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► **To cite this version:**

Alain Escassut, Ta Thi Hoai An. New applications of the p-adic Nevanlinna Theory. p-Adic Numbers, Ultrametric Analysis and Applications, 2018. hal-01921858

HAL Id: hal-01921858

<https://uca.hal.science/hal-01921858>

Submitted on 14 Nov 2018

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New applications of the p -adic Nevanlinna Theory

Alain Escassut and Ta Thi Hoai An

Abstract

Let \mathbb{K} be an algebraically closed field of characteristic 0 complete for an ultrametric absolute value. Following results obtained in complex analysis, here we examine problems of uniqueness for meromorphic functions having finitely many poles, sharing points or a pair of sets (C.M. or I.M.) defined either in the whole field \mathbb{K} or in an open disk, or in the complement of an open disk. Following previous works in \mathbb{C} , we consider functions $f^n(x)f^m(ax+b)$, $g^n(x)g^m(ax+b)$ with $|a|=1$ and $n \neq m$, sharing a rational function and we show that $\frac{f}{g}$ is a $n+m$ -th root of 1 whenever $n+m \geq 5$. Next, given a small function w , if $n, m \in \mathbb{N}$ are such that $|n-m|_\infty \geq 5$, then $f^n(x)f^m(ax+b) - w$ has infinitely many zeros. Finally, we examine branched values for meromorphic functions $f^n(x)f^m(ax+b)$.

1 Introduction

Let \mathbb{K} be a complete ultrametric algebraically closed field of characteristic 0 whose ultrametric absolute value is denoted by $|\cdot|$. The Nevanlinna theory, well known for complex meromorphic functions [19], was examined over \mathbb{K} by Ha Huy Khoai [9] and A. Boutabaa [3]. Next, in [4] a similar theory was made for unbounded meromorphic functions in an “open” disk of \mathbb{K} , taking into account Lazard’s problem. In [10], M. O. Hanyak and A. A. Kondratyuk constructed a Nevanlinna theory for meromorphic functions in a set of the form $\mathbb{C} \setminus \{a_1, \dots, a_m\}$, where the meromorphic functions can admit essential singularities at a_1, \dots, a_m [10].

Here we recall the Nevanlinna theory for meromorphic functions in the complement of an open disk. Next, we can apply this to obtain results on uniqueness and branched values as it was done in similar problems [1], [2], [3], [3], [4]. We then show new results on meromorphic functions sharing two sets and properties of meromorphic functions of the form $f^n(x)f^m(ax+b)$ with regards to branched values and Picard’s values.

⁰2010 Mathematics Subject Classification: 12J25; 30D35; 30G06; 46S10

⁰Keywords: p -adic meromorphic functions, Nevanlinna’s Theory, Values distribution, small functions, Picard values, Branched values.

⁰The second named author is supported by Vietnam’s National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2017.320 and Vietnam Institute Advanced Study in Mathematics

All results of Paragraphs 1 to 7 were already published with all proofs in [2], [4], [7], [8], [14]. Therefore we will only give the proofs of theorems presented in Paragraph 8.

Notation: Given $r > 0$, $a \in \mathbb{K}$ we denote by $d(a, r)$ the disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$, by $d(a, r^-)$ the disk $\{x \in \mathbb{K} \mid |x - a| < r\}$, and by $C(a, r)$ the circle $\{x \in \mathbb{K} \mid |x - a| = r\}$. Given $r'' > r'$, we put $\Delta(0, r', r'') = d(0, r'') \setminus d(0, r'^-)$.

Henceforth, we fix $R > 0$, we denote by S the disk $d(0, R^-)$ and put $D = \mathbb{K} \setminus S$.

Given a subset E of \mathbb{K} having infinitely many points, we denote by $R(E)$ the \mathbb{K} -algebra of rational functions $h \in \mathbb{K}(x)$ having no pole in E . We then denote by $H(E)$ the \mathbb{K} -vector space of analytic elements on E , i.e., the completion of $R(E)$ with respect to the topology of uniform convergence on E . By classical properties of analytic elements in [12], [7], given a circle $C(a, r)$ and an element f of $H(C(a, r))$, i.e., a Laurent series $f(x) = \sum_{n=-\infty}^{+\infty} c_n(x - a)^n$ converging whenever $|x - a| = r$, then $|f(x)|$ is equal to $\sup_{n \in \mathbb{Z}} |c_n| r^n$ in all classes of $C(a, r)$ except maybe in finitely many. When $a = 0$, we put $|f|(r) = \sup_{n \in \mathbb{Z}} |c_n| r^n$.

We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} , by $\mathcal{A}(d(a, R^-))$ the \mathbb{K} -algebra of power series $\sum_{n=0}^{\infty} c_n(x - a)^n$ converging in all $d(a, R^-)$ and by

$\mathcal{A}(D)$ the \mathbb{K} -algebra of Laurent series $\sum_{n=-\infty}^{\infty} c_n(x - a)^n$ converging in D . Similarly,

we denote by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} , i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$, by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions in $d(a, R^-)$ i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$, and by $\mathcal{M}(D)$ the field of meromorphic functions in D i.e. the field of fractions of $\mathcal{A}(D)$.

Next, we denote by $\mathcal{A}_b(d(a, R^-))$, the set of $f \in \mathcal{A}(d(a, R^-))$ that are bounded in $d(a, R^-)$ and we put $\mathcal{A}_u(d(a, R^-)) = \mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. We denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$ and put $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

We denote by $\mathcal{A}^z(D)$ the set of $f \in \mathcal{A}(D)$ admitting finitely many zeros in D and we put $\mathcal{A}^*(D) = \mathcal{A}(D) \setminus \mathcal{A}^z(D)$ and similarly, we denote by $\mathcal{M}^z(D)$ the field of fraction of $\mathcal{A}^z(D)$ and we put $\mathcal{M}^*(D) = \mathcal{M}(D) \setminus \mathcal{M}^z(D)$. So, $\mathcal{M}^*(D)$ is the set of meromorphic functions in D having at least infinitely many zeros or infinitely many poles in D .

2 Nevanlinna Theory in the classical p -adic context

Definition and notation: Let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(D)$) and let $\alpha \in d(a, R^-)$, resp. $\alpha \in D$. If f admits a zero of order q at α we set $\omega_\alpha(f) = q$ and if f has no zero and no pole at α , we set $\omega_\alpha(f) = 0$.

Let $f = \frac{h}{l} \in \mathcal{M}(d(a, R^-))$, (resp. $f \in \mathcal{M}(D)$). For each $\alpha \in \mathbb{K}$ (resp. $\alpha \in d(a, R^-)$, resp. $\alpha \in D$) the number $\omega_\alpha(h) - \omega_\alpha(l)$ does not depend on the

functions h, l choosed to make $f = \frac{h}{l}$. Thus, we can generalize the notation by setting $\omega_\alpha(f) = \omega_\alpha(h) - \omega_\alpha(l)$.

If $\omega_\alpha(f)$ is an integer $q > 0$, α is called a zero of f of order q . If $\omega_\alpha(f)$ is an integer $q < 0$, α is called a pole of f of order $-q$. If $\omega_\alpha(f) \geq 0$, f is said to be holomorphic at α .

Throughout the next paragraphs, we denote by I the interval $[t, +\infty[$, by J an interval of the form $[t, R]$ with $t > 0$. We denote by f a function that belongs either to $\mathcal{M}(\mathbb{K})$ or to $\mathcal{M}(S)$.

We have to introduce the counting function of zeros and poles of f , counting or not multiplicity. Here we will choose a presentation that avoids assuming that all functions we consider admit no zero and no pole at the origin.

Definitions: We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way.

Let (a_n) , $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq r$, of respective order s_n .

We set $Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|)$ and so, $Z(r, f)$ is called *the counting function of zeros of f in $d(0, r)$, counting multiplicity*.

In order to define the counting function of zeros of f without multiplicity, we put $\bar{\omega}_0(f) = 0$ if $\omega_0(f) \leq 0$ and $\bar{\omega}_0(f) = 1$ if $\omega_0(f) \geq 1$.

Now, set $\bar{Z}(r, f) = \bar{\omega}_0(f) \log r + \sum_{n=1}^{\sigma(r)} (\log r - \log |a_n|)$ and so, $\bar{Z}(r, f)$ is called *the counting function of zeros of f in $d(0, r)$ ignoring multiplicity*.

In the same way, considering the finite sequence (b_n) , $1 \leq n \leq \tau(r)$ of poles of f such that $0 < |b_n| \leq r$, with respective multiplicity order t_n , we put

$N(r, f) = \max(-\omega_0(f), 0) \log r + \sum_{n=1}^{\tau(r)} t_n (\log r - \log |b_n|)$ and then $N(r, f)$ is called *the counting function of the poles of f , counting multiplicity*.

Next, we put $\bar{\omega}_0(f) = 0$ if $\omega_0(f) \geq 0$ and $\bar{\omega}_0(f) = 1$ if $\omega_0(f) \leq -1$ and we set $\bar{N}(r, f) = \bar{\omega}_0(f) \log r + \sum_{n=1}^{\tau(r)} (\log r - \log |b_n|)$ and then $\bar{N}(r, f)$ is called *the counting function of the poles of f , ignoring multiplicity*.

Now, we can define the Nevanlinna function $T(r, f)$ in I or J as $T(r, f) = \max(Z(r, f), N(r, f))$ and the function $T(r, f)$ is called *characteristic function of f or Nevanlinna function of f* .

Finally, if Y is a subset of \mathbb{K} we will denote by $Z^Y(r, f')$ the counting function of zeros of f' , excluding those which are zeros of $f - a$ for any $a \in Y$.

Remark: If we change the origin, the functions Z, N, T are not changed, up to an additive constant.

Theorem 3.1: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) have no zero and no pole at 0. Then $\log(|f|(r)) = \log(|f(0)|) + Z(r, f) - N(r, f)$.*

Theorem 3.2 (First Main Theorem) : *Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(S)$). Then $Z(r, fg) \leq Z(r, f) + Z(r, g)$, $N(r, fg) \leq N(r, f) + N(r, g)$, and $T(r, f + b) = T(r, f) + O(1)$, $T(r, fg) \leq T(r, f) + T(r, g)$, $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$, $T(r, cf) = T(r, f) \forall c \in \mathbb{K}^*$, $T(r, \frac{1}{f}) = T(r, f)$, $T(r, \frac{f}{g}) \leq T(r, f) + T(r, g)$. Let $P(X) \in \mathbb{K}[X]$. Then $T(r, P(f)) = \deg(P)T(r, f) + O(1)$ and $T(r, f'P(f)) \geq T(r, P(f))$.*

Suppose now $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \mathcal{A}(S)$). Then $Z(r, fg) = Z(r, f) + Z(r, g)$, $T(r, f) = Z(r, f)$, $T(r, fg) = T(r, f) + T(r, g) + O(1)$ and $T(r, f + g) \leq \max(T(r, f), T(r, g))$.

Moreover, if $\lim_{r \rightarrow +\infty} T(r, f) - T(r, g) = +\infty$ then $T(r, f + g) = T(r, f)$ when r is big enough.

