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# On superperfection of edge-intersection graphs of paths\*

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## Abstract

The Routing and Spectrum Assignment problem in Flexgrid Elastic Optical Networks can be modeled in two phases: a selection of paths in the network and an interval coloring problem in the edge-intersection graph of these paths. The interval chromatic number equals the smallest size of a spectrum such that a proper interval coloring is possible, the weighted clique number is a natural lower bound. Graphs where both parameters coincide for all induced subgraphs and for all possible integral weights are called superperfect. We examine the question which minimal non-superperfect graphs can occur in the edge-intersection graphs of paths in different underlying networks and show that for any possible network (even if it is restricted to a path) the resulting edge-intersection graphs are not necessarily superperfect.

## 1 Introduction

Flexgrid Elastic Optical Networks constitute a new generation of optical networks in response to the sustained growth of data traffic volumes and demands in communication networks. In such networks, light is used as communication medium between sender and receiver nodes, and the frequency spectrum of an optical fiber is divided into narrow frequency slots of fixed spectrum width. Any sequence of consecutive slots can form a channel that can be switched in the network to create a lightpath (i.e., an optical connection represented by a route and a channel). The *Routing and Spectrum Assignment (RSA) problem* consists of establishing the lightpaths for a set of end-to-end traffic demands, which involves finding a route and assigning an interval of consecutive frequency slots for each demand such that the intervals of lightpaths using a same edge in the network are disjoint, see e.g. [15]. Thereby, the following constraints need to be respected when dealing with the RSA problem:

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1. *spectrum continuity*: the frequency slots remain the same on all the links of a route;
2. *spectrum contiguity*: the frequency slots allocated to a demand must be contiguous;
3. *non-overlapping spectrum*: a frequency slot can be allocated to at most one demand.

The RSA problem is a generalization of the well-studied *Routing and Wavelength Assignment (RWA) problem* that is associated with a fixed grid of frequencies [6]. The former problem has started to receive a lot of attention over the last few years. It has been shown to be NP-hard [3, 16]. In fact, if for each demand the route is already known, the RSA problem reduces to the so-called *Spectrum Assignment (SA) problem* and only consists of determining the demands' channels. The SA problem has been shown to be NP-hard on paths [14] which makes the SA problem (and thus also the RSA problem) much harder than the RWA problem which is well-known to be polynomially solvable on paths, see e.g. [6].

More precisely, we are given a communication network  $G$  and a set of end-to-end traffic demands between pairs  $u, v$  of nodes in  $G$  specifying the number  $d_{uv}$  of required frequency slots to satisfy this demand. The routing part of the RSA problem consists of selecting a route through  $G$  from  $u$  to  $v$ , i.e. a  $(u, v)$ -path  $P_{uv}$  in  $G$ , for each such traffic demand. The spectrum assignment can then be interpreted as an interval coloring of the edge-intersection graph  $I(\mathcal{P})$  of the set  $\mathcal{P}$  of selected paths:

- Each path  $P_{uv} \in \mathcal{P}$  becomes a node of  $I(\mathcal{P})$  and two nodes are joined by an edge if the corresponding paths in  $G$  are in conflict as they share an edge (note: we do not care whether they share nodes).
- Any interval coloring in this graph  $I(\mathcal{P})$  weighted with the demands  $d_{uv}$  correctly solves the spectrum assignment: we assign a frequency interval of  $d_{uv}$  consecutive frequency slots (*spectrum contiguity*) to every node of  $I(\mathcal{P})$  (and, thus, to every path  $P_{uv} \in \mathcal{P}$  (*spectrum continuity*)) in such a way that the intervals of adjacent nodes are disjoint (*non-overlapping spectrum*).

The interval chromatic number  $\chi_I(I(\mathcal{P}), d)$  is the smallest size of a spectrum such that  $I(\mathcal{P})$  weighted with the traffic demand  $d_{uv}$  for each path  $P_{uv}$  has a proper interval coloring. The weighted clique number  $\omega(I(\mathcal{P}), d)$ , also taking the traffic demands  $d_{uv}$  as weights, is a natural lower bound for  $\chi_I(I(\mathcal{P}), d)$ . Graphs where both parameters coincide for all induced subgraphs and for all possible non-negative integral weights are called *superperfect*.

