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# Complex meromorphic functions $f'P'(f)$ , $g'P'(g)$ sharing a small function

Kamal Boussaf, Alain Escassut and Jacqueline Ojeda

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## Abstract

Let  $f$ ,  $g$  be two transcendental meromorphic functions in  $\mathbb{C}$ , let  $P$  be a polynomial of uniqueness for meromorphic functions in  $\mathbb{C}$  and let  $\alpha$  be a small meromorphic function with regards to  $f$  and  $g$ . If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  counting multiplicity, then we show that  $f = g$  provided that the multiplicity order of zeroes of  $P'$  satisfy certain inequalities. There is no additional condition on  $\alpha$ . We consider the particular case of entire functions.

## 1 Introduction and Main Results

Following several previous studies, the problem we consider consists of determining polynomials  $P$  such that, if  $f$  and  $g$  are complex meromorphic functions and if  $\alpha$  is a small meromorphic function such that  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  counting multiplicity, then we can conclude  $f = g$ . The first (and hardest) step consists of showing that  $f'P'(f) = g'P'(g)$ , so  $P(f) - P(g)$  is a constant. The second step consists of showing that the constant is zero. And then, if we take for  $P$  a polynomial of uniqueness for complex meromorphic functions, we can get  $f = g$ .

Problems of uniqueness on meromorphic functions were examined first in  $\mathbb{C}$  [8], [9], [11], [15], [16], [17], [18], [23], [24], [26] and next in a p-adic field [1], [4], [5], [6], [7], [13], [14], [19], [20], [21], [22]. After examining problems of the form  $P(f) = P(g)$ , where  $P$  is a polynomial, several studies were made on the equality  $f'P'(f) = g'P'(g)$ , or value sharing questions: if  $P(f)$  and  $P(g)$  share a value, or a small function, do we have  $f = g$ ?

We denote by  $\mathcal{A}(\mathbb{C})$  the  $\mathbb{C}$ -algebra of entire functions in  $\mathbb{C}$ , by  $\mathcal{M}(\mathbb{C})$  the field of meromorphic functions in  $\mathbb{C}$ , i.e. the field of fractions of  $\mathcal{A}(\mathbb{C})$  and by  $\mathbb{C}(x)$  the field of rational functions.

The problem of value sharing a small function for functions of the form  $f'P'(f)$  was examined first when  $P$  was just of the form  $x^n$  [8], [19] in both complex and p-adic contexts. More recently it was examined when  $P$  was a polynomial such that  $P'$  had exactly two distinct zeros [16], [18], [21], both in complex analysis and in p-adic analysis. In [16], [18] the functions were meromorphic on  $\mathbb{C}$ , with a small function that was a constant or the identity. In [19], the problem was considered for analytic functions in a complete ultrametric algebraically closed field  $\mathbb{K}$  of characteristic 0 concerning entire functions and unbounded analytic functions in an open disk, sharing a rational

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function. In [2] the three authors considered the same problem for meromorphic functions in the  $p$ -adic field  $\mathbb{K}$ . The results obtained in that case have then suggested a similar study in  $\mathbb{C}$ . But several crucial lemmas obtained on a  $p$ -adic field have to be re-examined particularly because the characteristic Nevanlinna function  $T(r, f)$  is quite different for complex meromorphic functions and for  $p$ -adic ones, particularly all statements involving functions  $m(r, f)$  on  $\mathcal{M}(\mathbb{C})$ . So, we must develop complete proofs for all intermediate and final results.

Here we enjoy polynomials of uniqueness for complex meromorphic functions [1], [6] and we can finally obtain results similar to those obtained in  $p$ -adic analysis: see Theorem 1[2].

The definition of polynomials of uniqueness was introduced in [17] by P. Li and C. C. Yang and was studied in many papers [1], [10], [11], [24] for complex functions and [5], [6], [22] for  $p$ -adic functions.

**Definition.** Recall that a polynomial  $P \in \mathbb{C}[x]$  is called a *polynomial of uniqueness* for a class of functions  $\mathcal{F}$  if for any two functions  $f, g \in \mathcal{F}$  the property  $P(f) = P(g)$  implies  $f = g$ .

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field  $\mathbb{C}$ . Here, for convenience, we will use notation long ago used in  $p$ -adic analysis in order to denote the counting functions.

Let  $\log$  be a real logarithm function of base  $> 1$ . Given  $u \in \mathbb{R}_+^*$ , we denote by  $\log^+$  the real function defined as  $\log^+(u) = \max(\log(u), 0)$ .

Let  $f \in \mathcal{M}(\mathbb{C})$ . Given  $r > 0$ , we set  $m(r, f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ . Suppose  $f$  has no zero and no pole at 0. Let  $\gamma \in \mathbb{C}$ . If  $f$  has a zero of order  $n$  at  $\gamma$ , we set  $\omega_\gamma(f) = n$ . If  $f$  has a pole of order  $n$  at  $\gamma$ , we put  $\omega_\gamma(f) = -n$  and finally, if  $f(\gamma) \neq 0, \infty$ , we put  $\omega_\gamma(f) = 0$ .

We denote by  $Z(r, f)$  the *counting function of zeros of  $f$*  in the ball  $|z| \leq r$ , counting multiplicity, i.e. we set

$$Z(r, f) = \sum_{\omega_\gamma(f) > 0, |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by  $\bar{Z}(r, f)$  the counting function of zeros of  $f$  in the ball  $|z| \leq r$ , ignoring multiplicity, and set

$$\bar{Z}(r, f) = \sum_{\omega_\gamma(f) > 0, |\gamma| \leq r} (\log r - \log |\gamma|).$$

In the same way, we set  $N(r, f) = Z\left(r, \frac{1}{f}\right)$  (resp.  $\bar{N}(r, f) = \bar{Z}\left(r, \frac{1}{f}\right)$ ) to denote the *counting function of poles of  $f$*  in the ball  $|z| \leq r$ , counting multiplicity (resp. ignoring multiplicity).

If  $f$  admits a zero of order  $s$  at 0, we can make a change of origin or count the zero at 0 by adding  $s \log r$  and similarly, if  $f$  admits a pole at 0 of order  $s$ , we can make a change of origin or count the pole at 0 by adding  $-s \log r$ .

For  $f \in \mathcal{M}(\mathbb{C})$  having no zero and no pole at 0, the *Nevanlinna function* is defined by  $T(r, f) = m(r, f) + N(r, f)$ .

**Definition.** Let  $f \in \mathcal{M}(\mathbb{C})$  such that  $f(0) \neq 0, \infty$ . A function  $\alpha \in \mathcal{M}(\mathbb{C})$  having no zero and no pole at 0 is called a *small function with respect to  $f$* , if it satisfies  $\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0$ . We denote by  $\mathcal{M}_f(\mathbb{C})$  the set of small meromorphic functions with respect to  $f$  in  $\mathbb{C}$ . And we say that two functions  $f, g \in \mathcal{M}(\mathbb{C})$  share a function  $\alpha$  C.M. if  $f - \alpha$  and  $g - \alpha$  admit the same zeros with the same multiplicity.

**Remark 1.** For simplicity, we have kept the same notation as in  $p$ -adic analysis for the counting function of zeros of a meromorphic function.

Now, we can give some sufficient conditions to get polynomials  $P$  such that, if  $f'P(f)$  and  $g'P(g)$  share a small meromorphic function, then  $f = g$ .

Recall the following Theorems A, B that we can get from [18]:

**Theorem A.** Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental. Let  $n \in \mathbb{N}$ ,  $n \geq 13$  and let  $a \in \mathbb{C} \setminus \{0\}$ . If  $f^n(f-a)^2f'$  and  $g^n(g-a)^2g'$  share the identical function  $x$  C.M. then  $f = g$ .

**Theorem B.** Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $a \in \mathbb{C} \setminus \{0\}$  and let  $n \in \mathbb{N}$ ,  $n \geq 12$ . If  $f^n(f-a)f'$  and  $g^n(g-a)g'$  share the identical function  $x$  C.M. then either  $f = g$  or there exists  $h \in \mathcal{M}(\mathbb{C})$  such that  $f = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1}-1}{h^{n+2}-1} \right) h$  and  $g = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1}-1}{h^{n+2}-1} \right)$ .

In [2], we have obtained the following Theorem C.

**Theorem C.** Let  $\mathbb{K}$  be a complete ultrametric algebraically closed field of characteristic zero. Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$ , let  $P' = b(x-a_1)^n \prod_{i=2}^l (x-a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ , with  $l \geq 2$  and  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l-1$ , whenever  $l > 2$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

$$\text{if } l = 2, \text{ then } n \neq 2k, 2k + 1, 3k + 1,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

Next, several particular applications were given when the small function is a constant or a Moebius function. Here we can't get similar refinements because the complex Nevanlinna Theory is less accurate than the  $p$ -adic Nevanlinna Theory.

However, we aim at obtaining for complex meromorphic functions results almost similar to those obtained for  $p$ -adic functions, thanks to theorems of uniqueness in  $\mathbb{C}$ . Let us recall Theorems D and E that we can extract from Theorem 3 [1] and Corollary 9.1 [6] respectively.

**Notation:** Let  $L$  be an algebraically closed field and let  $P \in L[x] \setminus \mathbb{K}$  and let  $\Xi(P)$  be the set of zeros  $c$  of  $P'$  such that  $P(c) \neq P(d)$  for every zero  $d$  of  $P'$  other than  $c$ . We denote by  $\Phi(P)$  its cardinal.

In [1], Theorem 3 is stated under the general condition that the polynomial  $P$  we consider is injective on the set of zeros of its derivative. Here we reduce that hypothesis to a hypothesis concerning  $\Phi(P)$ . Particularly, the following Theorem D (concerning the case when  $P'$  only has two distinct zeros) comes from Theorem 3 [1], but actually, it is useless to assume the general hypothesis that the restriction of  $P$  on the set of zeros of  $P'$  is injective because in that case, such a polynomial  $P$  necessarily satisfies  $P(c_1) \neq P(c_2)$  with the two zeros  $c_1, c_2$  of  $P'$ , as shown in Lemma 10 of [6].