Theorem 3.3: *Let $f \in \mathcal{M}(\mathbb{K})$. Then f belongs to $\mathbb{K}(x)$ if and only if $T(r, f) = O(\log r)$.*

Corollary 3.3.a: *Let $f \in \mathcal{M}^*(\mathbb{K})$. Then f is transcendental over $\mathbb{K}(x)$.*

Theorem 3.4: *Let $f \in \mathcal{M}(S)$. Then f belongs to $\mathcal{M}_b(S)$ if and only if $T(r, f)$ is bounded in $[0, R[$.*

Corollary 3.4.a: *Let $f \in \mathcal{M}_u(S)$. Then f is transcendental over $\mathcal{M}_b(S)$.*

Theorem 3.5: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(S)$). Then for all $k \in \mathbb{N}^*$, we have $N(r, f^{(k)}) = N(r, f) + k\bar{N}(r, f)$ and $Z(r, f^{(k)}) \leq Z(r, f) + k\bar{N}(r, f) - k \log(r) + O(1)$.*

Theorem 3.6: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(S)$) and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct. Then $(q-1)T(r, f) \leq \max_{1 \leq k \leq q} \left(\sum_{j=1, j \neq k}^q Z(r, f - a_j) \right) + O(1)$.*

Remark: The last Theorem does not hold in complex analysis. Indeed, let f be a meromorphic function in \mathbb{C} omitting two values a and b , such as $f(x) = \frac{e^x}{e^x - 1}$. Then $Z(r, f - a) + Z(r, f - b) = 0$.

We can now state the Second Main Theorem.

Theorem 3.7: (Second Main Theorem) [3], [4], [7] *Let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$, with $q \geq 2$, let $Y = \{\alpha_1, \dots, \alpha_q\}$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$). Then*

$$(q-1)T(r, f) \leq \sum_{j=1}^q \bar{Z}(r, f - \alpha_j) + \bar{N}(r, f) - Z_0^Y(r, f') - \log r + O(1) \quad \forall r \in I$$

(resp. $\forall r \in J$).

Moreover, if f belongs to $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(S)$), then $(q-1)T(r, f) \leq \sum_{j=1}^q \bar{Z}(r, f - \alpha_j) - Z_0^Y(r, f') - \log(r) + O(1) \quad \forall r \in I$ (resp. $\forall r \in J$).

3 Nevanlinna Theory out of hole

Henceforth, we denote by L the interval $[R, +\infty[$. According to classical properties of analytic elements on infraconnected sets and by implicitly using [7], it is easy to have the following properties:

Proposition 4.1: *Let $f \in H(D)$ have no zero in D . Then $f(x)$ is of the form $\sum_{-\infty}^q a_n x^n$ with $|a_q|R^q > |a_n|R^n$ for all $n < q$.*

Definition and notation: Let $f \in H(D)$ have no zero in D , $f(x) = \sum_{-\infty}^q a_n x^n$ with $|a_q|R^q > |a_n|R^n$ for all $n < q$ and $a_q = 1$. Then f will be called a *Motzkin factor associated to S* and the integer q will be called *the Motzkin index of f* and will be denoted by $m(f, S)$ (see [2], [7], [14]). We put $L = [R, +\infty[$.

Theorem 4.2: *Let $f \in \mathcal{M}(D)$. We can write f in a unique way in the form $f^S f^0$ with $f^S \in H(D)$ a Motzkin factor associated to S and $f^0 \in \mathcal{M}(\mathbb{K})$, having no zero and no pole in S .*

Given $f \in \mathcal{M}(D)$, for $r > R$. If $\alpha_1, \dots, \alpha_m$ are the distinct zeros of f in $\Delta(0, R, r)$, with respective multiplicity u_j , $1 \leq j \leq m$, then *the counting function of zeros $Z_R(r, f)$* of f between R and r will be denoted by

$$Z_R(r, f) = \sum_{j=1}^m u_j (\log(r) - \log(|\alpha_j|)).$$

Similarly, if β_1, \dots, β_n are the distinct poles of f in $\Delta(0, R, r)$, with respective multiplicity v_j , $1 \leq j \leq n$, then *the counting function of poles $N_R(r, f)$* of f between R and r will be denoted by

$$N_R(r, f) = \sum_{j=1}^n v_j (\log(r) - \log(|\beta_j|)).$$

We put $T_R(r, f) = \max(Z_R(r, f), N_R(r, f))$.

The counting function of zeros without counting multiplicity $\bar{Z}_R(r, f)$ is defined as: $\bar{Z}_R(r, f) = \sum_{j=1}^m \log(r) - \log(|\alpha_j|)$, where $\alpha_1, \dots, \alpha_m$ are the distinct zeros of f in $\Delta(0, R, r)$. Similarly, *the counting function of poles without counting multiplicity $\bar{N}_R(r, f)$* is defined as: $\bar{N}_R(r, f) = \sum_{j=1}^n \log(r) - \log(|\beta_j|)$, where β_1, \dots, β_n are the distinct poles of f in $\Delta(0, R, r)$.

Finally, putting $Y = \{a_1, \dots, a_q\}$, we denote by $Z_R^Y(r, f')$ the counting function of zeros of f' on points x where $f(x) \notin Y$.

Theorem 4.3: *Let $f \in \mathcal{M}(D)$. Then, for all $r \in L$, $\log(|f|(r)) - \log(|f|(R)) = Z_R(r, f) - N_R(r, f) + m(f, S)(\log r - \log R)$.*

Corollary 4.3.a *Let $f \in \mathcal{M}(D)$. Then $T_R(r, f)$ is identically zero if and only if f is a Motzkin factor. Let $f, g \in \mathcal{A}(D)$ satisfy $\log(|f|(r)) \leq \log(|g|(r))$ for all $r \in L$. Then $Z_R(r, f) \leq Z_R(r, g) + (m(g, S) - m(f, S))(\log(r) - \log(R))$.*

Theorem 4.4: *Let $f \in \mathcal{A}(D)$. Then, $Z_R(r, f') \leq Z_R(r, f) - \log(r) + O(1)$, ($r \in L$).*

We can now characterize the set $\mathcal{M}^*(D)$:

Theorem 4.5: *Let $f \in \mathcal{M}(D)$. The three following statements are equivalent:*

- i) $\lim_{r \rightarrow +\infty} \frac{T_R(r, f)}{\log(r)} = +\infty$ for $r \in L$,
- ii) $\frac{T_R(r, f)}{\log(r)}$ is unbounded,
- iii) f belongs to $\mathcal{M}^*(D)$.

Operations on $\mathcal{M}(D)$ work almost like for meromorphic functions in the whole field [3], [7].

Theorem 4.6: *If $f, g \in \mathcal{M}(D)$. Then for every $b \in \mathbb{K}$ and $r \in L$, we have $T_R(r, f^n) = nT_R(r, f)$, $T_R(r, f.g) \leq T_R(r, f) + T_R(r, g) + O(\log(r))$,*

$$T_R(r, \frac{1}{f}) = T_R(r, f), \quad T_R(r, f + g) \leq T_R(r, f) + T_R(r, g) + O(\log(r)),$$

$T_R(r, f + b) = T_R(r, f) + O(\log(r))$, $T_R(r, h \circ f) = T_R(r, f) + O(\log(r))$, where h is a Moebius function.

Moreover, if both f and g belong to $\mathcal{A}(D)$, then

$$T_R(r, f + g) \leq \max(T_R(r, f), T_R(r, g)) + O(\log(r)), \quad T_R(r, fg) = T_R(r, f) + T_R(r, g).$$

Particularly, if $f \in \mathcal{A}^(D)$, then $T_R(r, f + b) = T_R(r, f) + O(1)$.*

Given a polynomial $P(x) \in \mathbb{K}[x]$, then $T_R(r, P \circ f) = \deg(P)T_R(r, f) + O(\log(r))$.

Theorem 4.7: *Every $f \in \mathcal{M}^*(D)$ is transcendental over $\mathcal{M}^z(D)$.*

Theorem 4.8: *Let $f \in \mathcal{M}(D)$. Then, for $r \in L$, $N_R(r, f^{(k)}) = N_R(r, f) + k\bar{N}_R(r, f)$ and $Z_R(r, f^{(k)}) \leq Z_R(r, f) + k\bar{N}_R(r, f) + O(\log(r))$.*

Like in the whole field, the Nevanlinna second Main Theorem is based on the following theorem:

Theorem 4.9: *Let $f \in \mathcal{M}(D)$ and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct. Then*

$$(q-1)T_R(r, f) \leq \max_{1 \leq k \leq q} \left(\sum_{j=1, j \neq k}^q Z_R(r, f - a_j) \right) + O(\log(r)).$$

Theorem 4.10: (Second Main Theorem) *Let $f \in \mathcal{M}(D)$, let $\alpha_1, \dots, \alpha_q \in \mathbb{K}$, with $q \geq 2$ and let $Y = \{\alpha_1, \dots, \alpha_q\}$. Then, for $r \in L$, $(q-1)T_R(r, f)$*

$$\leq \sum_{j=1}^q \bar{Z}_R(r, f - \alpha_j) + \bar{N}_R(r, f) - Z_R^Y(r, f') + O(\log(r)).$$

Particularly, if $f \in \mathcal{A}(D)$, then

$$(q-1)T_R(r, f) \leq \sum_{j=1}^q \bar{Z}_R(r, f - \alpha_j) - Z_R^Y(r, f') + O(\log(r)).$$

4 Immediate applications

Theorem 5.1: *Let $a_1, a_2 \in \mathbb{K}$ (with $a_1 \neq a_2$) and let $f, g \in \mathcal{A}^*(\mathbb{K})$ satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2$). Then $f = g$.*

Theorem 5.2: Let $a_1, a_2, a_3 \in \mathbb{K}$ (with $a_i \neq a_j \forall i \neq j$) and let $f, g \in \mathcal{A}_u(d(0, R^-))$ (resp. $f, g \in \mathcal{A}^*(D)$) satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3$). Then $f = g$.

Theorem 5.3: Let $a_1, a_2, a_3, a_4 \in \mathbb{K}$ (with $a_i \neq a_j \forall i \neq j$) and let $f, g \in \mathcal{M}^*(\mathbb{K})$ satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3, 4$). Then $f = g$.

Theorem 5.4: Let $a_1, a_2, a_3, a_4, a_5 \in \mathbb{K}$ (with $a_i \neq a_j \forall i \neq j$) and let $f, g \in \mathcal{M}_u(d(0, R^-))$ (resp. $f, g \in \mathcal{M}^*(D)$) satisfy $f^{-1}(\{a_i\}) = g^{-1}(\{a_i\})$ ($i = 1, 2, 3, 4, 5$). Then $f = g$.

Theorem 5.5: Let Λ be a non-degenerate elliptic curve of equation $y^2 = (x - a_1)(x - a_2)(x - a_3)$.

There do not exist $g, f \in \mathcal{M}(\mathbb{K})$ such that $g(t) = y, f(t) = x, t \in \mathbb{K}$.

There do not exist $g, f \in \mathcal{A}_u(d(0, R^-))$ such that $g(t) = y, f(t) = x, t \in d(0, R^-)$.

There do not exist $g, f \in \mathcal{A}^*(D)$ such that $g(t) = y, f(t) = x, t \in D$.

Theorem 5.6: Let Λ be a curve of equation $y^q = P(x)$, $q \geq 2$, with $P \in \mathbb{K}[x]$ admitting n distinct zeros of order 1 with $n \geq 4$. There do not exist $g, f \in \mathcal{M}(\mathbb{K})$ such that $g(t) = y, f(t) = x, t \in \mathbb{K}$.

Theorem 5.7: Let Λ be a curve of equation $y^q = P(x)$, $q \geq 2$, with $P \in \mathbb{K}[x]$ admitting n distinct zeros of order 1 with $n \geq 5$. There do not exist $g, f \in \mathcal{M}_u(d(0, R^-))$ (resp. $g, f \in \mathcal{M}^*(D)$) such that $g(t) = y, f(t) = x, t \in d(0, R^-)$ (resp. $t \in D$).