A graph is *perfect* if and only if we have  $\omega(G, d) = \chi_I(G, d)$  for every  $(0, 1)$ -weighting  $d$  of its nodes, thus every superperfect graph is perfect. A graph  $G = (V, E)$  is *comparability* if and only if there exists a partial order  $\mathcal{O}$  on  $V \times V$  such that  $uv \in E$  if and only if  $u$  and  $v$  are comparable w.r.t.  $\mathcal{O}$ . Comparability graphs form a subclass of superperfect graphs [10], but there are also superperfect non-comparability graphs such as e.g. even antiholes [8].

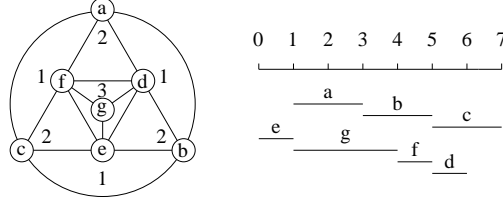


Figure 1: The graph  $\overline{A}_1$  together with node weights  $d$  and an interval coloring.

A complete list of minimal non-comparability graphs is presented in [7], the superperfect graphs from this list have been determined in [1]. The remaining non-comparability graphs from the list in [7] which are not superperfect by [1] are thus minimal non-superperfect: the graph  $\overline{A}_1$  shown in Fig. 1 and all

- odd holes  $C_{2k+1}$  and odd antiholes  $\overline{C}_{2k+1}$  for  $k \geq 2$ ,
- the graphs  $J_k$  and  $J'_k$  for  $k \geq 2$  (see Fig. 2),
- the complements of  $D_k$  for  $k \geq 2$  and of  $E_k, F_k$  for  $k \geq 1$  (see Fig. 3).

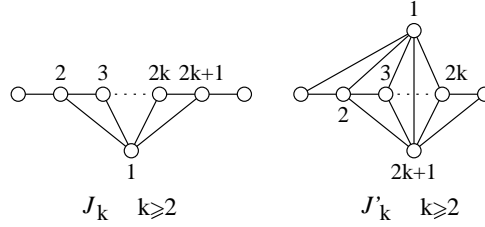


Figure 2: Minimal non-superperfect graphs:  $J_k, J'_k$  for  $k \geq 2$ .

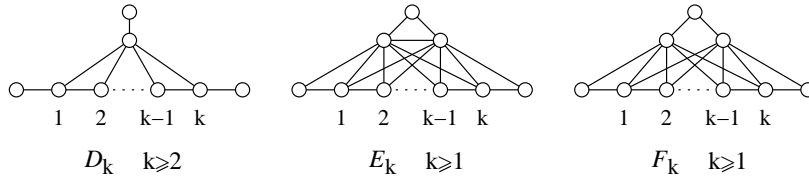


Figure 3: Minimal non-superperfect graphs: the complements of  $D_k, E_k, F_k$ .

Note that we have  $\omega(G, \mathbf{1}) < \chi_I(G, \mathbf{1})$  with  $\mathbf{1} = (1, \dots, 1)$  if  $G$  is an odd hole or an odd antihole (as they are not perfect by [4]), whereas the other minimal non-superperfect graphs from the list are perfect and, thus,  $\omega(G, d) < \chi_I(G, d)$  is attained for some  $d \neq \mathbf{1}$  (see Fig. 1).

We examine the question which of these minimal non-superperfect graphs can occur in edge-intersection graphs of paths in different underlying networks  $G$ : if  $G$  is a tree (see Section 2) or more generally if  $G$  is a 1-tree (see Section 3). We close with some concluding remarks and open problems.

## 2 If the network is a tree

If the underlying communication network  $G$  is a tree, then there exists exactly one  $(u, v)$ -path  $P_{uv}$  in  $G$  for every traffic demand between a pair  $u, v$  of nodes. Hence, if  $G$  is a tree, then  $\mathcal{P}$  and  $I(\mathcal{P})$  are uniquely determined for any set of end-to-end traffic demands, and the RSA problem reduces to the spectrum assignment part.

Next, we examine which minimal non-superperfect graphs can occur in  $I(\mathcal{P})$  if  $G$  is a tree:

**Theorem 1** *If  $\mathcal{P}$  is a set of paths in a tree, then  $I(\mathcal{P})$  can contain  $\overline{A}_1$  and*

- *odd holes  $C_{2k+1}$  for  $k \geq 2$ , but no odd antiholes  $\overline{C}_{2k+1}$  for  $k \geq 3$ ,*
- *the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$ ,*
- *the graphs  $\overline{D}_2, \overline{D}_3, \overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{F}_1, \overline{F}_2, \overline{F}_3$ , but none of  $\overline{D}_k, \overline{E}_k, \overline{F}_k$  for  $k \geq 4$ .*

**Proof (Sketch)** In order to prove the theorem, we will present according path collections for the affirmative cases and exhibit a  $\overline{P}_6$  (which cannot occur in  $I(\mathcal{P})$  if  $G$  is a tree) as common subgraph of the remaining cases.

