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Complex meromorphic functions $f'P'(f),\ g'P'(g)$ sharing a small function

Kamal Boussaf, Alain Escassut and Jacqueline Ojeda

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Abstract

Let $f,\ g$ be two transcendental meromorphic functions in $\mathbb{C}$, let P be a polynomial of uniqueness for meromorphic functions in $\mathbb{C}$ and let α be a small meromorphic function with regards to f and g. If $f'P'(f)$ and $g'P'(g)$ share α counting multiplicity, then we show that $f = g$ provided that the multiplicity order of zeroes of P' satisfy certain inequalities. There is no additional condition on α. We consider the particular case of entire functions.

1 Introduction and Main Results

Following several previous studies, the problem we consider consists of determining polynomials P such that, if f and g are complex meromorphic functions and if α is a small meromorphic function such that $f'P'(f)$ and $g'P'(g)$ share α counting multiplicity, then we can conclude $f = g$. The first (and hardest) step consists of showing that $f'P'(f) = g'P'(g)$, so $P(f) - P(g)$ is a constant. The second step consists of showing that the constant is zero. And then, if we take for P a polynomial of uniqueness for complex meromorphic functions, we can get $f = g$.

Problems of uniqueness on meromorphic functions were examined first in $\mathbb{C}$ [8], [9], [11], [15], [16], [17], [18], [23], [24], [26] and next in a p-adic field [1], [4], [5], [6], [7], [13], [14], [19], [20], [21], [22]. After examining problems of the form $P(f) = P(g)$, where P is a polynomial, several studies were made on the equality $f'P'(f) = g'P'(g)$, or value sharing questions: if $P(f)$ and $P(g)$ share a value, or a small function, do we have $f = g$?

We denote by $\mathcal{A}(\mathbb{C})$ the $\mathbb{C}$-algebra of entire functions in $\mathbb{C}$, by $\mathcal{M}(\mathbb{C})$ the field of meromorphic functions in $\mathbb{C}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{C})$ and by $\mathbb{C}(x)$ the field of rational functions.

The problem of value sharing a small function for functions of the form $f'P'(f)$ was examined first when P was just of the form x^n [8], [19] in both complex and p-adic contexts. More recently it was examined when P was a polynomial such that P' had exactly two distinct zeros [16], [18], [21], both in complex analysis and in p-adic analysis. In [16], [18] the functions where meromorphic on $\mathbb{C}$, with a small function that was a constant or the identity. In [19], the problem was considered for analytic functions in a complete ultrametric algebraically closed field $\mathbb{K}$ of characteristic 0 concerning entire functions and unbounded analytic functions in an open disk, sharing a rational

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function. In [2] the three authors considered the same problem for meromorphic functions in the p-adic fileld $\mathbb{K}$. The results obtained in that case have then suggested a similar study in $\mathbb{C}$. But several crucial lemmas obtained on a p-adic field have to be re-examined particularly because the characteristic Nevanlinna function $T(r,f)$ is quite different for complex meromorphic functions and for p-adic ones, particularly all statements involving functions $m(r,f)$ on $\mathcal{M}(\mathbb{C})$. So, we must develop complete proofs for all intermediate and final results.

Here we enjoy polynomials of uniqueness for complex meromorphic functions [1], [6] and we can finally obtain results similar to those obtained in p-adic analysis: see Theorem 1[2].

The definition of polynomials of uniqueness was introduced in [17] by P. Li and C. C. Yang and was studied in many papers [1], [10], [11] , [24] for complex functions and [5], [6], [22] for p-adic functions.

Definition. Recall that a polynomial $P \in \mathbb{C}[x]$ is called *a polynomial of uniqueness* for a class of functions $\mathcal{F}$ if for any two functions $f,\ g \in \mathcal{F}$ the property $P(f) = P(g)$ implies $f = g$.

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field $\mathbb{C}$. Here, for convenience, we will use notation long ago used in p-adic analysis in order to denote the counting functions.

Let log be a real logarithm function of base > 1. Given $u \in \mathbb{R}_+^*$, we denote by $\log^+$ the real function defined as $\log^+(u) = \max(\log(u), 0)$.

Let $f \in \mathcal{M}(\mathbb{C})$. Given r$> 0$, we set $m(r,f) = \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$. Suppose f has no zero and no pole at 0. Let $\gamma \in \mathbb{C}$. If f has a zero of order n at γ, we set $\omega_\gamma(f) = n$. If f has a pole of order n at γ, we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we put $\omega_\gamma(f) = 0$

We denote by $Z(r,f)$ the *counting function of zeros of* f in the ball $|z| \leq r$, counting multiplicity, i.e. we set

$$Z(r,f) = \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} \omega_\gamma(f)(\log r - \log|\gamma|).$$

Similarly, we denote by $\overline{Z}(r,f)$ the counting function of zeros of f in the ball $|z| \leq r$, ignoring multiplicity, and set

$$\overline{Z}(r,f) = \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} (\log r - \log|\gamma|).$$

In the same way, we set $N(r,f) = Z\Big(r, \frac{1}{f}\Big)$ $\Big($resp. $\overline{N}(r,f) = \overline{Z}\Big(r, \frac{1}{f}\Big)\Big)$ to denote the *counting function of poles of* f in the ball $|z| \leq r$, counting multiplicity (resp. ignoring multiplicity).

If f admits a zero of order s at 0, we can make a change of origin or count the zero at 0 by adding $s \log r$ and similarly, if f admits a pole at 0 of order s, we can make a change of origin or count the pole at 0 by adding $-s \log r$.

For $f \in \mathcal{M}(\mathbb{C})$ having no zero and no pole at 0, the *Nevanlinna function* is defined by $T(r,f) = m(r,f) + N(r,f)$.

Definition. Let $f \in \mathcal{M}(\mathbb{C})$ such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{C})$ having no zero and no pole at 0 is called *a small function with respect to* f, if it satisfies $\lim\limits_{r\to+\infty} \dfrac{T(r,\alpha)}{T(r,f)} = 0$. We denote by $\mathcal{M}_f(\mathbb{C})$ the set of small meromorphic functions with respect to f in $\mathbb{C}$. And we say that two functions $f,\ g \in \mathcal{M}(\mathbb{C})$ *share a function* α *C.M.* if $f - \alpha$ and $g - \alpha$ admit the same zeros with the same multiplicity.

Remark 1. *For simplicity, we have kept the same notation as in p-adic analysis for the counting function of zeros of a meromorphic function.*

Now, we can give some sufficient conditions to get polynomials P such that, if $f'P'(f)$ and $g'P'(g)$ share a small meromorphic function, then $f = g$.

Recall the following Theorems A, B that we can get from [18]:

Theorem A. *Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental. Let $n \in \mathbb{N},\ n \geq 13$ and let $a \in \mathbb{C} \setminus \{0\}$. If $f^n(f-a)^2 f'$ and $g^n(g-a)^2 g'$ share the identical function x C.M. then $f = g$.*

Theorem B. *Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $a \in \mathbb{C} \setminus \{0\}$ and let $\in \mathbb{N},\ n \geq 12$. If $f^n(f-a)f'$ and $g^n(g-a)g'$ share the identical function x C.M. then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{C})$ such that $f = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1}-1}{h^{n+2}-1}\Big)h$ and $g = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1}-1}{h^{n+2}-1}\Big)$.*

In [2], we have obtained the following Theorem C.

Theorem C. *Let $\mathbb{K}$ be a complete ultrametric algebraically closed field of characteristic zero. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, with $l \geq 2$ and $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$, whenever $l > 2$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq 10 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,

$n \geq k+2$,

if $l = 2$, then $n \neq 2k,\ 2k+1,\ 3k+1$,

if $l = 3$, then $n \neq 2k+1,\ 3k_i - k\ \forall i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Next, several particular applications were given when the small function is a constant or a Moebius function. Here we can't get similar refinements because the complex Nevanlinna Theory is less accurate than the p-adic Nevanlinna Theory.

However, we aim at obtaining for complex meromorphic functions results almost similar to those obtained for p-adic functions, thanks to theorems of uniqueness in $\mathbb{C}$. Let us recall Theorems D and E that we can extract from Theorem 3 [1] and Corollary 9.1 [6] respectively.

Notation: Let L be an algebraically closed field and let $P \in L[x] \setminus \mathbb{K}$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c. We denote by $\Phi(P)$ its cardinal.

In [1], Theorem 3 is stated under the general condition that the polynomial P we consider is injective on the set of zeros of its derivative. Here we reduce that hypothesis to a hypothesis concerning $\Phi(P)$. Particularly, the following Theorem D (concerning the case when P' only has two distinct zeros) comes fromTheorem 3 [1], but actually, it is useless to assume the general hypothesis that the restriction of P on the set of zeros of P' is injective because in that case, such a polynomial P necessarily satisfies $P(c_1) \neq P(c_2)$ with the two zeros $c_1,\ c_2$ of P', as shown in Lemma 10 of [6].

Remark 2. *The hypothesis $\Phi(P) \geq 4$ is weaker than the hypothesis: "the restriction of P on the set of zeros of P' is injective" as soon as the number of distinct zeros of P' is at least* 6.