**Remark 2.** The hypothesis  $\Phi(P) \geq 4$  is weaker than the hypothesis: "the restriction of  $P$  on the set of zeros of  $P'$  is injective" as soon as the number of distinct zeros of  $P'$  is at least 6.

**Theorem D** Let  $P \in \mathbb{C}[x]$  be such that  $P'$  has exactly two distinct zeros  $\gamma_1$  of order  $c_1$  and  $\gamma_2$  of order  $c_2$  with  $\min\{c_1, c_2\} \geq 2$  and  $\max(c_1, c_2) \geq 3$ . Then  $P$  is a polynomial of uniqueness for  $\mathcal{M}(\mathbb{C})$ .

**Theorem E** Let  $P \in \mathbb{C}[x]$  be such that  $P'$  has exactly three distinct zeros  $\gamma_1$  of order  $c_1$ ,  $\gamma_2$  of order  $c_2$ ,  $\gamma_3$  of order  $c_3$  with  $\max\{c_1, c_2, c_3\} \geq 2$ , satisfying further  $\Phi(P) = 3$ . Then  $P$  is a polynomial of uniqueness for  $\mathcal{M}(\mathbb{C})$ .

We can find Theorem F in [6] (Corollary 9.1, p. 226):

**Theorem F.** Let  $P \in \mathbb{C}[x]$ . If  $\Phi(P) \geq 4$  then  $P$  is a polynomial of uniqueness for  $\mathcal{M}(\mathbb{C})$ .

We can now state our main theorems on the problem  $f'P'(f)$ ,  $g'P'(g)$  sharing a small function.

**Theorem 1.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{A}(\mathbb{C})$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{C}^*$ ,  $l \geq 2$ , and  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l - 1$  when  $l > 2$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2 \\ n &\geq 5 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \end{aligned}$$

Let  $f, g \in \mathcal{A}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem F, we can derive Corollary 1.1:

**Corollary 1.1** Let  $P \in \mathbb{C}[x]$  be such that  $P'(x) = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{C}^*$ ,  $\Phi(P) \geq 4$ ,

and  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l - 1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2, \\ n &\geq 5 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \end{aligned}$$

Let  $f, g \in \mathcal{A}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem E, we can drive Corollary 1.2:

**Corollary 1.2** Let  $P \in \mathbb{C}[x]$  be such that  $P'(x) = b(x - a_1)^n (x - a_2)^{k_2} (x - a_3)^{k_3}$  with  $b \in \mathbb{C}^*$ ,  $k_2 \geq k_3$ ,  $\Phi(P) = 3$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq k_2 + k_3 + 2, \\ n &\geq 5 + \max(0, 4 - k_3) + \max(0, 5 - k_2). \end{aligned}$$

Let  $f, g \in \mathcal{A}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 2.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{A}(\mathbb{C})$  such that  $P'$  is of the form

$$b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{C}^*, \text{ satisfying, } n \geq l + 5.$$

Let  $f, g \in \mathcal{A}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem F we have Corollary 2.1:

**Corollary 2.1** Let  $P \in \mathbb{C}[x]$  be such that  $P'(x) = b(x - a_1)^n \prod_{i=2}^l (x - a_i)$  with  $\Phi(P) \geq 4$ ,  $b \in \mathbb{C}^*$ , satisfying  $n \geq l + 5$ .

Let  $f, g \in \mathcal{A}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

And by Theorem E we have Corollary 2.1:

**Corollary 2.2** Let  $P \in \mathbb{C}[x]$  be such that  $P'(x) = b(x - a_1)^n (x - a_2)^{k_2} (x - a_3)^{k_3}$  with  $\Phi(P) = 3$ ,  $b \in \mathbb{C}^*$ , satisfying  $n \geq 8$ .

Let  $f, g \in \mathcal{A}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem 3.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{C})$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{C}^*$ , and  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l - 1$  when  $l > 2$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 3,$$

$$\text{if } l = 2, \text{ then } n \neq 2k, 2k + 1, 3k + 1,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k, \forall i = 2, 3.$$

Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem F, we have Corollary 3.1:

**Corollary 3.1** Let  $P \in \mathbb{C}[x]$  be such that  $P'(x) = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{C}^*$ ,  $\Phi(P) \geq 4$ ,

$k_i \geq k_{i+1}$  and  $2 \leq i \leq l - 1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 3.$$

Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

And by Theorem D we also have Corollary 3.2

**Corollary 3.2** Let  $P \in \mathbb{C}[x]$  be such that  $P'$  is of the form  $b(x - a_1)^n (x - a_2)^k$  with  $k \geq 2$ . Suppose that  $P$  satisfies the further conditions:

$$n \geq 10 + \max(0, 5 - k),$$

$$n \geq k + 3,$$

$$n \neq 2k, 2k + 1, 3k + 1.$$

Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Example:** Let  $P(x) = \frac{x^{16}}{16} - \frac{2x^{15}}{15} + \frac{x^{14}}{14}$ . Then  $P'(x) = x^{13}(x-1)^2$ . Given  $f, g \in \mathcal{M}(\mathbb{C})$  transcendental such that  $f'P'(f)$  and  $g'P'(g)$  share a small function  $\alpha \in \mathcal{M}(\mathbb{C})$  C.M., we have  $f = g$ .

By Corollary 3.2 we can also derive Corollary 3.3:

**Corollary 3.3** Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. Let  $a \in \mathbb{C}^*$ . If  $f'f^n(f-a)^2$  and  $g'g^n(g-a)^2$  share the function  $\alpha$  C.M. and if  $n \geq 13$ , then  $f = g$ .

By Theorem E, we can also derive Corollary 3.4:

**Corollary 3.4** Let  $P \in \mathbb{C}[x]$  be such that  $P'(x) = b(x-a_1)^n(x-a_2)^{k_2}(x-a_3)^{k_3}$  with  $b \in \mathbb{C}^*$ ,  $\Phi(P) = 3$ ,  $k_i \geq k_{i+1}$  and  $i = 2, 3$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_2) + \max(0, 5 - k_3),$$

$$n \geq k_2 + k_3 + 3.$$

Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

When all  $k_i$  are equal to 1, we can obtain a better formulation:

**Theorem 4.** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{C})$  such that  $P'$  is of the form

$$b(x-a_1)^n \prod_{i=2}^l (x-a_i) \text{ with } l \geq 3, b \in \mathbb{C}^*, \text{ with } n \geq l + 10.$$

Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem F, we have Corollary 4.1:

**Corollary 4.1** Let  $P \in \mathbb{C}[x]$  be such that  $P'$  is of the form

$$P'(x) = b(x-a_1)^n \prod_{i=2}^l (x-a_i), \text{ with } \Phi(P) \geq 4, b \in \mathbb{C}^* \text{ satisfying:}$$

$$n \geq l + 10.$$

Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Example:** Let  $P(x) = \frac{x^{18}}{18} - \frac{2x^{17}}{17} - \frac{x^{16}}{16} + \frac{2x^{15}}{15}$ . Then  $P'(x) = x^{17} - 2x^{16} - x^{15} + 2x^{14} = x^{14}(x-1)(x+1)(x-2)$ . We check that:  $P(0)$ ,  $P(1)$ ,  $P(-1)$ ,  $P(2)$  are all distinct. Consequently  $\Phi(P) = 4$ . So,  $P$  is a polynomial of uniqueness for  $\mathcal{M}(\mathbb{C})$ . Moreover, we have  $n = 14$ ,  $l = 4$ , so we can apply Corollary 4.1.

Given  $f, g \in \mathcal{M}(\mathbb{C})$  transcendental such that  $f'P'(f)$  and  $g'P'(g)$  share a small function  $\alpha \in \mathcal{M}(\mathbb{C})$  C.M., we have  $f = g$ .

And by Theorem E we have Corollary 4.2

**Corollary 4.2** Let  $P \in \mathbb{C}[x]$  be such that  $P'$  is of the form

$P'(x) = b(x - a_1)^n(x - a_2)(x - a_3)$  with  $\Phi(P) = 3$ ,  $b \in \mathbb{C}^*$ , satisfying  $n \geq 13$ .

Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Example:** Consider the general form of a polynomial  $P$  such that  $P'$  has 3 distinct zeros, one of them being 0 of order  $n$  and the two others  $a$  and  $b$  of order 1. We can obviously suppose that  $P(0) = 0$ . And then we have

$$P(x) = x^{n+3}(n+2)(n+1) - x^{n+2}(a_1 + a_2)(n+3)(n+1) + x^{n+1}(a_1 a_2)(n+3)(n+2)$$

and therefore

$$P'(x) = (n+1)(n+2)(n+3)x^n(x - a_1)(x - a_2).$$

Now we can check that  $P(a_1) = P(0) = 0$  if and only if  $\frac{a_1}{a_2} = \frac{n+3}{n+1}$  and that  $P(a_1) = P(a_2)$  if and only if  $[a_1^{n+3} - a_2^{n+3}](n+1) = [a_1 a_2(a_1^{n+1} - a_2^{n+1})](n+3)$ . Particularly, the last equality is satisfied when  $n$  is odd and  $a_2 = -a_1$ .

Consequently, if  $\frac{a_1}{a_2} \neq \frac{n+3}{n+1}$  and  $\frac{a_1}{a_2} \neq \frac{n+1}{n+3}$  and if

$$[a_1^{n+3} - a_2^{n+3}](n+1) \neq [a_1 a_2(a_1^{n+1} - a_2^{n+1})](n+3)$$

then  $P$  is a polynomial of uniqueness for complex meromorphic functions. Moreover, if  $n \geq 13$ , then we can apply Corollary 4.2.

