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p-adic meromorphic functions $f'P'(f),\ g'P'(g)$ sharing a small function, ignoring multiplicity

Kamal Boussaf and Alain Escassut

Abstract. Let $\mathbb{K}$ be a complete algebraically closed p-adic field of characteristic zero. Let $f,\ g$ be two transcendental meromorphic functions in the whole field $\mathbb{K}$ or meromorphic functions in an open disk that are not quotients of bounded analytic functions. Let P be a polynomial of uniqueness for meromorphic functions in $\mathbb{K}$ or in an open disk and let α be a small meromorphic function with regard to f and g. If $f'P'(f)$ and $g'P'(g)$ share α ignoring multiplicity orders, then we show that $f = g$ provided that the multiplicity order of zeros of P' satisfies certain inequalities. If α is a Moebius function or a non-zero constant, we can obtain more general results on P and if f is an analytic funtion in $\mathbb{K}$ or in the disk, we also obtain more precise results. All results follow previous ones obtained for similar meromorphic functions sharing a small function, counting multiplicity. That comes after similar results obtained by Buy Thi Kieu Oanh and Ngo Thi Thu Thuy for complex functions and results obtained by the present authors with Jacqueline Ojeda, counting multiplicity.

1. Introduction and Main Results

Let $f,\ g$ be two meromorphic functions in a p-adic field. Here we study polynomials P such that, when $f'P'(f)$ and $g'P'(g)$ share a small function α, then $f = g$. Problems of uniqueness on meromorphic functions were examined first in $\mathbb{C}$ [3], [11], [14], [16], [17], [18], [23], [24], [25] and next in a p-adic field [1], [2], [5], [7], [8], [12], [13], [19], [20], [21], [22]. After examining problems of the form $P(f) = P(g)$, several studies were made on the equality $f'P'(f) = g'P'(g)$, or value sharing questions: if $f'P'(f)$ and $g'P'(g)$ share a value, or a small function *counting multiplicity*, do we have $f = g$? Here we will try to generalize results previously obtained in [2] by supposing that $f'P'(f)$ and $g'P'(g)$ share a value, or a small function *ignoring multiplicity*. That is similar to a work by Buy Thi Kieu Oanh and Ngo Thi Thu concerning complex meromorphic functions. But in the present paper as in [2] we can also examine the situation for meromorphic functions inside an open disk.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by $|\ .\ |$. We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$, by $\mathcal{M}(\mathbb{K})$ the

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field of meromorphic functions in $\mathbb{K}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions.

Let $a \in \mathbb{K}$ and $R \in]0, +\infty[$. We denote by $d(a, R)$ the closed disk $\{x \in \mathbb{K} : |x - a| \leq R\}$ and by $d(a, R^-)$ the "open" disk $\{x \in \mathbb{K} : |x - a| < R\}$. We denote by $\mathcal{A}(d(a, R^-))$ the set of analytic functions in $d(a, R^-)$, i.e. the $\mathbb{K}$-algebra of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converging in $d(a, R^-)$ and by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the $\mathbb{K}$ - subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n\in\mathbb{N}} |a_n|R^n < +\infty$. And we denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

The problem of value sharing a small function by functions of the form $f'P'(f)$, counting multiplicity, was examined first when P was just of the form x^n [17], [19]. Next, it was examined when P was a polynomial such that P' had exactly two distinct zeros [16], [18], [21], both in complex analysis and in p-adic analysis. In [16], [18] the functions were meromorphic on $\mathbb{C}$, with a small function that was a constant or the identity. In [21], the problem was considered for analytic functions in the field $\mathbb{K}$: on the one hand for entire functions and on the other hand for unbounded analytic functions in an open disk.

Actually solving a value sharing problem involving $f'P'(f),\ g'P'(g)$ requires to know polynomials of uniqueness P for meromorphic functions.

In [21] its author studied several problems of uniqueness and particularly the following:

Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental $\big($resp. Let $f, g \in \mathcal{A}_u(d(0, R^-))\big)$ and $\alpha \in \mathcal{A}(\mathbb{K})$ $\big($resp. $\alpha \in \mathcal{A}_u(d(0, R^-))\big)$ be a small function, such that $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share α, counting multiplicity, with $n, k \in \mathbb{N}$ and $a \in \mathbb{K} \setminus \{0\}$. Do we have $f = g$?