Theorem D *Let $P \in \mathbb{C}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 with $\min\{c_1, c_2\} \geq 2$ and $\max(c_1, c_2) \geq 3$. Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.*

Theorem E *Let $P \in \mathbb{C}[x]$ be such that P' has exactly three distinct zeros γ_1 of order c_1, γ_2 of order c_2, γ_3 of order c_3 with $\max\{c_1, c_2, c_3\} \geq 2$, satisfying further $\Phi(P) = 3$. Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.*

We can find Theorem F in [6] (Corollary 9.1, p. 226):

Theorem F. *Let $P \in \mathbb{C}[x]$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.*

We can now state our main theorems on the problem $f'P'(f),\ g'P'(g)$ sharing a small function.

Theorem 1. *Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{C}^*$, $l \geq 2$, and $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$ when $l > 2$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq k + 2$

$n \geq 5 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2)$,

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem F, we can derive Corollary 1.1:

Corollary 1.1 *Let $P \in \mathbb{C}[x]$ be such that $P'(x) = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{C}^*$, $\Phi(P) \geq 4$, and $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq k + 2$,

$n \geq 5 + \sum_{i=3}^{l} \max(0, 4 - k_i) + \max(0, 5 - k_2)$.

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem E, we can drive Corollary 1.2:

Corollary 1.2 *Let $P \in \mathbb{C}[x]$ be such that $P'(x) = b(x-a_1)^n(x-a_2)^{k_2}(x-a_3)^{k_3}$ with $b \in \mathbb{C}^*$, $k_2 \geq k_3$, $\Phi(P) = 3$. Suppose P satisfies the following conditions:*

$n \geq k_2 + k_3 + 2$,

$n \geq 5 + \max(0, 4 - k_3) + \max(0, 5 - k_2)$.

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 2. *Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$ such that P' is of the form $b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geq 3$, $b \in \mathbb{C}^*$, satisfying, $n \geq l + 5$.*

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem F we have Corollary 2.1:

Corollary 2.1 *Let $P \in \mathbb{C}[x]$ be such that $P'(x) = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $\Phi(P) \geq 4$, $b \in \mathbb{C}^*$, satisfying $n \geq l+5$.*
Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem E we have Corollary 2.1:

Corollary 2.2 *Let $P \in \mathbb{C}[x]$ be such that $P'(x) = b(x-a_1)^n(x-a_2)^{k_2}(x-a_3)^{k_3}$ with $\Phi(P) = 3$, $b \in \mathbb{C}^*$, satisfying $n \geq 8$.*
Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 3. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{C}^*$, and $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ when $l > 2$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq 10 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,
$n \geq k+3$,
if $l = 2$, then $n \neq 2k,\ 2k+1,\ 3k+1$,
if $l = 3$, then $n \neq 2k+1,\ 3k_i - k,\ \forall i = 2,\ 3$.
Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem F, we have Corollary 3.1:

Corollary 3.1 *Let $P \in \mathbb{C}[x]$ be such that $P'(x) = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{C}^*$, $\Phi(P) \geq 4$, $k_i \geq k_{i+1}$ and $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq 10 + \sum_{i=3}^{l} \max(0, 4-k_i) + \max(0, 5-k_2)$,
$n \geq k+3$.
Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem D we also have Corollary 3.2

Corollary 3.2 *Let $P \in \mathbb{C}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $k \geq 2$. Suppose that P satisfies the further conditions:*
$n \geq 10 + \max(0, 5-k)$,
$n \geq k+3$,
$n \neq 2k,\ 2k+1,\ 3k+1$.

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let $P(x) = \frac{x^{16}}{16} - \frac{2x^{15}}{15} + \frac{x^{14}}{14}$. Then $P'(x) = x^{13}(x-1)^2$. Given $f,\ g \in \mathcal{M}(\mathbb{C})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{M}(\mathbb{C})$ C.M., we have $f = g$.

By Corollary 3.2 we can also derive Corollary 3.3:

Corollary 3.3 *Let $f,\ g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. Let $a \in \mathbb{C}^*$. If $f'f^n(f-a)^2$ and $g'g^n(g-a)^2$ share the function α C.M. and if $n \geq 13$, then $f = g$.*

By Theorem E, we can also derive Corollary 3.4:

Corollary 3.4 *Let $P \in \mathbb{C}[x]$ be such that $P'(x) = b(x-a_1)^n(x-a_2)^{k_2}(x-a_3)^{k_3}$ with $b \in \mathbb{C}^*$, $\Phi(P) = 3$, $k_i \geq k_{i+1}$ and $i = 2,\ 3$. Suppose P satisfies the following conditions:*

$n \geq 10 + \sum_{i=3}^{l} \max(0, 4-k_2) + \max(0, 5-k_3)$,

$n \geq k_2 + k_3 + 3$.

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

When all k_i are equal to 1, we can obtain a better formulation:

Theorem 4. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$ such that P' is of the form*
$b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ *with $l \geq 3$, $b \in \mathbb{C}^*$, with $n \geq l + 10$.*
Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem F, we have Corollary 4.1:

Corollary 4.1 *Let $P \in \mathbb{C}[x]$ be such that P' is of the form*

$P'(x) = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$, *with $\Phi(P) \geq 4$, $b \in \mathbb{C}^*$ satisfying:*

$n \geq l + 10$.

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let $P(x) = \frac{x^{18}}{18} - \frac{2x^{17}}{17} - \frac{x^{16}}{16} + \frac{2x^{15}}{15}$. Then $P'(x) = x^{17} - 2x^{16} - x^{15} + 2x^{14} = x^{14}(x-1)(x+1)(x-2)$. We check that: $P(0),\ P(1),\ P(-1),\ P(2)$ are all distinct. Consequently $\Phi(P) = 4$. So, P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$. Moreover, we have $n = 14,\ l = 4$, so we can apply Corollary 4.1.

Given $f,\ g \in \mathcal{M}(\mathbb{C})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{M}(\mathbb{C})$ C.M., we have $f = g$.

And by Theorem E we have Corollary 4.2

Corollary 4.2 *Let $P \in \mathbb{C}[x]$ be such that P' is of the form*

$P'(x) = b(x-a_1)^n(x-a_2)(x-a_3)$ with $\Phi(P) = 3$, $b \in \mathbb{C}^*$, satisfying $n \geq 13$.
Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Consider the general form of a polynomial P such that P' has 3 distinct zeros, one of them being 0 of order n and the two others a and b of order 1. We can obviously suppose that $P(0) = 0$. And then we have

$$P(x) = x^{n+3}(n+2)(n+1) - x^{n+2}(a_1+a_2)(n+3)(n+1) + x^{n+1}(a_1a_2)(n+3)(n+2)$$

and therefore

$$P'(x) = (n+1)(n+2)(n+3)x^n(x-a_1)(x-a_2).$$

Now we can check that $P(a_1) = P(0) = 0$ if and only if $\dfrac{a_1}{a_2} = \dfrac{n+3}{n+1}$ and that $P(a_1) = P(a_2)$ if and only if $[a_1^{n+3} - a_2^{n+3}](n+1) = [a_1a_2(a_1^{n+1} - a_2^{n+1})](n+3)$. Particularly, the last equality is satisfied when n is odd and $a_2 = -a_1$.

Consequently, if $\dfrac{a_1}{a_2} \neq \dfrac{n+3}{n+1}$ and $\dfrac{a_1}{a_2} \neq \dfrac{n+1}{n+3}$ and if

$$[a_1^{n+3} - a_2^{n+3}](n+1) \neq [a_1a_2(a_1^{n+1} - a_2^{n+1})](n+3)$$

then P is a polynomial of uniqueness for complex meromorphic functions. Moreover, if $n \geq 13$, then we can apply Corollary 4.2.

Conversely, suppose for instance n is odd and $a_2 = -a_1$. Then we can check that $P(f) = P(-f)$ for every function, therefore P is not a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.

Theorem 5. *Let $f,\ g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. Let $a \in \mathbb{C}^*$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function α C.M. and if $n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{C})$ such that $f = \dfrac{a(n+2)}{n+1}\Big(\dfrac{h^{n+1}-1}{h^{n+2}-1}\Big)h$ and $g = \dfrac{a(n+2)}{n+1}\Big(\dfrac{h^{n+1}-1}{h^{n+2}-1}\Big)$.*

Remark 3. *In Theorem 5, the second conclusion is well known to occur [5], [24]. Indeed, let $h \in \mathcal{M}(\mathbb{C})$ and let us define f and g as: $g = (\dfrac{n+2}{n+1})\Big(\dfrac{h^{n+1}-1}{h^{n+2}-1}\Big)$ and $f = hg$. We can see that the polynomial $P(y) = \dfrac{1}{n+2}y^{n+2} - \dfrac{1}{n+1}y^{n+1}$ satisfies $P(f) = P(g)$, hence $f'P'(f) = g'P'(g)$, therefore $f'P'(f)$ and $g'P'(g)$ trivially share any function.*

Remark 4. *On a p-adic field, we obtained conclusions with hypotheses slightly more general when the small function α is either a Moebius function or a constant, thanks to a very accurate Second Main Theorem. On the field $\mathbb{C}$, we cannot do the same.*

2 Basic and specifc results:

Notation: Given $f \in \mathcal{M}(\mathbb{C})$, we will denote by $S_f(r)$ a function in r defined in $]0, +\infty[$ such that $\lim_{r\to+\infty} \frac{S_f(r)}{T(r,f)} = 0$ out of a subset J of $]0, +\infty[$ of finite measure.

