A Decomposition Method for a unilateral Contact Problem With Tresca Friction Arising in Electro-elastostatics

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Abstract

This paper is concerned with the numerical modeling of unilateral contact problems in an electro-elastic material with Tresca friction law and electrical conductivity condition. First, we prove the existence and uniqueness of the weak solution of the model. Rather than deriving a solution method for the full coupled problem, we present and study a successive iterative (decomposition) method. The idea is to solve successively a displacement subproblem and an electric potential subproblem, in block Gauss-Seidel fashion. The displacement subproblem leads to a constraint non-differentiable (convex) minimization problem for which we propose an augmented Lagrangian algorithm. The electric potential unknown is computed explicitly using the Riesz's representation theorem. The convergence of the iterative decomposition method is proved. Some numerical experiments are carried out to illustrate the performances of the proposed algorithm.

Key-words: Signorini contact problem, Electro-elastostatics, Tresca friction, augmented Lagrangian.

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1 Introduction

Currently there is a considerable interest in contact problems involving piezoelectric materials, i.e. materials characterized by the coupling between the mechanical and electrical properties. This coupling, in a piezoelectric material, leads to the appearance of electric potential when mechanical stress is present and, conversely, mechanical stress is generated when electric potential is applied. The piezoelectric materials can be divided in two main

groups: crystals and ceramics. The most well-known piezoelectric materials are quartz SiO2 and ceramics (BaTiO3, KNbO3, LiNbO3, etc.). General models for elastic materials with piezoelectric effects can be found in [15, 18] and, more recently, in [1, 7]. The static frictional contact problems for electro-elastic materials were studied in [16, 17], under the assumption that the foundation is insulated, and in [3] under the assumption that the foundation is electrically conductive. Recent modeling, analysis and numerical simulations of contact with or without friction for piezoelectric materials can be found in [2, 14] and references therein.

We propose in this paper a decomposition method for an unilateral contact problem with Tresca friction. Rather than deriving a solution method for the full coupled problem, we propose and study a successive iterative (decomposition) method. Let u and φ be the displacement vector field and electric potential, respectively. The idea is to solve successively a subproblem in u and a subproblem in φ, in block Gauss-Seidel fashion. The subproblem in u leads to a constraint non-differentiable (convex) minimization problem for which we propose an augmented Lagrangian (Uzawa) algorithm. The electric potential φ is computed explicitly using the Riesz's representation theorem. We prove the convergence of our method. Some numerical experiments show the behavior of our method.

The rest of the paper is structured as follows. In Section 2, we describe the model of the frictional contact process between electro-elastic body and a conductive foundation and derive the variational formulation. In Section 3, We state our main result, the existence and uniqueness of the weak solution of the model. The decomposition algorithm is detailed in Section 4 followed by the convergence result. In Section 5 we report some numerical experiments carried out with the new algorithm.

2 Mechanical and variational formulation

We consider a piezoelectric body that initially occupies an open bounded subset Ω in $\mathbb{R}^d$, $d = 2, 3$, with a sufficiently smooth boundary $\Gamma = \partial\Omega$. This boundary is divided into three open disjoint parts Γ_D, Γ_N and Γ_C on the one hand and a partition of $\Gamma_D \cup \Gamma_N$ into two open parts Γ_a and Γ_b on the other hand, such that $meas(\Gamma_D) > 0$ and $meas(\Gamma_a) > 0$. The body is submitted to the action of body forces of density f_0 and volume electric charges of density q_0. It is also submitted to mechanical and electric constraint on the boundary. On the boundary part Γ_D we assume that the body is clamped and therefore the displacement field vanishes there. Moreover, we assume that a density of traction forces, denoted by f_N , acts on the boundary part Γ_N. We also assume that the electrical potential vanishes on Γ_a and a surface electrical charge of density q_b is prescribed on Γ_b. In the deformed configuration the body may come in contact over Γ_C with an electrically conductive foundation, we assume that its potential is maintained at φ_F. The contact is frictional and there may be electrical charges on the contact surface. The normalized gap between Γ_C and the rigid foundation is denoted by g.

Here and below, to simplify the notation, we do not indicate the dependence of various functions on the spatial variable $x \in \overline{\Omega}$, the indices i, j, k, l take values in $\{1, \ldots, d\}$, the summation convention over repeated indices is used and the index that follows a comma

indicates a partial derivative with respect to the corresponding component of the spatial variable, i.e. $u_{i,j} = \partial u_i/\partial x_j$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ or equivalently, the space of real symmetric matrices of order d. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$\begin{aligned} u \cdot v &= u_i \cdot v_i \quad ; \ \|v\| = (v \cdot v)^{\frac{1}{2}} \quad \forall u,\ v \in \mathbb{R}^d \\ \sigma \cdot \tau &= \sigma_{ij} \cdot \tau_{ij} \ ; \ \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}} \quad \forall \sigma,\ \tau \in \mathbb{S}^d. \end{aligned}$$

Moreover, we denote by $\operatorname{Div} \sigma = (\sigma_{ij,j})$, $\operatorname{div} D = (D_{j,j})$ the divergence operator for tensor and vector valued functions, respectively.
The governing equations of piezoelectricity consist of the equilibrium equation, constitutive relations, strain-mechanical displacement and electric field-potential relations.
The elastic strain-displacement and electric field-potential relations are given by

$$\begin{aligned} \varepsilon(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^*\right) \quad \text{in } \Omega, \\ E(\varphi) &= -\nabla \varphi \quad \text{in } \Omega, \end{aligned}$$

where $\varepsilon(u) = (\varepsilon_{ij}(u))$, $E(\varphi) = (E_i(\varphi))$, $u = (u_i)$, φ are, respectively, the linear strain tensor, quasi-static electric field vector, displacement vector field and electric potential.
We suppose that the process is static. The equations of stress equilibrium and the equation of quasi-stationary electric field are, respectively, given by

$$\operatorname{Div} \sigma + f_0 = 0 \quad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} D = q_0 \quad \text{in } \Omega, \tag{2.2}$$

where $\sigma = (\sigma_{ij})$ and $D = (D_i)$ represent the stress tensor and the electric displacement field, respectively.
The constitutive equations of a linear piezoelectric material can be written as

$$\sigma = \mathfrak{F}\, \varepsilon(u) - \mathcal{E}^* E(\varphi) \quad \text{in } \Omega, \tag{2.3}$$

$$D = \mathcal{E}\, \varepsilon(u) + \beta\, E(\varphi) \quad \text{in } \Omega, \tag{2.4}$$

where $\mathfrak{F} = (f_{ijkl})$, $\mathcal{E} = (e_{ijk})$ and $\beta = (\beta_{ij})$ are respectively, (fourth-order) elastic, (third-order) piezoelectric, electric permittivity tensors and $\mathcal{E}^* = (e^*_{ijk})$ is the transpose of $\mathcal{E}$. We mention that $\mathcal{E}^* = (e_{kij})$ and satisfies

$$\mathcal{E}\sigma v = \sigma \mathcal{E}^* v, \quad \forall \sigma \in \mathbb{S}^d, v \in \mathbb{R}^d. \tag{2.5}$$

Next, to complete the mechanical model according to the description of the physical setting, we have the following boundary conditions

$$u = 0 \quad \text{on } \Gamma_D, \tag{2.6}$$

$$\sigma \nu = f_N \quad \text{on } \Gamma_N, \tag{2.7}$$

$$\varphi = 0 \quad \text{on } \Gamma_a, \tag{2.8}$$

$$D \cdot \nu = q_2 \quad \text{on } \Gamma_b, \tag{2.9}$$

Finally, we need to prescribe the contact conditions on Γ_C. To this end, it is usual to decompose the displacement and the stress vector in normal and tangential components as follows:

$$\begin{aligned} v_n &= v \cdot n \quad ; \; v_\tau = v - v_n n \\ \sigma_n &= \sigma n \cdot n \; ; \; \sigma_\tau = \sigma n - \sigma_n n. \end{aligned}$$

We model the frictional contact on the contact surface Γ_C with Signorini, Tresca's law and electrical conductivity conditions given by

$$\sigma_n(u) \le 0, \quad u_n - g \le 0, \quad \sigma_n(u)\,(u_n - g) = 0 \quad \text{on } \Gamma_C, \tag{2.10}$$

$$\left.\begin{array}{l} \|\sigma_\tau\| \le S \\ \|\sigma_\tau\| < S \Rightarrow u_\tau = 0 \\ \|\sigma_\tau\| = S \Rightarrow \exists \lambda \ne 0,\ \sigma = -\lambda u_\tau \end{array}\right\} \text{ on } \Gamma_C, \tag{2.11}$$

$$D \cdot \nu = \psi(u_\nu - g)\phi_L(\varphi - \varphi_f) \quad \text{on } \Gamma_C. \tag{2.12}$$

The boundary condition (2.12) represents a regularized electrical contact condition, similar to that used in [13], such that

$$\phi_L(s) = \begin{cases} -L & \text{if } s < -L, \\ s & \text{if } -L \le s \le L, \\ L & \text{if } s > L, \end{cases} \qquad \psi(r) = \begin{cases} 0 & \text{if } r < 0, \\ k_e \delta r & \text{if } 0 \le r \le 1/\delta, \\ k_e & \text{if } r > 1/\delta, \end{cases}$$

in which L is a large positive constant, $\delta > 0$ denotes a small parameter and $k_e \ge 0$ is the electrical conductivity coefficient. Note that when $\psi \equiv 0$, the equality (2.12) leads to

$$D \cdot \nu = 0 \quad \text{on } \Gamma_3. \tag{2.13}$$

The condition (2.13) models the case when the obstacle is a perfect insulator. Under these conditions, the mechanical problem may be formulated as follows.