Conversely, suppose for instance  $n$  is odd and  $a_2 = -a_1$ . Then we can check that  $P(f) = P(-f)$  for every function, therefore  $P$  is not a polynomial of uniqueness for  $\mathcal{M}(\mathbb{C})$ .

**Theorem 5.** Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$  be non-identically zero. Let  $a \in \mathbb{C}^*$ . If  $f'f^n(f - a)$  and  $g'g^n(g - a)$  share the function  $\alpha$  C.M. and if  $n \geq 12$ , then either  $f = g$  or there exists  $h \in \mathcal{M}(\mathbb{C})$  such that  $f = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right) h$  and  $g = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right)$ .

**Remark 3.** In Theorem 5, the second conclusion is well known to occur [5], [24]. Indeed, let  $h \in \mathcal{M}(\mathbb{C})$  and let us define  $f$  and  $g$  as:  $g = \left( \frac{n+2}{n+1} \right) \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right)$  and  $f = hg$ . We can see that the polynomial  $P(y) = \frac{1}{n+2}y^{n+2} - \frac{1}{n+1}y^{n+1}$  satisfies  $P(f) = P(g)$ , hence  $f'P'(f) = g'P'(g)$ , therefore  $f'P'(f)$  and  $g'P'(g)$  trivially share any function.

**Remark 4.** On a  $p$ -adic field, we obtained conclusions with hypotheses slightly more general when the small function  $\alpha$  is either a Moebius function or a constant, thanks to a very accurate Second Main Theorem. On the field  $\mathbb{C}$ , we cannot do the same.

## 2 Basic and specific results:

**Notation:** Given  $f \in \mathcal{M}(\mathbb{C})$ , we will denote by  $S_f(r)$  a function in  $r$  defined in  $]0, +\infty[$  such that  $\lim_{r \rightarrow +\infty} \frac{S_f(r)}{T(r, f)} = 0$  out of a subset  $J$  of  $]0, +\infty[$  of finite measure.

Let us recall a few classical lemmas [12]:



**Lemma 1.** *Let  $f, g \in \mathcal{M}(\mathbb{C})$ , let  $a \in \mathbb{C}$  and let  $P(f) \in \mathbb{C}[x]$  be of degree  $q$ . Then  $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$ ,  $T(r, fg) \leq T(r, f) + T(r, g)$ ,  $T(r, f - a) = T(r, f) + O(1)$ ,  $T(r, \frac{1}{f}) = T(r, f) + O(1)$ ,  $T(r, P(f)) = qT(r, f) + O(1)$ .*

**Lemma 2.** *Let  $f, g \in \mathcal{M}(\mathbb{C})$ . Then  $Z(r, f - a) \leq T(r, f) + O(1) \forall a \in \mathbb{C}$ ,  $m(r, fg) \leq m(r, f) + m(r, g)$ ,  $N(r, f') = N(r, f) + \overline{N}(r, f)$ ,  $Z(r, f') \leq Z(r, f) + \overline{N}(r, f) + S_f(r)$ . Let  $Q \in \mathbb{C}[x]$  be of degree  $q$ . Then  $Z(r, f'Q(f)) \geq Z(r, Q(f))$  and  $N(r, f'Q(f)) = (q + 1)N(r, f) + \overline{N}(r, f)$ .*

By Milloux's Theory we have the following lemma (Theorem 3.1 in [12]).

**Lemma 3.** *Let  $f \in \mathcal{M}(\mathbb{C})$ . Then  $m(r, \frac{f'}{f}) = S_f(r)$  and  $T(r, f') \leq T(r, f) + \overline{N}(r, f) + S_f(r)$ . Moreover, if  $f \in \mathcal{A}(\mathbb{C})$  then  $T(r, f') \leq T(r, f) + S_f(r)$ .*

The statements of Lemma 4 are known and used in [18]:

**Lemma 4.** *Let  $f \in \mathcal{M}(\mathbb{C})$ . Then  $m(r, \frac{1}{f'}) \geq m(r, \frac{1}{f}) + S_f(r)$ . Moreover,  $T(r, f) - Z(r, f) \leq T(r, f') - Z(r, f') + S_f(r)$ .*

**Notation:** Let  $f \in \mathcal{M}(\mathbb{C})$  satisfy  $f'(0) \neq 0, \infty$ . Let  $S$  be a finite subset of  $\mathbb{C}$  and  $r \in ]0, +\infty[$ . We denote by  $Z_0^S(r, f')$  the counting function of zeros of  $f'$  in  $\mathbb{C}$  which are not zeros of any  $f - s$  for  $s \in S$ . This is, if  $(\gamma_n)_{n \in \mathbb{N}}$  is the finite or infinite sequence of zeros of  $f'$  in  $\mathbb{C}$  that are not zeros of  $f - s$  for  $s \in S$ , with multiplicity order  $q_n$  respectively, we set

$$Z_0^S(r, f') = \sum_{|\gamma_n| \leq r} q_n (\log r - \log |\gamma_n|).$$

**Theorem N.** [12] *Let  $a_1, \dots, a_n \in \mathbb{C}$  with  $n \geq 2, n \in \mathbb{N}$ , and let  $f \in \mathcal{M}(\mathbb{C})$ . Let  $S = \{a_1, \dots, a_n\}$ . Assume that none of  $f, f'$  and  $f - a_j$  with  $1 \leq j \leq n$ , equals 0 or  $\infty$  at the origin. Then, for  $r > 0$  we have*

$$(n - 1)T(r, f) \leq \sum_{j=1}^n \overline{Z}(r, f - a_j) + \overline{N}(r, f) - Z_0^S(r, f') + S_f(r).$$

We will also need the following Lemma 5 which is specific to the complex case.

**Lemma 5.** *Let  $Q \in \mathbb{C}[x]$  and let  $f \in \mathcal{M}(\mathbb{C})$  be transcendental. Then  $T(r, Q(f)) \leq T(r, f'Q(f)) + m(r, \frac{1}{f'})$  and  $T(r, Q(f)) \leq T(r, f) + \overline{N}(r, f) + T(r, f'Q(f)) + S_f(r)$ ,  $r \in ]0, +\infty[$ . Moreover, if  $\deg(Q) \geq 3$ , then  $\mathcal{M}_f(\mathbb{C}) \subset \mathcal{M}_{f'Q(f)}(\mathbb{C})$ .*

**Proof:** We have  $T(r, Q(f)) = N(r, Q(f)) + m(r, f' \frac{Q(f)}{f'}) \leq N(r, Q(f)) + N(r, f') + m(r, f'Q(f)) + m(r, \frac{1}{f'})$ . Now, since the poles of  $f'$  are those of  $Q(f)$  (with different multiplicities), we have  $N(r, Q(f)) + N(r, f') = N(r, f'Q(f))$  and therefore  $N(r, Q(f)) + N(r, f') + m(r, f'Q(f)) + m(r, \frac{1}{f'}) = N(r, f'Q(f)) + m(r, f'Q(f)) + m(r, \frac{1}{f'}) = T(r, f'Q(f)) + m(r, \frac{1}{f'})$ , which proves the first inequality. Next,  $m(r, \frac{1}{f'}) \leq T(r, f') + S_f(r) = m(r, f') + N(r, f') + S_f(r)$ . But by Lemma 3  $m(r, f') \leq m(r, f) + S_f(r)$ , hence  $m(r, \frac{1}{f'}) \leq m(r, f) + N(r, f') + S_f(r) = m(r, f) + N(r, f) + \overline{N}(r, f) + S_f(r)$

and therefore  $T(r, Q(f)) \leq T(r, f'Q(f)) + T(r, f) + \overline{N}(r, f) + S_f(r) \leq T(r, f'Q(f)) + 2T(r, f) + S_f(r)$ , which finishes proving the second inequality. Particularly, if  $\deg(Q) \geq 3$ , the last statement is an immediate consequence of both inequalities.

On a p-adic field, the following lemma 6 is only proven for meromorphic functions [2]. Here we must consider the particular case of entire functions which will be useful in the proof of Theorem 1.

**Lemma 6.** *Let  $Q \in \mathbb{C}[x]$  and let  $f, g \in \mathcal{M}(\mathbb{C})$  (resp.  $f, g \in \mathcal{A}(\mathbb{C})$ ) be transcendental. Let  $P(x) = x^{n+1}Q(x)$  be such that  $n \geq \deg(Q) + 3$  (resp.  $n \geq \deg(Q) + 2$ ). If  $P'(f)f' = P'(g)g'$  then  $P(f) = P(g)$ .*

**Proof :** Set  $k = \deg(Q)$ . Since  $P'(f)f' = P'(g)g'$ , there exists  $c \in \mathbb{C}$  such that  $P(f) = P(g) + c$ . Suppose that  $c \neq 0$ . Then by Theorem N, we have

$$T(r, P(f)) \leq \overline{Z}(r, P(f)) + \overline{Z}(r, P(f) - c) + \overline{N}(r, P(f)) + S_f(r). \quad (1)$$

Obviously we see that  $\overline{Z}(r, P(f)) = \overline{Z}(r, f^{n+1}Q(f)) = \overline{Z}(r, fQ(f)) \leq T(r, fQ(f))$ . By Lemma 1 we have  $T(r, fQ(f)) = (k+1)T(r, f) + O(1)$  and then  $\overline{Z}(r, P(f)) \leq (k+1)T(r, f) + O(1)$ . We also have  $\overline{Z}(r, P(f) - c) = \overline{Z}(r, P(g)) \leq \overline{Z}(r, g) + \overline{Z}(r, Q(g)) \leq T(r, g) + T(r, Q(g))$ . Then by Lemma 1,  $\overline{Z}(r, P(f) - c) \leq (k+1)T(r, g) + O(1)$ . Notice that  $\overline{N}(r, P(f)) = \overline{N}(r, f) \leq T(r, f) + O(1)$  then by (1) we obtain

$$T(r, P(f)) \leq (k+2)T(r, f) + (k+1)T(r, g) + S_f(r) \quad (2)$$

(resp.