Let us recall a few classical lemmas [12]:

Lemma 1. *Let $f, g \in \mathcal{M}(\mathbb{C})$, let $a \in \mathbb{C}$ and let $P(f) \in \mathbb{C}[x]$ be of degree q. Then $T(r, f+g) \leq T(r,f) + T(r,g) + O(1)$, $T(r, fg) \leq T(r,f) + T(r,g)$, $T(r, f-a) = T(r,f) + O(1)$, $T(r, \frac{1}{f}) = T(r,f) + O(1)$, $T(r, P(f)) = qT(r,f) + O(1)$.*

Lemma 2. *Let $f,\ g \in \mathcal{M}(\mathbb{C})$. Then $Z(r, f-a) \leq T(r,f) + O(1)\ \forall a \in \mathbb{C}$, $m(r, fg) \leq m(r,f) + m(r,g)$, $N(r,f') = N(r,f) + \overline{N}(r,f)$, $Z(r,f') \leq Z(r,f) + \overline{N}(r,f) + S_f(r)$. Let $Q \in \mathbb{C}[x]$ be of degree q. Then $Z(r, f'Q(f)) \geq Z(r, Q(f))$ and $N(r, f'Q(f)) = (q+1)N(r,f) + \overline{N}(r,f)$.*

By Milloux's Theory we have the following lemma (Theorem 3.1 in [12]).

Lemma 3. *Let $f \in \mathcal{M}(\mathbb{C})$. Then $m(r, \frac{f'}{f}) = S_f(r)$ and $T(r,f') \leq T(r,f) + \overline{N}(r,f) + S_f(r)$. Moreover, if $f \in \mathcal{A}(\mathbb{C})$ then $T(r,f') \leq T(r,f) + S_f(r)$.*

The statements of Lemma 4 are known and used in [18]:

Lemma 4. *Let $f \in \mathcal{M}(\mathbb{C})$. Then $m(r, \frac{1}{f'}) \geq m(r, \frac{1}{f}) + S_f(r)$. Moreover, $T(r,f) - Z(r,f) \leq T(r,f') - Z(r,f') + S_f(r)$.*

Notation: Let $f \in \mathcal{M}(\mathbb{C})$ satisfy $f'(0) \neq 0, \infty$. Let S be a finite subset of $\mathbb{C}$ and $r \in]0, +\infty[$. We denote by $Z_0^S(r,f')$ *the counting function of zeros of f' in $\mathbb{C}$ which are not zeros of any $f-s$ for $s \in S$*. This is, if $(\gamma_n)_{n\in\mathbb{N}}$ is the finite or infinite sequence of zeros of f' in $\mathbb{C}$ that are not zeros of $f-s$ for $s \in S$, with multiplicy order q_n respectively, we set

$$Z_0^S(r,f') = \sum_{|\gamma_n| \leq r} q_n(\log r - \log|\gamma_n|).$$

Theorem N. [12] *Let $a_1, ..., a_n \in \mathbb{C}$ with $n \geq 2, n \in \mathbb{N}$, and let $f \in \mathcal{M}(\mathbb{C})$. Let $S = \{a_1, ..., a_n\}$. Assume that none of f, f' and $f - a_j$ with $1 \leq j \leq n$, equals 0 or ∞ at the origin. Then, for $r > 0$ we have*

$$(n-1)T(r,f) \leq \sum_{j=1}^{n} \overline{Z}(r, f-a_j) + \overline{N}(r,f) - Z_0^S(r,f') + S_f(r).$$

We will also need the following Lemma 5 which is specific to the complex case.

Lemma 5. *Let $Q \in \mathbb{C}[x]$ and let $f \in \mathcal{M}(\mathbb{C})$ be transcendental. Then $T(r, Q(f)) \leq T(r, f'Q(f)) + m(r, \frac{1}{f'})$ and $T(r,Q(f)) \leq T(r,f) + \overline{N}(r,f) + T(r, f'Q(f)) + S_f(r)$, $r \in]0, +\infty[$. Moreover, if $\deg(Q) \geq 3$, then $\mathcal{M}_f(\mathbb{C}) \subset \mathcal{M}_{f'Q(f)}(\mathbb{C})$.*

Proof : We have $T(r,Q(f)) = N(r,Q(f)) + m(r, f'\frac{Q(f)}{f'}) \leq N(r,Q(f)) + N(r,f') + m(r, f'Q(f)) + m(r, \frac{1}{f'})$. Now, since the poles of f' are those of $Q(f)$ (with different multiplicities), we have $N(r,Q(f)) + N(r,f') = N(r, f'Q(f))$ and therefore $N(r,Q(f)) + N(r,f') + m(r, f'Q(f)) + m(r, \frac{1}{f'}) = N(r, f'Q(f)) + m(r, f'Q(f)) + m(r, \frac{1}{f'}) = T(r, f'Q(f)) + m(r, \frac{1}{f'})$, which proves the first inequality. Next, $m(r, \frac{1}{f'}) \leq T(r,f') + S_f(r) = m(r,f') + N(r,f') + S_f(r)$. But by Lemma 3 $m(r,f') \leq m(r,f) + S_f(r)$, hence $m(r, \frac{1}{f'}) \leq m(r,f) + N(r,f') + S_f(r) = m(r,f) + N(r,f) + \overline{N}(r,f) + S_f(r)$

and therefore $T(r,Q(f)) \leq T(r,f'Q(f))+T(r,f)+\overline{N}(r,f)+S_f(r) \leq T(r,f'Q(f))+2T(r,f)+S_f(r)$, which finishes proving the second inequality. Particularly, if $\deg(Q) \geq 3$, the last statement is an immediate consequence of both inequalities.

On a p-adic field, the following lemma 6 is only proven for meromorphic functions [2]. Here we must consider the particular case of entire functions which will be useful in the proof of Theorem 1.

Lemma 6. *Let $Q \in \mathbb{C}[x]$ and let $f,\ g \in \mathcal{M}(\mathbb{C})$ (resp. $f,\ g \in \mathcal{A}(\mathbb{C})$) be transcendental. Let $P(x) = x^{n+1}Q(x)$ be such that $n \geq \deg(Q)+3$ (resp. $n \geq \deg(Q)+2$). If $P'(f)f' = P'(g)g'$ then $P(f) = P(g)$.*

Proof : Set $k = \deg(Q)$. Since $P'(f)f' = P'(g)g'$, there exists $c \in \mathbb{C}$ such that $P(f) = P(g)+c$. Suppose that $c \neq 0$. Then by Theorem N, we have

$$T(r,P(f)) \leq \overline{Z}(r,P(f)) + \overline{Z}(r,P(f)-c) + \overline{N}(r,P(f)) + S_f(r). \tag{1}$$

Obviously we see that $\overline{Z}(r,P(f)) = \overline{Z}(r,f^{n+1}Q(f)) = \overline{Z}(r,fQ(f)) \leq T(r,fQ(f))$. By Lemma 1 we have $T(r,fQ(f)) = (k+1)T(r,f)+O(1)$ and then $\overline{Z}(r,P(f)) \leq (k+1)T(r,f)+O(1)$. We also have $\overline{Z}(r,P(f)-c) = \overline{Z}(r,P(g)) \leq \overline{Z}(r,g)+\overline{Z}(r,Q(g)) \leq T(r,g)+T(r,Q(g))$. Then by Lemma 1, $\overline{Z}(r,P(f)-c) \leq (k+1)T(r,g)+O(1)$. Notice that $\overline{N}(r,P(f)) = \overline{N}(r,f) \leq T(r,f)+O(1)$ then by (1) we obtain

$$T(r,P(f)) \leq (k+2)T(r,f)+(k+1)T(r,g)+S_f(r) \tag{2}$$

(resp.

$$T(r,P(f)) \leq (k+1)T(r,f)+(k+1)T(r,g)+S_f(r) \tag{3})$$

According to Lemma 1 we have $T(r,P(f)) = (n+k+1)T(r,f)+O(1)$. Then by (2) we have

$$nT(r,f) \leq T(r,f)+(k+1)T(r,g)+S_f(r). \tag{4}$$

(resp.

$$nT(r,f) \leq T(r,f)+(k+1)T(r,g)+S_f(r). \tag{5})$$

Similarly we obtain

$$nT(r,g) \leq T(r,g)+(k+1)T(r,f)+S_f(r). \tag{6})$$

(resp.

$$nT(r,g) \leq (k+1)T(r,f)+S_f(r). \tag{7})$$

Hence adding (6) and (7) we have

$$n(T(r,f)+T(r,g)) \leq (k+2)\big(T(r,f)+T(r,g)\big)+S_f(r)$$

and then

$$0 \leq (k+2-n)\big(T(r,f)+T(r,g)\big)+S_f(r)$$

(resp.

$$0 \leq (k+1-n)\big(T(r,f)+T(r,g)\big)+S_f(r))$$

In the general case, that leads to a contradiction because $n \geq k+3$ and $\lim_{r\to+\infty}(T(r,f)+T(r,g)) = +\infty$ outside a subset J of $]0,+\infty[$ of finite measure. Thus $c = 0$ and consequently $P(f) = P(g)$. And when $f,\ g \in\in \mathcal{A}(\mathbb{C})$ we have contradiction when $n \geq k+2$.