Problem (P)**.** *Find a displacement field $u : \Omega \to \mathbb{R}^d$, a stress field $\sigma : \Omega \to \mathbb{S}$, an electric potential $\varphi : \Omega \to \mathbb{R}$ and an electric displacement field $D : \Omega \to \mathbb{R}^d$ such that* (2.1)-(2.12) *hold.*

In the next, we introduce the notations and recall some necessary definitions which we need later. Moreover, we introduce the following functional spaces:

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d,$$

$$\mathcal{H} = \{\tau = (\tau_{ij}) \,|\, \tau_{ij} = \tau_{ji} \in L^2(\Omega)\}, \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H} \,|\, \mathrm{Div}\, \sigma \in H\}.$$

These are real Hilbert spaces endowed with the inner products

$$(u, v)_H = \int_\Omega u_i v_i \, dx, \quad (u, v)_{H_1} = (u, v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

$$(\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}\,dx, \quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\mathrm{Div}\,\sigma, \mathrm{Div}\,\tau)_{\mathcal{H}},$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_{\mathcal{H}_1}$, respectively.

Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace map. For every element $v \in H_1$, we also use the notation v_γ to denote the trace γv of v on Γ.

Let $H_\Gamma^{'}$ be the dual of H_Γ and let $\langle\cdot,\cdot\rangle$ denote the duality pairing between $H_\Gamma^{'}$ and H_Γ. For every $\sigma \in \mathcal{H}_1$, $\sigma\nu$ can be defined as the element in $H_\Gamma^{'}$ which satisfies

$$\langle \sigma\nu, \gamma v\rangle = (\sigma, \varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}\,\sigma, v)_H, \quad \forall v \in H_1. \tag{2.14}$$

Moreover, if σ is continuously differentiable on $\overline{\Omega}$, then

$$\langle \sigma\nu, \gamma v\rangle = \int_\Gamma \sigma\nu \cdot v\; da. \tag{2.15}$$

for all $v \in H_1$, where da is the surface measure element. Keeping in mind the boundary condition (2.6), we introduce the closed subspace of H_1 defined by

$$V = \{v \in H_1 \,|\, v = 0 \text{ on } \Gamma_D\},$$

and K be the set of admissible displacements

$$K = \{v \in V \,|\, v_\nu - g \le 0 \text{ on } \Gamma_C\}.$$

Since $meas(\Gamma_D) > 0$ and Korn's inequality holds,

$$\|\varepsilon(v)\|_{\mathcal{H}} \ge c_k \|v\|_{H_1}, \quad \forall v \in V, \tag{2.16}$$

where $c_k > 0$ is a constant which depends only on Ω and Γ_D. Over the space V we consider the inner product given by

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \quad \|u\|_V = (u,u)_V^{\frac{1}{2}}, \tag{2.17}$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (2.16) that the norms $\|\cdot\|_{H_1}$and $\|\cdot\|_V$ are equivalent on V. Therefore $(V, \|\cdot\|_V)$ is a Hilbert space. Moreover, by the Sobolev trace theorem, (2.16) and (2.17) there exists a constant $c_0 > 0$ which depends only on the domain Ω, Γ_C and Γ_D such that

$$\|v\|_{L^2(\Gamma)^d} \le c_0 \|v\|_V, \quad \forall v \in V. \tag{2.18}$$

We also introduce the spaces

$$\begin{aligned} W &= \{\psi \in H^1(\Omega) \,|\, \psi = 0 \text{ on } \Gamma_a\}, \\ \mathcal{W} &= \{D = (D_i) \in H^1(\Omega) \,|\, (D_i) \in L^2(\Omega),\ \ \mathrm{div}\, D \in L^2(\Omega)\}. \end{aligned}$$

The spaces W and $\mathcal{W}$ are real Hilbert spaces with the inner products

$$(\varphi,\psi)_W = (\varphi,\psi)_{H^1(\Omega)}, \quad (D,E)_{\mathcal{W}} = (D,E)_{L^2(\Omega)^d} + (\mathrm{div}\, D, \mathrm{div}\, E)_{L^2(\Omega)}.$$

The associated norms will be denoted by $\|\cdot\|_W$ and $\|\cdot\|_{\mathcal{W}}$, respectively. Notice also that, since $meas(\Gamma_a) > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla\psi\|_{\mathcal{W}} \geq c_F\|\psi\|_W, \quad \forall\psi \in W, \tag{2.19}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. Moreover, by the Sobolev trace theorem, there exists a constant c_1, depending only on Ω, Γ_a and Γ_C, such that

$$\|\xi\|_{L^2(\Gamma_3)} \leq c_1\|\xi\|_W, \quad \forall\xi \in W. \tag{2.20}$$

When $D \in \mathcal{W}$ is a sufficiently regular function, the following Green's type formula holds,

$$(D, \nabla\xi)_{L^2(\Omega)^d} + (\operatorname{div} D, \xi)_{L^2(\Omega)} = \int_\Gamma D \cdot \nu\xi \; da, \quad \forall\xi \in H^1(\Omega). \tag{2.21}$$

We assume that the elasticity tensor $\mathfrak{F}$, the piezoelectric tensor $\mathcal{E}$, the electric permittivity tensor β and The surface electrical conductivity function ψ satisfy the following assumptions.

- (h_1) The elasticity operator $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies $f_{ijkl} = f_{jikl} = f_{lkij} \in L^\infty(\Omega)$ and $f_{ijkl}(x)\ \xi_k\xi_l \geq m_{\mathfrak{F}}\ \|\xi\|^2 \quad \forall\xi \in \mathbb{S}^d\ ,\ \forall x \in \Omega$ where $m_{\mathfrak{F}}$ is positive constant.
- (h_2) The piezoelectric tensor $\mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies $\mathcal{E} = (e_{ijk})$, $e_{ijk} = e_{ikj} \in L^\infty(\Omega)$.
- (h_3) The electric permittivity tensor $\beta : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies $\beta_{ij} = \beta_{ji} \in L^\infty(\Omega)$ and $\beta_{ij}\ E_iE_j \geq m_\beta\ \|E\|^2$ for all $E \in \mathbb{R}^d$ and $x \in \Omega$, where m_β is a positive constant.
- (h_4) The surface electrical conductivity $\psi : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ is a bounded function by a positive constant M_ψ, such as, $x \to \psi(x, u)$ is measurable on Γ_3, for all $u \in \mathbb{R}$ and is zero for all $u \leq 0$.
- (h_5) The function $u \to \psi(x, u)$ is a Lipschitz function on $\mathbb{R}$ for all $x \in \Gamma_3$; $|\psi(x, u_1) - \psi(x, u_2)| \leq L_\psi|u_1 - u_2| \quad \forall u_1, u_2 \in \mathbb{R}$, where L_ψ is a positive constant.

The forces, traction, volume and surface free charge densities satisfy

- (h_6) $f_0 \in L^2(\Omega)^d, \qquad f_N \in L^2(\Gamma_N)^d,$
- (h_7) $q_0 \in L^2(\Omega), \qquad q_b \in L^2(\Gamma_b).$

The potential of the contact surface satisfies

- (h_8) $\varphi_F \in L^2(\Gamma_C).$

Finally, the friction bound function satisfies

- (h_9) $S \in L^\infty(\Gamma_C), \quad S \geq 0.$

Next, we use Riesz's representation theorem, consider the elements $f \in V$, and $q \in W$ given by

$$(f, v)_V = \int_\Omega f_0 \cdot v \; dx + \int_{\Gamma_N} f_N \cdot v \; da, \qquad \forall v \in V, \tag{2.22}$$

$$(q, \xi)_W = \int_\Omega q_0 \xi \; dx - \int_{\Gamma_b} q_b \xi \; da, \qquad \forall \xi \in W, \tag{2.23}$$

and, we define the mappings $j : V \times V \to \mathbb{R}$ and $\ell : V \times W \times W \to \mathbb{R}$, respectively, by

$$j(v) = \int_{\Gamma_C} S \; |v_\tau| \; da, \qquad \forall v \in V. \tag{2.24}$$

$$\ell(u, \varphi, \xi) = \int_{\Gamma_C} \psi(u_\nu - g)\phi_L(\varphi - \varphi_0)\xi \; da, \qquad \forall u \in V, \; \forall \varphi, \xi \in W, \tag{2.25}$$

where in (2.24), $|\cdot|$ stands for the Euclidean norm. Keeping in mind assumptions (h_4)-(h_8)-(h_9), it follows that the integrals in (2.24)-(2.25) are well-defined.