If  $\mathcal{P}$  is a set of paths in a tree, then  $I(\mathcal{P})$  can contain

- odd holes  $C_{2k+1}$  for  $k \geq 2$ ,
- the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$ ,
- the graphs  $\overline{A}_1, \overline{D}_2, \overline{D}_3, \overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{F}_1, \overline{F}_2, \overline{F}_3$ ,

see the corresponding collections of paths in Fig. 4 and Fig. 5 (which can be easily extended to all cases for  $k \geq 2$ ), and Fig. 6, Fig. 7, Fig. 8, Fig. 9 and Fig. 10 for the remaining graphs.

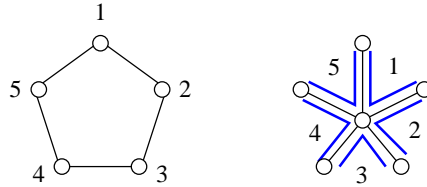


Figure 4: The odd hole  $C_5 = I(\mathcal{P})$  with  $\mathcal{P}$  in a star.

Furthermore, we can show by case analysis that  $\overline{P}_6$  cannot occur in the edge-intersection graph of paths in a tree. This implies that  $I(\mathcal{P})$  cannot contain

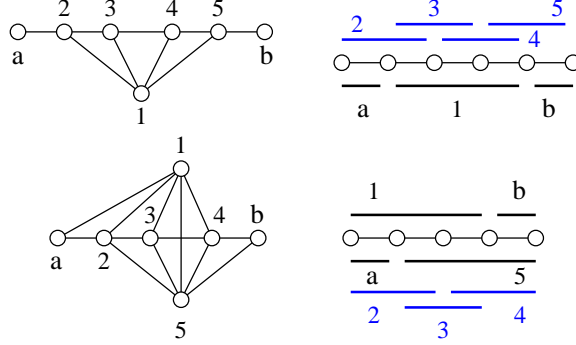


Figure 5: The graphs  $J_2 = I(\mathcal{P})$  and  $J'_2 = I(\mathcal{P})$  with  $\mathcal{P}$  in a path.

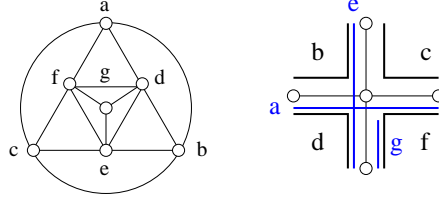


Figure 6: The graph  $\overline{A}_1 = I(\mathcal{P})$  with  $\mathcal{P}$  in a tree.

- odd antiholes  $\overline{C}_{2k+1}$  for  $k \geq 3$ ,
- the graphs  $\overline{D}_k, \overline{E}_k, \overline{F}_k$  for  $k \geq 4$ ,

as their complements all contain a  $P_6$  (which is clear for odd holes  $C_{2k+1}$  for  $k \geq 3$  and for the graphs  $D_k, E_k, F_k$  for all  $k \geq 4$  from their definition, see Fig. 3).  $\square$

This implies that edge-intersection graphs of paths in a tree are not necessarily perfect (as they may contain odd holes).

We next examine the situation when we restrict the tree further. A graph is *triangulated* if it does not have holes  $C_k$  with  $k \geq 4$  as induced subgraph. We can show the following:

**Lemma 2** *If  $\mathcal{P}$  is a set of paths in a tree with maximum degree 3, then  $I(\mathcal{P})$  is triangulated and can contain the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$  and  $\overline{D}_2, \overline{E}_1, \overline{E}_2$ .*

**Proof (Sketch)** For that, we can prove (again by case analysis) that  $C_k$  with  $k \geq 4$  can occur in the edge-intersection graph of paths in a tree only if the tree contains a star  $K_{1,k}$ . This implies for the case where the network is a tree with maximum degree 3 that  $I(\mathcal{P})$  cannot contain any hole  $C_k$  with  $k \geq 4$ ,