Notation: Given two meromorphic functions $f,\ g \in \mathcal{M}(\mathbb{C})$. We set

$$\Psi_{f,g} = \frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}.$$

That function was already used in several previous papers, particularly [18].

The following Lemma was proved in [25].

Lemma 7. *Let $F, G \in \mathcal{M}(\mathbb{C})$ be non-constant, having no zero and no pole at 0 and sharing the value 1 C.M. Suppose that $\Psi_{f,g} = 0$ and that*

$$\limsup_{r\to+\infty}\left(\frac{\overline{Z}(r,f)+\overline{Z}(r,g)+\overline{N}(r,f)+\overline{N}(r,g)}{\max(T(r,f),T(r,g))}\right) < 1$$

outside a set of finite measure. Then either $f = g$ or $fg = 1$.

Lemma 8 is similar to Lemma 9 [2] for complex meromorphic functions. But here, we can get a better result for entire functions.

Lemma 8. *Let $P(x) = (x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i} \in \mathbb{C}[x]$ $(a_i \neq a_j,\ \forall i \neq j)$ with $l \geq 2$ and $n \geq \max\{k_2, .., k_l\}$ and let $k = \sum_{i=2}^{l} k_i$. Let $f,\ g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\theta = P(f)f'P(g)g'$. If θ belongs to $\mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$, then we have the following :*
if $l = 2$ then n belongs to $\{k,\ k+1,\ 2k,\ 2k+1,\ 3k+1\}$,
if $l = 3$ then n belongs to $\{\frac{k}{2}, k+1, 2k+1, 3k_2-k, 3k_3-k\}$,
if $l \geq 4$ then $n = k+1$.
Moreover, if $f,\ g$ belong to $\mathcal{A}(\mathbb{C})$, then θ does not belong to $\mathcal{A}_f(\mathbb{C})$.

Proof: Without loss of generality, we can assume $a_1 = 0$. Suppose $f, g \in \mathcal{M}(\mathbb{C})$ satisfy

$$f^n(\prod_{i=2}^{l}(f-a_i)^{k_i})f'g^n(\prod_{i=2}^{l}(g-a_i)^{k_i})g' = \theta \tag{1}$$

Let Σ be the set of zeros and poles of θ. We will first show that $P'(f)$ and $P'(g)$ admit zeros or poles out of Σ. Indeed, suppose that none of the zeros and the poles of $f'P'(f)$ belong to Σ. So, every zero and every pole of $f'P'(f)$ must be either a zero or a pole of θ. Therefore, $\overline{Z}(r, f'P'(f)) + \overline{N}(r, f'P'(f)) \leq 2T(r,\theta)$. Actually, each zero of $f'P'(f)$ is either a zero of f' or of $P'(f)$ because the poles of $P'(f)$ are the poles of f'. Consequently, $\overline{Z}(r, P'(f)) + \overline{N}(r, f) \leq 2T(r,\theta)$ and therefore $\sum_{i=1}^{l}\overline{Z}(r, f-a_i)) + \overline{N}(r, f) \leq 2T(r,\theta)$ But now by Theorem N we have

$$(l-1)T(r,f) \leq \sum_{i=1}^{l}\overline{Z}(r, f-a_i) + \overline{N}(r,f) + S_f(r)$$

hence $(l-1)T(r,f) \leq 2T(r,\theta) + S_f(r)$, a contradiction. Consequently, $P'(f)$ admits zeros or poles that do not belong to Σ. Similarly, $P'(g)$ admits zeros or poles that do not belong to Σ.

Now, suppose that $f,\ g$ belong to $\mathcal{A}(\mathbb{C})$. Since both $f,\ g$ have no pole, all zeros of both must belong to Σ. Then, we have

$$Z(r, f'P'(f)g'P'(g)) = Z(r,f') + \sum_{i=1}^{l} k_i Z(r, f-a_i) + Z(r,g') + \sum_{i=1}^{l} k_i Z(r, g-a_i),$$

therefore

$$Z(r,\theta) \geq \sum_{i=1}^{l} \overline{Z}(r, f-a_i) \geq (l-1)T(r,f) + S_f(r) + S_g(r)$$

a contradiction to hypothesis again. Consequently, $f'P'(f)g'P'(g)$ cannot belong to $\mathcal{A}_f(\mathbb{C})$ or $\mathcal{A}_g(\mathbb{C})$.

Now, let us go back to the general case and suppose that $f,\ g$ are meromorphic functions. Let $\gamma \in \mathbb{C} \setminus \Sigma$ be a zero of g of order s. Clearly, by (1), γ is a pole of f of order, for example, t. And since γ is neither a zero nor a pole of θ we can derive the following relation

$$s(n+1) = t(n+k+1) + 2 \tag{2}$$

Now, suppose that for $i \in \{2,..,l\}$, $g-a_i$ has a zero $\gamma \in \mathbb{C} \setminus \Sigma$ of order s_i. It is a pole of f of order t_i. So, by (2), we obtain

$$s_i(k_i+1) = t_i(n+k+1) + 2 \tag{3}$$

By (2) and (3) it is obvious that $s > t$ and $s_i > t_i$.

Consider now a pole $\gamma \in \mathbb{C} \setminus \Sigma$ of f. Either it is a zero of g, or it is a zero of $g-a_i$ for some $i \in \{2,..,l\}$, or it is a zero of g' that is neither a zero of g nor a zero of $g-a_i$ ($\forall i \in \{2,..,l\}$). Let $Z_0(r,g')$ be the counting function of zeros of g' that are neither a zero of g nor a zero of $g-a_i$ for all $i \in \{2,..,l\}$ (counting multiplicity) and let $\overline{Z}_0(r,g')$ be the counting function of zeros of g' that are neither a zero of g nor a zero of $g-a_i$ for all $i \in \{2,..,l\}$, ignoring multiplicity. Since $T(r,\theta) = S_f(r) = S_g(r)$, we have

$$\overline{N}(r,f) \leq \overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r, g-a_i) + \overline{Z}_0(r,g') + S_f(r) + S_g(r)(r) \tag{4}$$

Now, by Theorem N, we have $(l-1)T(r,f) \leq \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i) + \overline{N}(r,f) - Z_0(r,f') + S_f(r)$, hence by (4), we obtain

$$(l-1)T(r,f) \leq \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i) + \overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}_0(r,g') - Z_0(r,f') + S_f(r) + S_g(r) \tag{5}$$

And similarly,

$$(l-1)T(r,g) \leq \overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i) + \overline{Z}_0(r,f') - Z_0(r,g') + S_f(r) + S_g(r) \tag{6}$$

Hence, adding (5) and (6), we obtain

$$(l-1)(T(r,f)+T(r,g)) \leq 2\big(\overline{Z}(r,g) + \sum_{i=2}^{l} \overline{Z}(r,g-a_i) + \overline{Z}(r,f) + \sum_{i=2}^{l} \overline{Z}(r,f-a_i)\big) + S_f(r) + S_g(r) \tag{7}$$

Case $l = 2$:

Without loss of generality, we can assume $a_2 = 1$. Suppose now that all zeros of $f,\ f-1,\ g,\ g-1$ are at least of order 5, except maybe those lying in Σ: then

$$\overline{Z}(r,f) \leq \frac{1}{5}T(r,f) + S_f(r) + S_g(r),\ \overline{Z}(r,f-1) \leq \frac{1}{5}T(r,f) + S_f(r) + S_g(r),,$$

$$\overline{Z}(r,g) \leq \frac{1}{5}T(r,g) + S_f(r) + S_g(r),\ \overline{Z}(r,g-1) \leq \frac{1}{5}T(r,g) + S_f(r) + S_g(r),$$

a contradiction to (7), proving the statement of the Lemma. So, we will look for pairs (n,k) leading to zeros of f or g out of Σ, of order ≤ 4.

Consequently, we will examine all situations leading to zeros of order ≤ 4 for $f,\ f-1,\ g,\ g-1$. Actually, since f and g play the same role with respect to n and k, it is sufficient to examine the situation, for instance, when g or $g-1$ has a zero of order $s \leq 4$. In each case we denote by t the order of the pole of f which is a zero of g or $g-1$. Recall that when f has a pole of order 4, g or $g-1$, if it has a zero, must have a zero of order ≥ 5. Consequently, we only have to examine zeros of g or $g-1$ that are poles of f of order 1, 2, 3.

Supppose first g has a zero $\gamma \notin \Sigma$ of order $s=2$. Then

$$2(n+1) = t(k+n+1) + 2 \tag{8}$$

By (8) if $t=1$ we find a solution:

$$n = k+1 \tag{9}$$

Next, if $t \geq 2$, we check that $2n+2 < t(k+n+1)+2$, hence (9) is the only solution.