Another application concerns analytic functions:

Theorem 5.8: Let $f, g \in \mathcal{M}(d(0, R^-))$ (resp. $f, g \in \mathcal{M}(D)$) satisfy $g^m + f^n = 1$, with $\min(m, n) \geq 3$ and $\max(m, n) \geq 4$. Then f and g belong to $\mathcal{M}_b(d(0, R^-))$ (resp. to $\mathcal{M}^z(D)$). Moreover, if $f, g \in \mathcal{A}(d(0, R^-))$ (resp. if $f, g \in \mathcal{A}(D)$) satisfy $g^m + f^n = 1$, with $\min(m, n) \geq 2$ and $(m, n) \neq (2, 2)$, then f and g belong to $\mathcal{A}_b(d(0, R^-))$, (resp to $\mathcal{A}^z(D)$).

5 Small functions

Definitions: For each $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(S)$, resp. $f \in \mathcal{M}(D)$), we will denote by $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(S)$, $\mathcal{M}_f(D)$) the set of functions $h \in \mathcal{M}(\mathbb{K})$ (resp. $h \in \mathcal{M}(S)$, $h \in \mathcal{M}(D)$) such that $T(r, h) = o(T(r, f))$, $r \in I$ (resp. $T(r, h) = o(T(r, f))$, $r \in J$, resp. $T_R(r, h) = o(T_R(r, f))$, $r \in L$). Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(S)$, resp. $f \in \mathcal{A}(D)$) we will denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(S)$, resp. $\mathcal{A}_f(D)$) the set $\mathcal{M}_f(\mathbb{K}) \cap \mathcal{A}(\mathbb{K})$, (resp. $\mathcal{M}_f(S) \cap \mathcal{A}(S)$, resp. $\mathcal{M}_f(D) \cap \mathcal{A}(D)$).

The elements of $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(S)$, resp. $\mathcal{M}_f(D)$) are called *small meromorphic functions with respect to f* . Similarly, if $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(S)$, resp. $f \in \mathcal{A}(D)$) these functions are called *small analytic functions with respect to f* .

A small function w with respect to a function $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(S)$, resp. $f \in \mathcal{M}(D)$) will be called a *quasi-exceptional small function for f* if $f - w$ has finitely many zeros in D .

Theorem 7.1: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$). Then f admits at most one quasi-exceptional small function. Moreover, if f has finitely many poles, then f admits no quasi-exceptional small function.*

Corollary 7.1.a: *Let $f \in \mathcal{A}^*(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(S)$, resp. $f \in \mathcal{A}^*(D)$). Then f has no quasi-exceptional small function.*

The Second Main Theorem for Three Small Functions holds as well as in complex analysis. Notice that this theorem was generalized to any finite set of small functions by K. Yamanoi in complex analysis [16], through methods that have no equivalent on a p-adic field.

Theorem 7.2: *Let $f \in \mathcal{M}^*(\mathbb{K})$, (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$) and let $w_1, w_2, w_3 \in \mathcal{M}_f(\mathbb{K})$ (resp. $\in \mathcal{M}_f(d(0, R^-))$, resp. $\in \mathcal{M}_f(D)$) be pairwise distinct. Then: $T(r, f) \leq \sum_{j=1}^3 \overline{Z}(r, f - w_j) + o(T(r, f))$
(resp. $T(r, f) \leq \sum_{j=1}^3 \overline{Z}(r, f - w_j) + o(T(r, f))$, resp. $T_R(r, f) \leq \sum_{j=1}^3 \overline{Z}_R(r, f - w_j) + o(T_R(r, f))$).*

Corollary 7.2.a: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$ resp. $f \in \mathcal{M}^*(D)$) and let $w_1, w_2 \in \mathcal{A}_f(\mathbb{K})$ (resp. $w_1, w_2 \in \mathcal{A}_f(S)$ resp. $w_1, w_2 \in \mathcal{A}_f(D)$) be distinct. Then $T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{N}(r, f) + o(T(r, f))$,
(resp. $T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + \overline{N}(r, f) + o(T_R(r, f))$, resp. $T_R(r, f) \leq \overline{Z}_R(r, f - w_1) + \overline{Z}_R(r, f - w_2) + \overline{N}_R(r, f) + o(T_R(r, f))$).*

Corollary 7.2.b: *Let $f \in \mathcal{A}^*(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(S)$, resp. $f \in \mathcal{A}^*(D)$) and let $w_1, w_2 \in \mathcal{A}_f(S)$, resp. $w_1, w_2 \in \mathcal{A}_f(D)$ be distinct. Then
 $T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + o(T(r, f))$,
(resp. $T(r, f) \leq \overline{Z}(r, f - w_1) + \overline{Z}(r, f - w_2) + o(T(r, f))$,
resp. $T_R(r, f) \leq \overline{Z}_R(r, f - w_1) + \overline{Z}_R(r, f - w_2) + o(T_R(r, f))$).*

Definitions: Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$, resp. $f \in \mathcal{M}^*(D)$) and let $w \in \mathcal{M}_f(\mathbb{K})$ (resp. $w \in \mathcal{M}_f(d(0, R^-))$, resp. $w \in \mathcal{M}_f(D)$). Then w is called a *perfectly branched function with respect to f* if all zeros of $f - w$ are multiple except maybe finitely many [7] and w is called a *totally branched function with respect to f* if all zeros of $f - w$ are multiple, without exception [5]. Particularly, the definition applies to constants. The following Theorem 7.3, 7.4, 7.5 are proved in [7] concerning the sets $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$), $f \in \mathcal{A}^*(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$). The proofs concerning $\mathcal{M}^*(D)$ and $\mathcal{A}^*(D)$ are similar to those concerning $\mathcal{M}^*(\mathbb{K})$ and $\mathcal{A}^*(\mathbb{K})$ respectively.

Theorem 7.3: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$, resp. $f \in \mathcal{M}^*(D)$). Then f admits at most four perfectly branched values.*

Theorem 7.4: *Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}^*(D)$) having finitely many poles. Then f admits at most one perfectly branched rational function.*

Corollary 7.4.a: *Let $f \in \mathcal{A}^*(\mathbb{K})$ (resp. $f \in \mathcal{A}^*(D)$). Then f admits at most one perfectly branched rational function.*

Theorem 7.5: *Let $f \in \mathcal{M}_u(d(0, R^-))$, having finitely many poles. Then f admits at most two perfectly branched rational functions.*

Corollary 7.5.a: *Let $f \in \mathcal{A}_u(d(0, R^-))$. Then f admits at most two perfectly branched rational functions.*

6 New applications of the Nevanlinna Theory

Definitions: Recall that two functions f and g meromorphic in a set B are said to share a set $X \subset \mathbb{K}$, counting multiplicity, or *C.M.* in brief, if for each $b \in X$, when $f(x) - b$ has a zero of order q at a point $a \in B$, then there exists $c \in X$ such that $g(x) - c$ also has a zero of order q at a . And the functions f and g are said to share X ignoring multiplicity or *I.M.* in brief, if $f^{-1}(X) = g^{-1}(X)$.

In the same way, two functions f and g meromorphic in a set B are said to share a nonidentically zero function h from B to \mathbb{K} , counting multiplicity, or *C.M.* in brief, if $\frac{f}{h}$ and $\frac{g}{h}$ share 1 *C.M.*, i.e. if $f - h$ and $g - h$ have the same zeros with the same multiplicity.

By Theorem 5.3, two meromorphic functions in \mathbb{K} sharing 4 points *I.M.* are identical, by Theorem 5.4 two meromorphic functions in S or in D sharing 5 points *I.M.* are identical, by Theorem 5.1, two entire functions sharing 2 points *I.M.* are identical and by Theorem 5.2 two meromorphic functions in S or in D sharing 3 points *I.M.* are identical. Here we will first examine two meromorphic functions sharing a few points *C.M.*

Theorem 8.1: Let $f, g \in \mathcal{M}_u(S)$ (resp. let $f, g \in \mathcal{M}^*(D)$) share *C.M.* 4 points $a_j \in \mathbb{K} \cup \{\infty\}$, $j = 1, 2, 3, 4$. Then $f \equiv g$.

Theorem 8.2: Let $f, g \in \mathcal{M}_u(S)$ (resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles and share *C.M.* 3 points $a_j \in \mathbb{K} \cup \{\infty\}$, $j = 1, 2, 3$. Then $f \equiv g$.

Corollary 8.2.a: Let $f, g \in \mathcal{A}_u(S)$ (resp. $f, g \in \mathcal{A}^*(D)$) share *C.M.* 3 points $a_j \in \mathbb{K}$, $j = 1, 2, 3$. Then $f \equiv g$.

Theorem 8.3 is not immediate and has a similar version in complex analysis for meromorphic functions provided with a finite growth order that is not integral. Here, we don't need any hypothesis on the growth order.

Theorem 8.3: Let $f, g \in \mathcal{M}^*(\mathbb{K})$ share *C.M.* 3 points $a_j \in \mathbb{K} \cup \{\infty\}$, $j = 1, 2, 3$. Then $f \equiv g$.

In the particular case of functions $f, g \in \mathcal{M}_u(S)$ or functions $f, g \in \mathcal{M}^*(D)$ having finitely many poles and sharing poles *C.M.*, we can add this theorem:

Theorem 8.4: Let $f, g \in \mathcal{M}_u(S)$, (resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in S (resp. in D) and share *C.M.* two values a, b and poles. Then $f \equiv g$.

Our main theorems are Theorems 8.5 and 8.6 that follow the same kind of reasoning as in [6]. We denote by $Y_1 = \{\alpha_1, \dots, \alpha_k\}$ and $Y_2 = \{\beta_1, \beta_2\}$ the two sets satisfying

$$(\mathcal{H}) \quad \left(\prod_{j=1}^k (\beta_1 - \alpha_j) \right)^2 \neq \left(\prod_{j=1}^k (\beta_2 - \alpha_j) \right)^2.$$

Theorem 8.5: Let $f, g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(S)$, resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in \mathbb{K} (resp. in S , resp. in D) and share Y_1 *C.M.* and Y_2 *I.M.* Then $f \equiv g$.

Corollary 8.5.a: *Let $f, g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(S)$, resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in \mathbb{K} (resp. in S , resp. in D) and share a value α C.M. and Y_2 I.M. If $(\alpha - \beta_1)^2 \neq (\alpha - \beta_2)^2$, then $f \equiv g$.*

Corollary 8.5.b: *Let $f, g \in \mathcal{A}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}_u(S)$, resp. let $f, g \in \mathcal{A}^*(D)$) and share Y_1 C.M. and Y_2 I.M. Then $f \equiv g$.*

Corollary 8.5.c: *Let $f, g \in \mathcal{A}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}_u(S)$, resp. let $f, g \in \mathcal{A}^*(D)$) and share a value α C.M. and Y_2 I.M. If $(\alpha - \beta_1)^2 \neq (\alpha - \beta_2)^2$, then $f \equiv g$.*

Theorem 8.6: *Let $f, g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(S)$, resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in \mathbb{K} (resp. in S , resp. in D) and share Y_1 I.M. and Y_2 C.M. Then $f \equiv g$.*

Corollary 8.6.a: *Let $f, g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(S)$, resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in \mathbb{K} (resp. in S , resp. in D) and share a value α I.M. and Y_2 C.M. If $(\alpha - \beta_1)^2 \neq (\alpha - \beta_2)^2$, then $f \equiv g$.*

Corollary 8.6.b: *Let $f, g \in \mathcal{A}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}_u(S)$, resp. let $f, g \in \mathcal{A}^*(D)$) and share Y_1 I.M. and Y_2 C.M. Then $f \equiv g$.*

Corollary 8.6.c: *Let $f, g \in \mathcal{A}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}_u(S)$, resp. let $f, g \in \mathcal{A}^*(D)$) and share a value α I.M. and Y_2 C.M. If $(\alpha - \beta_1)^2 \neq (\alpha - \beta_2)^2$, then $f \equiv g$.*

It is known that if two functions $f, g \in \mathcal{A}(\mathbb{K})$ share separately two values $a, b \in \mathbb{K}$ C.M., then $f \equiv g$ [7] (Theorem 41.1). However, here the hypothesis f, g share Y_1 and share Y_2 cannot be compared: for example, concerning Y_2 , f and g are not supposed to share β_1 or β_2 separately. The same remark applies to meromorphic functions having finitely many poles.