$$T(r, P(f)) \leq (k+1)T(r, f) + (k+1)T(r, g) + S_f(r) \quad (3))$$

According to Lemma 1 we have  $T(r, P(f)) = (n+k+1)T(r, f) + O(1)$ . Then by (2) we have

$$nT(r, f) \leq T(r, f) + (k+1)T(r, g) + S_f(r). \quad (4)$$

(resp.

$$nT(r, f) \leq T(r, f) + (k+1)T(r, g) + S_f(r). \quad (5))$$

Similarly we obtain

$$nT(r, g) \leq T(r, g) + (k+1)T(r, f) + S_f(r). \quad (6)$$

(resp.

$$nT(r, g) \leq (k+1)T(r, f) + S_f(r). \quad (7))$$

Hence adding (6) and (7) we have

$$n(T(r, f) + T(r, g)) \leq (k+2)(T(r, f) + T(r, g)) + S_f(r)$$

and then

$$0 \leq (k+2-n)(T(r, f) + T(r, g)) + S_f(r)$$

(resp.

$$0 \leq (k+1-n)(T(r, f) + T(r, g)) + S_f(r))$$

In the general case, that leads to a contradiction because  $n \geq k+3$  and  $\lim_{r \rightarrow +\infty} (T(r, f) + T(r, g)) = +\infty$  outside a subset  $J$  of  $]0, +\infty[$  of finite measure. Thus  $c = 0$  and consequently  $P(f) = P(g)$ . And when  $f, g \in \mathcal{A}(\mathbb{C})$  we have contradiction when  $n \geq k+2$ .

**Notation:** Given two meromorphic functions  $f, g \in \mathcal{M}(\mathbb{C})$ . We set

$$\Psi_{f,g} = \frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}.$$

That function was already used in several previous papers, particularly [18].

The following Lemma was proved in [25].

**Lemma 7.** *Let  $F, G \in \mathcal{M}(\mathbb{C})$  be non-constant, having no zero and no pole at 0 and sharing the value 1 C.M. Suppose that  $\Psi_{f,g} = 0$  and that*

$$\limsup_{r \rightarrow +\infty} \left( \frac{\overline{Z}(r, f) + \overline{Z}(r, g) + \overline{N}(r, f) + \overline{N}(r, g)}{\max(T(r, f), T(r, g))} \right) < 1$$

outside a set of finite measure. Then either  $f = g$  or  $fg = 1$ .

Lemma 8 is similar to Lemma 9 [2] for complex meromorphic functions. But here, we can get a better result for entire functions.

**Lemma 8.** *Let  $P(x) = (x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i} \in \mathbb{C}[x]$  ( $a_i \neq a_j, \forall i \neq j$ ) with  $l \geq 2$  and  $n \geq \max\{k_2, \dots, k_l\}$  and let  $k = \sum_{i=2}^l k_i$ . Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental and let  $\theta = P(f)f'P'(g)g'$ . If  $\theta$  belongs to  $\mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ , then we have the following :*

*if  $l = 2$  then  $n$  belongs to  $\{k, k+1, 2k, 2k+1, 3k+1\}$ ,*

*if  $l = 3$  then  $n$  belongs to  $\{\frac{k}{2}, k+1, 2k+1, 3k_2 - k, 3k_3 - k\}$ ,*

*if  $l \geq 4$  then  $n = k+1$ .*

*Moreover, if  $f, g$  belong to  $\mathcal{A}(\mathbb{C})$ , then  $\theta$  does not belong to  $\mathcal{A}_f(\mathbb{C})$ .*

**Proof:** Without loss of generality, we can assume  $a_1 = 0$ . Suppose  $f, g \in \mathcal{M}(\mathbb{C})$  satisfy

$$f^n \left( \prod_{i=2}^l (f - a_i)^{k_i} \right) f' g^n \left( \prod_{i=2}^l (g - a_i)^{k_i} \right) g' = \theta \quad (1)$$

Let  $\Sigma$  be the set of zeros and poles of  $\theta$ . We will first show that  $P'(f)$  and  $P'(g)$  admit zeros or poles out of  $\Sigma$ . Indeed, suppose that none of the zeros and the poles of  $f'P'(f)$  belong to  $\Sigma$ . So, every zero and every pole of  $f'P'(f)$  must be either a zero or a pole of  $\theta$ . Therefore,  $\overline{Z}(r, f'P'(f)) + \overline{N}(r, f'P'(f)) \leq 2T(r, \theta)$ . Actually, each zero of  $f'P'(f)$  is either a zero of  $f'$  or of  $P'(f)$  because the poles of  $P'(f)$  are the poles of  $f'$ . Consequently,  $\overline{Z}(r, P'(f)) + \overline{N}(r, f) \leq 2T(r, \theta)$  and therefore  $\sum_{i=1}^l \overline{Z}(r, f - a_i) + \overline{N}(r, f) \leq 2T(r, \theta)$  But now by Theorem N we have

$$(l-1)T(r, f) \leq \sum_{i=1}^l \overline{Z}(r, f - a_i) + \overline{N}(r, f) + S_f(r)$$

hence  $(l-1)T(r, f) \leq 2T(r, \theta) + S_f(r)$ , a contradiction. Consequently,  $P'(f)$  admits zeros or poles that do not belong to  $\Sigma$ . Similarly,  $P'(g)$  admits zeros or poles that do not belong to  $\Sigma$ .

Now, suppose that  $f, g$  belong to  $\mathcal{A}(\mathbb{C})$ . Since both  $f, g$  have no pole, all zeros of both must belong to  $\Sigma$ . Then, we have

$$Z(r, f'P'(f)g'P'(g)) = Z(r, f') + \sum_{i=1}^l k_i Z(r, f - a_i) + Z(r, g') + \sum_{i=1}^l k_i Z(r, g - a_i),$$

therefore

$$Z(r, \theta) \geq \sum_{i=1}^l \bar{Z}(r, f - a_i) \geq (l-1)T(r, f) + S_f(r) + S_g(r)$$

a contradiction to hypothesis again. Consequently,  $f'P(f)g'P(g)$  cannot belong to  $\mathcal{A}_f(\mathbb{C})$  or  $\mathcal{A}_g(\mathbb{C})$ .

Now, let us go back to the general case and suppose that  $f$ ,  $g$  are meromorphic functions. Let  $\gamma \in \mathbb{C} \setminus \Sigma$  be a zero of  $g$  of order  $s$ . Clearly, by (1),  $\gamma$  is a pole of  $f$  of order, for example,  $t$ . And since  $\gamma$  is neither a zero nor a pole of  $\theta$  we can derive the following relation

$$s(n+1) = t(n+k+1) + 2 \quad (2)$$

Now, suppose that for  $i \in \{2, \dots, l\}$ ,  $g - a_i$  has a zero  $\gamma \in \mathbb{C} \setminus \Sigma$  of order  $s_i$ . It is a pole of  $f$  of order  $t_i$ . So, by (2), we obtain

$$s_i(k_i+1) = t_i(n+k+1) + 2 \quad (3)$$

By (2) and (3) it is obvious that  $s > t$  and  $s_i > t_i$ .

Consider now a pole  $\gamma \in \mathbb{C} \setminus \Sigma$  of  $f$ . Either it is a zero of  $g$ , or it is a zero of  $g - a_i$  for some  $i \in \{2, \dots, l\}$ , or it is a zero of  $g'$  that is neither a zero of  $g$  nor a zero of  $g - a_i$  ( $\forall i \in \{2, \dots, l\}$ ). Let  $Z_0(r, g')$  be the counting function of zeros of  $g'$  that are neither a zero of  $g$  nor a zero of  $g - a_i$  for all  $i \in \{2, \dots, l\}$  (counting multiplicity) and let  $\bar{Z}_0(r, g')$  be the counting function of zeros of  $g'$  that are neither a zero of  $g$  nor a zero of  $g - a_i$  for all  $i \in \{2, \dots, l\}$ , ignoring multiplicity. Since  $T(r, \theta) = S_f(r) = S_g(r)$ , we have

$$\bar{N}(r, f) \leq \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}_0(r, g') + S_f(r) + S_g(r) \quad (4)$$

Now, by Theorem N, we have  $(l-1)T(r, f) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{N}(r, f) - Z_0(r, f') + S_f(r)$ , hence by (4), we obtain

$$(l-1)T(r, f) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}_0(r, g') - Z_0(r, f') + S_f(r) + S_g(r) \quad (5)$$

And similarly,

$$(l-1)T(r, g) \leq \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}_0(r, f') - Z_0(r, g') + S_f(r) + S_g(r) \quad (6)$$

Hence, adding (5) and (6), we obtain

$$(l-1)(T(r, f) + T(r, g)) \leq 2(\bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i)) + S_f(r) + S_g(r) \quad (7)$$

**Case  $l = 2$ :**

Without loss of generality, we can assume  $a_2 = 1$ . Suppose now that all zeros of  $f$ ,  $f-1$ ,  $g$ ,  $g-1$  are at least of order 5, except maybe those lying in  $\Sigma$ : then

$$\bar{Z}(r, f) \leq \frac{1}{5}T(r, f) + S_f(r) + S_g(r), \quad \bar{Z}(r, f-1) \leq \frac{1}{5}T(r, f) + S_f(r) + S_g(r),$$

$$\overline{Z}(r, g) \leq \frac{1}{5}T(r, g) + S_f(r) + S_g(r), \quad \overline{Z}(r, g-1) \leq \frac{1}{5}T(r, g) + S_f(r) + S_g(r),$$

a contradiction to (7), proving the statement of the Lemma. So, we will look for pairs  $(n, k)$  leading to zeros of  $f$  or  $g$  out of  $\Sigma$ , of order  $\leq 4$ .