Supppose now g has a zero $\gamma \notin \Sigma$ of order $s=3$. Then

$$3(n+1) = t(k+n+1) + 2 \tag{10}$$

By (10) if $t=1$ we find no solution because $k \leq n$.
If $t=2$ we find

$$n = 2k+1 \tag{11}$$

If $t \geq 3$ we have $3(n+3) < 3(k+n+1)+2$ hence (11) is the only solution.

Supppose now g has a zero of order $s=4$. Then

$$4(n+1) = t(k+n+1) + 2 \tag{12}$$

If $t=1$, since $k \leq n$, we have $4(n+1) > t(k+n+1)+2$.
If $t=2$, by (12) we have a solution

$$n = k \tag{13}$$

If $t=3$, we have another solution

$$n = 3k+1 \tag{14}$$

Consequently, by (9), (11), (12), (14), all possibilities for g to have a zero of order $s \leq 4$ are as follows:

$$n = k+1,\ s=2, \quad n = 2k+1, s=3, \quad n = k, s=4, \quad n = 3k+1, s=4. \tag{15}$$

Now, we will examine zeros of $g-1$ $\gamma \notin \Sigma$ of order ≤ 4. So, the order s of $g-1$ satisfies

$$s(k+1) = t(k+n+1) + 2 \tag{16}$$

Supppose first $g-1$ has a zero $\gamma \notin \Sigma$ of order $s=2$. Then by (16), we have

$$2(k+1) = t(k+n+1) + 2 \tag{17}$$

Since $k \leq n$, we find no solution neither when $t = 1$ that would lead to $k = n + 1$, nor when $t \geq 2$ because $2(k + 1) < t(k + n + 1) + 2$.

Suppose now that $s = 3$.
If $t = 1$ we find a solution:

$$n = 2k \tag{18}$$

If $t \geq 2$, we have no solution with $k \leq n$ because $3(k + 1) < t(k + n + 1) + 2$

Suppose now that $s = 4$.
If $t = 1$ we find a solution:

$$n = 3k + 1 \tag{19}$$

If $t = 2$ we find another solution:

$$n = k \tag{20}$$

If $t \geq 3$, we find no solution with $k \leq n$ because $4(k + 1) < t(k + n + 1) + 2$.

Consequently, by (18), (19), (20), all possibilities for $g - 1$ to have a zero $\gamma \notin \Sigma$ of order $s \leq 4$ are as follows:

$$n = 2k,\ s = 3, \quad n = 3k + 1, s = 4, \quad n = k, s = 4. \tag{21}$$

Thus, we have proved that when $n \neq k,\ k + 1, 2k,\ 2k + 1, 3k + 1$, none of the zeros of $f,\ f - 1$, $g,\ g - 1$ out of Σ is of order ≤ 4 and therefore the statement of the Lemma is proved in the case $l = 2$.

Case $l = 3$ **:** Suppose that all zeros of $f,\ g,\ f - a_i,\ g - a_i\ \forall i = 2,\ 3$ are at least of order 4, except maybe those lying in Σ: then

$$\overline{Z}(r, f) \leq \frac{1}{4}T(r, f) + S_f(r) + S_g(r), \quad \text{and } \forall i = 2,\ 3,\ \overline{Z}(r, f - a_i) \leq \frac{1}{4}T(r, f) + S_f(r) + S_g(r),,$$

$$\overline{Z}(r, g) \leq \frac{1}{4}T(r, g) + S_f(r) + S_g(r), \quad \text{and } \forall i = 2, 3\ \overline{Z}(r, g - a_i) \leq \frac{1}{4}T(r, g) + S_f(r) + S_g(r).$$

Then by (7) we obtain $l \leq 2$, a contradiction.

Consequently, we will examine all n and k_i $(i \in \{2, 3\})$ leading to zeros out of Σ of order ≤ 3 for $f,\ g,\ f - a_i,\ g - a_i$ for $i = 2,\ 3$. Actually, since f and g play the same role, it is sufficient to examine the situation, for instance, when g or some $g - a_i$ has a zero of order less than 3. In each case we denote by t the order of the pole of f which is a zero of g or $g - a_i$ for some i. Recall that when f has a pole of order 3, g or $g - a_i$, if it has a zero, must have a zero of order ≥ 4. Consequently, we only have to examine zeros of g or $g - a_i$ $(\forall i \in \{2, 3\})$ that are poles of f of order $1,\ 2$.

Supppose first g has a zero $\gamma \notin \Sigma$ of order $s = 2$. By (2) we have

$$2(n + 1) = t(k + n + 1) + 2 \tag{22}$$

By (22) if $t = 1$ we find a solution:

$$n = k + 1 \tag{23}$$

Next, if $t = 2$, we check that $2n + 2 < 2(k + n + 1) + 2$, hence (23) is the only solution.

Supppose now g has a zero $\gamma \notin \Sigma$ of order $s = 3$. Then

$$3(n + 1) = t(k + n + 1) + 2 \tag{24}$$

By (24) if $t = 1$ we find a solution :

$$n = \frac{k}{2} \tag{25}$$

If $t = 2$ we find

$$n = 2k + 1 \tag{26}$$

Consequently, by (23), (25), (26) all possibilities for g to have a zero of order $s \leq 3$ are as follows:

$$n = k + 1,\ s = 2,\ t = 1, \quad n = \frac{k}{2},\ s = 3,\ t = 1, \quad n = 2k + 1, s = 3,\ t = 2.$$

Consider now zeros of $g - a_i\ \gamma \notin \Sigma$ of order $s_i \leq 3$. So, the order s_i of $g - a_i$ satisfies

$$s_i(k_i + 1) = t(k + n + 1) + 2 \tag{27}$$

Supppose first $g - a_i$ has a zero $\gamma \notin \Sigma$ of order $s_i = 2$. Then by (28), we have

$$2(k_i + 1) = t(k + n + 1) + 2 \tag{28}$$

Since $k_i \leq n$ and $k_i \leq k$ we have $2(k_i + 1) < t(k + n + 1) + 2$. Hence we find no solution for (28).

Suppose now that $s = 3$.
If $t = 1$ we find a solution:

$$3k_i = n + k \tag{29}$$

If $t = 2$, we have no solution because $3k_i < 2(n + k)$.

Consequently, the unique possibility for $g - a_i$ to have a zero $\gamma \notin \Sigma$ of order $s_i \leq 3$ is :

$$n + k = 3k_i,\ s = 3,\ t = 1$$

Thus, we have proved that when $n \neq k+1, \frac{k}{2},\ 2k+1,\ 3k_i - k$ none of the zeros of f, g, $f - a_i$, $g - a_i$ $(\forall i = 2,\ 3)$ out of Σ is of order ≤ 3 and therefore the statement of the Lemma is proved in the case $l = 3$.

Case $l \geq 4$ **:**

Suppose now that all zeros of $f,\ g,\ f - a_i,\ g - a_i\ \forall i \in \{2, .., l\}$ are at least of order 3, except maybe those lying in Σ: then

$$\overline{Z}(r, f) \leq \frac{1}{3}T(r, f) + S_f(r) + S_g(r), \quad \text{and } \forall i \in \{2, .., l\},\ \overline{Z}(r, f - a_i) \leq \frac{1}{3}T(r, f) + S_f(r) + S_g(r),$$

$$\overline{Z}(r, g) \leq \frac{1}{3}T(r, g) + S_f(r) + S_g(r), \quad \text{and } \forall i \in \{2, .., l\},\ \overline{Z}(r, g - a_i) \leq \frac{1}{3}T(r, g) + S_f(r) + S_g(r).$$

Then using (7) we obtain $l \leq 3$ a contradiction.

Consequently, we will examine all n and k_i $(i \in \{2, .., l\})$ leading to zeros out of Σ of order ≤ 2 for $f,\ g,\ f - a_i,\ g - a_i$ for all $i \in \{2, .., l\}$. Actually, since f and g play the same role, it is sufficient to examine the situation, for instance, when g or some $g - a_i$ has a zero of order less than 2. In each case we denote by t the order of the pole of f which is a zero of g or $g - a_i$ for some i. Recall that when f has a pole of order 2, g or $g - a_i$, if it has a zero, must have a zero of order ≥ 3.

Consequently, we only have to examine zeros of g or $g - a_i$ $(\forall i \in \{2, .., l\})$ that are poles of f of order 1.

Supppose first g has a zero $\gamma \notin \Sigma$ of order $s = 2$. Then γ is a pole of f of order $t = 1$. Then by (2) we have

$$2(n + 1) = (k + n + 1) + 2 \tag{30}$$

We find a solution :

$$n = k + 1 \tag{31}$$

Let $i \in \{2, .., l\}$ and suppose $g - a_i$ has a zero $\gamma \notin \Sigma$ of order $s_i = 2$. Then γ is a pole of f of order $t = 1$. So by (3) we have :

$$2(k_i + 1) = (n + k + 1) + 2$$

That means $2k_i = n + k + 1$. Since $k_i \leq n$ and $k_i \leq k$, we find no solution when $s_i = 2$ and $t = 1$.

Consequently, by (31), the only possibility for g or some $g - a_i$ to have a zero $\gamma \notin \Sigma$ of order ≤ 3 is :

$$n = k + 1.$$

This completes the proof of Lemma 8.

