cardinality $\lceil \frac{n+1}{2} \rceil$. $\square$

Now, if $X = LD$ or $X = LTD$, it can be checked that $\gamma_{LD}(P_4^2) = \gamma_{LTD}(P_4^2) = \gamma_{LD}(P_5^2) = \gamma_{LTD}(P_5^2) = 2$ and $\gamma_{LD}(P_6^2) = \gamma_{LTD}(P_6^2) = 3$.

Remark 7 *Combining relation (1) with the lower bound for $\gamma_{LD}(P_n^2)$ in [4], we have that for every $n \geq 1$*

$$\gamma_{LTD}(P_n^2) \geq \gamma_{LD}(P_n^2) \geq \left\lceil \frac{n+1}{3} \right\rceil.$$

Theorem 8 *For P_n^2 with $n \geq 7$ we have that*

$$\left\lceil \frac{n+1}{3} \right\rceil \leq \gamma_{LTD}(P_n^2) \leq \left\lceil \frac{n+1}{3} \right\rceil + 1.$$

Moreover, the lower bound is attained if $n = 6k$.

Proof (Sketch) For $n \geq 7$, let $n = 6k + r$, with $k \geq 1$ and $r \in \{0, 1, \ldots, 5\}$. We can show that

- $\{4\} \cup \{6i + 3, 6i + 5 : 0 \leq i \leq k - 1\}$ for $r = 0$

is an LTD-set of P_n^2 of cardinality $\lceil \frac{n+1}{3} \rceil$, and that

- $\{4, n\} \cup \{6i+3, 6i+5 : 0 \leq i \leq k-1\}$ for $r = 1$,
- $\{4, n-1\} \cup \{6i+3, 6i+5 : 0 \leq i \leq k-1\}$ for $r = 2$,
- $\{4, n-1, n\} \cup \{6i+3, 6i+5 : 0 \leq i \leq k-1\}$ for $r = 3$,
- $\{4, n-3, n-1\} \cup \{6i+3, 6i+5 : 0 \leq i \leq k-1\}$ for $r = 4$,
- $\{4, n-2, n\} \cup \{6i+3, 6i+5 : 0 \leq i \leq k-1\}$ for $r = 5$,

are LTD-sets of P_n^2 of cardinality $\lceil \frac{n+1}{3} \rceil + 1$. $\square$

As a consequence of Remark 7 and the above result, we have:

Corollary 9 *For P_n^2 with $n \geq 7$ we have that*

$$\left\lceil \frac{n+1}{3} \right\rceil \leq \gamma_{LD}(P_n^2) \leq \left\lceil \frac{n+1}{3} \right\rceil + 1.$$

Moreover, the lower bound is attained if $n = 6k$ or $n = 6k+3$.

Proof If $n \geq 7$, let $n = 6k + r$, with $k \geq 1$ and $r \in \{0, 1, \ldots, 5\}$. We only need to observe that if $r = 3$, the set $\{3, 4, 5, n\} \cup \{6h+3, 6h+5, 1 \leq h \leq k-1\}$ is an LD-sets of P_n^2 of cardinality $\lceil \frac{n+1}{3} \rceil$. $\square$

Finally, it is not hard to check that $\gamma_{OLD}(P_5^2) = 3$ and $\gamma_{OLD}(P_n^2) = 4$ when $n = 6, 7, 8, 9$.

Theorem 10 *For P_n^2 with $n \geq 10$, $n = 10k + r$ with $r \in \{0, \ldots, 9\}$ we have that*

$$\gamma_{OLD}(P_n^2) \leq \begin{cases} 4k+1 & \textit{if} \quad r = 0, 1 \\ 4k+2 & \textit{if} \quad r = 2, 3 \\ 4k+3 & \textit{if} \quad r = 4, 5 \\ 4k+4 & \textit{if} \quad r = 6, 7, 8, 9 \end{cases}$$

Proof (Sketch) Let $n = 10k + r$ with $k \geq 1$ and $r \in \{0, \ldots, 9\}$. We can show that

- $\{2i : 1 \leq i \leq k\} - \{10i : 1 \leq i \leq k-1\}$ when $r \in \{0, \ldots, 7\}$,
- $\{2i : 1 \leq i \leq k\} - \{10i : 1 \leq i \leq k\}$ when $r = 8, 9$

are OLD-sets with cardinality $4k + 1 + \lfloor \frac{r}{2} \rfloor$ in the first case and $4k + \lfloor \frac{r}{2} \rfloor$ in the second case. $\square$

Finally, computational evidence encourages us to conjecture that Thm. 10 in fact gives the exact values for $\gamma_{OLD}(P_n^2)$.