Using Green's formula (2.14), (2.15) and (2.21) it is straightforward to see that if $(u, \; \sigma, \; \varphi, \; D)$ are sufficiently regular function which satisfy (2.3)-(2.12), then

$$\big(\sigma, \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{H}} + j(v) - j(u) \geq (f, v - u)_V, \qquad \forall v \in K, \tag{2.26}$$

$$\big(D, \nabla\xi\big)_{L^2(\Omega)^d} = \ell(u, \varphi, \xi) - (q, \xi)_W, \qquad \forall \xi \in W. \tag{2.27}$$

We plug (2.1) in (2.26), (2.2) in (2.27) and use the notation $E = -\nabla\varphi$ to obtain the following variational formulation of Problem (P), in terms of displacements field and electric potential.

Problem (PV) *Find a displacement field $u \in K$ and an electric potential $\varphi \in W$*, such that :

$$\begin{aligned} \big(\mathfrak{F}\varepsilon(u), \varepsilon(v) - \varepsilon(u)\big)_{\mathcal{H}} + \big(\mathcal{E}^*\nabla\varphi, \varepsilon(v) - \varepsilon(u)\big)_{L^2(\Omega)^d} \; &+ \; j(v) - j(u) \\ &\geq \; (f, v - u)_V, \quad \forall v \in K, \end{aligned} \tag{2.28}$$

$$\big(\beta\nabla\varphi, \nabla\xi\big)_{L^2(\Omega)^d} - \big(\mathcal{E}\varepsilon(u), \nabla\xi\big)_{L^2(\Omega)^d} + \ell(u, \varphi, \xi) \; = \; (q, \xi)_W, \quad \forall \xi \in W. \tag{2.29}$$

3 Existence and uniqueness results

The existence of a unique weak solution of problem (PV) is stated in the following

Theorem 3.1. *Assume that (h_1)-(h_4) and (h_8)-(h_9) hold. Then :*

(1) Problem (PV) has at least one solution $(u, \varphi) \in K \times W$;

(2) Under the assumption (h_5), there exists $L^* > 0$ such that if $L_\psi L + M_\psi < L^*$. Then Problem (PV) has a unique solution.

The proof of Theorem 3.1 will be carried out in several steps and it is based on arguments for elliptic variational inequalities and fixed point. We suppose in the sequel that the assumption of Theorem 3.1 are fulfilled and we consider the product spaces $X = V \times W$ and $Y = L^2(\Gamma_3) \times L^2(\Gamma_3)$ together with the inner products

$$(x, y)_X = (u, v)_V + (\varphi, \xi)_W, \qquad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{3.1}$$

$$(\eta, \theta)_Y = (g, \lambda)_{L^2(\Gamma_3)} + (z, \zeta)_{L^2(\Gamma_3)}, \qquad \forall \eta = (g, z),\ \theta = (\lambda, \zeta) \in Y, \tag{3.2}$$

and the associated norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $U = K \times W$ be non-empty closed convex subset of X. We define the operator $A : X \to X$, the functions $\widetilde{j}$ on X, $\widetilde{\ell}$ on $X \times X$ and the element $f^e \in X$ by equalities:

$$\begin{aligned}(Ax, y)_X = {} & (\mathfrak{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}} + (\beta\nabla\varphi, \nabla\xi)_{L^2(\Omega)^d} + (\mathcal{E}^*\nabla\varphi, \varepsilon(v))_{L^2(\Omega)^d} \\ & - (\mathcal{E}\varepsilon(u), \nabla\xi)_{L^2(\Omega)^d}, \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X,\end{aligned} \tag{3.3}$$

$$\widetilde{j}(y) = j(v), \quad \forall y = (v, \xi) \in X, \tag{3.4}$$

$$\widetilde{\ell}(x, y) = \int_{\Gamma_3} \psi(u_\nu)\phi_L(\varphi - \varphi_0)\xi \, da, \quad \forall x = (u, \varphi),\ y = (v, \xi) \in X, \tag{3.5}$$

$$f^e = (f, q) \in X. \tag{3.6}$$

We start by the following equivalence result

Lemma 3.2. *The pair $x = (u, \varphi)$ is a solution of problem (PV) if and only if:*

$$(Ax, y - x)_X + \widetilde{j}(y) - \widetilde{j}(x) + \widetilde{\ell}(x, y - x) \geq (f^e, y - x)_X, \forall y = (v, \xi) \in U. \tag{3.7}$$

Proof. Let $x = (u, \varphi) \in U$ be a solution to problem PV and let $y = (v, \xi) \in U$. We use the test function $\xi - \varphi$ in (2.29), add the corresponding inequality to (2.28) and use (3.1) and (3.3)-(3.6) to obtain (3.7). Conversely, let $x = (u, \varphi) \in U$ be a solution to the elliptic variational inequalities (3.7). We take $y = (v, \varphi)$ in (3.7) where v is an arbitrary element of K and obtain (2.28). Then for any $\xi \in W$, we take successively $y = (v, \varphi + \xi)$ and $y = (v, \varphi - \xi)$ in (3.7) to obtain (2.29), which concludes the proof of lemma 3.2. □

Let $\eta \in L^2(\Gamma_C)$ be given, and we define the closed convex set

$$\mathcal{K} = \{\eta \in L^2(\Gamma_C) \ / \ \|\eta\|_{L^2(\Gamma_C)} \leq k\},$$

where the constants k is given by

$$k \ = \ M_\psi \, L \, meas(\Gamma_C)^{\frac{1}{2}}. \tag{3.8}$$

We also define the function

$$\ell_\eta(\xi) = \int_{\Gamma_C} \eta \ \xi da, \quad \forall \xi \in W, \tag{3.9}$$

and we construct the following intermediate problem.

Problem (PV_η). *Find* $x_\eta \in U$ *such that*:

$$(Ax_\eta, y - x_\eta)_X + \widetilde{j}(y) - \widetilde{j}(x_\eta) \geq (f_\eta^e, y - x_\eta)_X, \quad \forall\, y = (v, \xi) \in U. \tag{3.10}$$

Where

$$(f_\eta^e, y - x_\eta)_X = (f^e, y - x_\eta)_X - \ell_\eta(\xi) \quad \forall\, y = (v, \xi) \in U. \tag{3.11}$$

We have the following existence and uniqueness result.

Lemma 3.3. *For any* $\eta \in \mathcal{K}$, *assume that* (h_1)-(h_3) *hold. Then :*

(i) The problem PV_η *has a unique solution* $x_\eta = (u_\eta, \varphi_\eta) \in K \times W$ *which depends Lipschitz continuously on* $\eta \in L^2(\Gamma_C)$.

(ii) There exists a constant $c_2 > 0$ *such that the solution of problem* PV_η *satisfies*

$$\|x_\eta\|_X \leq c_2 \|f_\eta^e\|_X. \tag{3.12}$$

Proof. Let the operator A and the functional $\widetilde{j}$ given by (3.3) and (3.4), respectively. We prove that :

(a) The operator $A : X \to X$ is strongly monotone and Lipschitz continuous.

(b) The functional $\widetilde{j}$ is proper, convex and continuous.

First, consider two elements $x_1 = (u_1, \varphi_1),\ x_2 = (u_2, \varphi_2) \in X$, using (3.3) we have

$$\begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X &= \big(\mathfrak{F}\varepsilon(u_1) - \mathfrak{F}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2)\big)_{\mathcal{H}} \\ &\quad + \big(\beta\nabla\varphi_1 - \beta\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2\big)_{L^2(\Omega)^d} \\ &\quad + \big(\mathcal{E}^*\nabla\varphi_1 - \mathcal{E}^*\nabla\varphi_1, \varepsilon(u_1) - \varepsilon(u_2)\big)_{L^2(\Omega)^d} \\ &\quad - \big(\mathcal{E}\varepsilon(u_1) - \mathcal{E}\varepsilon(u_2), \nabla\varphi_1 - \nabla\varphi_2\big)_{L^2(\Omega)^d},\end{aligned}$$

and, since if follows by (2.5) that $(\mathcal{E}^*\nabla\varphi, \varepsilon(u))_{\mathcal{H}} = (\mathcal{E}\varepsilon(u), \nabla\varphi)_{L^2(\Omega)^d}$ for all $x = (u, \varphi)$, we find

$$\begin{aligned}(Ax_1 - Ax_2, x_1 - x_2)_X &= \big(\mathfrak{F}\varepsilon(u_1) - \mathfrak{F}\varepsilon(u_2), \varepsilon(u_1) - \varepsilon(u_2)\big)_{\mathcal{H}} \\ &\quad + \big(\beta\nabla\varphi_1 - \beta\nabla\varphi_2, \nabla\varphi_1 - \nabla\varphi_2\big)_{L^2(\Omega)^d}.\end{aligned}$$

We use now (2.20), (h_1) and (h_3) there exists $m_A > 0$ which depends only on $\mathfrak{F}$, β, Ω and Γ_a such that

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A(\|u_1 - u_2\|_V^2 + \|\varphi_1 - \varphi_2\|_W^2),$$

and, keeping in mind (3.1), we obtain

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A \|x_1 - x_2\|_X^2. \tag{3.13}$$