Consequently, we will examine all situations leading to zeros of order  $\leq 4$  for  $f$ ,  $f-1$ ,  $g$ ,  $g-1$ . Actually, since  $f$  and  $g$  play the same role with respect to  $n$  and  $k$ , it is sufficient to examine the situation, for instance, when  $g$  or  $g-1$  has a zero of order  $s \leq 4$ . In each case we denote by  $t$  the order of the pole of  $f$  which is a zero of  $g$  or  $g-1$ . Recall that when  $f$  has a pole of order 4,  $g$  or  $g-1$ , if it has a zero, must have a zero of order  $\geq 5$ . Consequently, we only have to examine zeros of  $g$  or  $g-1$  that are poles of  $f$  of order 1, 2, 3.

Suppose first  $g$  has a zero  $\gamma \notin \Sigma$  of order  $s = 2$ . Then

$$2(n+1) = t(k+n+1) + 2 \quad (8)$$

By (8) if  $t = 1$  we find a solution:

$$n = k + 1 \quad (9)$$

Next, if  $t \geq 2$ , we check that  $2n + 2 < t(k + n + 1) + 2$ , hence (9) is the only solution.

Suppose now  $g$  has a zero  $\gamma \notin \Sigma$  of order  $s = 3$ . Then

$$3(n+1) = t(k+n+1) + 2 \quad (10)$$

By (10) if  $t = 1$  we find no solution because  $k \leq n$ .

If  $t = 2$  we find

$$n = 2k + 1 \quad (11)$$

If  $t \geq 3$  we have  $3(n+3) < 3(k+n+1) + 2$  hence (11) is the only solution.

Suppose now  $g$  has a zero of order  $s = 4$ . Then

$$4(n+1) = t(k+n+1) + 2 \quad (12)$$

If  $t = 1$ , since  $k \leq n$ , we have  $4(n+1) > t(k+n+1) + 2$ .

If  $t = 2$ , by (12) we have a solution

$$n = k \quad (13)$$

If  $t = 3$ , we have another solution

$$n = 3k + 1 \quad (14)$$

Consequently, by (9), (11), (12), (14), all possibilities for  $g$  to have a zero of order  $s \leq 4$  are as follows:

$$n = k + 1, s = 2, \quad n = 2k + 1, s = 3, \quad n = k, s = 4, \quad n = 3k + 1, s = 4. \quad (15)$$

Now, we will examine zeros of  $g-1$   $\gamma \notin \Sigma$  of order  $\leq 4$ . So, the order  $s$  of  $g-1$  satisfies

$$s(k+1) = t(k+n+1) + 2 \quad (16)$$

Suppose first  $g-1$  has a zero  $\gamma \notin \Sigma$  of order  $s = 2$ . Then by (16), we have

$$2(k+1) = t(k+n+1) + 2 \quad (17)$$

Since  $k \leq n$ , we find no solution neither when  $t = 1$  that would lead to  $k = n + 1$ , nor when  $t \geq 2$  because  $2(k + 1) < t(k + n + 1) + 2$ .

Suppose now that  $s = 3$ .

If  $t = 1$  we find a solution:

$$n = 2k \quad (18)$$

If  $t \geq 2$ , we have no solution with  $k \leq n$  because  $3(k + 1) < t(k + n + 1) + 2$

Suppose now that  $s = 4$ .

If  $t = 1$  we find a solution:

$$n = 3k + 1 \quad (19)$$

If  $t = 2$  we find another solution:

$$n = k \quad (20)$$

If  $t \geq 3$ , we find no solution with  $k \leq n$  because  $4(k + 1) < t(k + n + 1) + 2$ .

Consequently, by (18), (19), (20), all possibilities for  $g - 1$  to have a zero  $\gamma \notin \Sigma$  of order  $s \leq 4$  are as follows:

$$n = 2k, s = 3, \quad n = 3k + 1, s = 4, \quad n = k, s = 4. \quad (21)$$

Thus, we have proved that when  $n \neq k, k + 1, 2k, 2k + 1, 3k + 1$ , none of the zeros of  $f, f - 1, g, g - 1$  out of  $\Sigma$  is of order  $\leq 4$  and therefore the statement of the Lemma is proved in the case  $l = 2$ .

**Case  $l = 3$  :** Suppose that all zeros of  $f, g, f - a_i, g - a_i \forall i = 2, 3$  are at least of order 4, except maybe those lying in  $\Sigma$ : then

$$\overline{Z}(r, f) \leq \frac{1}{4}T(r, f) + S_f(r) + S_g(r), \quad \text{and } \forall i = 2, 3, \overline{Z}(r, f - a_i) \leq \frac{1}{4}T(r, f) + S_f(r) + S_g(r),$$

$$\overline{Z}(r, g) \leq \frac{1}{4}T(r, g) + S_f(r) + S_g(r), \quad \text{and } \forall i = 2, 3, \overline{Z}(r, g - a_i) \leq \frac{1}{4}T(r, g) + S_f(r) + S_g(r).$$

Then by (7) we obtain  $l \leq 2$ , a contradiction.

Consequently, we will examine all  $n$  and  $k_i$  ( $i \in \{2, 3\}$ ) leading to zeros out of  $\Sigma$  of order  $\leq 3$  for  $f, g, f - a_i, g - a_i$  for  $i = 2, 3$ . Actually, since  $f$  and  $g$  play the same role, it is sufficient to examine the situation, for instance, when  $g$  or some  $g - a_i$  has a zero of order less than 3. In each case we denote by  $t$  the order of the pole of  $f$  which is a zero of  $g$  or  $g - a_i$  for some  $i$ . Recall that when  $f$  has a pole of order 3,  $g$  or  $g - a_i$ , if it has a zero, must have a zero of order  $\geq 4$ . Consequently, we only have to examine zeros of  $g$  or  $g - a_i$  ( $\forall i \in \{2, 3\}$ ) that are poles of  $f$  of order 1, 2.

Suppose first  $g$  has a zero  $\gamma \notin \Sigma$  of order  $s = 2$ . By (2) we have

$$2(n + 1) = t(k + n + 1) + 2 \quad (22)$$

By (22) if  $t = 1$  we find a solution:

$$n = k + 1 \quad (23)$$

Next, if  $t = 2$ , we check that  $2n + 2 < 2(k + n + 1) + 2$ , hence (23) is the only solution.

Suppose now  $g$  has a zero  $\gamma \notin \Sigma$  of order  $s = 3$ . Then

$$3(n + 1) = t(k + n + 1) + 2 \quad (24)$$

By (24) if  $t = 1$  we find a solution :

$$n = \frac{k}{2} \quad (25)$$

If  $t = 2$  we find

$$n = 2k + 1 \quad (26)$$

Consequently, by (23), (25), (26) all possibilities for  $g$  to have a zero of order  $s \leq 3$  are as follows:

$$n = k + 1, s = 2, t = 1, \quad n = \frac{k}{2}, s = 3, t = 1, \quad n = 2k + 1, s = 3, t = 2.$$

Consider now zeros of  $g - a_i$   $\gamma \notin \Sigma$  of order  $s_i \leq 3$ . So, the order  $s_i$  of  $g - a_i$  satisfies

$$s_i(k_i + 1) = t(k + n + 1) + 2 \quad (27)$$

Suppose first  $g - a_i$  has a zero  $\gamma \notin \Sigma$  of order  $s_i = 2$ . Then by (28), we have

$$2(k_i + 1) = t(k + n + 1) + 2 \quad (28)$$

Since  $k_i \leq n$  and  $k_i \leq k$  we have  $2(k_i + 1) < t(k + n + 1) + 2$ . Hence we find no solution for (28).

Suppose now that  $s = 3$ .

If  $t = 1$  we find a solution:

$$3k_i = n + k \quad (29)$$

If  $t = 2$ , we have no solution because  $3k_i < 2(n + k)$ .

Consequently, the unique possibility for  $g - a_i$  to have a zero  $\gamma \notin \Sigma$  of order  $s_i \leq 3$  is :

$$n + k = 3k_i, \quad s = 3, \quad t = 1$$

Thus, we have proved that when  $n \neq k + 1, \frac{k}{2}, 2k + 1, 3k_i - k$  none of the zeros of  $f, g, f - a_i, g - a_i$  ( $\forall i = 2, 3$ ) out of  $\Sigma$  is of order  $\leq 3$  and therefore the statement of the Lemma is proved in the case  $l = 3$ .

**Case  $l \geq 4$  :**

Suppose now that all zeros of  $f, g, f - a_i, g - a_i$   $\forall i \in \{2, \dots, l\}$  are at least of order 3, except maybe those lying in  $\Sigma$ : then

$$\overline{Z}(r, f) \leq \frac{1}{3}T(r, f) + S_f(r) + S_g(r), \quad \text{and } \forall i \in \{2, \dots, l\}, \overline{Z}(r, f - a_i) \leq \frac{1}{3}T(r, f) + S_f(r) + S_g(r),$$

$$\overline{Z}(r, g) \leq \frac{1}{3}T(r, g) + S_f(r) + S_g(r), \quad \text{and } \forall i \in \{2, \dots, l\}, \overline{Z}(r, g - a_i) \leq \frac{1}{3}T(r, g) + S_f(r) + S_g(r).$$

Then using (7) we obtain  $l \leq 3$  a contradiction.

Consequently, we will examine all  $n$  and  $k_i$  ( $i \in \{2, \dots, l\}$ ) leading to zeros out of  $\Sigma$  of order  $\leq 2$  for  $f, g, f - a_i, g - a_i$  for all  $i \in \{2, \dots, l\}$ . Actually, since  $f$  and  $g$  play the same role, it is sufficient to examine the situation, for instance, when  $g$  or some  $g - a_i$  has a zero of order less than 2. In each case we denote by  $t$  the order of the pole of  $f$  which is a zero of  $g$  or  $g - a_i$  for some  $i$ . Recall that when  $f$  has a pole of order 2,  $g$  or  $g - a_i$ , if it has a zero, must have a zero of order  $\geq 3$ .