Lemma 9 is known and easily checked (Theorems 2,3 [5] and [24]):

Lemma 9. *Let $f,\ g \in \mathcal{M}(\mathbb{C})$ be transcendental satisfy $(f - a)f^n = (g - a)g^n$ with $a \in \mathbb{C}$ and let $h = \dfrac{f}{g}$. If h is not identically 1, then*

$$g = \frac{h^n - 1}{h^{n+1} - 1},\ f = \frac{h^{n+1} - h}{h^{n+1} - 1}.$$

Notation: Let $f \in \mathcal{M}(\mathbb{C})$ be such that $f(0) \neq 0, \infty$. We denote by $Z_{[2]}(r, f)$ the counting function of the zeros of f each being counted with multiplicity when it is at most 2 and with multiplicity 2 when it is bigger.

We can extract the following Lemma 10 from a result that is proved in several papers and particularly in Lemma 3 [15].

Lemma 10. *Let $f, g \in \mathcal{M}(\mathbb{C})$ be such that $f(0) \neq 0, \infty$ and share the value 1 CM. If $\Psi_{f,g}$ is not identically zero, then,*

$$\max(T(r, f), T(r, g)) \leq N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) + S_f(r) + S_g(r)$$

3 Proof of Theorems

The polynomial P is the one we considered in theorems 1, 2, 3, 4 and we can assume $a_1 = 0$. In Theorem 5 we call P the polynomial such that $P(0) = 0$ and $P'(x) = x^n(x - a)$. Set $F = \dfrac{f'P'(f)}{\alpha}$ and $G = \dfrac{g'P'(g)}{\alpha}$. Clearly F and G share the value 1 C.M. Since $f,\ g$ are transcendental, we notice that so are F and G. Recall that

$$\Psi_{F,G} = \frac{F''}{F'} - \frac{2F'}{F - 1} - \frac{G''}{G'} + \frac{2G'}{G - 1}$$

We will prove that under the hypotheses of each theorem, $\Psi_{F,G}$ is identically zero.

Set $\widehat{F} = P(f),\ \widehat{G} = P(g)$. We notice that $P(x)$ is of the form $x^{n+1}Q(x)$ with $Q \in \mathbb{C}[x]$ of degree k. Now, by Lemma 4, we have

$$T(r,\widehat{F}) - Z(r,\widehat{F}) \leq T(r,\widehat{F}') - Z(r,\widehat{F}') + S_F(r)$$

Consequently, since $(\widehat{F})' = \alpha F$, we have

$$T(r,\widehat{F}) \leq T(r,F) + Z(r,\widehat{F}) - Z(r,F) + T(r,\alpha) + S_f(r), \tag{1}$$

hence, by (1), we obtain

$$T(r,\widehat{F}) \leq T(r,F)+(n+1)Z(r,f)+Z\big(r,Q(f)\big)-nZ(r,f)-\sum_{i=2}^{l} k_i Z(r,f-a_i)-Z(r,f')+T(r,\alpha)+S_f(r)$$

i.e.

$$T(r,\widehat{F}) \leq T(r,F) + Z(r,f) + Z\big(r,Q(f)\big) - \sum_{i=2}^{l} k_i Z(r,f-a_i) - Z(r,f') + T(r,\alpha) + S_f(r). \tag{2}$$

and similarly,

$$T(r,\widehat{G}) \leq T(r,G) + Z(r,g) + Z\big(r,Q(g)\big) - \sum_{i=2}^{l} k_i Z(r,g-a_i) - Z(r,g') + T(r,\alpha) + S_f(r). \tag{3}$$

Now, it follows from the definition of F and G that

$$Z_{[2]}(r,F) + N_{[2]}(r,F) \leq 2Z(r,f) + 2\sum_{i=2}^{l} Z(r,f-a_i) + Z(r,f') + 2\overline{N}(r,f) + T(r,\alpha) + S_f(r) \tag{4}$$

and similarly

$$Z_{[2]}(r,G) + N_{[2]}(r,G) \leq 2Z(r,g) + 2\sum_{i=2}^{l} Z(r,g-a_i) + Z(r,g') + 2\overline{N}(r,g) + T(r,\alpha) + S_g(r) \tag{5}$$

And particularly, if $k_i = 1,\ \forall i \in \{2,..,l\}$, then

$$Z_{[2]}(r,F) + N_{[2]}(r,F) \leq 2Z(r,f) + \sum_{i=2}^{l} Z(r,f-a_i) + Z(r,f') + 2\overline{N}(r,f) + T(r,\alpha) + S_f(r) \tag{6}$$

and similarly

$$Z_{[2]}(r,G) + N_{[2]}(r,G) \leq 2Z(r,g) + \sum_{i=2}^{l} Z(r,g-a_i) + Z(r,g') + 2\overline{N}(r,g) + T(r,\alpha) + S_g(r) \tag{7}$$

Suppose now that $\Psi_{F,G}$ is not identically zero. Now, by Lemma 10, we have

$$T(r,F) \leq Z_{[2]}(r,F) + N_{[2]}(r,F) + Z_{[2]}(r,G) + N_{[2]}(r,G) + S_f(r) + S_g(r)$$

hence by (2), we obtain

$$T(r,\widehat{F}) \leq Z_{[2]}(r,F) + N_{[2]}(r,F) + Z_{[2]}(r,G) + N_{[2]}(r,G) + Z(r,f) + Z(r,Q(f))$$
$$-\sum_{i=2}^{l} k_i Z(r,f-a_i) - Z(r,f') + T(r,\alpha) + S_f(r) + S_g(r)$$

and hence by (4) and (5):

$$T(r,\widehat{F}) \leq 2Z(r,f)+2\sum_{i=2}^{l} Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+2Z(r,g)+2\sum_{i=2}^{l} Z(r,g-a_i)+Z(r,g')+$$

$$2\overline{N}(r,g) + Z(r,f) + Z(r,Q(f)) - \sum_{i=2}^{l} k_i Z(r,f-a_i) - Z(r,f') + 2T(r,\alpha) + S_f(r) + S_g(r) \quad (8)$$

and similarly,

$$T(r,\widehat{G}) \leq 2Z(r,g)+2\sum_{i=2}^{l} Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+2Z(r,f)+2\sum_{i=2}^{l} Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)$$

$$+Z(r,g) + Z(r,Q(g)) - \sum_{i=2}^{l} k_i Z(r,g-a_i) - Z(r,g') + 2T(r,\alpha) + S_f(r) + S_g(r) \quad (9)$$

Consequently,

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq 5(Z(r,f)+Z(r,g))+\sum_{i=2}^{l}(4-k_i)(Z(r,f-a_i)+Z(r,g-a_i))+(Z(r,f')+Z(r,g'))+$$

$$4(\overline{N}(r,f) + \overline{N}(r,g)) + (Z(r,Q(f)) + Z(r,Q(g))) + S_f(r) + S_g(r). \quad (10)$$

Moreover, if $k_i = 1,\ \forall i \in \{2,..,l\}$, then by (6) and (7) we have

$$T(r,\widehat{F}) \leq 2Z(r,f) + \sum_{i=2}^{l} Z(r,f-a_i) + Z(r,f') + 2\overline{N}(r,f) + 2Z(r,g) + \sum_{i=2}^{l} Z(r,g-a_i)+$$

$$Z(r,g') + 2\overline{N}(r,g) + Z(r,f) + Z(r,Q(f)) - \sum_{i=2}^{l} Z(r,f-a_i) - Z(r,f') + S_f(r) + S_g(r)$$

and similarly,

$$T(r,\widehat{G}) \leq 2Z(r,g) + \sum_{i=2}^{l} Z(r,g-a_i) + Z(r,g') + 2\overline{N}(r,g) + 2Z(r,f) + \sum_{i=2}^{l} Z(r,f-a_i)+$$

$$Z(r,f') + 2\overline{N}(r,f) + Z(r,g) + Z(r,Q(g))) - \sum_{i=2}^{l} Z(r,g-a_i) - Z(r,g') + S_f(r) + S_g(r).$$

Consequently,

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq 5(Z(r,f)+Z(r,g))+\sum_{i=2}^{l}(Z(r,f-a_i)+Z(r,g-a_i))+Z(r,Q(f))+Z(r,Q(g))+$$

$$(Z(r,f')+Z(r,g'))+4(\overline{N}(r,f)+\overline{N}(r,g))+S_f(r)+S_g(r) \tag{11}$$

Now, let us go back to the general case. By Lemma 2, we can write $Z(r,f')+Z(r,g') \leq Z(r,f-a_2)+Z(r,g-a_2)+\overline{N}(r,f)+\overline{N}(r,g)+S_f(r)$. Hence, in general, by (10) we obtain

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq$$

$$5(Z(r,f)+Z(r,g))+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)+(5-k_2)\big((Z(r,f-a_2)+Z(r,g-a_2))$$

$$+5(\overline{N}(r,f)+\overline{N}(r,g))+(Z(r,Q(f))+Z(r,Q(g)))+S_f(r)+S_g(r).$$

And hence, since $T(r,Q(f))=kT(r,f)+O(1)$ and $T(r,Q(g))=kT(r,g)+O(1)$, we have

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq$$

$$5(T(r,f)+T(r,g))+\sum_{i=3}^{l}(4-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)+(5-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big)$$

$$+5(\overline{N}(r,f)+\overline{N}(r,g))+k(T(r,f)+T(r,g))+S_f(r)+S_g(r). \tag{12}$$