Here we consider functions $f,\ g \in \mathcal{M}(\mathbb{K})$ or $f,\ g \in \mathcal{M}(d(a, R^-))$ and polynomials of uniqueness P: we must only assume certain hypotheses on the multiplicity order of the zeros of P'. The method for the various theorems we will show is the following: assuming that $f'P'(f)$ and $g'P'(g)$ share a small function, we first prove that $f'P'(f) = g'P'(g)$. Next, we derive $P(f) = P(g)$. And then, when P is a polynomial of uniqueness for the functions we consider, we can conclude $f = g$. Similar reasonings were made in [2], where the major hypothesis was that $f'P'(f)$ and $g'P'(g)$ share a small function *counting multiplicity*. Here we erase that hypothesis and only suppose that $f'P'(f)$ and $g'P'(g)$ shared a small function *ignoring multiplicity*.

Now, in order to define small functions, we have to briefly recall definitions of the classical Nevanlinna theory in the field $\mathbb{K}$ and a few specific properties of ultrametric analytic or meromorphic functions.

Let log be a real logarithm function of base $b > 1$ and let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. $f \in \mathcal{M}(d(0, R^-))\big)$ having no zero and no pole at 0. Let $r \in]0, +\infty[$ $\big($resp. $r \in]0, R[\big)$ and let $\gamma \in d(0, r)$. If f has a zero of order n at γ, we put $\omega_\gamma(f) = n$. If f has a pole of order n at γ, we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$.

We denote by $Z(r, f)$ the *counting function of zeros of f* in $d(0, r)$, counting multiplicity, i.e. we set

$$Z(r, f) = \sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r} \omega_\gamma(f)(\log r - \log|\gamma|).$$

Similarly, we denote by $\overline{Z}(r,f)$ the counting function of zeros of f in $d(0,r)$, ignoring multiplicity, and set

$$\overline{Z}(r,f)=\sum_{\omega_\gamma(f)>0,\ |\gamma|\leq r}(\log r-\log|\gamma|).$$

In the same way, we set $N(r,f)=Z\Big(r,\frac{1}{f}\Big)$ $\Big(\text{resp. } \overline{N}(r,f)=\overline{Z}\Big(r,\frac{1}{f}\Big)\Big)$ to denote the *counting function of poles of* f in $d(0,r)$, counting multiplicity (resp. ignoring multiplicity).

For $f\in\mathcal{M}(d(0,R^-))$ having no zero and no pole at 0, the *Nevanlinna function* is defined by $T(r,f)=\max\big\{Z(r,f)+\log|f(0)|,N(r,f)\big\}$.

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f\in\mathcal{M}(\mathbb{K})$ $\big(\text{resp. let } f\in\mathcal{M}(d(0,R^-))\big)$ such that $f(0)\neq 0,\infty$. A function $\alpha\in\mathcal{M}(\mathbb{K})$ $\big(\text{resp. } \alpha\in\mathcal{M}(d(0,R^-))\big)$ having no zero and no pole at 0 is called *a small function with respect to* f, if it satisfies $\lim\limits_{r\to+\infty}\dfrac{T(r,\alpha)}{T(r,f)}=0$ $\Big(\text{resp. } \lim\limits_{r\to R^-}\dfrac{T(r,\alpha)}{T(r,f)}=0\Big)$.

If 0 is a zero or a pole of f or α, we can make a change of variable such that the new origin is not a zero or a pole for both f and α. Thus it is easily seen that the last relation does not really depend on the origin.

We denote by $\mathcal{M}_f(\mathbb{K})$ $\big(\text{resp. } \mathcal{M}_f(d(0,R^-))\big)$ the set of small meromorphic functions with respect to f in $\mathbb{K}$ $\big(\text{resp. in } d(0,R^-)\big)$.

Remark 1. *Thanks to classical properties of the Nevanlinna function $T(r,f)$ with respect to the operations in a field of meromorphic functions, such as $T(r,f+g)\leq T(r,f)+T(r,g)$ and $T(r,fg)\leq T(r,f)+T(r,g)$, for $f,g\in\mathcal{M}(\mathbb{K})$ and $r>0$, it is easily proved that $\mathcal{M}_f(\mathbb{K})$ $\big($resp. $\mathcal{M}_f(d(0,R^-))\big)$ is a subfield of $\mathcal{M}(\mathbb{K})$ $\big($resp. $\mathcal{M}(d(0,R^-))\big)$ and that $\mathcal{M}(\mathbb{K})$ $\big($resp. $\mathcal{M}(d(0,R))\big)$ is a transcendental extension of $\mathcal{M}_f(\mathbb{K})$ $\big($resp. of $\mathcal{M}_f(d(0,R^-))\big)$ [9].*

Let us remember the following definition.

Definition. Let $f,g,\alpha\in\mathcal{M}(\mathbb{K})$ $\big(\text{resp. let } f,g,\alpha\in\mathcal{M}(d(0,R^-))\big)$. We say that f and g *share the function* α *I.M.*, if $f-\alpha$ and $g-\alpha$ have the same zeros in $\mathbb{K}$ $\big(\text{resp. in } d(0,R^-)\big)$. Similarly, we say f and g *share the function* α *C.M.*, if $f-\alpha$ and $g-\alpha$ have the same zeros in $\mathbb{K}$ $\big($ resp. in $d(0,R^-)\big)$ with the same multiplicity, in $\mathbb{K}$ (resp. in $d(0,R^-)$).