Consequently, we only have to examine zeros of  $g$  or  $g - a_i$  ( $\forall i \in \{2, \dots, l\}$ ) that are poles of  $f$  of order 1.

Suppose first  $g$  has a zero  $\gamma \notin \Sigma$  of order  $s = 2$ . Then  $\gamma$  is a pole of  $f$  of order  $t = 1$ . Then by (2) we have

$$2(n+1) = (k+n+1) + 2 \quad (30)$$

We find a solution :

$$n = k + 1 \quad (31)$$

Let  $i \in \{2, \dots, l\}$  and suppose  $g - a_i$  has a zero  $\gamma \notin \Sigma$  of order  $s_i = 2$ . Then  $\gamma$  is a pole of  $f$  of order  $t = 1$ . So by (3) we have :

$$2(k_i + 1) = (n + k + 1) + 2$$

That means  $2k_i = n + k + 1$ . Since  $k_i \leq n$  and  $k_i \leq k$ , we find no solution when  $s_i = 2$  and  $t = 1$ .

Consequently, by (31), the only possibility for  $g$  or some  $g - a_i$  to have a zero  $\gamma \notin \Sigma$  of order  $\leq 3$  is :

$$n = k + 1.$$

This completes the proof of Lemma 8.

Lemma 9 is known and easily checked (Theorems 2,3 [5] and [24]):

**Lemma 9.** *Let  $f, g \in \mathcal{M}(\mathbb{C})$  be transcendental satisfy  $(f - a)f^n = (g - a)g^n$  with  $a \in \mathbb{C}$  and let  $h = \frac{f}{g}$ . If  $h$  is not identically 1, then*

$$g = \frac{h^n - 1}{h^{n+1} - 1}, \quad f = \frac{h^{n+1} - h}{h^{n+1} - 1}.$$

**Notation:** Let  $f \in \mathcal{M}(\mathbb{C})$  be such that  $f(0) \neq 0, \infty$ . We denote by  $Z_{[2]}(r, f)$  the counting function of the zeros of  $f$  each being counted with multiplicity when it is at most 2 and with multiplicity 2 when it is bigger.

We can extract the following Lemma 10 from a result that is proved in several papers and particularly in Lemma 3 [15].

**Lemma 10.** *Let  $f, g \in \mathcal{M}(\mathbb{C})$  be such that  $f(0) \neq 0, \infty$  and share the value 1 C.M. If  $\Psi_{f,g}$  is not identically zero, then,*

$$\max(T(r, f), T(r, g)) \leq N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) + S_f(r) + S_g(r)$$

### 3 Proof of Theorems

The polynomial  $P$  is the one we considered in theorems 1, 2, 3, 4 and we can assume  $a_1 = 0$ . In Theorem 5 we call  $P$  the polynomial such that  $P(0) = 0$  and  $P'(x) = x^n(x - a)$ . Set  $F = \frac{f'P'(f)}{\alpha}$  and  $G = \frac{g'P'(g)}{\alpha}$ . Clearly  $F$  and  $G$  share the value 1 C.M. Since  $f, g$  are transcendental, we notice that so are  $F$  and  $G$ . Recall that

$$\Psi_{F,G} = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$$



We will prove that under the hypotheses of each theorem,  $\Psi_{F,G}$  is identically zero.

Set  $\widehat{F} = P(f)$ ,  $\widehat{G} = P(g)$ . We notice that  $P(x)$  is of the form  $x^{n+1}Q(x)$  with  $Q \in \mathbb{C}[x]$  of degree  $k$ . Now, by Lemma 4, we have

$$T(r, \widehat{F}) - Z(r, \widehat{F}) \leq T(r, \widehat{F}') - Z(r, \widehat{F}') + S_F(r)$$

Consequently, since  $(\widehat{F})' = \alpha F$ , we have

$$T(r, \widehat{F}) \leq T(r, F) + Z(r, \widehat{F}) - Z(r, F) + T(r, \alpha) + S_f(r), \quad (1)$$

hence, by (1), we obtain

$$T(r, \widehat{F}) \leq T(r, F) + (n+1)Z(r, f) + Z(r, Q(f)) - nZ(r, f) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + S_f(r)$$

i.e.

$$T(r, \widehat{F}) \leq T(r, F) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + S_f(r). \quad (2)$$

and similarly,

$$T(r, \widehat{G}) \leq T(r, G) + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l k_i Z(r, g - a_i) - Z(r, g') + T(r, \alpha) + S_g(r). \quad (3)$$

Now, it follows from the definition of  $F$  and  $G$  that

$$Z_{[2]}(r, F) + N_{[2]}(r, F) \leq 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + T(r, \alpha) + S_f(r) \quad (4)$$

and similarly

$$Z_{[2]}(r, G) + N_{[2]}(r, G) \leq 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + T(r, \alpha) + S_g(r) \quad (5)$$

And particularly, if  $k_i = 1$ ,  $\forall i \in \{2, \dots, l\}$ , then

$$Z_{[2]}(r, F) + N_{[2]}(r, F) \leq 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + T(r, \alpha) + S_f(r) \quad (6)$$

and similarly

$$Z_{[2]}(r, G) + N_{[2]}(r, G) \leq 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + T(r, \alpha) + S_g(r) \quad (7)$$

Suppose now that  $\Psi_{F,G}$  is not identically zero. Now, by Lemma 10, we have

$$T(r, F) \leq Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) + S_f(r) + S_g(r)$$

hence by (2), we obtain

$$T(r, \widehat{F}) \leq Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) + Z(r, f) + Z(r, Q(f)) \\ - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + S_f(r) + S_g(r)$$

and hence by (4) and (5):

$$T(r, \widehat{F}) \leq 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + \\ 2\overline{N}(r, g) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + 2T(r, \alpha) + S_f(r) + S_g(r) \quad (8)$$

and similarly,

$$T(r, \widehat{G}) \leq 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + \\ + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l k_i Z(r, g - a_i) - Z(r, g') + 2T(r, \alpha) + S_f(r) + S_g(r) \quad (9)$$

Consequently,

$$T(r, \widehat{F}) + T(r, \widehat{G}) \leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) + (Z(r, f') + Z(r, g')) + \\ 4(\overline{N}(r, f) + \overline{N}(r, g)) + (Z(r, Q(f)) + Z(r, Q(g))) + S_f(r) + S_g(r). \quad (10)$$

Moreover, if  $k_i = 1$ ,  $\forall i \in \{2, \dots, l\}$ , then by (6) and (7) we have

$$T(r, \widehat{F}) \leq 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + \\ Z(r, g') + 2\overline{N}(r, g) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l Z(r, f - a_i) - Z(r, f') + S_f(r) + S_g(r)$$

and similarly,

$$T(r, \widehat{G}) \leq 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + \\ Z(r, f') + 2\overline{N}(r, f) + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l Z(r, g - a_i) - Z(r, g') + S_f(r) + S_g(r).$$

Consequently,

$$T(r, \widehat{F}) + T(r, \widehat{G}) \leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (Z(r, f - a_i) + Z(r, g - a_i)) + Z(r, Q(f)) + Z(r, Q(g)) +$$

$$(Z(r, f') + Z(r, g')) + 4(\overline{N}(r, f) + \overline{N}(r, g)) + S_f(r) + S_g(r) \quad (11)$$

Now, let us go back to the general case. By Lemma 2, we can write  $Z(r, f') + Z(r, g') \leq Z(r, f - a_2) + Z(r, g - a_2) + \overline{N}(r, f) + \overline{N}(r, g) + S_f(r)$ . Hence, in general, by (10) we obtain

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq \\ 5(Z(r, f) + Z(r, g)) &+ \sum_{i=3}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + (Z(r, Q(f)) + Z(r, Q(g))) + S_f(r) + S_g(r). \end{aligned}$$

And hence, since  $T(r, Q(f)) = kT(r, f) + O(1)$  and  $T(r, Q(g)) = kT(r, g) + O(1)$ , we have

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq \\ 5(T(r, f) + T(r, g)) &+ \sum_{i=3}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + k(T(r, f) + T(r, g)) + S_f(r) + S_g(r). \end{aligned} \quad (12)$$

Now, if  $k_i = 1$ ,  $\forall i \in \{2, \dots, l\}$ , by (11) and Lemma 2 we have

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (Z(r, f - a_i) + Z(r, g - a_i)) + (l - 1)(T(r, f) + T(r, g)) \\ &+ (Z(r, f - a_2) + Z(r, g - a_2)) + 5(\overline{N}(r, f) + \overline{N}(r, g)) + S_f(r) + S_g(r) \end{aligned}$$

hence

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq \\ 5(T(r, f) + T(r, g)) &+ \sum_{i=2}^l (T(r, f - a_i) + T(r, g - a_i)) + (l - 1)(T(r, f) + T(r, g)) + (T(r, f - a_2) + T(r, g - a_2)) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + S_f(r) + S_g(r) \end{aligned}$$

and hence

$$T(r, \widehat{F}) + T(r, \widehat{G}) \leq (9 + 2l)(T(r, f) + T(r, g)) + S_f(r) + S_g(r) \quad (13)$$

Now, let us go back to the general case. Since  $\widehat{F}$  is a polynomial in  $f$  of degree  $n + k + 1$ , we have  $T(r, \widehat{F}) = (n + k + 1)T(r, f) + O(1)$  and similarly,  $T(r, \widehat{G}) = (n + k + 1)T(r, g) + O(1)$ , hence by (12) we can derive

$$\begin{aligned} (n + k + 1)(T(r, f) + T(r, g)) &\leq \\ 5(T(r, f) + T(r, g)) &+ (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + \sum_{i=3}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + k(T(r, f) + T(r, g)) + S_f(r) + S_g(r). \end{aligned} \quad (14)$$

Hence

$$(n + k + 1)(T(r, f) + T(r, g)) \leq (10 + k)(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i))$$

$$+(5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + S_f(r) + S_g(r)$$

and hence

$$\begin{aligned} n(T(r, f) + T(r, g)) &\leq 9(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) \\ &\quad + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + S_f(r) + S_g(r). \end{aligned} \quad (15)$$

Then at least, for each  $i = 3, \dots, l$  we have  $(4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) \leq \max(0, 4 - k_i)(T(r, f) + T(r, g)) + O(1)$  and

$$(5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \leq \max(0, 5 - k_2)(T(r, f) + T(r, g)) + O(1).$$

Consequently, by (15) we have

$$\begin{aligned} n(T(r, f) + T(r, g)) &\leq 9(T(r, f) + T(r, g)) + \\ &\quad + \sum_{i=3}^l \max(0, 4 - k_i)(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(T(r, f) + T(r, g)) + O(1) \end{aligned}$$

and hence,

$$n \leq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \quad (16)$$

Thus, we have proved that  $\Psi_{F,G} = 0$  when

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

concerning Theorem 3.