Now, if $k_i=1$, $\forall i\in\{2,..,l\}$, by (11) and Lemma 2 we have

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq 5(Z(r,f)+Z(r,g))+\sum_{i=2}^{l}(Z(r,f-a_i)+Z(r,g-a_i))+(l-1)(T(r,f)+T(r,g))$$

$$+(Z(r,f-a_2)+Z(r,g-a_2))+5(\overline{N}(r,f)+\overline{N}(r,g))+S_f(r)+S_g(r)$$

hence

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq$$

$$5(T(r,f)+T(r,g))+\sum_{i=2}^{l}(T(r,f-a_i)+T(r,g-a_i))+(l-1)(T(r,f)+T(r,g))+(T(r,f-a_2)+T(r,g-a_2))$$

$$+5(\overline{N}(r,f)+\overline{N}(r,g))+S_f(r)+S_g(r)$$

and hence

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq (9+2l)(T(r,f)+T(r,g))+S_f(r)+S_g(r) \tag{13}$$

Now, let us go back to the general case. Since $\widehat{F}$ is a polynomial in f of degree $n+k+1$, we have $T(r,\widehat{F})=(n+k+1)T(r,f)+O(1)$ and similarly, $T(r,\widehat{G})=(n+k+1)T(r,g)+O(1)$, hence by (12) we can derive

$$(n+k+1)(T(r,f)+T(r,g)) \leq$$

$$5(T(r,f)+T(r,g))+(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))+\sum_{i=3}^{l}(4-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)$$

$$+5(\overline{N}(r,f)+\overline{N}(r,g))+k(T(r,f)+T(r,g))+S_f(r)+S_g(r). \tag{14}$$

Hence

$$(n+k+1)(T(r,f)+T(r,g)) \leq (10+k)(T(r,f)+T(r,g))+\sum_{i=3}^{l}(4-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)$$

$$+(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))+S_f(r)+S_g(r)$$

and hence

$$n(T(r,f)+T(r,g))\leq 9(T(r,f)+T(r,g))+\sum_{i=3}^{l}(4-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)$$

$$+(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))+S_f(r)+S_g(r). \tag{15}$$

Then at least, for each $i=3,..,l$ we have
$(4-k_i)(Z(r,f-a_i)+Z(r,g-a_i))\leq \max(0,4-k_i)(T(r,f)+T(r,g))+O(1)$ and

$$(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))\leq \max(0,5-k_2)(T(r,f)+T(r,g))+O(1).$$

Consequently, by (15) we have

$$n(T(r,f)+T(r,g))\leq 9(T(r,f)+T(r,g))+$$

$$+\sum_{i=3}^{l}\max(0,4-k_i)(T(r,f)+T(r,g))+\max(0,5-k_2)(T(r,f)+T(r,g))+O(1)$$

and hence,

$$n\leq 9+\sum_{i=3}^{l}\max(0,4-k_i)+\max(0,5-k_2). \tag{16}$$

Thus, we have proved that $\Psi_{F,G}=0$ when

$$n\geq 10+\sum_{i=3}^{l}\max(0,4-k_i)+\max(0,5-k_2),$$

concerning Theorem 3.

And, if $k_i=1$, $\forall i\in\{2,..,l\}$, by (13) we have $n+k+1\leq 9+2l$, hence

$$n\leq 9+l. \tag{17}$$

So, if $k_i=1$, $\forall i\in\{2,..,l\}$, when $n\geq l+10$ we have $\Psi_{F,G}=0$ which concerns Theorems 4 and 5.

Now, suppose that $f,\ g,\ \alpha$ belong to $\mathcal{A}(\mathbb{C})$. By (14), we have

$$(n+k+1)(T(r,f)+T(r,g))\leq (5+k)(T(r,f)+T(r,g))+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)$$

$$+(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))+S_f(r)+S_g(r)$$

and hence

$$n(T(r,f)+T(r,g))\leq 4(T(r,f)+T(r,g))+\sum_{i=3}^{l}(4-k_i)\big((Z(r,f-a_i)+Z(r,g-a_i))\big)$$

$$+(5-k_2)(Z(r,f-a_2)+Z(r,g-a_2))+S_f(r)+S_g(r)$$

and hence

$$n(T(r,f)+T(r,g)) \leq 4(T(r,f)+T(r,g))$$

$$+\sum_{i=3}^{l}\max(0,4-k_i)(T(r,f)+T(r,g))+\max(0,5-k_2)(T(r,f)+T(r,g))+O(1),$$

and hence,

$$n \leq 4+\sum_{i=3}^{l}\max(0,4-k_i)+\max(0,5-k_2).$$

Consequently, if $n \geq 5+\sum_{i=3}^{l}\max(0,4-k_i)+\max(0,5-k_2)$ when $f,\ g,\ \alpha$ belong to $\mathcal{A}(\mathbb{C})$, we have $\Psi_{F,G}=0$, which concerns Theorem 1.

Finally, suppose that $f,\ g,\ \alpha$ belong to $\mathcal{A}(\mathbb{C})$ and $k_i=1\ \forall i=1,...,l$. We have $n+k+1 \leq 4+2l$, hence $n \leq 4+l$. Consequently, if $n \geq 5+l$ when $f,\ g,\ \alpha$ belong to $\mathcal{A}(\mathbb{C})$, we have $\Psi_{F,G}=0$, which concerns Theorem 2.

Thus, henceforth, we can assume that $\Psi_{F,G}=0$ in each theorem. Note that we can write $\Psi_{F,G}=\dfrac{\phi'}{\phi}$ with $\phi=\Big(\dfrac{F'}{(F-1)^2}\Big)\Big(\dfrac{(G-1)^2}{G'}\Big)$. Since $\Psi_{F,G}=0$, there exist $A,B\in\mathbb{C}$ such that

$$\frac{1}{G-1}=\frac{A}{F-1}+B \tag{18}$$

and $A\neq 0$.

We notice $\overline{Z}(r,f)\leq T(r,f),\ \overline{N}(r,f)\leq T(r,f),\ \overline{Z}(r,f-a_i)\leq T(r,f-a_i)\leq T(r,f)+O(1),\ i=2,...,l$ and $\overline{Z}(r,f')\leq T(r,f')\leq 2T(r,f)+O(1)$. Similarly for g and g'. Moreover, by Lemma 5, we have

$$T(r,F)\geq (n+k)T(r,f)-m(r,\frac{1}{f'})+S_f(r). \tag{19}$$

Will consider the following two cases: $B=0$ and $B\neq 0$.

Case 1: $B=0$.

Suppose $A\neq 1$. Then, by (18), we have $F=AG+(1-A)$. Applying Theorem N to F, we obtain

$$T(r,F)\leq \overline{Z}(r,F)+\overline{Z}\Big(r,F-\Big(1-A\Big)\Big)+\overline{N}(r,F)+S_F(r))\leq \overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)$$

$$+\overline{Z}(r,f')+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')+\overline{N}(r,f)+S_f(r)+S_g(r). \tag{20}$$

Consequently, by (19) and (20) we obtain

$$(n+k)T(r,f)-m(r,\frac{1}{f'})\leq \overline{Z}(r,F)+\overline{Z}\Big(r,F-\Big(1-A\Big)\Big)+\overline{N}(r,F)+S_F(r)$$

$$\leq \overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')+\overline{N}(r,f)+S_f(r)+S_g(r)$$

therefore

$$(n+k)T(r,f) \le \overline{Z}(r,F)+\overline{Z}\Big(r,F-\Big(1-A\Big)\Big)+\overline{N}(r,F)+m(r,\frac{1}{f'})+S_F(r) \le \overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)$$

$$+\overline{Z}(r,f')+m(r,\frac{1}{f'})+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')+\overline{N}(r,f)+S_f(r)+S_g(r).$$

Here we notice that $\overline{Z}(r,f')+m(r,\frac{1}{f'}) \le T(r,\frac{1}{f'}) = T(r,f')+O(1)$, hence $(n+k)T(r,f) \le$

$$\le \overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{N}(r,f)+\overline{Z}(r,g')+T(r,f')+S_f(r)+S_g(r). \tag{21}$$

Then, considering all the previous inequalities in (21), by Lemma 3 we can derive

$$(n+k)T(r,f) \le (l+3)T(r,f)+(l+2)T(r,g)+S_f(r)+S_g(r). \tag{22}$$

Since f and g satisfy the same hypothesis, we also have

$$(n+k)T(r,g) \le (l+3)T(r,g)+(l+2)T(r,f)+S_f(r)+S_g(r). \tag{23}$$

Hence, adding (22) and (23), we have

$$(n+k)\big[T(r,f)+T(r,g)\big] \le (2l+5)\big[T(r,f)+T(r,g)\big]+S_f(r)+S_g(r),$$

which leads to a contradiction whenever $n+k \ge (2l+6)$.