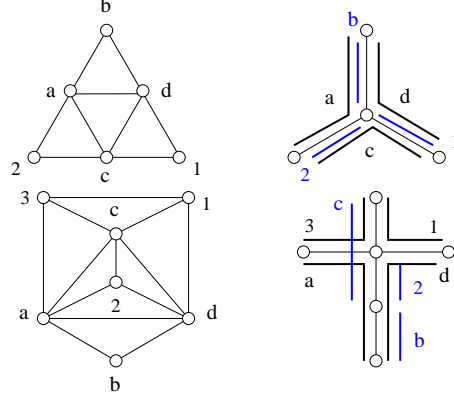


Figure 7: The graphs  $\overline{D}_2 = I(\mathcal{P})$  and  $\overline{D}_3 = I(\mathcal{P})$  with  $\mathcal{P}$  in a tree.

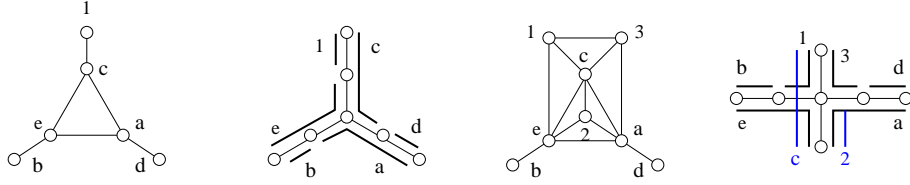


Figure 8: The graphs  $\overline{E}_1 = I(\mathcal{P})$  and  $\overline{E}_3 = I(\mathcal{P})$  with  $\mathcal{P}$  in a tree.

hence  $I(\mathcal{P})$  is triangulated and from the above list, we can exclude the following graphs:

- all odd holes  $C_{2k+1}$  for  $k \geq 2$ ,
- the graphs  $\overline{A}_1, \overline{D}_3, \overline{E}_3, \overline{F}_1, \overline{F}_2, \overline{F}_3$  (that all contain a  $C_4$ ),

whereas the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$  and  $\overline{D}_2, \overline{E}_1, \overline{E}_2$  have an according representation.  $\square$

This implies that edge-intersection graphs of paths in a tree with maximum degree 3 are perfect (as they neither contain odd holes nor odd antiholes), but not necessarily superperfect.

Moreover, an *interval graph* is the intersection graph of intervals in a line.

**Lemma 3** *If  $\mathcal{P}$  is a set of paths in a path, then  $I(\mathcal{P})$  is an interval graph and can contain the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$  and  $\overline{E}_2$ .*

**Proof** In this case,  $I(\mathcal{P})$  is clearly an interval graph and only

- $\overline{E}_2$  and the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$

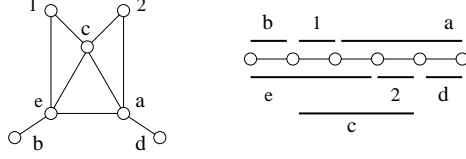


Figure 9: The graph  $\overline{E}_2 = I(\mathcal{P})$  with  $\mathcal{P}$  in a path.

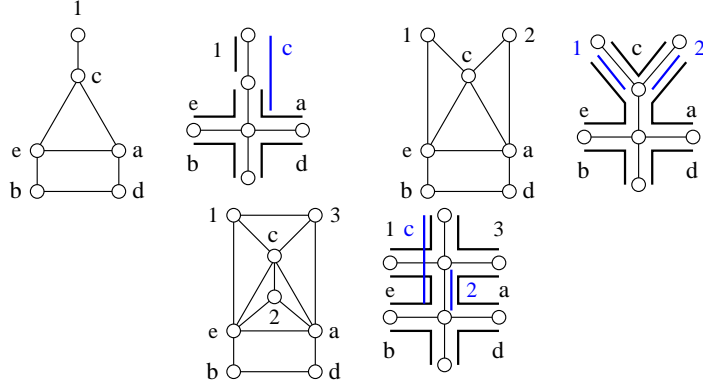


Figure 10: The graphs  $\overline{F}_1 = I(\mathcal{P})$ ,  $\overline{F}_2 = I(\mathcal{P})$  and  $\overline{F}_3 = I(\mathcal{P})$  with  $\mathcal{P}$  in a tree.

can occur (see their according representations in Fig. 5 and Fig. 9), whereas  $\overline{D}_2$ ,  $\overline{E}_1$  are excluded (as known examples of non-interval graphs from [12]).  $\square$

This implies that even edge-intersection graphs of paths in a path are not necessarily superperfect.