In the same way, using (h_1)-(h_3), after some algebra it follows that there exists $c_A > 0$ which depends only on $\mathfrak{F}$, β and $\mathcal{E}$ such that

$$\begin{aligned}(Ax_1 - Ax_2, y)_X &\leq c_A\big(\|u_1 - u_2\|_V\|v\|_V + \|\varphi_1 - \varphi_2\|_W\|v\|_V \\ &\quad + \|u_1 - u_2\|_V\|\xi\|_W + \|\varphi_1 - \varphi_2\|_W\|\xi\|_W\big),\end{aligned}$$

for all $y = (v, \xi) \in X$. We use (3.1) and the previous inequality to obtain

$$(Ax_1 - Ax_2, y)_X \leq 4c_A(\|x_1 - x_2\|_X \ \|y\|_X), \quad \forall\, y \in X,$$

and, taking $y = Ax_1 - Ax_2 \in X$, we find

$$\|Ax_1 - Ax_2\|_X \leq M_A\|x_1 - x_2\|_X, \tag{3.14}$$

where $M_A = 4c_A$. Now (a) is a consequence of inequalities (3.13) and (3.14). Next we investigate the properties of the functional $\widetilde{j}$ given by (3.4), (2.25). We first remark that $\widetilde{j_g}$ is proper and convex on U. Let $x_1 = (u_1, \varphi_1),\ x_1 = (u_2, \varphi_2) \in U$, we have

$$\begin{aligned}|\widetilde{j}(x_1) - \widetilde{j}(x_2)| &= |\int_{\Gamma_C} S(|u_{1\tau}| - |u_{2\tau}|)da| \leq \int_{\Gamma_C} S|u_{1\tau} - u_{2\tau}|da \\ &\leq \|S\|_{L^\infty(\Gamma_C)} meas(\Gamma_C)^{\frac{1}{2}}\|u_1 - u_2\|_{L^2(\Gamma_C)^d}.\end{aligned}$$

Using (2.19), we obtain

$$|\widetilde{j}(x_1) - \widetilde{j}(x_2)| \leq c_0\ \|S\|_{L^\infty(\Gamma_C)}\, meas(\Gamma_C)^{\frac{1}{2}}\ \|u - v\|_V.$$

Now, by (3.1), we find that

$$|\widetilde{j}(x_1) - \widetilde{j}(x_2)| \leq c_0\, \|S\|_{L^\infty(\Gamma_C)}\, meas(\Gamma_C)^{\frac{1}{2}}\ \|x_1 - x_2\|_X.$$

Thus $\widetilde{j}$ is Lipschitz continuous and, therefore, $\widetilde{j}$ is a fortiori lower semi-continuous function.

Moreover, using (3.9) and (3.6) it is easy to see that the function f_η^e defined by (3.11) is an element of X. Lemma 3.3 result now from (a), (b) and standard arguments of elliptic variational inequalities.

We show next that this solution depends Lipschitz continuously on $\eta \in L^2(\Gamma_C)$. Let $\eta_1, \eta_2 \in L^2(\Gamma_C)$ be given and denote the corresponding solution of the problem PV_η by $x_{\eta_1} = (u_{\eta_1}, \varphi_{\eta_1})$ and $x_{\eta_2} = (u_{\eta_2}, \varphi_{\eta_2})$. Then we have

$$(Ax_{\eta_1}, y - x_{\eta_1})_X + \widetilde{j}(y) - \widetilde{j}(x_{\eta_1}) \geq (f^e_{\eta_1}, y - x_{\eta_1})_X, \quad \forall\, y \in U,$$

$$(Ax_{\eta_2}, y - x_{\eta_2})_X + \widetilde{j}(y) - \widetilde{j}(x_{\eta_2}) \geq (f^e_{\eta_2}, y - x_{\eta_2})_X, \quad \forall\, y \in U.$$

We take $y = x_{\eta_2}$ in the first inequality and $y = x_{\eta_1}$ in the second inequality, successively, we obtain

$$\begin{aligned}(Ax_{\eta_1} - Ax_{\eta_2}, x_{\eta_1} - x_{\eta_2}) &\leq \int_{\Gamma_C} (\eta_1 - \eta_2)(\varphi_{\eta_1} - \varphi_{\eta_2})\, da \\ &\leq \|\eta_1 - \eta_2\|_{L^2(\Gamma_C)}\|\varphi_{\eta_1} - \varphi_{\eta_2}\|_{L^2(\Gamma_C)}.\end{aligned}$$

Thus, using (2.21) we deduce

$$(Ax_{\eta_1} - Ax_{\eta_2}, x_{\eta_1} - x_{\eta_2})_X \leq c_1 \|\eta_1 - \eta_2\|_{L^2(\Gamma_C)} \|\varphi_{\eta_1} - \varphi_{\eta_2}\|_W,$$

and, using (3.1) and (3.13)

$$\|x_{\eta_1} - x_{\eta_2}\|_X \leq \tfrac{c_1}{m_A} \|\eta_1 - \eta_2\|_{L^2(\Gamma_C)},$$

thus, there exists a positive constant $c_3 = \frac{c_1}{m_A}$ such that

$$\|x_{\eta_1} - x_{\eta_2}\|_X \leq c_3 \|\eta_1 - \eta_2\|_{L^2(\Gamma_C)}, \tag{3.15}$$

whence (i) follows. We turn now to the proof of (ii). Let $\eta \in \mathcal{K}$, we take $y = 0$ in the inequality (3.15), we have

$$(Ax_\eta, x_\eta)_X + \widetilde{j}_g(x_\eta) \leq (f_\eta^e, x_\eta)_X, \quad \forall\, x_\eta \in X.$$

As $S \geq 0$, we obtain

$$(Ax_\eta, x_\eta)_X \leq (f_\eta^e, x_\eta)_X, \quad \forall\, x_\eta \in X,$$

using (3.13), we deduce

$$\|x_\eta\|_X \leq \frac{1}{c_A} \|f_\eta^e\|_X.$$

□

We now consider the operator $\Lambda : L^2(\Gamma_C) \to L^2(\Gamma_C)$ such that for all $\eta \in L^2(\Gamma_C)$, we have

$$\Lambda\eta = \psi(u_{\eta\nu})\phi_L(\varphi_\eta - \varphi_0), \quad \forall\, \eta \in L^2(\Gamma_C), \tag{3.16}$$

it follows from assumptions (h_4) that the operator Λ is well-defined. In order to prove that Λ has a fixed point, we will need the following result

Lemma 3.4. *The mapping $\eta \to x_\eta$, where x_η is the solution to PV_η, is weakly continuous from $L^2(\Gamma_C)$ to X.*

Proof. Let a sequence (η_n) in $L^2(\Gamma_C)$ converging weakly to η, we denote by $x_{\eta_n} = (u_{\eta_n}, \varphi_{\eta_n}) \in U$ the solution of PV_η corresponding to η_n, then we have

$$(Ax_{\eta_n}, y - x_{\eta_n})_X + \widetilde{j}(y) - \widetilde{j}(x_{\eta_n}) \geq (f_{\eta_n}^e, y - x_{\eta_n})_X, \quad \forall\, y = (v, \xi) \in U, \tag{3.17}$$

where

$$(f_{\eta_n}^e, y - x_{\eta_n}^n)_X = (f, v - u_{\eta_n})_V + (q, \xi - \varphi_{\eta_n})_W - \ell_{\eta_n}(\xi - \varphi_{\eta_n}),$$

taking $y = 0$ in (3.17) and using (3.13), (2.20) and $S \geq 0$, we deduce

$$\|x_{\eta_n}\|_X \leq c\left(\|f\|_V + \|q\|_W + \|\eta_n\|_{L^2(\Gamma_C)}\right),$$

that is, the sequence (x_{η_n}) is bounded in X, then, there exists $\widetilde{x} = (\widetilde{u}, \widetilde{\varphi}) \in X$ and a subsequence, denote again (x_{η_n}), such that

$$x_{\eta_n} \rightharpoonup \widetilde{x} \in X, \quad \text{as } n \to +\infty.$$

Moreover, U is closed convex set in a real Hilbert space X, therefor U is weakly closed, then $\widetilde{x} \in U$.

We next prove that $\widetilde{x}$ is solution of (3.10). First we prove that

$$(f^e_{\eta_n}, y - x_{\epsilon\eta_n})_X \to (f^e_{\eta}, y - \widetilde{x})_X, \quad \text{as } n \to +\infty\cdot \tag{3.18}$$

We have

$$\begin{aligned} |\ell_{\eta_n}(\xi - \widetilde{\varphi}) - \ell_{\eta_n}(\xi - \varphi_{\eta_n})| &\leq \|\eta_n\|_{L^2(\Gamma_C)} \|\widetilde{\varphi} - \varphi_{\eta_n}\|_{L^2(\Gamma_C)} \\ &\leq \| \underbrace{\eta_n}_{bounded} \|_{L^2(\Gamma_C)} \|\widetilde{x} - x_{\eta_n}\|_{L^2(\Gamma_C)\times L^2(\Gamma_C)}. \end{aligned}$$

Since the trace map $\gamma : X \to L^2(\Gamma_C)^d \times L^2(\Gamma_C)$ is compact operator, from the weak convergence $x_{\eta_n} \rightharpoonup \widetilde{x}$ in X, we obtain the convergence $x_{\eta_n} \to \widetilde{x}$ strongly in $L^2(\Gamma_C)^d \times L^2(\Gamma_C)$. So we have (3.18).