And, if  $k_i = 1, \forall i \in \{2, \dots, l\}$ , by (13) we have  $n + k + 1 \leq 9 + 2l$ , hence

$$n \leq 9 + l. \quad (17)$$

So, if  $k_i = 1, \forall i \in \{2, \dots, l\}$ , when  $n \geq l + 10$  we have  $\Psi_{F,G} = 0$  which concerns Theorems 4 and 5.

Now, suppose that  $f, g, \alpha$  belong to  $\mathcal{A}(\mathbb{C})$ . By (14), we have

$$\begin{aligned} (n + k + 1)(T(r, f) + T(r, g)) &\leq (5 + k)(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) \\ &\quad + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + S_f(r) + S_g(r) \end{aligned}$$

and hence

$$\begin{aligned} n(T(r, f) + T(r, g)) &\leq 4(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) \\ &\quad + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + S_f(r) + S_g(r) \end{aligned}$$

and hence

$$n(T(r, f) + T(r, g)) \leq 4(T(r, f) + T(r, g)) + \sum_{i=3}^l \max(0, 4 - k_i)(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(T(r, f) + T(r, g)) + O(1),$$

and hence,

$$n \leq 4 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2).$$

Consequently, if  $n \geq 5 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2)$  when  $f, g, \alpha$  belong to  $\mathcal{A}(\mathbb{C})$ , we have  $\Psi_{F,G} = 0$ , which concerns Theorem 1.

Finally, suppose that  $f, g, \alpha$  belong to  $\mathcal{A}(\mathbb{C})$  and  $k_i = 1 \forall i = 1, \dots, l$ . We have  $n + k + 1 \leq 4 + 2l$ , hence  $n \leq 4 + l$ . Consequently, if  $n \geq 5 + l$  when  $f, g, \alpha$  belong to  $\mathcal{A}(\mathbb{C})$ , we have  $\Psi_{F,G} = 0$ , which concerns Theorem 2.

Thus, henceforth, we can assume that  $\Psi_{F,G} = 0$  in each theorem. Note that we can write  $\Psi_{F,G} = \frac{\phi'}{\phi}$  with  $\phi = \left(\frac{F'}{(F-1)^2}\right)\left(\frac{(G-1)^2}{G'}\right)$ . Since  $\Psi_{F,G} = 0$ , there exist  $A, B \in \mathbb{C}$  such that

$$\frac{1}{G-1} = \frac{A}{F-1} + B \quad (18)$$

and  $A \neq 0$ .

We notice  $\bar{Z}(r, f) \leq T(r, f)$ ,  $\bar{N}(r, f) \leq T(r, f)$ ,  $\bar{Z}(r, f - a_i) \leq T(r, f - a_i) \leq T(r, f) + O(1)$ ,  $i = 2, \dots, l$  and  $\bar{Z}(r, f') \leq T(r, f') \leq 2T(r, f) + O(1)$ . Similarly for  $g$  and  $g'$ . Moreover, by Lemma 5, we have

$$T(r, F) \geq (n+k)T(r, f) - m(r, \frac{1}{f'}) + S_f(r). \quad (19)$$

Will consider the following two cases:  $B = 0$  and  $B \neq 0$ .

**Case 1:**  $B = 0$ .

Suppose  $A \neq 1$ . Then, by (18), we have  $F = AG + (1 - A)$ . Applying Theorem N to  $F$ , we obtain

$$\begin{aligned} T(r, F) &\leq \bar{Z}(r, F) + \bar{Z}\left(r, F - (1 - A)\right) + \bar{N}(r, F) + S_F(r) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) \\ &\quad + \bar{Z}(r, f') + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) + S_f(r) + S_g(r). \end{aligned} \quad (20)$$

Consequently, by (19) and (20) we obtain

$$\begin{aligned} (n+k)T(r, f) - m(r, \frac{1}{f'}) &\leq \bar{Z}(r, F) + \bar{Z}\left(r, F - (1 - A)\right) + \bar{N}(r, F) + S_F(r) \\ &\leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, f') + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) + S_f(r) + S_g(r) \end{aligned}$$

therefore

$$(n+k)T(r, f) \leq \bar{Z}(r, F) + \bar{Z}\left(r, F - \left(1-A\right)\right) + \bar{N}(r, F) + m\left(r, \frac{1}{f'}\right) + S_F(r) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) \\ + \bar{Z}(r, f') + m\left(r, \frac{1}{f'}\right) + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) + S_f(r) + S_g(r).$$

Here we notice that  $\bar{Z}(r, f') + m\left(r, \frac{1}{f'}\right) \leq T\left(r, \frac{1}{f'}\right) = T(r, f') + O(1)$ , hence  $(n+k)T(r, f) \leq$

$$\leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{N}(r, f) + \bar{Z}(r, g') + T(r, f') + S_f(r) + S_g(r). \quad (21)$$

Then, considering all the previous inequalities in (21), by Lemma 3 we can derive

$$(n+k)T(r, f) \leq (l+3)T(r, f) + (l+2)T(r, g) + S_f(r) + S_g(r). \quad (22)$$

Since  $f$  and  $g$  satisfy the same hypothesis, we also have

$$(n+k)T(r, g) \leq (l+3)T(r, g) + (l+2)T(r, f) + S_f(r) + S_g(r). \quad (23)$$

Hence, adding (22) and (23), we have

$$(n+k)[T(r, f) + T(r, g)] \leq (2l+5)[T(r, f) + T(r, g)] + S_f(r) + S_g(r),$$

which leads to a contradiction whenever  $n+k \geq (2l+6)$ .

In the hypothesis of Theorems 3, we have

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2).$$

That implies

$$n+k \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2) + \sum_{i=2}^l k_i \geq 10 + \sum_{i=2}^l \max(k_i, 4) \geq 10 + 4(l-1)$$

hence

$$n+k \geq 4l+6. \quad (24)$$

That contradiction proves that  $A = 1$  in Theorem 3.

In the hypothesis of Theorems 4 and 5 we have  $n \geq l+10$  with  $k = l-1$  hence

$$n+k \geq 2l+9 \quad (25)$$

therefore  $A = 1$  again in Theorems 4 and 5.

Consider now the situation in Theorem 1. First we notice that given a positive number  $c$  we have  $\max(0, 4-c) + c \geq 4$ . By (21), with help of Lemma 3, we can derive

$$(n+k)T(r, f) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + T(r, f') + S_f(r) + S_g(r),$$

$$\leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + T(r, g) + T(r, f) + S_f(r) + S_g(r),$$

with a similar inequality for  $g$ , thereby

$$(n+k)(T(r, f) + T(r, g)) \leq (2l+2)(T(r, f) + T(r, g))$$

and hence  $n+k \leq 2l+2$ , which leads to a contradiction as soon as  $n+k \geq 2l+3$ . Instead of (24) we now have  $n+k \geq 4l+1$ . Consequently, the relation  $n+k \geq 2l+3$  is easily satisfied in Theorem 1, proving  $A=1$  again.

And now, consider the hypothesis of Theorem 2. We have  $n \geq l+5$ , hence  $n+k \geq 2l+4$  assuring that  $A=1$  again.

Thus, in case  $B=0$ , we have proved that  $F=G$ . Now,  $\alpha F = \alpha G$ , i.e.  $(\widehat{F})' = (\widehat{G})'$ . We assume  $n \geq k+3$  in Theorem 3 and this is automatically satisfied in Theorems 4 and 5. Next, we assume  $n \geq k+2$  in Theorem 1 and this is automatically satisfied in Theorems 2. Consequently, by Lemma 6, we have  $\widehat{F} = \widehat{G}$ , i.e.  $P(f) = P(g)$ . But in Theorems 3 and 4,  $P$  is a polynomial of uniqueness for the family of meromorphic functions we consider, hence we have  $f=g$ . And in Theorem 5, the conclusion comes from Lemma 9. Finally, in Theorems 1 and 2, we assume  $P$  is a polynomial of uniqueness for  $\mathcal{A}(\mathbb{C})$ , we have  $f=g$  again.