### 3 If the network is a 1-tree

Modern optical networks have clearly not a tree-like structure due to survivability aspects concerning node or edge failures in the network  $G$ . At least the subset of “core nodes” has to lie on a cycle (to have the choice between two paths, see e.g. [11]). We wonder which minimal non-superperfect graphs from the list in [7] can occur in edge-intersection graphs of paths in *1-trees* (that are graphs obtained from a tree by adding one edge, i.e., graphs having exactly one cycle).

The situation when the network  $G$  is a 1-tree includes in particular the case when  $G$  is a cycle and, thus, when  $I(\mathcal{P})$  is a *circular-arc graph* (as it is the intersection graph of arcs in a cycle). Making use of this fact, we can prove:

**Theorem 4** *If  $\mathcal{P}$  is a set of paths in a cycle, then  $I(\mathcal{P})$  can contain  $\overline{A}_1$  and*

- all odd holes  $C_{2k+1}$  and odd antiholes  $\overline{C}_{2k+1}$  for  $k \geq 2$ ,
- the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$ ,
- the graphs  $\overline{D}_k$  for all  $k \geq 2$ ,
- $\overline{E}_1$  and  $\overline{E}_2$ , but not  $\overline{E}_k$  for  $k \geq 3$ ,
- $\overline{F}_2$ , but not  $\overline{F}_1$  neither  $\overline{F}_k$  for  $k \geq 3$ .

**Proof (Sketch)** If  $\mathcal{P}$  is a set of paths in a cycle, then  $I(\mathcal{P})$  can contain

- odd holes  $C_{2k+1}$  and odd antiholes  $\overline{C}_{2k+1}$  for  $k \geq 2$  (as they are well-known examples of circular-arc graphs, see e.g. [5] and Fig. 11 for illustration),
- the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$  and  $\overline{E}_2$  (as they are, by Theorem 1, examples of interval graphs, which form by construction a subclass of circular-arc graphs, see Fig. 5 and Fig. 9 for illustration),
- $\overline{A}_1$ , the graphs  $\overline{D}_k$  for all  $k \geq 2$ ,  $\overline{E}_1$  and  $\overline{F}_2$  by presenting the corresponding collections of paths, see Fig. 12 and Fig. 13.

Note that from the path representation of  $\overline{D}_2$  in Fig. 12, we can get the corresponding collections of paths for the graphs  $\overline{D}_k$  for all  $k \geq 3$  as follows:  $\overline{P}_{k+2} = a, 1, \dots, k, d \subseteq \overline{D}_k$  can be embedded in the cycle (e.g. by using the  $k+2$  corresponding paths of a path representation of an sufficiently large odd antihole, see Fig. 11), then we add a long path for  $c$  and a short path for  $b$  (only sharing an edge with the paths of  $a$  and  $d$ ).

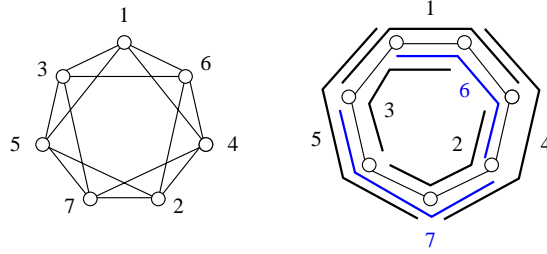


Figure 11: The graph  $\overline{C}_7 = I(\mathcal{P})$  with  $\mathcal{P}$  in a cycle.

However,  $I(\mathcal{P})$  cannot contain  $\overline{F}_1$ , neither  $\overline{F}_k$  nor  $\overline{E}_k$  for  $k \geq 3$  because

- $\overline{E}_3$  is not a circular-arc graph, as there is no corresponding collection of paths (which can be shown by case analysis);
- each of  $\overline{E}_k$  for  $k \geq 4$  and  $\overline{F}_1$  contains  $C_4 \cup K_1$  as induced subgraph ( $1, 3, 4, a, b$  in  $\overline{E}_k$  and  $1, a, b, d, e$  in  $\overline{F}_1$ , see Fig. 10) which is a well-known minimal non-circular-arc graph by [2];
- each of  $\overline{F}_k$  for  $k \geq 3$  contains a domino induced by  $1, k, a, b, d, e$  as subgraph (see  $\overline{F}_3$  in Fig. 10 for illustration), which is another well-known minimal non-circular-arc graph by [5].