Now, from (3.17), we have

$$\begin{aligned} (Ax_{\eta_n}, y - x_{\eta_n})_X &\geq (f^e_{\eta_n}, y - x_{\eta_n})_X - \big(\widetilde{j}(y) - \widetilde{j}(\widetilde{x})\big) \\ &\quad - \big(\widetilde{j}(\widetilde{x}) - \widetilde{j}(x_{\eta_n})\big), \quad \forall y = (v, \xi) \in U. \end{aligned}$$

Since

$$\begin{aligned} |\widetilde{j}(\widetilde{x}) - \widetilde{j}(x_{\eta_n})| &\leq \|S\|_{L^\infty(\Gamma_C)}\, meas(\Gamma_C)^{\frac{1}{2}}\, \|\widetilde{u} - u_{\eta_n}\|_{L^2(\Gamma_C)} \\ &\leq c \underbrace{\|\widetilde{x} - x_{\eta_n}\|_{L^2(\Gamma_C)\times L^2(\Gamma_C)}}_{\to 0}. \end{aligned}$$

By pseudomonotonicity of A and (3.17)-(3.18), we get

$$\begin{cases} \widetilde{x} \in U \\ (A\widetilde{x}, y - \widetilde{x})_X + \widetilde{j_g}(y) - \widetilde{j_g}(\widetilde{x}) \geq (f_\eta, y - \widetilde{x})_X, \quad \forall y = (v, \xi) \in U, \end{cases} \tag{3.19}$$

from (3.19) we find that $\widetilde{x}$ is a solution of problem PV_η and from the uniqueness of the solution for this variational inequality we obtain $\widetilde{x} = x_\eta$. Since x_η is the unique weak limit of any subsequence of (x_{η_n}), we deduce that the whole sequence (x_{η_n}) is weakly convergent in X to x_η, ensures that the weak continuous mapping $\eta \to x_\eta$, from $L^2(\Gamma_C) \times L^2(\Gamma_C)$ to X. □

Lemma 3.5. *Λ is an operator of $\mathcal{K}$ into itself and has at least one fixed point.*

Proof. Let $\eta \in \mathcal{K}$, i.e.

$$\|\eta\|_{L^2(\Gamma_C)} \leq k.$$

By (3.16), we have

$$\|\Lambda\eta\|_{L^2(\Gamma_C)} \leq \|\psi(u_{\eta\nu})\phi_L(\varphi_\eta - \varphi_0)\|_{L^2(\Gamma_C)},$$

using the properties of ψ and ϕ_L we obtain

$$\|\Lambda\eta\|_{L^2(\Gamma_C)} \leq M_\psi\, L\, meas(\Gamma_C)^{\frac{1}{2}},$$

and keeping in mind (3.8), we get

$$\|\Lambda\eta\|_{L^2(\Gamma_C)} \leq k,$$

then Λ is an operator of $\mathcal{K}$ into itself, and note that $\mathcal{K}$ is a nonempty, convex and closed subset of $L^2(\Gamma_C)$. Since $L^2(\Gamma_C)$ is a reflexive space, $\mathcal{K}$ is weakly compact. Using continuity of ϕ_L, ψ and lemma 3.4, we deduce that Λ is weakly continuous. Hence, by Schauder's fixed point theorem the operator Λ has at least one fixed point. □

Proof of theorem 3.1 :

1) Existence. Let η^* be the fixed point of operator Λ. We denote by $(u^*,\, \varphi^*)$ the solution of the variational problem PV_η for $\eta = \eta^*$. Using (3.10) and (3.16), it is easy to see that $(u^*,\, \varphi^*)$ is a solution of PV. This proves the existence part of theorem 3.1.

2) Uniqueness. We show next that if $L_\psi L + M_\psi < L^*$ the solution is unique.
Let $x_1 = (u_1, \varphi_1)$, $x_2 = (u_2, \varphi_2) \in U$ the solution of problem (3.7) we have

$$(Ax_1, y - x_1)_X + \widetilde{j}(y) - \widetilde{j}(x_1) + \widetilde{\ell}(x_1, y - x_1) \geq (f^e, y - x_1)_X, \tag{3.20}$$

$$(Ax_2, y - x_2)_X + \widetilde{j}(y) - \widetilde{j}(x_2) + \widetilde{\ell}(x_2, y - x_2) \geq (f^e, y - x_2)_X. \tag{3.21}$$

We take $y = x_2$ in the first inequality, $y = x_1$ in the second, and add the two inequality to obtain

$$(A_1x_1 - Ax_2, x_1 - x_2)_X \leq G, \tag{3.22}$$

where

$$G = \widetilde{\ell}(x_1, x_2 - x_1) + \widetilde{\ell}(x_2, x_1 - x_2). \tag{3.23}$$

From (3.23) and (3.5), we find

$$\begin{aligned} G = &\int_{\Gamma_C} \psi(u_{2\nu})\Big(\phi_L(\varphi_2 - \varphi_0) - \phi_L(\varphi_1 - \varphi_0)\Big)\Big(\varphi_1 - \varphi_2\Big)\, da \\ &+ \int_{\Gamma_C} \phi_L(\varphi_2 - \varphi_0)\Big(\psi(u_{2\nu}) - \psi(u_{1\nu})\Big)\Big(\varphi_1 - \varphi_2\Big)\, da, \end{aligned}$$

thus, by using (h_5), the bounds $|\phi_L(\varphi_2 - \varphi_0)| \leq L$, the Lipschitz continuity of the function ϕ_L, (2.18), (2.20) and (3.1) we deduce

$$G \leq (M_\psi\, c_1^2 + L\, L_\psi\, c_0\, c_1)\|x_1 - x_2\|_X^2. \tag{3.24}$$

Using (3.22) and (3.24) hence there exists a constant $c_* > 0$ such that

$$\|x_1 - x_2\|_X^2 \leq c_*(L_\psi\, L + M_\psi)\, \|x_1 - x_2\|_X^2.$$

Let $L^* = \frac{1}{c_*}$, then if $L_\psi\, L + M_\psi < L^*$ therefore $x_1 = x_2$.

Algorithm 1 Successive iterative decomposition method

Step 1 Find $u^{(n+1)} \in K$ such that

$$\begin{aligned}(\mathfrak{F}\varepsilon(u^{(n+1)}),\varepsilon(v)-\varepsilon(u^{(n+1)}))_{\mathcal{H}} \;+\; &(\mathcal{E}^*\nabla\varphi^{(n)},\varepsilon(v)-\varepsilon(u^{(n+1)}))_{L^2}+j(v)-j(u^{(n+1)})\\ &\geq\quad (f,v-u^{(n+1)})_V,\quad \forall v\in K \end{aligned}\tag{4.1}$$

Step 2 Find $\varphi^{(n+1)} \in W$ such that

$$\begin{aligned}(\beta\nabla\varphi^{(n+1)},\nabla\xi)_{L^2}-(\mathcal{E}\varepsilon(u^{(n+1)}),\nabla\xi)_{L^2} \quad &+\quad \ell(u^{(n+1)},\varphi^{(n+1)},\xi)\\ &=\quad (q,\xi)_W,\quad \forall\xi\in W. \end{aligned}\tag{4.2}$$

4 Successive iterative decomposition method

Rather than derive a solution method for the full coupled problem (2.28)-(2.29), we study a successive iterative (decomposition) method. The idea is to solve successively a subproblem in u and a subproblem in φ, in block Gauss-Seidel fashion. Given an initial guess $(u^{(0)},\varphi^{(0)})$, we define the sequence $((u^{(n)},\varphi^{(n)})) \in K\times W$ as in Algorithm 1.

4.1 Convergence

To study the convergence of the problem (4.1)-(4.2) we make the following smallness assumption

$$\inf(m_{\mathfrak{F}},m_\beta) > LL_\psi c_0c_1+c_{\mathcal{E}},\tag{4.3}$$

where $m_{\mathfrak{F}}$, m_β, L_ψ , c_0 and c_1 are given in (h_1), (h_2)(3.13), (h_5), (2.18), (2.20), respectively, and $c_{\mathcal{E}}$ is a positive constant which depends on the piezoelectric tensor $\mathcal{E}$.

We have the following convergence result.