Recall that a polynomial $P\in\mathbb{K}[x]$ is called *a polynomial of uniqueness* for a class of functions $\mathcal{F}$ if for any two functions $f,\ g\in\mathcal{F}$ the property $P(f)=P(g)$ implies $f=g$.

The definition of polynomials of uniqueness was introduced in [17] by P. Li and C. C. Yang and was studied in many papers [10], [11] for complex functions and [1], [5], [7], [12], [13], [15], [22], for p-adic functions.

Actually, in a p-adic field, we can obtain various results, not only for functions defined in the whole field $\mathbb{K}$ but also for functions defined inside an open disk because the p-adic Nevanlinna Theory works inside a disk, for functions of $\mathcal{M}_u(d(0,R^-))$.

Let us recall Theorem A [7], [9], [22]:

Theorem A. *Let $P\in\mathbb{K}[x]$ be such that P' has exactly two distinct zeros a_1 of order k_1 and a_2 of order k_2. Then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. Moreover, if $\min\{k_1,k_2\}\geq 2$, then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.*

Theorem A was first proved in [22] with the addiditional hypothesis $P(a_1) \neq P(a_2)$. Actually this hypothesis is useless because, as showed in Lemma 10 [7], (see also [9]) the equality $P(a_1) = P(a_2)$ is impossible since P' only has two distinct zeros.

Notation: Let $P \in \mathbb{K}[x] \setminus \mathbb{K}$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(b)$ for every zero b of P' other than c. We denote by $\Phi(P)$ the cardinal of $\Xi(P)$.

REMARK 2. *If* $\deg(P) = q$ *then* $\Phi(P) \leq q - 1$.

From [7] we have the following results:

Theorem B. *Let* $d(a, R^-)$ *be an open disk in* $\mathbb{K}$ *and* $P \in \mathbb{K}[x]$. *If* $\Phi(P) \geq 2$ *then* P *is a polynomial of uniqueness for* $\mathcal{A}(\mathbb{K})$. *If* $\Phi(P) \geq 3$ *then* P *is a polynomial of uniqueness for both* $\mathcal{A}_u(d(a, R^-))$ *and* $\mathcal{M}(\mathbb{K})$. *If* $\Phi(P) \geq 4$ *then* P *is a polynomial of uniqueness for* $\mathcal{M}_u(d(a, R^-))$.

And from [21] we have:

Theorem C. *Let* $P \in \mathbb{K}[x]$ *be of degree* $n \geq 6$ *be such that* P' *only has two distinct zeros, one of them being of order* 2. *Then* P *is a polynomial of uniqueness for* $\mathcal{M}_u(d(0, R^-))$.

We can now state our main theorems.

THEOREM 1. *Let* P *be a polynomial of uniqueness for* $\mathcal{M}(\mathbb{K})$, *let* $P' = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ *and let* $k = \sum_{i=2}^{l} k_i$. *Suppose* P *satisfies the following conditions:*

$n \geq 18 + \max(0, 10 - k_2) + \sum_{i=3}^{l} \max(0, 7 - k_i)$,

$n \geq k + 2$,

if $l = 2$, *then* $n \neq 2k,\ 2k + 1,\ 3k + 1$,

if $l = 3$, *then* $n \neq 2k + 1,\ 3k_i - k\ \forall i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ *be non-identically zero. If* $f'P'(f)$ *and* $g'P'(g)$ *share* α *I.M., then* $f = g$.

By Theorem B, we have Corollary 1.1:

Corollary 1.1 *Let* $P \in \mathbb{K}[x]$ *satisfy* $\Phi(P) \geq 3$, *let* $P' = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ *and let* $k = \sum_{i=2}^{l} k_i$. *Suppose* P *satisfies the following conditions:*

$n \geq 18 + \max(0, 10 - k_2) + \sum_{i=3}^{l} \max(0, 7 - k_i)$,

$n \geq k + 2$,

if $l = 3$, *then* $n \neq 2k + 1,\ 3k_i - k\ \forall i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ *be transcendental and let* $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ *be non-identically zero. If* $f'P'(f)$ *and* $g'P'(g)$ *share* α *I.M., then* $f = g$.