□

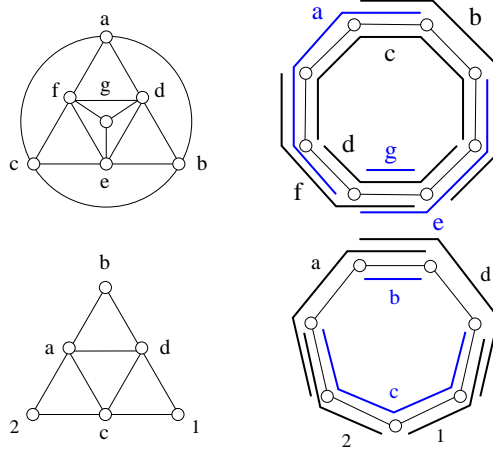


Figure 12: The graphs  $\overline{A}_1 = I(\mathcal{P})$  and  $\overline{D}_2 = I(\mathcal{P})$  with  $\mathcal{P}$  in a cycle.

**Remark 5** Note that  $\overline{E}_3$  is, to the best of our knowledge, a new example of a minimal non-circular-arc graph (see e.g. the results on circular-arc graphs surveyed in [5]).

If the network is a 1-tree, then all minimal non-superperfect graphs occurring in  $I(\mathcal{P})$  when the network is a tree or a cycle can clearly be present. In addition, we can further show, by presenting according path collections, that also the families  $\overline{E}_k$  and  $\overline{F}_k$  can occur in such edge-intersection graphs. Thus, we obtain:

**Theorem 6** If  $\mathcal{P}$  is a set of paths in a 1-tree, then  $I(\mathcal{P})$  can contain  $\overline{A}_1$  and

- all odd holes  $C_{2k+1}$  and odd antiholes  $\overline{C}_{2k+1}$  for  $k \geq 2$ ,
- the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$ ,
- the graphs  $\overline{D}_k$  for all  $k \geq 2$ ,
- the graphs  $\overline{E}_k$  and  $\overline{F}_k$  for all  $k \geq 1$ .

**Proof (Sketch)** In the case where the network  $G$  is a 1-tree, all studied minimal non-superperfect graphs occurring in  $I(\mathcal{P})$  when the network is a tree or a cycle can be present, hence we conclude from Theorem 1 and Theorem 4 that we have

- $\overline{A}_1$  (can occur in both cases)
- all odd holes  $C_{2k+1}$  for  $k \geq 2$  (can occur in both cases),
- all odd antiholes  $\overline{C}_{2k+1}$  for  $k \geq 2$  (can occur if  $G$  is a cycle),
- the graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$  (can occur in both cases),
- the graphs  $\overline{D}_k$  for all  $k \geq 2$  (can occur if  $G$  is a cycle),
- $\overline{E}_1, \overline{E}_2, \overline{E}_3, \overline{F}_1, \overline{F}_2, \overline{F}_3$  (can occur if  $G$  is a tree).

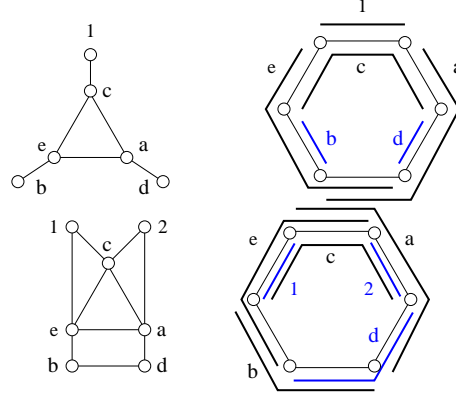


Figure 13: The graphs  $\overline{E}_1 = I(\mathcal{P})$  and  $\overline{F}_2 = I(\mathcal{P})$  with  $\mathcal{P}$  in a cycle.

Hence, it is left to address the two families  $\overline{E}_k, \overline{F}_k$  for  $k \geq 4$ . We can show that the graphs  $\overline{E}_k$  and  $\overline{F}_k$  for all  $k \geq 4$  can occur if  $G$  is a 1-tree, by presenting the corresponding collections of paths. For that, we first note that  $\overline{E}_k$  and  $\overline{F}_k$  differ only in the presence of edge  $bd$ . The path representation of the subgraphs induced by  $a, b, c, d, e$  are shown in Fig. 14.