Theorem 4.1. *Under the assumptions of theorem* (3.1) *and assumption* (4.3)*, the iteration method* (4.1)-(4.2) *converges, i.e.*

$$\begin{aligned}&\|u^{(n)}-u\|_V\to 0 \quad as \quad n\to\infty.\\ &\|\varphi^{(n)}-\varphi\|_W\to 0 \quad as \quad n\to\infty.\end{aligned}$$

Furthermore, for some constant $0<k<1$*, we have the estimate*

$$\begin{aligned}&\|u^{(n)}-u\|_V\leq ck^n.\\ &\|\varphi^{(n)}-\varphi\|_W\leq ck^n.\end{aligned}\tag{4.4}$$

Proof. We take $v=u^{(n+1)}$ in (2.28), $\xi=\varphi^{(n+1)}-\varphi$ in (2.29), and adding, we have

$$\begin{aligned}&\big(\mathfrak{F}\varepsilon(u),\varepsilon(u^{(n+1)})-\varepsilon(u)\big)_{\mathcal{H}}+\big(\beta\nabla\varphi,\nabla(\varphi^{(n+1)}-\varphi)\big)_{L^2(\Omega)^d}\\ +\;&\big(\mathcal{E}^*\nabla\varphi,\varepsilon(u^{(n+1)})-\varepsilon(u)\big)_{L^2(\Omega)^d}-\big(\mathcal{E}\varepsilon(u),\nabla(\varphi^{(n+1)}-\varphi)\big)_{L^2(\Omega)^d}\\ +\;&j(u^{(n+1)})-j(u)+\ell(u,\varphi,\varphi^{(n+1)}-\varphi)\geq(f,u^{(n+1)}-u)_V+(q,\varphi^{(n+1)}-\varphi)_W,\end{aligned}\tag{4.5}$$

and, taking $v = u$ in (4.1), $\xi = \varphi - \varphi^{(n+1)}$ in (4.2), we find

$$\begin{aligned}
&(\mathfrak{F}\varepsilon(u^{(n+1)}), \varepsilon(u) - \varepsilon(u^{(n+1)}))_{\mathcal{H}} + (\beta\nabla\varphi^{(n+1)}, \nabla(\varphi - \varphi^{(n+1)}))_{L^2(\Omega)^d} \\
+ \quad & (\mathcal{E}^*\nabla\varphi^{(n)}, \varepsilon(u) - \varepsilon(u^{(n+1)}))_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u^{(n+1)}), \nabla(\varphi - \varphi^{(n+1)}))_{L^2(\Omega)^d} \\
+ \quad & j(u) - j(u^{(n+1)}) + \ell(u^{(n+1)}, \varphi^{(n+1)}, \varphi - \varphi^{(n+1)}) \\
\geq \quad & (f, u - u^{(n+1)})_V + (q, \varphi - \varphi^{(n+1)})_W. \qquad (4.6)
\end{aligned}$$

We subtract (4.5) from (4.6), we have

$$\begin{aligned}
&\big(\mathfrak{F}\varepsilon(u^{(n+1)} - \mathfrak{F}\varepsilon(u)), \varepsilon(u^{(n+1)}) - \varepsilon(u)\big)_{\mathcal{H}} + \big(\beta\nabla\varphi^{(n+1)} - \beta\nabla\varphi, \nabla\varphi^{(n+1)} - \nabla\varphi)\big)_{L^2(\Omega)^d} \\
&+\ell(u, \varphi^{(n+1)}, \varphi^{(n+1)} - \varphi) - \ell(u, \varphi, \varphi^{(n+1)} - \varphi) \\
\leq \quad & (\mathcal{E}(\varepsilon(u^{(n+1)}) - \varepsilon(u)), \nabla\varphi^{(n+1)} - \nabla\varphi)_{L^2(\Omega)^d} - \big(\mathcal{E}^*(\nabla\varphi^{(n)} - \nabla\varphi), \varepsilon(u^{(n+1)}) - \varepsilon(u)\big)_{L^2(\Omega)^d} \\
&+\ell(u, \varphi^{(n+1)}, \varphi^{(n+1)} - \varphi) - \ell(u^{(n+1)}, \varphi^{(n+1)}, \varphi^{(n+1)} - \varphi),
\end{aligned}$$

and, by (h_4) combined with the monotonicity of the function ϕ_L, we obtain

$$\begin{aligned}
&\big(\mathfrak{F}\varepsilon(u^{(n+1)} - \mathfrak{F}\varepsilon(u)), \varepsilon(u^{(n+1)}) - \varepsilon(u)\big)_{\mathcal{H}} + \big(\beta\nabla\varphi^{(n+1)} - \beta\nabla\varphi, \nabla\varphi^{(n+1)} - \nabla\varphi)\big)_{L^2(\Omega)^d} \\
&\leq (\mathcal{E}(\varepsilon(u^{(n+1)}) - \varepsilon(u)), \nabla\varphi^{(n+1)} - \nabla\varphi)_{L^2(\Omega)^d} - \big(\mathcal{E}^*(\nabla\varphi^{(n)} - \nabla\varphi), \varepsilon(u^{(n+1)}) - \varepsilon(u)\big)_{L^2(\Omega)^d} \\
&+ \int_{\Gamma_C} \phi_L(\varphi - \varphi_0)\big(\psi(u_\nu^{(n+1)} - g) - \psi(u_\nu - g)\big)\big(\varphi^{(n+1)} - \varphi\big)\, da.
\end{aligned}$$

Using (h_1)-(h_3), (h_5), (2.18) and (2.20) we find

$$\begin{aligned}
&m_{\mathfrak{F}}\|u^{(n+1)} - u\|_V^2 + m_\beta\|\varphi^{(n+1)} - \varphi\|_W^2 \\
\leq\, & c_{\mathcal{E}}\|u^{(n+1)} - u\|_V\|\varphi^{(n+1)} - \varphi\|_W + c_{\mathcal{E}}\|u^{(n+1)} - u\|_V\|\varphi^{(n)} - \varphi\|_W \\
& + LL_\psi c_0 c_1\|u^{(n+1)} - u\|_V\|\varphi^{(n+1)} - \varphi\|_W.
\end{aligned}$$

Thus, using (3.1) we deduce

$$\|x^{(n+1)} - x\|_X \leq \frac{c_{\mathcal{E}}}{(\inf(m_{\mathfrak{F}}, m_\beta) - LL_\psi c_0 c_1 - c_{\mathcal{E}})}\|x^{(n)} - x\|_X. \qquad (4.7)$$

where $x = (u, \varphi)$ and $x^{(n+1)} = (u^{(n+1)}, \varphi^{(n+1)})$.
Under the stated assumption, $k = c_{\mathcal{E}}/(\inf(m_{\mathfrak{F}}, m_\beta) - LL_\psi c_0 c_1 - c_{\mathcal{E}}) < 1$, and we have the estimate (4.4). □

4.2 Augmented Lagrangian method for (4.1)

To formulate (4.1) as a constrained minimization problem, we set

$$f^{(n)}(v) = (f, v)_V - (\mathcal{E}^*\nabla\varphi^{(n)}, \varepsilon(v))_{L^2}$$

and introduce the energy functional due to non-frictional effects

$$J(v) = \frac{1}{2}(\mathfrak{F}\varepsilon(v), \varepsilon(v))_{\mathcal{H}} - f^{(n)}(v).$$

The quadratic functional J is strictly convex, coercive and Gâteaux differentiable on V. Moreover, the friction functional j is convex and lower semi-continuous on V. We can therefore replace the variational inequality (4.1) by the convex minimization problem

Find $u \in K$ such that

$$J(u) + j(u) \leq J(v) + j(v), \quad \forall v \in K. \tag{4.8}$$

Since the functional $J + j$ is strictly convex and coercive ($\text{mes}(\Gamma_D) > 0$), there exists a unique solution to (4.8), see e.g. [8, 10.3]. Note that the functional j is non-differentiable, then standard optimization methods cannot be used for (4.8). In [10, 11, 12], Uzawa block relaxation methods are proposed for solving contact problems with Tresca friction. Uzawa block relaxation through augmented Lagrangian was initiated by Glowinski and Marocco [6], Fortin and Glowinski [4], Glowinski and Le Tallec [5] who systematically develop augmented Lagrangian methods as technique for solving nonlinear partial differential equations.

To achieve a solution of (4.8) by a Uzawa block relaxation method, we introduce auxiliary unknowns $p = (p_c, p_f)$ (c for contact and f for friction) and the set

$$C = \left\{ p_c \in L^2(\Gamma_c), \quad p_c - g \leq 0 \text{ on } \Gamma_c \right\}$$

and its characteristic functional

$$I_C(p_c) = \begin{cases} 0 & \text{if } p_c \in C, \\ +\infty & \text{if } p_c \notin C. \end{cases}$$

The constrained minimization problem (4.8) is equivalent to the following minimization problem

Find $(u, p_c, p_f) \in V \times (L^2(\Gamma_c))^2$ such that

$$J(u) + j(p_f) + I_C(p_c) \leq J(v) + j(q_f) + I_C(q_c) \quad \forall (v, q_c, q_f) \in V \times (L^2(\Gamma_c))^2, \tag{4.9}$$

$$u_\nu - p_c = 0 \text{ on } \Gamma_c, \tag{4.10}$$

$$u_\tau - p_f = 0 \text{ on } \Gamma_c, \tag{4.11}$$

We associate with (4.9)-(4.11) the augmented Lagrangian functional $\mathscr{L}_r$ defined over $V \times (L^2(\Gamma_c))^2 \times (L^2(\Gamma_c))^2$ by

$$\begin{aligned} \mathscr{L}_r(v, p; \mu) = {} & J(v) + j(q_f) + I_C(q_c) + (\mu_c, v_\nu - q_c)_{\Gamma_c} + (\mu_f, v_\tau - q_f)_{\Gamma_c} \\ & + \frac{r}{2} \parallel v_\nu - q_c \parallel^2_{0,\Gamma_c} + \frac{r}{2} \parallel v_\tau - q_f \parallel^2_{0,\Gamma_c} \end{aligned} \tag{4.12}$$

where we have set $q = (q_c, q_f)$ and $\mu = (\mu_c, \mu_f)$, and $r > 0$ is the penalty parameter.