In a similar way, we will study now the squares of cycles. Note that C_n^2 equals a clique when $n \leq 5$ so that no ID-codes exist and $\gamma_X(C_n^2)$ is known for $X \in \{LD, OLD, LTD\}$. If $X = ID$, $\gamma_{ID}(C_n^2) = \frac{n}{2}$ if n is even and, if n is odd, $\gamma_{ID}(C_n^2) = \frac{n+1}{2}$ if $n = 5k, 5k+2, 5k+3$ and $\gamma_{ID}(C_n^2) = \frac{n+1}{2} + 1$ if $n = 5k+1, 5k+4$ (see [7]).

If $X = LD$, and $n = 6k + r$, $k \geq 1$, $r = 0, 1, \ldots 5$, $\gamma_{LD}(C_n^2) = \lceil \frac{n}{3} \rceil + 1$ if $r = 3$ and $\gamma_{LD}(C_n^2) = \lceil \frac{n}{3} \rceil$ otherwise (see [7]). If $X = LTD$, it can be checked that $\gamma_{LTD}(C_6^2) = 3$.

Theorem 11 *For C_n^2 with $n \geq 7$, we have that*

$$\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{LTD}(C_n^2) \leq \left\lceil \frac{n}{3} \right\rceil + 1.$$

Moreover, the lower bound is attained if $n = 6k, 6k+1, 6k+2, 6k+4$.

Proof If $n \geq 7$, let $n = 6k + r$, with $k \geq 1$ and $r \in \{0, 1, \ldots, 5\}$. It is not hard to check that:

- $\{6i+3, 6i+5 : 0 \leq i \leq k-1\}$ if $r = 0$,
- $\{6i+3, 6i+5 : 0 \leq i \leq k-1\} \cup \{n-1\}$ if $r = 1, 2$, and
- $\{6i+3, 6i+5 : 0 \leq i \leq k-1\} \cup \{n-3, n-1\}$ if $r = 4$,

are LTD-sets of C_n^2 of cardinality $\lceil \frac{n}{3} \rceil$, and

- $\{6i+3, 6i+5 : 0 \leq i \leq k-1\} \cup \{n-1, n\}$ if $r = 3$,
- $\{6i+3, 6i+5 : 0 \leq i \leq k-1\} \cup \{4, n-2, n\}$ if $r = 5$,

are LTD-sets of C_n^2 of cardinality $\lceil \frac{n}{3} \rceil + 1$, but not necessarily minimum. $\square$

If $X = OLD$, it can be easily seen that C_6^2 has false twins and, thus, no OLD-set and that $\gamma_{OLD}(C_7^2), \gamma_{OLD}(C_8^2) = 4$ holds. Moreover, we can show:

Theorem 12 *For C_n^2 with $n \geq 9$, we have that*

$$\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{LTD}(C_n^2) \leq \gamma_{OLD}(C_n^2) \leq \left\lceil \frac{n-2}{2} \right\rceil + 1.$$

Proof (Sketch) From the general relation (1) and the lower bound for $\gamma_{LTD}(C_n^2)$ given in Thm. 11, we conclude the lower bound for $\gamma_{OLD}(C_n^2)$. The upper bound is true as we can show that

- $\{2i-1 : 1 \leq i < k\} \cup \{2k-2\}$ if $n = 2k+1$,
- $\{2i-1 : 1 \leq i \leq k\}$ if $n = 2k+2$

form OLD-codes of size k in C_n^2. $\square$

Note that this implies $\gamma_{LTD}(C_n^2) = \gamma_{OLD}(C_n^2)$ for $9 \leq n \leq 11$. For $12 \leq n \leq 15$, we know that the upper bound is tight and conjecture this also for all $n \geq 16$.

To conclude, we showed that adding a universal vertex changes the studied X-numbers by at most one (but remain the same in the case of paths and cycles), whereas taking the square of a graph can result in very different X-numbers. Moreover, the studied X-numbers of generalized coronas of a graph depend in most cases only on the corona, but not on the graph.

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