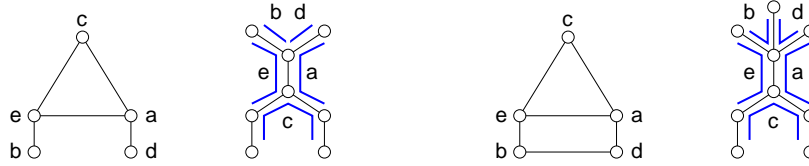


Figure 14: The paths representations of the subgraphs induced by  $a, b, c, d, e$  in  $\overline{E}_k$  and  $\overline{F}_k$ .

Moreover, the  $\overline{P}_{k+2} = a, 1, \dots, k, e \subseteq \overline{E}_k, \overline{F}_k$  can be embedded in a cycle running through the edges used by the paths of  $a, c, e$  (e.g. again by using for  $1, \dots, k$  the corresponding paths of a path representation of an sufficiently large odd antihole in such a way that the paths for  $a$  and  $e$  fit). The path of  $c$  needs to be enlarged to meet all paths of  $1, \dots, k$ , see Fig. 15 for illustration.

This finally proves the theorem.  $\square$

Hence, we can finally conclude that *all* the studied minimal non-superperfect graphs can occur in edge-intersection graphs of paths, as soon as the network  $G$  is a 1-tree and satisfies minimal survivability conditions concerning edge or node failures.

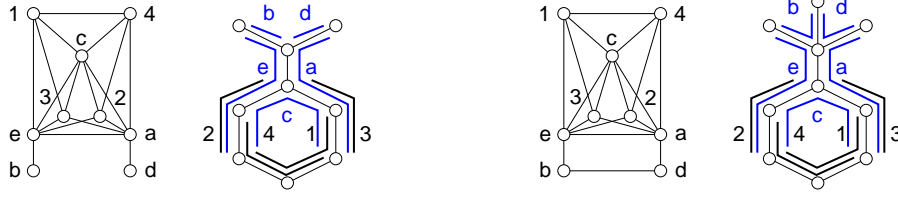


Figure 15: The graphs  $\overline{E}_4 = I(\mathcal{P})$  and  $\overline{F}_4 = I(\mathcal{P})$  with  $\mathcal{P}$  in a 1-tree.

## 4 Concluding remarks

We showed that edge-intersection graphs of paths in communication networks are not necessarily perfect (as they can contain odd holes and odd antiholes) and are, thus, also not necessarily superperfect. If we restrict the networks to trees having maximum degree 3, then  $I(\mathcal{P})$  is triangulated (and, thus, perfect), but not necessarily superperfect, and even if we restrict the networks to paths, then  $I(\mathcal{P})$  is an interval graph, but still not necessarily superperfect (as the minimal non-superperfect graphs  $J_k$  and  $J'_k$  for all  $k \geq 2$  and  $\overline{E}_1$  can occur). This is in accordance with the fact that the SA problem has been showed to be NP-hard on paths [14].

Hence, in all networks, it depends on the weights  $d$  induced by the traffic demands whether there is a gap between the weighted clique number  $\omega(I(\mathcal{P}), d)$  and the interval chromatic number  $\chi_I(I(\mathcal{P}), d)$ . To determine the size of this gap, we propose to extend the concept of  $\chi$ -binding functions introduced in [9] for usual coloring to interval coloring in weighted graphs, that is, to  $\chi_I$ -binding functions  $f$  with

$$\chi_I(I(\mathcal{P}), d) \leq f(\omega(I(\mathcal{P}), d))$$

for edge-intersection graphs  $I(\mathcal{P})$  in a certain class of networks and all possible non-negative integral weights  $d$ . We can identify for one of the studied families of minimal non-superperfect graphs such a  $\chi_I$ -binding function:

**Lemma 7** *If  $I(\mathcal{P})$  is an odd hole, then  $\chi_I(I(\mathcal{P}), d) \leq \frac{3}{2}\omega(I(\mathcal{P}), d)$  for all non-negative integral weights  $d$ .*

Note that a network of Spain together with its demands (see e.g. [13]) is a real instance where a natural routing  $\mathcal{P}$  yields an edge-intersection graph  $I(\mathcal{P})$  with several non-superperfect subgraphs, including an odd hole  $C_7$  that attains the worst-case bound of the  $\chi_I$ -binding function.

It is clearly of interest to study such  $\chi_I$ -binding functions for the other families of minimal non-superperfect graphs and to identify a hierarchy of graph classes between trees and sparse planar graphs resembling the structure of modern optical networks in terms of the gap between  $\omega_I(I(\mathcal{P}), d)$  and  $\chi_I(I(\mathcal{P}), d)$ .