Uzawa block relaxation method is obtained by minimizing $\mathscr{L}_r$, successively, over u and p, as follows (starting with p^0 and λ^0)

$$u^{(k+1)} = \arg\min_v \mathscr{L}_r(v, p^{(k)}; \lambda^{(k)}) \tag{4.13}$$

$$p^{(k+1} = \arg\min_q \mathscr{L}_r(u^{(k+1)}, q; \lambda^{(k)}) \tag{4.14}$$

$$\lambda^{(k+1)} = \lambda^{(k)} + r(u^{(k+1} - p^{(k+1)}) \tag{4.15}$$

Since $v \mapsto \mathscr{L}_r(v, p^{(k)}; \lambda^{(k)})$ is convex and differentiable, the solution of (4.13) can be characterized by the Euler-Lagrange equation (see [10, 12])

$$\begin{aligned}(\mathfrak{F}\varepsilon(u^{(k+1)}), \varepsilon(v))_{\mathcal{H}} + r(u_\nu^{(k+1)}, v_\nu)_{\Gamma_c} + r(u_\tau^{(k+1)}, v_\tau)_{\Gamma_c} &= f^{(n)}(v) + (\tilde{p}_c^{(k)}, v_\nu)_{\Gamma_c} \\ &+ (\tilde{p}_f^{(k)}, v_\tau)_{\Gamma_c}, \ \forall v \in V \end{aligned} \tag{4.16}$$

where we have set $\tilde{p}_c^{(k)} = rp_c^{(k)} - \lambda_c^{(k)}$ and $\tilde{p}_f^{(k)} = rp_f^{(k)} - \lambda_f^{(k)}$.

Problem (4.14) is uncoupled in q_c and q_f. For the contact subproblem, straightforward calculations using Karush-Kuhn-Tucker optimality conditions yield to (see [10, 12])

$$p_c^{(k+1)} = u_\nu^{(k+1)} + \frac{1}{r}\left[\lambda_c^{(k)} - \max(0, \lambda_c^{(k)} + r(u_\nu^{(k+1)} - g))\right]. \tag{4.17}$$

For the friction subproblem, using the Fenchel duality theory we get (see [10, 12])

$$p_f^{(k+1)} = \begin{cases} \frac{|\lambda_f^{(k)} + ru_\tau^{(k+1)}| - s}{r|\lambda_f^{(k)} + ru_\tau^{(k+1)}|}\left(\lambda_f^{(k)} + ru_\tau^{(k+1)}\right) & \text{if } |\lambda_f^{(k)} + ru_\tau^{(k+1)}| > s, \\ 0 & \text{if } |\lambda_f^{(k)} + ru_\tau^{(k+1)}| \le s. \end{cases} \tag{4.18}$$

Gathering the above results, we obtain Algorithm 2. We iterate until the relative error on u^k, $p_c^{(k)}$ and $p_f^{(k)}$ is sufficiently "small", i.e.

$$\frac{\| u^{(k)} - u^{(k-1)} \|_{0,\Omega}^2 + \| p_c^{(k)} - p_c^{(k-1)} \|_{0,\Gamma_c}^2 + \| p_f^{(k)} - p_f^{(k-1)} \|_{0,\Gamma_c}^2}{\| u^{(k)} \|_{0,\Omega}^2 + \| p_c^{(k)} \|_{0,\Gamma_c}^2 + \| p_f^{(k)} \|_{0,\Gamma_c}^2} < \varepsilon^2. \tag{4.19}$$

4.3 Solution for (4.2)

The solution of (4.2) is straightforward. Indeed, we use Riesz's representation theorem to define the operators $\mathcal{B} : W \to W$and $\mathcal{C} : V \to W$ by:

$$(\mathcal{B}\varphi, \psi)_W = (\beta\nabla\varphi, \nabla\psi) + \ell(u, \varphi, \xi), \quad \forall\psi \in W. \tag{4.20}$$

$$(\mathcal{C}u, \psi)_W = (\mathcal{E}\varepsilon(u), \nabla\psi), \quad \forall\psi \in W. \tag{4.21}$$

Algorithm 2 Uzawa block relaxation algorithm for (4.8)

Initialization: $r > 0$, $p^{(0)} = (p_c^{(0)}, p_f^{(0)})$ and $\lambda^{(0)} = (\lambda_c^{(0)}, \lambda_f^{(0)})$ are given.

Iteration: $k \geq 0$. Compute successively $u^{(k+1)}$, $p^{(k+1)} = (p_c^{(k+1)}, p_f^{(k+1)})$ and $\lambda^{(k+1)} = (\lambda_c^{(k+1)}, \lambda_f^{(k+1)})$ as follows

Step 1 Set $\tilde{p}_c^{(k)} = rp_c^{(k)} - \lambda_c^{(k)}$, $\tilde{p}_f^{(k)} = rp_f^{(k)} - \lambda_f^{(k)}$. Find $u^{(k+1)} \in V$ such that

$$\begin{aligned}(\mathfrak{F}\varepsilon(u^{(k+1)}), \varepsilon(v))_{\mathcal{H}} + r(u_\nu^{(k+1)}, v_\nu)_{\Gamma_c} + r(u_\tau^{(k+1)}, v_\tau)_{\Gamma_c} &= f^{(n)}(v) + (\tilde{p}_c^{(k)}, v_\nu)_{\Gamma_c} \\ &+ (\tilde{p}_f^{(k)}, v_\tau)_{\Gamma_c}, \ \forall v \in V.\end{aligned}$$

Step 2 Compute the auxiliary contact and friction variables

$$p_c^{(k+1)} = u_\nu^{(k+1)} + \frac{1}{r}\left[\lambda_c^{(k)} - \max(0, \lambda_c^{(k)} + r(u_\nu^{(k+1)} - g))\right],$$

$$p_f^{(k+1)} = \begin{cases} \frac{|\lambda_f^{(k)} + ru_\tau^{(k+1)}| - s}{r|\lambda_f^{(k)} + ru_\tau^{(k+1)}|}\left(\lambda_f^{(k)} + ru_\tau^{(k+1)}\right) & \text{if } |\lambda_f^{(k)} + ru_\tau^{(k+1)}| > s, \\ 0 & \text{if } |\lambda_f^{(k)} + ru_\tau^{(k+1)}| \leq s. \end{cases}$$

Step 3 Update the Lagrange multipliers

$$\begin{aligned}\lambda_c^{(k+1)} &= \lambda_c^{(k)} + r(u_\nu^{(k+1)} - p_c^{(k+1)}), \\ \lambda_f^{(k+1)} &= \lambda_f^{(k)} + r(u_\tau^{(k+1)} - p_f^{(k+1)}).\end{aligned}$$

Hence, Equation (4.2) can be write in the following form

$$(\mathcal{B}\varphi^{(n+1)}, \psi)_W = (\mathcal{C}u^{(n+1)}, \xi)_W + (q, \xi)_W, \quad \forall \xi \in W. \tag{4.22}$$

Then, we deduce that

$$\mathcal{B}\varphi^{(n+1)} = \mathcal{C}u^{(n+1)} + q, \tag{4.23}$$

It is easy to see that the operator $\mathcal{B}$ is a strongly monotone Lipschitz continuous operator on W and therefore

$$\varphi^{(n+1)} = \mathcal{B}^{-1}\mathcal{C}u^{(n+1)} + \mathcal{B}^{-1}q, \tag{4.24}$$

In the case when the obstacle is a perfect insulator, the operator $\mathcal{B}$ is given by

$$(\mathcal{B}\varphi, \psi)_W = (\beta\nabla\varphi, \nabla\psi), \quad \forall \psi \in W. \tag{4.25}$$

4.4 Algorithm

With the above results, the solution method for (4.1)-(4.2) is presented in Algorithm 2. We iterate until relative error on $(u^{(n)}, \varphi^{(n)})$ becomes "sufficiently" small.

Algorithm 3 Solution method for (4.1)-(4.2)

Initialization: $u^{(0)}$ and $\varphi^{(0)}$ are given.

Iteration: $n \geq 0$. Compute successively $u^{(n+1)}$ and $\varphi^{(n+1)}$ as follows

Step 1 Compute $u^{(n+1)}$ using Algorithm 2

Step 2 Compute $\varphi^{(n+1)}$ as solution of

$$\mathcal{B}\varphi^{(n+1)} = \mathcal{C}u^{(n+1)} + q$$

5 Numerical experiments

Algorithm 3 of the previous section was implemented in Matlab, using piece-wise linear finite element and vectorized codes [9], on a Linux workstation with 2.67GHz clock frequency and 12GB RAM. The test problem used is designed to illustrate the behavior of the algorithm more than to model contact actual phenomena. The deformed configuration is plotted with a magnification of the displacement.

The stopping criterion in Algorithm 3 is

$$\frac{\| u^{(n)} - u^{(n-1)} \|^2_{L^2(\Omega)} + \| \varphi^{(n)} - \varphi^{(n-1)} \|^2_{L^2(\Omega)}}{\| u^{(n)} \|^2_{L^2(\Omega)} + \| \varphi^{(n)} \|^2_{L^2(\Omega)}} < 10^{-12}.$$

In Algorithm 2, the stopping criterion is

$$\frac{\| u^{(k)} - u^{(k-1)} \|^2_{L^2(\Omega)} + \| p^{(k)} - p^{(k-1)} \|^2_{L^2(\Gamma_c)}}{\| u^{(k)} \|^2_{L^2(\Omega)} + \| p^{(k)} \|^2_{L^2(\Gamma_c)}} < 10^{-12},$$

where we have set $p = (p_c, p_f)$.