Results of [11] and [13] showed the interest of complex functions of the form $f(x)f(x+b)$. Similar studies were made in [11] and [13] in a p -adic field. Here we will generalize that kind of study on the field \mathbb{K} . In [13], it is proven that if two complex entire functions f and g are such that $f(x)^n f(x+c)$ and $g(x)^n g(x+c)$ share 1 C.M. with $n \geq 6$, then either fg is a constant t_1 such that $t_1^{n+1} = 1$, or $\frac{f}{g}$ is a constant t_2 such that $t_2^{n+1} = 1$. Here, on the field \mathbb{K} , we can obtain better results.

On the other hand, in [11], we can find similar results of uniqueness for meromorphic functions on a p -adic field involving derivatives, sharing 1 C.M. or I.M., also involving derivatives. Here we will examine functions of the form $f(x)^n (f(x+c))^m, g(x)^n (g(x+c))^m$ sharing a rational function and we will look for branched values and quasi-exceptional values of such functions.

Notation: We denote by \mathbb{N}^* the set of strictly positive integers. On \mathbb{Z} , we denote by $|\cdot|_\infty$ the Archimedean absolute value.

Theorem 8.7: *Let $a \in C(0, 1)$, let $b \in \mathbb{K}$ and let $f, g \in \mathcal{M}^*(\mathbb{K})$ have finitely many poles and take $m, n \in \mathbb{N}^*$ with $m \neq n$. If $f^n(x)f^m(ax+b)$ and $g^n(x)g^m(ax+b)$ share C.M. a rational function $Q \in \mathbb{K}(x)$, $Q \neq 0$ and if $n+m \geq 5$, then $\frac{f}{g}$ is*

a constant t such that $t^{n+m} = 1$. Moreover, if $f, g \in \mathcal{A}^*(\mathbb{K})$, if $f^n(x)f^m(ax+b)$ and $g^n(x)g^m(ax+b)$ share C.M. a constant $l \neq 0$ and if $n+m \geq 4$, then $\frac{f}{g}$ is a constant t such that $t^{n+m} = 1$.

Theorem 8.8: Let $a \in C(0,1)$ and let $b \in S$ (resp. let $b \in \mathbb{K}$) and let $f, g \in \mathcal{M}_u(S)$ (resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in S (resp. in D) and take $n, m \in \mathbb{N}^*$ with $n \neq m$. If $f^n(x)f^m(ax+b)$ and $g^n(x)g^m(ax+b)$ share C.M. a function $\theta \in \mathcal{M}(S)$ (resp. a function $\theta \in \mathcal{M}(D)$) having finitely many zeros and poles in S (resp. in D), if $n \geq 5$, then $\frac{f}{g}$ is a constant t such that $t^{n+m} = 1$.

In [15] it was shown that given a complex entire function f and $b \in \mathbb{C} \setminus \{0\}$, a function of the form $f^n(x)f(x+b) - c$ (with $c \neq 0$) has infinitely many zeros in \mathbb{C} provided $n \geq 3$. On the field \mathbb{K} , such a result is trivial since an entire functions and even a meromorphic function with finitely many poles (which is not a rational function) takes every value infinitely many times. But we can ask the question regarding in general functions $f \in \mathcal{M}^*(\mathbb{K})$, $f \in \mathcal{M}_u(S)$, $f \in \mathcal{M}^*(D)$.

Theorem 8.9: Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$), let $a \in C(0,1)$, let $b \in \mathbb{K}$ (resp. $b \in S$, resp. $b \in \mathbb{K}$) and let $w \in \mathcal{M}(\mathbb{K})$ (resp. $w \in \mathcal{M}(S)$, resp. $w \in \mathcal{M}(D)$) be a non identically zero small function with respect to f . If $n, m \in \mathbb{N}$ are such that $|n-m|_\infty \geq 5$, then $f^n(x)f^m(ax+b) - w$ has infinitely many zeros in \mathbb{K} (resp. in S , resp. in D).

Corollary 8.9.a: Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$), let $a \in C(0,1)$, let $b \in \mathbb{K}$ (resp. $b \in S$, resp. $b \in \mathbb{K}$). If $n, m \in \mathbb{N}$ are such that $|n-m|_\infty \geq 5$, then $f^n(x)f^m(ax+b)$ takes every nonzero value infinitely many times in \mathbb{K} (resp. in S , resp. in D).

Remarks: 1) Of course, the hypothesis $w \neq 0$ must not be excluded. Indeed, let $h \in \mathcal{A}^*(\mathbb{K})$ and let $f(x) = \frac{1}{h(x)}$. Then a function of the form $f^n(x)f(ax+b)$ has no zero in \mathbb{K} .

2) On the other hand, it is known and easily seen that if $a = 1$ and $b = 0$, a function $f^n - w$ has infinitely many zeros for every $n \geq 3$ and that f^2 takes every nonzero value c infinitely many times because given a square root l of c , then $f^2 - c = (f-l)(f+l)$ and at least one of the two values l and $-l$ is taken infinitely many times. We can ask whether a meromorphic function of the form $f^2(x) - w$ always has infinitely many zeros when w is not a constant.

3) Concerning Theorem 8.9, it is easily proven that if f is a meromorphic function with finitely many poles and w a small function, then $f-w$ has infinitely many zeros. So it is useless here to add a corollary concerning $f^n(x)f^m(ax+b) - w$ when f has finitely many poles.

Theorem 8.10: Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$), let $a \in C(0,1)$, let $b \in \mathbb{K}$ (resp. $b \in S$, resp. $b \in \mathbb{K}$) and let $n, m \in \mathbb{N}^*$. If $3|n-m|_\infty > 2(n+m) + 4$, then $f^n(x)f^m(ax+b)$ does not admit 4 distinct perfectly branched values. Moreover, if $4|n-m|_\infty > 3(n+m) + 4$, then $f^n(x)f^m(ax+b)$ does not admit 3 distinct perfectly branched values.

Further, if f belongs to $\mathcal{M}^*(\mathbb{K})$ and if $3|n - m|_\infty \geq 2(n + m + 1)$, then $f^n(x)f^m(ax + b)$ does not admit 4 distinct totally branched values; moreover, if $4|n - m|_\infty \geq 3(n + m) + 4$, then $f^n(x)f^m(ax + b)$ does not admit 3 distinct totally branched values.

7 Tools and Lemmas

In the proofs of our main theorems, we will use the following lemmas:

Lemma 1: *Let $P(x) \in \mathbb{K}[x]$ have all its zeros of order 1. Let $f \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f \in \mathcal{M}_u(S)$, resp. let $f \in \mathcal{M}^*(D)$) and suppose that $f - b$ admits a zero of order q at a point $u \in \mathbb{K}$ (resp. at a point $u \in S$, resp. at a point $u \in D$). Then $P(f) - P(b)$ admits u as a zero of order q .*

Lemmas 2, 3 and 4 are well known [7], Lemma 32.3, Corollary 32.6.

Lemma 2: *Let $f \in \mathcal{M}^*(\mathbb{K})$ have finitely many poles. Then for every $Q \in \mathbb{K}(x)$, $f - Q$ has infinitely many zeros.*

Lemma 3: *Let $f \in \mathcal{M}^*(\mathbb{K})$. There exist ϕ and $\psi \in \mathcal{A}(\mathbb{K})$ having no common zero, such that $f = \frac{\phi}{\psi}$. Moreover, if f has no zero and no pole then f is a constant.*

Lemma 4 is a consequence of Theorem 40.10 and Theorem 43.1 in [7]:

Lemma 4: *Let $w \in \mathcal{M}(S)$ have finitely many zeros and poles in S . Then w belongs to $\mathcal{M}_b(S)$. Moreover, given $f \in \mathcal{M}_u(S)$ having finitely many poles in S , then $f - w$ belongs to $\mathcal{M}_u(S)$ and has infinitely many zeros.*

Lemma 5: *Let $f(x) \in \mathcal{A}(D)$. Then f belongs to $\mathcal{A}^z(D)$ if and only if it is of the form $\sum_{n=0}^q a_n x^n$ with $q \in \mathbb{Z}$ and $\lim_{n \rightarrow -\infty} |a_n| R^n = 0$. Moreover, given $f \in \mathcal{M}^*(D)$ having finitely many poles in D and $w \in \mathcal{M}^z(D)$, then $f - w$ has infinitely many zeros.*

Lemma 6 is immediate :

Lemma 6: *Let $u \in C(0, 1)$ and $c \in \mathbb{K}$ (resp. $c \in S$, resp. $c \in \mathbb{K}$) and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(S)$, resp. $f \in \mathcal{M}(D)$). Then $T(r, f) = T(r, f(ux+c))$, $Z(r, f) = Z(r, f(ux+c))$, $N(r, f) = N(r, f(ux+c))$ whenever $r \geq |c|$ (resp. $T(r, f) = T(r, f(ux+c))$, $Z(r, f) = Z(r, f(ux+c))$, $N(r, f) = N(r, f(ux+c))$) whenever $r \geq |c|$, resp. $T_R(r, f) = T_R(r, f(ux+c))$, $Z_R(r, f) = Z_R(r, f(ux+c))$, $N_R(r, f) = N_R(r, f(ux+c))$ whenever $r \geq |c|$.*

Notation: Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}(S)$, resp. let $f, g \in \mathcal{M}(D)$). We set $M(r, f) = \max((Z(r, f), N(r, f)) - N(r, f))$ (resp. $M(r, f) = \max((Z(r, f), N(r, f)) - N(r, f))$, resp. $M_R(r, f) = \max((Z_R(r, f), N_R(r, f)) - N_R(r, f))$).

Lemma 7 is also easily proven by the property: $M(r, f) = \max(\log(|f|(r)), 0)$, on $\mathcal{M}(\mathbb{K})$ and on $\mathcal{M}(S)$ and $M_R(r, f) = \max(\log(|f|(r)), 0)$ on $\mathcal{M}(D)$.

Lemma 7: *Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}(S)$). Then $M(r, fg) \leq M(r, f) + M(r, g)$, $M(r, f+g) \leq M(r, f) + M(r, g)$, $M(r, \frac{f'}{f}) \leq O(1)$.*

Let $f, g \in \mathcal{M}(D)$. Then $M_R(r, fg) \leq M_R(r, f) + M_R(r, g)$, $M_R(r, f+g) \leq M_R(r, f) + M_R(r, g)$, $M_R(r, \frac{f'}{f}) \leq O(1)$.