Furthermore, in networks different from trees, the routing part of the RSA problem is crucial and raises the question whether it is possible to select the

routes in  $\mathcal{P}$  in such a way that neither non-superperfect graphs nor unnecessarily large weighted cliques occur in  $I(\mathcal{P})$ .

Note that the studied families of minimal non-superperfect graphs are not an exhaustive list. In fact, giving a complete list of such graphs is an open problem, so that our future work comprises to find more minimal non-superperfect graphs and to examine the here addressed questions for them.

Finally, edge-intersection graphs of paths in a path (resp. a cycle) correspond to the well-studied class of interval graphs (resp. circular-arc graphs). Can we characterize (by some combinatorial properties) the graph class that corresponds to all edge-intersection graphs of paths in a tree? The results from Section 2 imply that this needs to be a superclass of interval graphs and a subclass of  $\overline{P}_6$ -free graphs.

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## Appendix

We here provide a network of Spain together with its demands (taken from e.g. [13]) as a real instance of the RSA problem, see Fig. 16 for the network and Table 1 for a subset of demands.

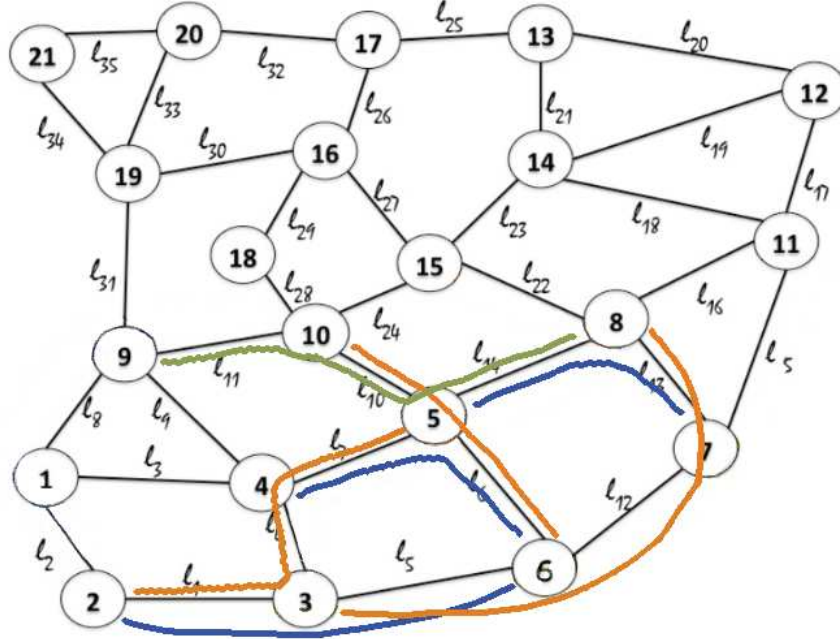


Figure 16: The network of Spain together with a routing  $\mathcal{P}$  of a subset of demands with  $I(\mathcal{P}) = C_7$ .

index	origin $s$	destination $t$	demand $d_{st}$
1	6	2	3
2	3	8	3
3	5	7	3
4	8	9	3
5	10	6	3
6	4	6	3
7	2	5	3

Table 1: The considered subset of demands

A natural routing  $\mathcal{P}$  of this subset of demands (see again Fig. 16) yields an edge-intersection graph  $I(\mathcal{P})$  equal to an odd hole  $C_7$ :

- $P_{6,2}$  and  $P_{3,8}$  share link  $l_5$ ,
- $P_{3,8}$  and  $P_{5,7}$  share link  $l_{13}$ ,
- $P_{5,7}$  and  $P_{8,9}$  share link  $l_{14}$ ,
- $P_{8,9}$  and  $P_{10,6}$  share link  $l_{10}$ ,
- $P_{10,6}$  and  $P_{4,6}$  share link  $l_6$ ,
- $P_{4,6}$  and  $P_{2,5}$  share link  $l_7$ ,
- $P_{2,5}$  and  $P_{6,2}$  share link  $l_1$ .

The odd hole  $C_7$  together with the weights  $d = (3, \dots, 3)$  attains the worst-case bound of the  $\chi_I$ -binding function as it has  $\omega(C_7, d) = 6 < 9 = \chi_I(C_7, d)$ .