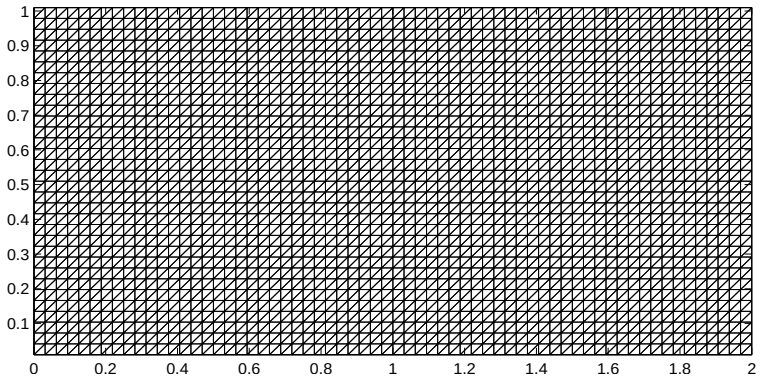


Figure 1: Mesh sample for $\Omega = (0,\ 2) \times (0,\ 1)$

We consider the two-dimensional domain (figure 1) $\Omega = [0,\ 2] \times [0,\ 1]$ with $\Gamma_D = \{0\} \times [0,\ 1]$, $\Gamma_c = [0,\ 2] \times \{0\}$ and $\Gamma_N = [0,\ 2] \times \{1\}$. On Γ_D, $u = 0$ and $\varphi = 0$ are prescribed. On Γ_N, non-homogeneous Neumann boundary conditions are prescribed,

$$\sigma(u) \cdot n = -2x.$$

The normalized gap between Γ_c and the foundation is $g(x) = 0.01$. The material constants are

- Elasticity: $E = 58.7102$GPA, $\nu = 0.3912$, $\nu_f = 0.6$ (friction coefficient);
- Piezoelectric (C/m^2): $e_{22} = -5.4$, $e_{33} = 15.8$, $e_{24} = 12.3$;
- Dielectric (nF/m): $\beta_{22} = 8.11$, $\beta_{33} = 7.35$

In the plane deformations setting, the constitutive equations (2.3)-(2.4) can be expressed as follows

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ D_1 \\ D_2 \end{bmatrix} = \begin{bmatrix} \lambda + 2\mu & \lambda & 0 & 0 & e_{32} \\ \lambda & \lambda + 2\mu & 0 & 0 & e_{33} \\ 0 & 0 & \mu & e_{24} & 0 \\ 0 & 0 & -e_{24} & \beta_{22} & 0 \\ -e_{32} & -e_{33} & 0 & 0 & \beta_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ 2\varepsilon_{23} \\ -E_2 \\ -E_3 \end{bmatrix},$$

where λ and μ denote Lamé (positive) constants. These coefficients are related (in plane deformations) to the Young modulus E and the Poisson coefficient ν by

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}.$$

The penalty parameter is $r = 3.25E$ for all mesh sizes.

To determine the known contact pressure S, we first solve a pure contact problem in the initialization step. Then we set $S = \nu_f|\sigma_n|$. Note that in Algorithm 2, $\lambda_c \approx \sigma_n$ and $\lambda_f \approx \sigma_t$, see e.g., [10, 12].

We first consider a uniform mesh with size $h = 1/32$ (i.e. 2145 nodes and 4096 triangles). With $\varphi_f = 8V$ (the potential of the conductive foun dation), Algorithm 3 stops after 2 iterations. The deformed configuration is shown in Figure 2 while the contour plots of the electric potential φ are shown in Figure 3. Figure 4 shows the stress distributions on Γ_c. The sticking zone ($|\sigma_t| < S$) and sliding zone ($|\sigma_t| = S$) are clearly identified. Table 1 shows the performances of Algorithm 3 using various values of the potential of the conductive foundation. One can notice that, the number of iterations is virtually independent of the potential of the conductive foundation φ_f, for a fixed mesh size.

We now study the scalability of Algorithm 3 by studying the evolution of the number of iterations versus the mesh size. The largest mesh has 131,841 nodes and 262,144 triangles. We report in Table 2 the performances of Algorithm 3 in terms of number of iterations and CPU time. One can notice that the number of iterations is virtually independent of the mesh size. Note that the timing results in Table 2 do not include the time to assemble the matrix systems; this time can be significant for large size meshes.

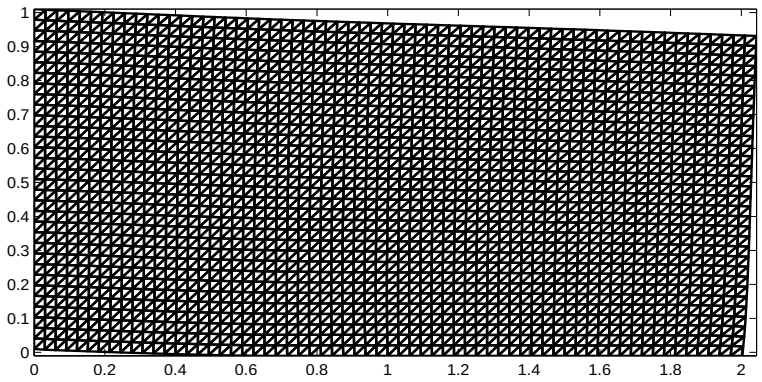


Figure 2: Deformed configuration

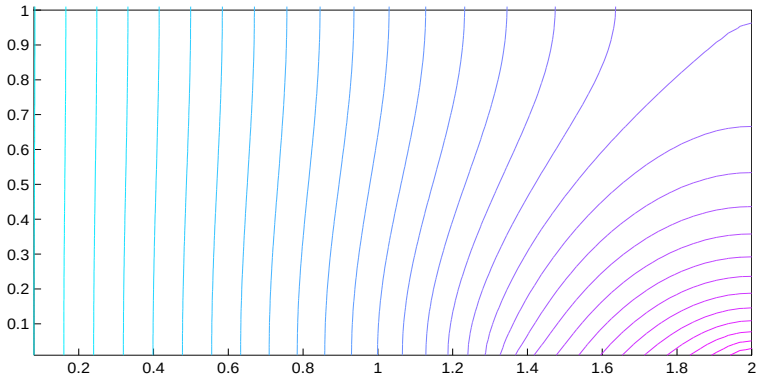


Figure 3: Contour plots of the electric potential

φ_f(V)	2	4	8	16	32
$\max \varphi(\times 10^{-5}\text{V})$	1.6958	3.3916	6.7832	13.5667	27.1338
Iterations	2	2	2	2	2

Table 1: Performances of Algorithm 3 with various values of φ_f, $h = 1/32$.

Mesh size h	1/8	1/16	1/32	1/64	1/128	1/256
Iterations	2	2	2	2	2	2
CPU Time (in Sec.)	0.025	0.034	0.190	1.587	9.618	54.391

Table 2: Performances of Algorithm 3 with various mesh sizes

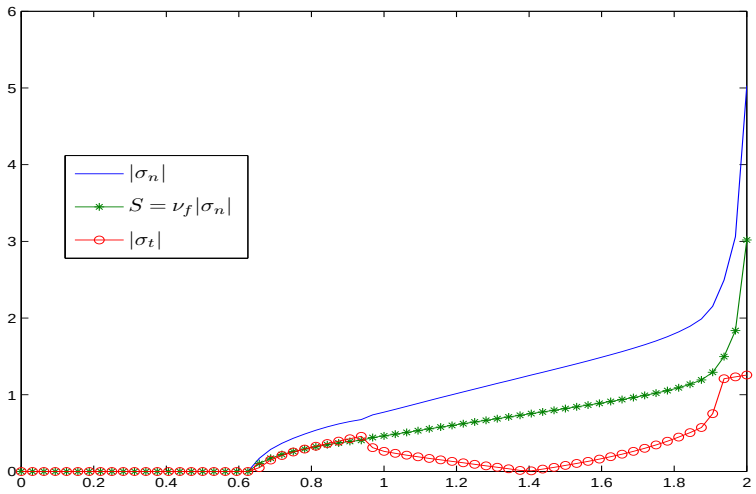


Figure 4: Stress distribution on Γ_c: σ_n contact pressure, σ_t tangential stress

6 Conclusion

We have investigate theoretically and numerically a decomposition method for unilateral contact problem with Tresca friction arising in electro-elastostatics. Rather than solving the coupled problem, we proposed a method solving successively a displacement subproblem and electric potential subproblem. Numerical experiments have shown that the method is robust and scalable (i.e. the number of iterations is virtually independent of the mesh size).

The practical implementation of our algorithm still faces the problem of the optimal choice of the penalty parameter r. Indeed, the number of iterations in the Uzawa block relaxation Algorithm 2 is highly dependent of the penalty parameter r. Further work is underway for the automatic penalty adjustment procedure. The problem of a deformable foundation is also under study.

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