Lemmas 8 and 9 are easily proven:

Lemma 8: *Let $f, g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(S)$, resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in \mathbb{K} (resp. in S , resp. in D). Then $T(r, \lambda f + \mu g) \leq \max(T(r, f), T(r, g)) + O(1)$ (resp. $T(r, \lambda f + \mu g) \leq \max(T(r, f), T(r, g)) + O(1)$, resp. $T_R(r, \lambda f + \mu g) \leq \max(T_R(r, f), T_R(r, g)) + O(\log(r))$).*

Lemma 9: *Let $f, g \in \mathcal{M}^*(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(S)$, resp. let $f, g \in \mathcal{M}^*(D)$) have finitely many poles in \mathbb{K} (resp. in S , resp. in D). Then $Z(r, \frac{1}{f} - \frac{1}{g}) \leq Z(r, f - g) + O(\log(r))$, (resp. $Z(r, \frac{1}{f} - \frac{1}{g}) \leq Z(r, f - g) + O(1)$, resp. $Z_R(r, \frac{1}{f} - \frac{1}{g}) \leq Z_R(r, f - g) + O(\log(r))$).*

8 Proof of Theorems

Proofs of Theorems 8.1 and 8.2: We will first prove Theorems 8.1 when f and g belong to $\mathcal{M}_u(S)$. So, let $f, g \in \mathcal{M}_u(S)$ sharing C.M. q points $a_j \in \mathbb{K}$, $1 \leq j \leq q$. By Theorem 3.7, we have

$$(q-1)T(r, f) \leq \sum_{j=1}^q Z(r, f - a_j) + O(1) \leq Z(r, f - g) + O(1)$$

$$(1) \quad \leq T(r, f) + T(r, g) + O(1)$$

and similarly

$$(q-1)T(r, g) \leq \sum_{j=1}^q Z(r, g - a_j) + O(1) \leq Z(r, f - g) + O(1)$$

$$(2) \quad \leq T(r, f) + T(r, g) + O(1)$$

hence

$$(3) \quad (q-1)(T(r, f) + T(r, g)) \leq 2(T(r, f) + T(r, g)) + O(1).$$

hence (3) implies $q \leq 3$.

Suppose now that f and g have finitely many poles. If f and g belong to $\mathcal{M}_u(S)$, by Lemma 8, we have $Z(r, f - g) \leq \max(T(r, f), T(r, g))$, hence by (1) and (2), now we obtain

$$(q-1) \max(T(r, f), T(r, g)) \leq \max(T(r, f), T(r, g)) + O(1)$$

and hence $q \leq 2$. Consequently, since $q = 3$, we have $f \equiv g$ in Theorem 8.2.

If $f, g \in \mathcal{M}^*(D)$, the same reasoning holds.

Proof of Theorem 8.3: Let $f, g \in \mathcal{M}^*(\mathbb{K})$ be distinct and share three points $a_j \in \mathbb{K}$, $j = 1, 2, 3$ C.M. Without loss of generality, we can assume that $a_1 a_2 \neq 0$. Suppose first that $f(x) \neq a_3 \forall x \in \mathbb{K}$, hence $g(x) \neq a_3 \forall x \in \mathbb{K}$. Let $F = \frac{1}{f}$, $G = \frac{1}{g}$ and let $b_j = \frac{1}{a_j}$ $j = 1, 2$. Then F and G belong to $\mathcal{A}(\mathbb{K})$ and share b_1 and b_2 C.M and hence, by Theorem 5.1 we have $F \equiv G$, hence $f \equiv g$.

Thus, henceforth, we can assume that f and g take each value a_j , $j = 1, 2, 3$ at least one time, therefore

$$(1) \quad Z(r, f - a_j) \geq \log(r) + O(1), \quad r \in]0, +\infty[, j = 1, 2, 3.$$

Now we notice that

$$(2) \quad \sum_{j=1}^3 Z(r, f - a_j) \leq Z(r, f - g) \leq T(r, f) + T(r, g).$$

Suppose first that $\lim_{r \rightarrow +\infty} \frac{Z(r, f - a_j)}{T(r, f)} = 1 \forall j = 1, 2, 3$. Then there exists $s > 0$

such that $\frac{Z(r, f - a_j)}{T(r, f)} \geq \frac{3}{4} \forall j = 1, 2, 3, \forall r > s$.

Then for every $r > s$, we have

$$\frac{9}{4}T(r, f) \leq \sum_{j=1}^3 Z(r, f - a_j) \leq Z(r, f - g) + O(1) \leq T(r, f) + T(r, g) + O(1)$$

and similarly

$$\frac{9}{4}T(r, g) \leq \sum_{j=1}^3 Z(r, g - a_j) \leq Z(r, f - g) + O(1) \leq T(r, f) + T(r, g) + O(1).$$

Therefore, we obtain $\frac{9}{4}(T(r, f) + T(r, g)) \leq 2(T(r, f) + T(r, g)) + O(1)$, a contradiction showing that $\liminf_{r \rightarrow +\infty} \frac{Z(r, f - a_j)}{T(r, f)} < 1$ holds for at least one index.

Consequently, we are now led to assume there exists $\epsilon > 0$ such that for a certain k , there exists a sequence of intervals $([s_n, t_n])_{n \in \mathbb{N}}$ such that $s_n < t_n < s_{n+1}$, $\lim_{n \rightarrow +\infty} s_n = +\infty$ and $\frac{Z(r, f - a_k)}{T(r, f)} \leq 1 - \epsilon \forall r \in [s_n, t_n], \forall n \in \mathbb{N}$. With-

out loss of generality, we can suppose $k = 3$ and set $M = \bigcup_{n=1}^{+\infty} [s_n, t_n]$. So, when r belongs to M , we have

$$(3) \quad Z(r, f - a_3) \leq (1 - \epsilon)T(r, f) \quad \forall r \in M$$

By Lemma 3 we can write f in the form $\frac{\phi}{\psi}$ with $\phi, \psi \in \mathcal{A}(\mathbb{K})$ having no common zero. Then $N(r, f - a_3) = N(r, f) = Z(r, \psi)$, hence by (3), we have

$$T(r, f - a_3) = T(r, f) = N(r, f) \quad \forall r \in M.$$

Now $Z(r, f - a_j) = Z(r, \phi - a_j\psi)$, $j = 1, 2, 3$, $N(r, f) = Z(r, \psi)$. Therefore by (3), we have

$$(4) \quad Z(r, \phi - a_3\psi) \leq (1 - \epsilon)Z(r, \psi).$$

By (4) and by Theorem 3.1, we can derive $\log(|\phi - a_3\psi|(r)) \leq (1 - \epsilon)\log(|(a_3 - a_1)\psi|(r)) + O(1)$ and hence $\log(|\phi - a_3\psi|(r)) < \log(|(a_3 - a_1)\psi|(r))$ when r is big enough in M . So, there exists $u \geq s$ such that $\log(|\phi - a_3\psi|(r)) < \log(|(a_3 - a_1)\psi|(r)) \forall r \in M, r \geq u$. Consequently, $\log(|\phi - a_1\psi|(r)) = \log(|(a_3 - a_1)\psi|(r)) \forall r \in J, r \geq u$ and hence $Z(r, f - a_1) \geq Z(r, \psi) + O(1) \forall r \in M, r \geq u$, therefore

$$(5) \quad Z(r, f - a_1) = T(r, f) + O(1) \forall r \in M, r \geq u.$$

Similarly, we have

$$(6) \quad Z(r, f - a_2) = T(r, f) + O(1) \forall r \in M, r \geq u.$$

Consequently, by (5) and (6), we obtain $Z(r, f - a_1) + Z(r, f - a_2) \geq 2T(r, f) + O(1) r \in J, r \geq u$ and hence $\sum_{j=1}^3 Z(r, f - a_j) \geq 2T(r, f) + Z(r, f - a_3) + O(1) r \in M, r \geq u$, therefore by (2)

$$2T(r, f) + Z(r, f - a_3) \leq T(r, f) + T(r, g) + O(1) r \in M, r \geq u$$

and similarly

$$2T(r, g) + Z(r, g - a_3) \leq T(r, f) + T(r, g) + O(1) r \in M, r \geq u.$$

Thus,

$$2(T(r, f) + T(r, g)) + Z(r, f - a_3) + Z(r, g - a_3) \leq 2(T(r, f) + T(r, g)) + O(1), r \in M, r \geq u.$$

But then, by (1) we have a contradiction that finishes the proof.

Proof of Theorem 8.4: Without loss of generality, we can assume $ab \neq 0$.

Let $F = \frac{1}{f}$, $G = \frac{1}{g}$ and let $a' = \frac{1}{a}$, $b' = \frac{1}{b}$. Then F and G share 0, a, b C.M.

Suppose that F and G are not identical. By Theorems 3.7 and 4.10, we have $2T(r, F) \leq Z(r, F) + Z(r, F - a') + Z(r, F - b') + O(1) \leq Z(r, F - G) + O(1)$ (resp. $2T_R(r, F) \leq Z_R(r, F) + Z_R(r, F - a') + Z_R(r, F - b') + O(\log(r)) \leq Z_R(r, F - G) + O(\log(r))$) hence by Lemma 9, $2T(r, F) \leq T(r, f - g) + O(1)$ (resp. $2T_R(r, F) \leq T_R(f - g)$) and similarly, $2T(r, G) \leq T(r, f - g) + O(1)$, (resp. $2T_R(r, G) \leq T_R(r, f - g) + O(\log(r))$). Consequently, by Theorem 3.2 and 4.6 and Lemma 8, we have $2 \max(T(r, f), T(r, g)) \leq \max(T(r, f), T(r, g)) + O(1)$ (resp. $2 \max(T_R(r, f), T_R(r, g)) \leq \max(T_R(r, f), T_R(r, g)) + O(\log(r))$), a contradiction since f, g belong to $\mathcal{M}_u(S)$, (resp. to $\mathcal{M}^*(D)$). Consequently, $F \equiv G$, and hence $f \equiv g$.

Proof of Theorem 8.5: Let $H \in \mathbb{K}(x)$ (resp. let $H \in \mathcal{M}_b(S)$, resp. $H \in \mathcal{M}^z(D)$) be such that the function $V(x) = \frac{H(x) \prod_{j=1}^k (f(x) - \alpha_j)}{\prod_{j=1}^k (g(x) - \alpha_j)}$ have no pole

and no zero in \mathbb{K} (resp. in S , resp. in D). Then by Lemma 3, V is a constant in \mathbb{K} (resp. V belongs to $\mathcal{M}_b(S)$, resp. V belongs to $\mathcal{M}^z(D)$).