In the hypothesis of Theorems 3, we have

$$n \ge 10+\sum_{i=3}^{l}\max(0,4-k_i)+\max(0,5-k_2).$$

That implies

$$n+k \ge 10+\sum_{i=3}^{l}\max(0,4-k_i)+\max(0,5-k_2)+\sum_{i=2}^{l}k_i \ge 10+\sum_{i=2}^{l}\max(k_i,4) \ge 10+4(l-1)$$

hence

$$n+k \ge 4l+6. \tag{24}$$

That contradiction proves that $A=1$ in Theorem 3.

In the hypothesis of Theorems 4 and 5 we have $n \ge l+10$ with $k=l-1$ hence

$$n+k \ge 2l+9 \tag{25}$$

therefore $A=1$ again in Theorems 4 and 5.

Consider now the situation in Theorem 1. First we notice that given a positive number c we have $\max(0,4-c)+c \ge 4$. By (21), with help of Lemma 3, we can derive

$$(n+k)T(r,f) \le \overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')+T(r,f')+S_f(r)+S_g(r),$$

$$\leq \overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+T(r,g)+T(r,f)+S_f(r)+S_g(r),$$

with a similar inequality for g, thereby

$$(n+k)(T(r,f)+T(r,g))\leq(2l+2)(T(r,f)+T(r,g))$$

and hence $n+k\leq 2l+2$, which leads to a contradiction as soon as $n+k\geq 2l+3$. Instead of (24) we now have $n+k\geq 4l+1$. Consequently, the relation $n+k\geq 2l+3$ is easily satisfied in Theorem 1, proving $A=1$ again.

And now, consider the hypothesis of Theorem 2. We have $n\geq l+5$, hence $n+k\geq 2l+4$ assuring that $A=1$ again.

Thus, in case $B=0$, we have proved that $F=G$. Now, $\alpha F=\alpha G$, i.e. $(\widehat{F})'=(\widehat{G})'$. We assume $n\geq k+3$ in Theorem 3 and this is automatically satisfied in Theorems 4 and 5. Next, we assume $n\geq k+2$ in Theorem 1 and this is automatically satisfied in Theorems 2. Consequently, by Lemma 6, we have $\widehat{F}=\widehat{G}$, i.e. $P(f)=P(g)$. But in Theorems 3 and 4, P is a polynomial of uniqueness for the family of meromorphic functions we consider, hence we have $f=g$. And in Theorem 5, the conclusion comes from Lemma 9. Finally, in Theorems 1 and 2, we assume P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$, we have $f=g$ again.

Case 2: $B\neq 0$.

We have $\overline{Z}(r,F)\leq\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')+S_f(r)$ and $\overline{N}(r,F)\leq\overline{N}(r,f)+S_f(r)$ and similarly for G, so we can derive

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+\overline{Z}(r,f')$$

$$+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+\overline{Z}(r,g')+\overline{N}(r,f)+\overline{N}(r,g)+S_f(r)+S_g(r). \tag{26}$$

Moreover, by (19) we have
$(n+k)T(r,f)\leq T(r,F)+m(r,\frac{1}{f'})+S_f(r)$ and $(n+k)T(r,g)\leq T(r,G)+m(r,\frac{1}{g'})+S_g(r)$. Consequently, by (26) we obtain

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq\overline{Z}(r,f)+\sum_{i=2}^{l}\overline{Z}(r,f-a_i)+[\overline{Z}(r,f')+m(r,\frac{1}{f'})]-m(r,\frac{1}{f'})$$

$$+\overline{Z}(r,g)+\sum_{i=2}^{l}\overline{Z}(r,g-a_i)+[\overline{Z}(r,g')+m(r,\frac{1}{g'})]-m(r,\frac{1}{g'})+\overline{N}(r,f)+\overline{N}(r,g)+S_f(r)+S_g(r)$$

$$\leq(l+1)\big[T(r,f)+T(r,g)\big]+T(r,f')+T(r,g')-m(r,\frac{1}{f'})-m(r,\frac{1}{g'})+S_f(r)+S_g(r)$$

hence by Lemma 3,

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq(l+3)(T(r,f)+T(r,g))-m(r,\frac{1}{f'})-m(r,\frac{1}{g'})+S_f(r)+S_g(r) \tag{28}$$

Now, in Theorems 3, 4, 5 we have $n+k \geq \min(4l+6, 2l+9)$ hence $\frac{2l+6}{k+n} < 1$, therefore by (19) we notice that

$$2(n+k)(\frac{T(r,f)+T(r,g)}{2}) - m(r,\frac{1}{f'}) - m(r,\frac{1}{g'}) \leq T(r,F) + T(r,G) + S_f(r) + S_g(r)$$

hence (28) yields

$$\overline{Z}(r,F) + \overline{Z}(r,G) + \overline{N}(r,F) + \overline{N}(r,G) \leq (\frac{2l+6}{n+k})(\frac{T(r,F)+T(r,G)}{2}) + S_f(r) + S_g(r).$$

Now by Lemma 5 we can derive

$$\overline{Z}(r,F) + \overline{Z}(r,G) + \overline{N}(r,F) + \overline{N}(r,G) \leq (\frac{2l+6}{n+k})(\frac{T(r,F)+T(r,G)}{2}) + S_F(r) + S_G(r) \qquad (29)$$

Consequently, by (29), we obtain

$$\limsup_{r\to+\infty}\left(\frac{\overline{Z}(r,F) + \overline{Z}(r,G) + \overline{N}(r,F) + \overline{N}(r,G)}{\max(T(r,F),T(r,G))}\right) < 1$$

outside a set of finite measure and hence, by Lemma 7, we have $F = G$ or $FG = 1$.

Suppose $FG = 1$, hence $f'P'(f)g'P'(g) = \alpha^2$. In Theorem 3 we have assumed that $n \geq k+3$ and in Theorems 4 and 5 this hypothesis is automatically satisfied. Next, if $l = 2$, then $n \neq 2k, 2k+1, 3k+1$ and if $l = 3$ then $n \neq 2k+1, 3k_i - k,\ i = 2,\ 3$. Moreover, these conditions are automatically satisfied in Theorems 4 and 5. So, we have a contradiction to Lemma 8 in Theorems 3, 4, 5. Consequently, $F = G$ and therefore we can conclude as in the case $B = 0$, that $P(f) = P(g)$. Now, in Theorems 3 and 4 P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$, hence we have $f = g$. And in Theorem 5 we know that either $f = g$ or f and g are linked by the relation mentionned in Theorems 2, 3 [5] and in [24].

Consider now the situation in Theorems 1 and 2. By (26) and Lemma 3, here we can write

$$\begin{aligned}\overline{Z}(r,F) + \overline{Z}(r,G) + \overline{N}(r,F) + \overline{N}(r,G) &\leq \overline{Z}(r,f) + \sum_{i=2}^{l}\overline{Z}(r,f-a_i) + \overline{Z}(r,f') + \overline{Z}(r,g') + \overline{Z}(r,g)\\ &\quad + \sum_{i=2}^{l}\overline{Z}(r,g-a_i) + S_f(r) + S_g(r)\\ &\leq \overline{Z}(r,f) + \sum_{i=2}^{l}\overline{Z}(r,f-a_i) + T(r,f) + T(r,g) - m(r,\frac{1}{f'}) - m(r,\frac{1}{g'}) + \overline{Z}(r,g)\\ &\quad + \sum_{i=2}^{l}\overline{Z}(r,g-a_i) + S_f(r) + S_g(r)\end{aligned}$$

hence finally,

$$\overline{Z}(r,F) + \overline{Z}(r,G) + \overline{N}(r,F) + \overline{N}(r,G) \leq (l+1)(T(r,f)+T(r,g)) - m(r,\frac{1}{f'}) - m(r,\frac{1}{g'}).$$

As in (29) here, thanks to Lemma 5 we can obtain (30):

$$\overline{Z}(r,F)+\overline{Z}(r,G)+\overline{N}(r,F)+\overline{N}(r,G)\leq(\frac{2l+2}{n+k})(\frac{T(r,F)+T(r,G)}{2})+S_F(r)+S_G(r) \qquad (30)$$

On the other hand, in Theorem 1, instead of (24) here we have $n+k\geq 4l+1$ hence $\frac{2l+2}{n+k}<1$ and hence we can apply Lemma 7 proving that either $F=G$ or $FG=1$. But since $\alpha F, \alpha G$ and α belong to $\mathcal{A}(\mathbb{C})$, by Lemma 8 we know that $(\alpha F)(\alpha G)=(\alpha)^2$ is impossible. Consequently $F=G$.

Now in Threorem 2, we have $n+k\geq 2l+4$ hence $\frac{2l+2}{n+k}<1$ and hence we can go on as in Theorem 1. So, finally we have $F=G$ in Theorems 1 and 2 again. Next, we can conclude as in the case $B=0$.

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Kamal Boussaf and Alain Escassut
Laboratoire de Mathématiques CNRS UMR 6620
Université Blaise Pascal
Les Cézeaux
63171 Aubière
FRANCE
kamal.boussaf@math.univ-bpclermont.fr
alain.escassut@math.univ-bpclermont.fr

Jacqueline Ojeda
Departamento de Matematica
Facultad de Ciencias Fsicas y Matematicas
Universidad de Concepcion
Concepción
CHILE
mail: jacqojeda@udec.cl
Jacqueline.Ojeda@math.univ-bpclermont.fr