**Case 2:**  $B \neq 0$ .

We have  $\bar{Z}(r, F) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, f') + S_f(r)$  and  $\bar{N}(r, F) \leq \bar{N}(r, f) + S_f(r)$  and similarly for  $G$ , so we can derive

$$\begin{aligned} \bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) &\leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, f') \\ &+ \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) + \bar{N}(r, g) + S_f(r) + S_g(r). \end{aligned} \quad (26)$$

Moreover, by (19) we have

$(n+k)T(r, f) \leq T(r, F) + m(r, \frac{1}{f'}) + S_f(r)$  and  $(n+k)T(r, g) \leq T(r, G) + m(r, \frac{1}{g'}) + S_g(r)$ . Consequently, by (26) we obtain

$$\begin{aligned} \bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) &\leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + [\bar{Z}(r, f') + m(r, \frac{1}{f'})] - m(r, \frac{1}{f'}) \\ &+ \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + [\bar{Z}(r, g') + m(r, \frac{1}{g'})] - m(r, \frac{1}{g'}) + \bar{N}(r, f) + \bar{N}(r, g) + S_f(r) + S_g(r) \\ &\leq (l+1)[T(r, f) + T(r, g)] + T(r, f') + T(r, g') - m(r, \frac{1}{f'}) - m(r, \frac{1}{g'}) + S_f(r) + S_g(r) \end{aligned}$$

hence by Lemma 3,

$$\bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) \leq (l+3)(T(r, f) + T(r, g)) - m(r, \frac{1}{f'}) - m(r, \frac{1}{g'}) + S_f(r) + S_g(r) \quad (28)$$

Now, in Theorems 3, 4, 5 we have  $n+k \geq \min(4l+6, 2l+9)$  hence  $\frac{2l+6}{k+n} < 1$ , therefore by (19) we notice that

$$2(n+k)\left(\frac{T(r,f)+T(r,g)}{2}\right) - m(r, \frac{1}{f'}) - m(r, \frac{1}{g'}) \leq T(r,F) + T(r,G) + S_f(r) + S_g(r)$$

hence (28) yields

$$\bar{Z}(r,F) + \bar{Z}(r,G) + \bar{N}(r,F) + \bar{N}(r,G) \leq \left(\frac{2l+6}{n+k}\right)\left(\frac{T(r,F)+T(r,G)}{2}\right) + S_f(r) + S_g(r).$$

Now by Lemma 5 we can derive

$$\bar{Z}(r,F) + \bar{Z}(r,G) + \bar{N}(r,F) + \bar{N}(r,G) \leq \left(\frac{2l+6}{n+k}\right)\left(\frac{T(r,F)+T(r,G)}{2}\right) + S_F(r) + S_G(r) \quad (29)$$

Consequently, by (29), we obtain

$$\limsup_{r \rightarrow +\infty} \left( \frac{\bar{Z}(r,F) + \bar{Z}(r,G) + \bar{N}(r,F) + \bar{N}(r,G)}{\max(T(r,F), T(r,G))} \right) < 1$$

outside a set of finite measure and hence, by Lemma 7, we have  $F = G$  or  $FG = 1$ .

Suppose  $FG = 1$ , hence  $f'P'(f)g'P'(g) = \alpha^2$ . In Theorem 3 we have assumed that  $n \geq k+3$  and in Theorems 4 and 5 this hypothesis is automatically satisfied. Next, if  $l = 2$ , then  $n \neq 2k, 2k+1, 3k+1$  and if  $l = 3$  then  $n \neq 2k+1, 3k_i - k$ ,  $i = 2, 3$ . Moreover, these conditions are automatically satisfied in Theorems 4 and 5. So, we have a contradiction to Lemma 8 in Theorems 3, 4, 5. Consequently,  $F = G$  and therefore we can conclude as in the case  $B = 0$ , that  $P(f) = P(g)$ . Now, in Theorems 3 and 4  $P$  is a polynomial of uniqueness for  $\mathcal{M}(\mathbb{C})$ , hence we have  $f = g$ . And in Theorem 5 we know that either  $f = g$  or  $f$  and  $g$  are linked by the relation mentioned in Theorems 2, 3 [5] and in [24].

Consider now the situation in Theorems 1 and 2. By (26) and Lemma 3, here we can write

$$\begin{aligned} \bar{Z}(r,F) + \bar{Z}(r,G) + \bar{N}(r,F) + \bar{N}(r,G) &\leq \bar{Z}(r,f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, f') + \bar{Z}(r, g') + \bar{Z}(r, g) \\ &\quad + \sum_{i=2}^l \bar{Z}(r, g - a_i) + S_f(r) + S_g(r) \\ &\leq \bar{Z}(r,f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + T(r,f) + T(r,g) - m(r, \frac{1}{f'}) - m(r, \frac{1}{g'}) + \bar{Z}(r, g) \\ &\quad + \sum_{i=2}^l \bar{Z}(r, g - a_i) + S_f(r) + S_g(r) \end{aligned}$$

hence finally,

$$\bar{Z}(r,F) + \bar{Z}(r,G) + \bar{N}(r,F) + \bar{N}(r,G) \leq (l+1)(T(r,f) + T(r,g)) - m(r, \frac{1}{f'}) - m(r, \frac{1}{g'}).$$



As in (29) here, thanks to Lemma 5 we can obtain (30):

$$\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \left(\frac{2l+2}{n+k}\right) \left(\frac{T(r, F) + T(r, G)}{2}\right) + S_F(r) + S_G(r) \quad (30)$$

On the other hand, in Theorem 1, instead of (24) here we have  $n+k \geq 4l+1$  hence  $\frac{2l+2}{n+k} < 1$  and hence we can apply Lemma 7 proving that either  $F = G$  or  $FG = 1$ . But since  $\alpha F, \alpha G$  and  $\alpha$  belong to  $\mathcal{A}(\mathbb{C})$ , by Lemma 8 we know that  $(\alpha F)(\alpha G) = (\alpha)^2$  is impossible. Consequently  $F = G$ .

Now in Theorem 2, we have  $n+k \geq 2l+4$  hence  $\frac{2l+2}{n+k} < 1$  and hence we can go on as in Theorem 1. So, finally we have  $F = G$  in Theorems 1 and 2 again. Next, we can conclude as in the case  $B = 0$ .

## References

- [1] T.T.H. An, J.T.Y. Wang and P.M. Wong, Strong uniqueness polynomials: the complex case. *Complex Variables* Vol. 49, No. 1, pp. 25-54 (2004).
- [2] K. Boussaf, A. Escassut, J. Ojeda, p-adic meromorphic functions  $f'P'(f)$ ,  $g'P'(g)$  sharing a small function, *to appear in Bulletin des Sciences Mathématiques*.
- [3] A. Boutabaa, Théorie de Nevanlinna p-adique, *Manuscripta Math.* 67, 251 - 269 (1990).
- [4] A. Boutabaa and A. Escassut, URS and URSIMS for p-adic meromorphic functions inside a disc, *Proc. of the Edinburgh Mathematical Society* 44, 485 - 504 (2001).
- [5] A. Escassut, L.Haddad, R. Vidal, Urs, Ursim, and nonurs, *Journal of Number Theory* 75, p. 133-144 (1999).
- [6] A. Escassut, Meromorphic functions of uniqueness, *Bulletin des Sciences Mathématiques* 131(3), 219 - 241 (2007).
- [7] A. Escassut, J. Ojeda and C. C. Yang, Functional equations in a p-adic context, *Journal of Mathematical Analysis and Applications*, 351, no. 1, p.350- 359 (2009)
- [8] M. Fang and X.H. Hua, Entire functions that share one value. *J. Nanjing University Math. Biq.* 13 (1), 44-48 (1996).
- [9] M. Fang and W. Hong, A unicity theorem for entire functions concerning differential polynomials, *Indian J. Pure Appl. Math.* 32(9), 1343 - 1348 (2001).
- [10] G. Frank and M. Reinders, A unique range set for meromorphic functions with 11 elements, *Complex Variable Theory Applic.* 37, 185 - 193 (1998).
- [11] H. Fujimoto, On uniqueness of Meromorphic Functions sharing finite sets, *Amer. J. Math.* 122 no. 6, 1175-1203 (2000).
- [12] W.K. Hayman, Meromorphic Functions, Oxford University Press, (1975)
- [13] N. T. Hoa, On the functional equation  $P(f) = Q(g)$  in non-archimedean field, *Acta Math. Vietnam.* 31(2), 167 - 180 (2006).
- [14] P. C. Hu and C. C. Yang, Meromorphic functions over non archimedean fields, *Kluwer Academy Publishers* (2000).

- [15] X. Hua and C. C. Yang, Uniqueness and value-sharing of meromorphic functions, *Ann. Acad. Sci. Fenn. Math.* 22, 395 - 406 (1997).
- [16] I. Lahiri and N. Mandal, Uniqueness of nonlinear differential polynomials sharing simple and double 1-points, *Int. J. Math. Math. Sci.* 12, 1933 - 1942 (2005).
- [17] P. Li and C.C. Yang, Some further results on the unique range sets of meromorphic functions. *Kodai Math. J.* 18, 437 - 450 (1995).
- [18] W. Lin and H. Yi, Uniqueness theorems for meromorphic functions concerning fixed-points. *Complex Var. Theory Appl.* 49(11), 793 - 806 (2004).
- [19] J. Ojeda, Applications of the p-adic Nevanlinna theory to problems of uniqueness, Advances in p-adic and Non-Archimedean analysis. *Contemporary Mathematics* 508, p.161-179 (2010).
- [20] J. Ojeda, zeros of ultrametric meromorphic functions  $f'f^n(f-a)^k-\alpha$ , *Asian-European Journal of mathematics* Vol.1, n.3, p. 415-429, (2008).
- [21] J. Ojeda, Uniqueness for ultrametric analytic functions. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)* 54(102), no. 2, 153-165 (2011)
- [22] J. T.Y. Wang, Uniqueness polynomials and bi-unique range sets, *Acta Arithmetica.* 104, p. 183-200. (2002).
- [23] Y. Xu and H. Qu, Entire functions sharing one value I.M. *Indian Journal Pure Appl. Math* 31 (7), 849-855 (2000).
- [24] C.C. Yang and X. Hua , Unique polynomials of entire and meromorphic functions. *Matematicheskaja Fizika Analys Geometriye*, v. 4, n.3, p. 391-398, (1997).
- [25] H.X. Yi, Meromorphic functions that share one or two values. *Complex variables and applications.* 28, p. 1-11. (1995).
- [26] H.X. Yi and C.C. Yang, Uniqueness theorems of meromorphic functions. Science Press, China (1995).

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