On the other hand, since f and g have finitely many poles, in all hypotheses, we note that $N(r, f) = O(\log(r))$, $N(r, g) = O(\log(r))$. Since f and g share S_2 I.M., we can derive that each zero of $(f(x) - \beta_1)(f(x) - \beta_2)$ in \mathbb{K} (resp. in S , resp. in D) is a zero of one of the following three functions:

$$H^{-1}V - 1, \quad H^{-1}V - \prod_{j=1}^k \frac{\beta_1 - \alpha_j}{\beta_2 - \alpha_j}, \quad H^{-1}V - \prod_{j=1}^k \frac{\beta_2 - \alpha_j}{\beta_1 - \alpha_j}.$$

We can then derive that one of the following three cases $a)$, $b)$, $c)$ must occur:

$$\begin{aligned} a) \quad & \prod_{j=1}^k \frac{f(x) - \alpha_j}{g(x) - \alpha_j} \equiv 1 \\ b) \quad & \prod_{j=1}^k \frac{f(x) - \alpha_j}{g(x) - \alpha_j} \equiv \prod_{j=1}^k \frac{\beta_1 - \alpha_j}{\beta_2 - \alpha_j} \\ c) \quad & \prod_{j=1}^k \frac{f(x) - \alpha_j}{g(x) - \alpha_j} \equiv \prod_{j=1}^k \frac{\beta_2 - \alpha_j}{\beta_1 - \alpha_j}. \end{aligned}$$

Indeed, suppose that none of the cases $a)$, $b)$, $c)$ occurs. Then the following three inequalities are satisfied:

$$\begin{aligned} (i) \quad & \prod_{j=1}^k \left(\frac{f(x) - \alpha_j}{g(x) - \alpha_j} \right) \not\equiv 1 \\ (ii) \quad & \prod_{j=1}^k \left(\frac{f(x) - \alpha_j}{g(x) - \alpha_j} \right) \not\equiv \prod_{j=1}^k \left(\frac{\beta_1 - \alpha_j}{\beta_2 - \alpha_j} \right) \\ (iii) \quad & \prod_{j=1}^k \left(\frac{f(x) - \alpha_j}{g(x) - \alpha_j} \right) \not\equiv \prod_{j=1}^k \left(\frac{\beta_2 - \alpha_j}{\beta_1 - \alpha_j} \right). \end{aligned}$$

Suppose first, for simplicity, that f and g belong to $\mathcal{M}^*(\mathbb{K})$. Then, by Theorem 3.7, we have:

$$\begin{aligned} T(r, f) &\leq \overline{Z}(r, f - \beta_1) + \overline{Z}(r, f - \beta_2) + \overline{N}(r, f) + O(\log(r)) \\ &\leq \overline{Z}(r, H^{-1}V - 1) + \overline{Z}\left(r, H^{-1}V - \prod_{j=1}^k \left(\frac{\beta_1 - \alpha_j}{\beta_2 - \alpha_j} \right)\right) + \overline{Z}\left(r, H^{-1}V - \prod_{j=1}^k \left(\frac{\beta_2 - \alpha_j}{\beta_1 - \alpha_j} \right)\right) + O(\log(r)) \\ &\leq 3T(r, H^{-1}V) + O(\log(r)) \leq O(\log(r)) \end{aligned}$$

Consequently, $T(r, f) = O(\log(r))$ i.e. $f \in \mathbb{K}(x)$, a contradiction. Similarly, if f and g belong to $\mathcal{M}_u(S)$ or to $\mathcal{M}^*(D)$, following the same calculation, we obtain $f \in \mathcal{M}_b(S)$, $f \in \mathcal{M}^z(D)$, the same contradiction. Therefore one of the cases $a)$, $b)$, $c)$ must occur. We will examine them successively.

Suppose $a)$ is satisfied. Since f and g share S_2 I.M., thanks to Hypothesis (\mathcal{H}) , we can see that $f(x) = \beta_1$ if and only if $g(x) = \beta_1$ and $f(x) = \beta_2$ if and only if $g(x) = \beta_2$. Consequently, f and g share both β_1 and β_2 I.M. Then, by Lemma 1 and Relation $a)$, this implies that f and g share β_1 C.M. and that they share β_2 C.M. Moreover, relation $a)$ shows that each pole of f is a pole of g of same order. Then by Theorem 8.4 we have $f \equiv g$.

Suppose case $b)$ is satisfied. Since f and g share S_2 I.M., thanks to Hypothesis (\mathcal{H}) , we can check again that $f(x) = \beta_1$ if and only if $g(x) = \beta_2$ and similarly, $f(x) = \beta_2$ if and only if $g(x) = \beta_1$. Next, by Lemmas 2, 4, 5, f takes every value infinitely many times. Hence there exists $u \in \mathbb{K}$ (resp. $u \in S$, resp. $u \in D$) such that $f(u) = \beta_1$ and hence $g(u) = \beta_2$. But then, by relation $b)$, we have

$$\left(\prod_{j=1}^k (\beta_1 - \alpha_j) \right)^2 = \left(\prod_{j=1}^k (\beta_2 - \alpha_j) \right)^2$$

a contradiction.

Suppose finally that case $c)$ occurs: this is exactly symmetric to case $b)$ by permuting β_1 and β_2 . Consequently, case $a)$ must occur and therefore $f = g$. That finishes the proof of Theorem 8.5.

Proof of Theorem 8.6. Let $F(x) = \prod_{j=1}^k (f(x) - \alpha_j)$ and $G(x) = \prod_{j=1}^k (g(x) - \alpha_j)$,

let $\gamma_1 = \prod_{j=1}^k (\beta_1 - \alpha_j)$ and $\gamma_2 = \prod_{j=1}^k (\beta_2 - \alpha_j)$. By hypotheses of Theorem 8.6, f and g share the set $\{\alpha_1, \dots, \alpha_k\}$ I.M. and the set $\{\beta_1, \beta_2\}$ C.M. hence F and G share 0 I.M. and share the set $\{\gamma_1, \gamma_2\}$ C.M. Moreover, by hypothesis, we have $\gamma_1 \gamma_2 \neq 0$ and $\gamma_1^2 \neq \gamma_2^2$. We notice that $\gamma_1 + \gamma_2 \neq 0$, hence when F has a zero, since G also has a zero, $F(x) + G(x)$ cannot be equal to $\gamma_1 + \gamma_2$. Consequently,

$$(1) \quad F(x) + G(x) \neq \gamma_1 + \gamma_2.$$

We will show that F and G share 0 C.M. The proof is obviously trivial when $F \equiv G$ and the case when F or G is constant is excluded by the hypotheses $f \in \mathcal{M}^*(\mathbb{K})$ and $g \in \mathcal{M}^*(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(S)$ and $g \in \mathcal{M}_u(S)$, resp. $f \in \mathcal{M}^*(D)$ and $g \in \mathcal{M}^*(D)$). Thus, in the following, we assume that neither F nor G are constant and that $F \neq G$.

Since f and g have finitely many poles in \mathbb{K} (resp. in S , resp. in D), and since F and G share $\{\gamma_1, \gamma_2\}$ C.M., it is easily seen that the function $\frac{(F(x) - \gamma_1)(F(x) - \gamma_2)}{(G(x) - \gamma_1)(G(x) - \gamma_2)}$ has finitely many zeros and poles which only come from the poles of f and g . Consequently, there exists a function $E \in \mathbb{K}(x)$ such that the function $W(x) = \frac{E(x)(F(x) - \gamma_1)(F(x) - \gamma_2)}{(G(x) - \gamma_1)(G(x) - \gamma_2)}$ have no zero and no pole in \mathbb{K} (resp. in S , resp. in D). Consequently, W belongs to $\mathbb{K}(x)$ (resp. to $\mathcal{M}_b(S)$, resp to $\mathcal{M}^z(D)$).

We must now introduce three auxiliary functions. Put $M(x) = (F(x) - G(x))(F(x) + G(x) - \gamma_1 - \gamma_2)$ and

$$\begin{aligned}\phi_1(x) &= M(x) \left(\frac{\gamma_1(\gamma_1 - \gamma_2)F'(x)}{F(x)(F(x) - \gamma_1)(F(x) - \gamma_2)} - \frac{\gamma_2(\gamma_2 - \gamma_1)G'(x)}{G(x)(G(x) - \gamma_1)(G(x) - \gamma_2)} \right), \\ \phi_2(x) &= M(x) \left(\frac{\gamma_2(\gamma_2 - \gamma_1)F'(x)}{F(x)(F(x) - \gamma_1)(F(x) - \gamma_2)} - \frac{\gamma_1(\gamma_1 - \gamma_2)G'(x)}{G(x)(G(x) - \gamma_1)(G(x) - \gamma_2)} \right) \\ \phi_3(x) &= M(x) \left(\frac{F'(x)}{F(x)(F(x) - \gamma_1)(F(x) - \gamma_2)} - \frac{G'(x)}{G(x)(G(x) - \gamma_1)(G(x) - \gamma_2)} \right).\end{aligned}$$

We will first prove that ϕ_1 and ϕ_2 are not identically zero. Suppose for simplicity that f and g belong to $\mathcal{M}(\mathbb{K})$.

Indeed, suppose that $\phi_1 \equiv 0$. By (1) and due to the fact that $F \not\equiv G$, we have

$$\frac{\gamma_1(\gamma_1 - \gamma_2)F'(x)}{F(x)(F(x) - \gamma_1)(F(x) - \gamma_2)} \equiv \frac{\gamma_2(\gamma_2 - \gamma_1)G'(x)}{G(x)(G(x) - \gamma_1)(G(x) - \gamma_2)}.$$

Let us remark that, by Lemma 2, F takes every value infinitely many times because F has finitely many poles. Let u be a zero of $F - \gamma_1$ of multiplicity j . Since F and G share $\{\gamma_1, \gamma_2\}$ C.M., either u is a zero of $G - \gamma_1$ of same multiplicity, or it is a zero of $G - \gamma_2$ of same multiplicity.

Suppose first that u is a zero of $G - \gamma_1$. By developing at u , we have $F(x) - \gamma_1 = a_j(x - u)^j + a_{j+1}(x - u)^{j+1}\varphi(x)$ and $G(x) - \gamma_1 = b_j(x - u)^j + b_{j+1}(x - u)^{j+1}\psi(x)$ with $\varphi, \psi \in \mathcal{M}(\mathbb{K})$, having no pole at u .

Then by computing we can check that the residue of $\frac{\gamma_1(\gamma_1 - \gamma_2)F'}{F(x)(F(x) - \gamma_1)(F(x) - \gamma_2)}$ at u is j because $\frac{\gamma_1(\gamma_1 - \gamma_2)}{F(u)(F(u) - \gamma_2)} = 1$ and this of $\frac{\gamma_2(\gamma_2 - \gamma_1)G'}{G(x)(G(x) - \gamma_1)(G(x) - \gamma_2)}$ is $-\frac{j\gamma_2}{\gamma_1}$ because $\frac{\gamma_2(\gamma_2 - \gamma_1)}{G(u)(G(u) - \gamma_2)} = -\frac{\gamma_2}{\gamma_1}$. So, we have $\frac{\gamma_2}{\gamma_1} = -1$, hence $\gamma_1^2 = \gamma_2^2$, a contradiction to the hypothesis. Therefore, u must be a zero of $G - \gamma_2$ of multiplicity j . So, by developing F and G at u we obtain again expressions of the form $F(x) - \gamma_1 = a_j(x - u)^j + a_{j+1}(x - u)^{j+1}\varphi(x)$ and $G(x) - \gamma_2 = b_j(x - u)^j + b_{j+1}(x - u)^{j+1}\psi(x)$ with $\varphi, \psi \in \mathcal{M}(\mathbb{K})$ having no pole at u , hence we can check that the residue at u of $\frac{\gamma_1(\gamma_1 - \gamma_2)F'}{F(x)(F(x) - \gamma_1)(F(x) - \gamma_2)}$ is $-\frac{j\gamma_1}{\gamma_2}$ and this of $\frac{\gamma_2(\gamma_2 - \gamma_1)G'}{G(x)(G(x) - \gamma_1)(G(x) - \gamma_2)}$ is $-\frac{j\gamma_2}{\gamma_1}$, hence we obtain again $\gamma_1^2 = \gamma_2^2$, the same contradiction. This finishes proving that $\phi_1 \not\equiv 0$. And similarly, we can see that $\phi_2 \not\equiv 0$.

We will now prove that $\phi_3 \equiv 0$. Indeed, suppose that $\phi_3 \not\equiv 0$. We will prove that the poles of ϕ_3 only come from the poles of F and of G . Indeed, let a be a pole of ϕ_3 and suppose that it is not a pole of F and G . Then it is a zero of F and G or a zero of $(F - \gamma_1)(F - \gamma_2)$ leading to a zero of $(G - \gamma_1)(G - \gamma_2)$.

Suppose first that a is a zero of F . Then it is also a zero of G and hence that makes a pole of order 1 or zero for

$$\frac{F'}{F(F - \gamma_1)(F - \gamma_2)} - \frac{G'}{G(G - \gamma_1)(G - \gamma_2)}.$$

But since a is also a zero for $F - G$, it is not a pole for ϕ_3 . Similarly, let b be a zero for $F - \gamma_1$. It is then a pole of order 1 or zero for

$$\frac{F'}{F(F - \gamma_1)(F - \gamma_2)} - \frac{G'}{G(G - \gamma_1)(G - \gamma_2)}.$$

But since F and G share $\{\gamma_1, \gamma_2\}$ C.M, we can check that the function

$$(F - G)(F + G - \gamma_1 - \gamma_2)$$

has a zero at b , hence ϕ_3 has no pole at b . Similarly, of course, if c is a zero of $F - \gamma_2$, then ϕ_3 has no pole at c .

This finishes proving that each pole of ϕ_3 must be a pole of f or g . Next, if f and g belongs to $\mathcal{M}(S)$ or to $\mathcal{M}(D)$ we can make the same reasoning. Thus, in each hypothesis, we have $N(r, \phi_3) \leq O(\log(r))$ (resp. $N(r, \Phi_3) \leq O(1)$, resp. $N_R(r, \phi_3) \leq O(\log(r))$).

Henceforth, we will gather the two first hypotheses: $f, g \in \mathcal{M}(\mathbb{K})$ and $f, g \in \mathcal{M}(S)$ and we will consider next the hypothesis $f, g \in \mathcal{M}(D)$.

Let us develop ϕ_3 . We have:

$$\begin{aligned} \phi_3 &= \left(\frac{(F - \gamma_1)(F - \gamma_2) - (G - \gamma_1)(G - \gamma_2)}{(F - \gamma_1)(F - \gamma_2)} \right) \frac{F'}{F} + \left(\frac{(G - \gamma_1)(G - \gamma_2) - (F - \gamma_1)(F - \gamma_2)}{(G - \gamma_1)(G - \gamma_2)} \right) \frac{G'}{G} \\ &= (1 - EW^{-1}) \frac{F'}{F} + (1 - E^{-1}W) \frac{G'}{G}. \end{aligned}$$

Therefore, by Lemma 7, we have

$$\begin{aligned} M(r, \phi_3) &\leq M(r, 1 - EW^{-1}) + M(r, \frac{F'}{F}) + M(r, 1 - E^{-1}W) + M(r, \frac{G'}{G}) \leq 2T(r, EW^{-1}) + O(1) \\ &\leq O(\log(r)) \end{aligned}$$

Now since $N(r, \phi_3) \leq O(\log(r))$, we can conclude that $T(r, \phi_3) \leq O(\log(r))$ (resp. since $N_R(r, \phi_3) \leq O(\log(r))$, we can conclude that $T_R(r, \phi_3) \leq O(\log(r))$).

Now, let us develop ϕ_1 : we have

$$\begin{aligned} \phi_1 &= \left(\frac{\gamma_1(\gamma_1 - \gamma_2)(F - \gamma_1)(F - \gamma_2) - (G - \gamma_1)(G - \gamma_2)}{(F - \gamma_1)(F - \gamma_2)} \right) \frac{F'}{F} \\ &\quad + \left(\frac{\gamma_2(\gamma_2 - \gamma_1)(G - \gamma_1)(G - \gamma_2) - (F - \gamma_1)(F - \gamma_2)}{(G - \gamma_1)(G - \gamma_2)} \right) \frac{G'}{G} \\ &= \gamma_1(\gamma_1 - \gamma_2)(1 - EW^{-1}) \frac{F'}{F} + \gamma_1(\gamma_1 - \gamma_2)(1 - E^{-1}W) \frac{G'}{G}. \end{aligned}$$

Therefore, as for ϕ_3 we have $T(r, \phi_1) \leq O(\log(r))$ and similarly, $T(r, \phi_2) \leq O(\log(r))$.

We will estimate $\bar{Z}(r, F - \gamma_1)$. Let u be a zero of $F - \gamma_1$ of multiplicity j : either it is a zero of multiplicity j of $G - \gamma_1$ or it is a zero of multiplicity j of $G - \gamma_2$ because F and G share the set $\{\gamma_1, \gamma_2\}$ C.M.

Suppose first that u is a zero of $G - \gamma_1$ of multiplicity j . As it is a zero of multiplicity j of $F - \gamma_1$, it is a zero of multiplicity at least j of $F + G + \gamma_1 + \gamma_2$.

On the othe hand, by developing $\frac{F'}{F(F - \gamma_1)(F - \gamma_2)} - \frac{G'}{G(G - \gamma_1)(G - \gamma_2)}$ at the point u , we can check that $\frac{F'}{F(F - \gamma_1)(F - \gamma_2)} - \frac{G'}{G(G - \gamma_1)(G - \gamma_2)}$ has no pole at u because u is a pole of same order for $\frac{F'}{F(F - \gamma_1)(F - \gamma_2)}$ and for

$\frac{G'}{G(G - \gamma_1)(G - \gamma_2)}$. Consequently, ϕ_3 has a zero of order at least j at u .

Suppose now that u is a zero of order j of $G - \gamma_1$. Then u is a zero of order at least j for $F + G - \gamma_1 - \gamma_2$. Consider now ϕ_1 : we can check that $\frac{\gamma_1(\gamma_1 - \gamma_2)F'}{F(F - \gamma_1)(F - \gamma_2)} - \frac{\gamma_2(\gamma_2 - \gamma_1)G'}{G(G - \gamma_1)(G - \gamma_2)}$ has no pole at u and hence, as previously, u is a zero of order at least j for ϕ_1 .

Consequently, if f, g belongs to $\mathcal{M}^*(\mathbb{K})$ or to $\mathcal{M}_u(S)$, we have

$$(2) \quad \overline{Z}(r, F - \gamma_1) \leq Z(r, \phi_1) + Z(r, \phi_3) \leq O(\log(r)).$$

Similarly and symmetrically, we obtain

$$(3) \quad \overline{Z}(r, F - \gamma_2) \leq Z(r, \phi_2) + Z(r, \phi_3) \leq O(\log(r)).$$

Consequently, by (2) and (3), and by the second Main Theorem, we have

$$T(r, F) \leq \overline{Z}(r, F - \gamma_1) + \overline{Z}(r, F - \gamma_2) + \overline{N}(r, F) + O(\log(r)) \leq O(\log(r))$$

and therefore F belongs to $\mathbb{K}(x)$, a contradiction. Thus the hypothesis made above: $\phi_3 \neq 0$ is wrong.

And similarly, if $\mathcal{M}^*(D)$, by the same reasoning, we obtain $\overline{Z}_R(r, F - \gamma_1) \leq Z_R(r, \phi_1) + Z_R(r, \phi_3) \leq O(\log(r))$, and $\overline{Z}_R(r, F - \gamma_2) \leq Z_R(r, \phi_2) + Z_R(r, \phi_3) \leq O(\log(r))$, therefore

$$T_R(r, F) \leq \overline{Z}_R(r, F - \gamma_1) + \overline{Z}_R(r, F - \gamma_2) + \overline{N}_R(r, F) + O(\log(r)) \leq O(\log(r))$$

and hence F belongs to $\mathcal{M}^z(D)$, a contradiction. Thus the hypothesis made above: $\phi_3 \neq 0$ is wrong again.

Consequently, now we have

$$(4) \quad \frac{F'(x)}{F(x)(F(x) - \gamma_1)(F(x) - \gamma_2)} \equiv \frac{G'(x)}{G(x)(G(x) - \gamma_1)(G(x) - \gamma_2)}.$$

Consider a zero u of F of order j which by hypothesis is also a zero of G . The development at u shows that G also admits u as a zero of order j and therefore F and G share 0 C.M. But then by Theorem 8.5, we have $F \equiv G$. Therefore, f and g now share the set S_1 C.M., and hence, by Theorem 8.5, we can conclude $f = g$.

Proof of Theorems 8.7: Suppose that $f^n(x)f^m(ax+b)$ and $g^n(x)g^m(ax+b)$ share a rational function $Q \in \mathbb{K}(x)$ C.M. and that f and g are two distinct meromorphic functions with finitely many poles. Set $F(x) = f^n(x)f^m(ax+b)$ and $G(x) = g^n(x)g^m(ax+b)$. Thus $F - Q$ and $G - Q$ are two meromorphic functions with finitely many poles having the same zeros with respectively the same multiplicity. Therefore, by Lemma 3, $\frac{F - Q}{G - Q}$ is a rational function $h \in \mathbb{K}(x)$, $h \neq 1$. Consequently,

$$(1) \quad G(x) = hF(x) + (1 - h)Q(x).$$

and of course

$$(2) \quad F(x) = \frac{1}{h}G(x) + \frac{Q}{h}(h - 1).$$

Applying Corollary 7.2.b, by (1) and (2) we have

$$\begin{aligned} T(r, F) &\leq \bar{Z}(r, \frac{1}{h}G) + \bar{Z}(r, \frac{1}{h}G + (h-1)\frac{Q}{h}) + \bar{N}(r, F) + o(T(r, F)) \\ (3) \qquad &= \bar{Z}(r, G) + \bar{Z}(r, F) + o(T(r, F)). \end{aligned}$$

Now, by Lemma 1, we have $\bar{Z}(r, G) \leq 2Z(r, g) \forall r \geq |c|$ and $\bar{Z}(r, F) \leq 2Z(r, f) \forall r \geq |c|$, hence by Lemma 6 we obtain $T(r, F) \leq 2(T(r, g) + T(r, f)) \forall r \geq |c|$. Next, by Theorems 3.2 and 4.6, we have $T(r, F) = (n+m)T(r, f) + O(\log(r))$, hence $(n+m)T(r, f) \leq 2(T(r, g) + T(r, f)) + O(\log(r))$. Similarly, we have $(n+m)T(r, g) \leq 2(T(r, f) + T(r, g)) + O(\log(r))$. Therefore $(n+m)(T(r, f) + T(r, g)) \leq 4(T(r, f) + T(r, g)) + o(T(r, F)) + o(T(r, G)) + O(\log(r))$. But by Theorems 3.2 $o(T(r, F))$ is equivalent to $o(T(r, F))$, $o(T(r, G))$ is equivalent to $o(T(r, g))$, therefore $(n+m)(T(r, f) + T(r, g)) \leq 4(T(r, f) + T(r, g)) + o(T(r, f)) + o(T(r, g))$, which implies $n+m \leq 4$. Consequently, $F \equiv G$ since $n+m \geq 5$.

Now, suppose that Q is a nonzero constant and that f and g are entire functions. Then $\frac{F-Q}{G-Q}$ is a constant $\lambda \neq 1$ and we have

$$(4) \qquad G(x) = \lambda F(x) + Q(1 - \lambda).$$

and of course

$$(5) \qquad F(x) = \frac{1}{\lambda}G(x) + \frac{Q}{\lambda}(\lambda - 1).$$

We can apply Theorem 3.7 on $\mathcal{M}(\mathbb{K})$ and obtain

$$(6) \qquad T(r, F) \leq \bar{Z}(r, G) + \bar{Z}(r, G + \lambda - 1) - \log(r) + O(1) = \bar{Z}(r, G) + \bar{Z}(r, F) - \log(r) + O(1).$$

On the other hand, since f, g belong to $\mathcal{A}^*(K)$, by Theorems 3.2, we now have $T(r, F) = T(r, f^n) + T(r, f^m(ax+b)) = (n+m)T(r, f) + O(1)$ and similarly, $T(r, G) = (n+m)T(r, g) + O(1)$. Consequently, $(n+m)(T(r, f) + T(r, g)) \leq 4(T(r, f) + T(r, g)) - 2\log(r) + O(1)$, which implies $n+m < 4$. Consequently, if f and g belong to $\mathcal{A}^*(\mathbb{K})$, if Q is a nonzero constant and if $n+m \geq 4$, then $F \equiv G$.

So, we have proven that in each case, we have $F \equiv G$. Now, set $\phi(x) = \frac{f(x)}{g(x)}$ and $\psi(x) = \phi(ax+b)$. Then we have $\phi^n(x)\psi^m(x) \equiv 1$, hence $\frac{1}{\psi^m} \equiv \phi^n$ and hence

$$(7) \qquad T(r, \frac{1}{\psi^m}) = T(r, \phi^n).$$

But $T(r, \frac{1}{\psi}) = T(r, \psi) = T(r, \phi) \forall r \geq |c|$. On the other hand, by Theorems 3.2 $T(r, \phi^n) = nT(r, \phi)$, $T(r, \psi^m) = mT(r, \psi)$ hence by (7), since $m \neq n$, we can see that $T(r, \phi) = 0$. Therefore, ϕ is a constant t and hence $f = tg$.

Now, $F - Q$ belongs to $\mathcal{M}^*(\mathbb{K})$, has finitely many poles and, by Lemma 2, admits infinitely many zeros that are not zeros of Q . Let c be such a zero. Then

c is also a zero of $G - Q$ and hence we have $f^n(c)f^m(ac + b) = g^n(c)g^m(ac + b)$, therefore $f^n(c)f^m(ac + b) = t^n f^n(c)t^m f^m(ac + b)$ i.e. $F(c) = t^{n+m}F(c)$. But since c is not a zero of Q , we have $F(c) \neq 0$, hence $t^{n+m} = 1$, which ends the proof.

Proof of Theorems 8.8: The proof of Theorem 8.8 roughly follows that of Theorem 8.7. Suppose first that $f^n(x)f^m(ax + b)$ and $g^n(x)g^m(ax + b)$ share C.M. a function $\theta \in \mathcal{M}(S)$ (resp. $\theta \in \mathcal{M}(D)$) having finitely many zeros and poles in S (resp. in D), and are two distinct functions lying in $\mathcal{M}_u(S)$ (resp. in $\mathcal{M}^*(D)$) and have finitely many poles in S (resp. in D). Set again $F(x) = f^n(x)f^m(ax + b)$ and $G(x) = g^n(x)g^m(ax + b)$. By Lemma 4, $F - \theta$ and $G - \theta$ belong to $\mathcal{M}_u(S)$ (resp. to $\mathcal{M}^*(D)$) and are two functions having the same zeros with respectively the same multiplicity and have finitely many poles. Therefore $\frac{F - \theta}{G - \theta}$ belongs to $\mathcal{M}(S)$ (resp. to $\mathcal{M}(D)$) and has finitely many zeros and poles in S (resp. in D). Consequently, it is a function w that belongs to $\mathcal{M}_b(S)$ (resp. to $\mathcal{M}^z(D)$).

Suppose first $f, g \in \mathcal{M}(S)$. Then by Corollary 7.2.b, similarly to the proof of Theorem 8.7, we obtain

$$T(r, F) \leq \bar{Z}(r, G) + \bar{Z}(r, G + (\frac{1}{w} - 1)\theta + o(T(r, F))) (r \rightarrow R) = \bar{Z}(r, G)$$

$$(1) \quad + \bar{Z}(r, F) + o(T(r, F)) (r \rightarrow R).$$

Now, by Lemma 1, we have again $\bar{Z}(r, G) \leq 2Z(r, g) + O(1)$ and $\bar{Z}(r, F) \leq 2Z(r, f) + O(1)$, hence by Lemma 6 we obtain $T(r, F) \leq 2(T(r, g) + T(r, f)) + O(1)$ hence, by Theorems 3.2, $(n + m)T(r, g) \leq 2(T(r, g) + T(r, f)) + O(1)$ and similarly $(n + m)T(r, f) \leq 2(T(r, f) + T(r, g)) + O(1)$. On the other hand, $o(T(r, F)) = o(T(r, f))$, $o(T(r, G)) = o(T(r, g))$, therefore by (1), we obtain

$$(n + m)(T(r, f) + T(r, g)) \leq 4(T(r, f) + T(r, g)) + o(T(r, f)) + o(T(r, g))$$

and hence $n + m < 5$, which is excluded by hypotheses. Consequently, we get $F \equiv G$. Putting again $\phi(x) = \frac{f(x)}{g(x)}$ and $\psi(x) = \phi(ax + b)$, we have $\frac{1}{\psi^m} \equiv \phi^n$, therefore in S we get $T(r, \frac{1}{\psi^m}) = T(r, \phi^n)$ which implies that $T(r, \phi) = 0$ since $n \neq m$. Thus ϕ is a constant t again and hence, since θ has finitely many zeros in S , by Lemma 4 we can conclude as in Theorem 8.7.

Similarly, if $f, g \in \mathcal{M}(D)$ we can make the same reasoning thanks to Lemma 5.

Proof of Theorem 8.9: Suppose first that f and w belong to $\mathcal{M}^*(\mathbb{K})$ and set $h(x) = f(ax + b)$, $F(x) = f(x)^n h^m(x)$. By Corollary 7.2.a we have

$$(1) \quad T(r, F) \leq \bar{Z}(r, F) + \bar{Z}(r, F - w) + \bar{N}(r, F) + o(T(r, F)) (r \rightarrow +\infty).$$

Clearly,

$$(2) \quad \bar{Z}(r, F) \leq Z(r, f) + Z(r, h).$$

and

$$(3) \quad \overline{N}(r, F) \leq N(r, f) + N(r, h) + N(r, w) = N(r, f) + N(r, h) + o(T(r, f)) \quad (r \rightarrow +\infty).$$

On the other hand, by Theorems 3.2, $T(r, f^n h^m) \geq |T(r, f^n) - T(r, h^m)|_\infty$ and by Lemma 6, for every $r \geq |b|$ we have $T(r, h) = T(r, f)$, $Z(r, f) = Z(r, h)$, $N(r, f) = N(r, h)$, hence

$T(r, F) \geq |n - m|_\infty T(r, f)$. Moreover, $o(T(r, F)) = o(T(r, f))$. Therefore, by (1), (2) and (3) we obtain $(|n - m|_\infty - 4)T(r, f) \leq \overline{Z}(r, F - w) + o(T(r, f)) \quad (r \rightarrow +\infty)$ which proves the claim whenever $|n - m|_\infty \geq 5$.

Suppose now that f and w belong to $\mathcal{M}(S)$. Keeping the same notations, we can get the same relations (1), (2), (3) and next $(|n - m|_\infty - 4)T(r, f) \leq \overline{Z}(r, F - w) + o(T(r, f)) \quad (r \rightarrow R)$, which proves again the claim whenever $|n - m|_\infty \geq 5$.

Finally, suppose that f and w belong to $\mathcal{M}(D)$ and keep the same notations. By Corollary 7.2.a now we have

$$(4) \quad T_R(r, F) \leq \overline{Z}_R(r, F) + \overline{Z}_R(r, F - w) + \overline{N}_R(r, F) + o(T(r, F)).$$

Next, using Theorem 4.6 instead of 3.2, we obtain here: $(|n - m|_\infty - 4)T_R(r, f) \leq \overline{Z}_R(r, F - w) + o(T_R(r, f)) \quad (r \rightarrow +\infty)$. Thus, if $|n - m|_\infty \geq 5$ we have $T_R(r, f) \leq \overline{Z}_R(r, F - w) + o(T_R(r, f)) \quad (r \rightarrow +\infty)$ which shows again that $F - w$ has infinitely many zeros .

Proof of Theorem 8.10: The beginning of the proof is similar to that of Theorem 8.9. Suppose first that $f \in \mathcal{M}^*(\mathbb{K})$ and set $h(x) = f(ax + b)$, $F(x) = f(x)^n h^m(x)$. Suppose that F admits 4 distinct perfectly branched values a_1, a_2, a_3, a_4 . By Theorem 3.7, we have

$$3T(r, F) \leq \sum_{j=1}^4 \overline{Z}(r, F - a_j) + \overline{N}(r, F) - \log(r) + O(1) \quad (r \rightarrow +\infty),$$

hence

$$(1) \quad 3T(r, F) \leq \left(\frac{1}{2}\right) \sum_{j=1}^4 Z(r, F - a_j) + N(r, f) + N(r, h) - \log(r) + O(1) \quad (r \rightarrow +\infty),$$

and hence

$$3(|n - m|_\infty)T(r, f) \leq \left(\frac{1}{2}\right)4(n + m)T(r, f) + 2T(r, f) + O(\log(r)) \quad (r \rightarrow +\infty),$$

therefore $3|n - m|_\infty \leq 2(n + m) + 2 + \lambda(r)$, with $\lim_{r \rightarrow +\infty} \lambda(r) = 0$. Consequently, if $3|n - m|_\infty > 2(n + m + 1)$, the function F does not admit 4 perfectly branched values. Moreover, if F admits 4 totally branched values a_1, a_2, a_3, a_4 , then by (1) now we can get

$$3(|n - m|_\infty)T(r, f) \leq \left(\frac{1}{2}\right)4(n + m)T(r, f) + 2T(r, f) - \log(r) + O(1) \quad (r \rightarrow +\infty),$$

therefore $3|n - m|_\infty < 2(n + m) + 2$. Consequently, if $3|n - m|_\infty \geq 2(n + m + 1)$, the function F does not admit 4 totally branched values.

Suppose now that F admits 3 distinct perfectly branched values a_1, a_2, a_3 . By Theorem 3.7, we have

$$(2) \quad 2T(r, F) \leq \sum_{j=1}^3 \bar{Z}(r, F - a_j) + \bar{N}(r, F) - \log(r) + O(1) \quad (r \rightarrow +\infty),$$

and hence $2(|n - m|_\infty)T(r, f) \leq \left(\frac{1}{2}\right)3(n + m)T(r, f) + 2T(r, f) + \lambda(r)$ with $\lim_{r \rightarrow +\infty} \lambda(r) = 0$, ($r \rightarrow +\infty$), therefore $2|n - m|_\infty \leq \left(\frac{3}{2}\right)(n + m) + 2$. Consequently, if $4|n - m|_\infty > 3(n + m) + 4$, the function F does not admit 3 perfectly branched values.

Moreover if F admits 3 totally branched values a_1, a_2, a_3 , then by (2) now we can get

$$2(|n - m|_\infty)T(r, f) \leq \left(\frac{1}{2}\right)3(n + m)T(r, f) + 2T(r, f) - \log(r) + O(1) \quad (r \rightarrow +\infty),$$

therefore $4|n - m|_\infty < 3(n + m) + 4$. Consequently, if $4|n - m|_\infty \geq 3(n + m) + 4$, the function F does not admit 3 totally branched values.

Finally, if f belongs to $\mathcal{M}_u(S)$ or to $\mathcal{M}^*(D)$, using in each case Theorem 3.7 or 4.10, we can do the same reasoning for perfectly branched values. But Theorem 7.2 here does not show the term $-\log(r) + O(1)$ which is essential when we consider totally branched values. This is why we must replace \geq by $>$ in each statement for functions in $\mathcal{M}_u(S)$ or in $\mathcal{M}^*(D)$ concerning totally branched values.

Acknowledgements We are grateful to the referee for pointing out to us many misprints.

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