Example: Let $P'(x) = x^{19}(x - 1)^{10}(x - 2)^7$. Consider the polynomial function $P(t)$ defined in $[0, +\infty[$. Clearly, $P'(t) < 0\ \forall t \in]0, 1[$ hence $P(1) < 0$. At the point 1, P' has a zero of even order, hence P remains decreasing in $]1, 2[$ hence $P(2) < P(1) < P(0)$. Therefore, $\Phi(P) = 3$ and hence,

P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$. Now, consider two functions $f,\ g \in \mathcal{M}(\mathbb{K})$ and a small function α. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

And by Theorem A we also have Corollary 1.2.

Corollary 1.2 *Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $k \geq 2$ and*
$n \geq 18 + \max(0, 10-k)$,
$n \geq k+2$,
$n \neq 2k,\ 2k+1,\ 3k+1$.
Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(x) = x^{18}(x-1)^{10}$. By Theorem A, P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$. Now, consider two functions $f,\ g \in \mathcal{M}(\mathbb{K})$ and a small function $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

THEOREM 2. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let*

$$P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$$

with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:

$n \geq 17 + \max(0, 10-k_2) + \sum_{i=3}^{l} \max(0, 7-k_i)$,
$n \geq k+2$,
if $l = 2$, then $n \neq 2k,\ 2k+1,\ 3k+1$,
if $l = 3$, then $n \neq 2k+1,\ 3k_i - k\ \forall i = 2, 3$.
Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

By Theorem B, we have Corollary 2.1.

Corollary 2.1 *Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq 17 + \max(0, 10-k_2) + \sum_{i=3}^{l} \max(0, 7-k_i)$,
$n \geq k+2$,
if $l = 3$, then $n \neq 2k+1,\ 3k_i - k\ \forall i = 2, 3$.
Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(x) = x^{18}(x-1)^{10}(x-2)^6$. Consider the polynomial function $P(t)$ defined in $[0, +\infty[$. Clearly, $P'(t) > 0\ \forall t \in]0, 1[\cup]1, 2[$ hence $P(2) > P(1) > P(0)$. Therefore, $\Phi(P) = 3$ and hence, P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$. On the other hand, P satisfies the hypotheses of Corollary 2.1. Now, consider two functions $f,\ g \in \mathcal{M}(\mathbb{K})$ and a Moebius function α. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

And by Theorem A, we have Corollary 2.2.

Corollary 2.2 *Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $\min(k,n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:*

$n \geq 17 + \max(0, 10-k)$,

$n \geq k+2$,

$n \neq 2k,\ 2k+1,\ 3k+1$,

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(x) = x^{17}(x-1)^{10}$. By Theorem A, P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ and it satisfies all hypotheses of Corollary 2.2. Now, consider two functions $f,\ g \in \mathcal{M}(\mathbb{K})$ and a Moebius function α. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

THEOREM 3. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let*

$$P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$$

with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:

$n \geq k+2$,

$n \geq 17 + \max(0, 10-k_2) + \sum_{i=3}^{l} \max(0, 7-k_i)$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

By Theorem B, we have Corollary 3.1.

Corollary 3.1 *Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1},\ 2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq k+2$,

$n \geq 17 + \max(0, 10-k_2) + \sum_{i=3}^{l} \max(0, 7-k_i)$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(x) = x^{23}(x-1)^6(x-2)^5$. Consider the polynomial function $P(t)$ defined in $[0, +\infty[$. Clearly, $P'(t) < 0\ \forall t \in]0,1[\cup]1,2[$ hence $P(2) < P(1) < P(0)$. Therefore, $\Phi(P) = 3$ and hence, P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$. On the other hand, P satisfies the hypotheses of Corollary 3.1. Now, consider two functions $f,\ g \in \mathcal{M}(\mathbb{K})$ and a constant α. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$. Here we notice that $n = 2k+1 = 23$ but this hypothesis is not excluded in Theorem 3.

And by Theorem A, we have Corollary 3.2

Corollary 3.2 *Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $k \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:*

$n \geq 17 + \max(0, 10-k)$,

$n \geq k+2$,

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(x) = x^{18}(x-1)^9$. By Theorem A, P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ and it satisfies all hypotheses of Corollary 3.2. Now, consider two functions $f,\ g \in \mathcal{M}(\mathbb{K})$ and a constant $\alpha \neq 0$. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$. Here we notice that $n = 2k$ by this hypothesis is not excluded in Corollary 3.2.

THEOREM 4. *Let $a \in \mathbb{K}$ and $R > 0$. Let P be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$ and let*

$$P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$$

with $b \in \mathbb{K}^*$, $l \geq 2,\ k_i \geq k_{i+1},\ \ 2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:

$n \geq 18 + \max(0, 10 - k_2) + \sum_{i=3}^{l} \max(0, 7 - k_i)$,

$n \geq k + 3$,

if $l = 2$, then $n \neq 2k,\ 2k+1,\ 3k+1$,

if $l = 3$, then $n \neq 2k+1,\ 3k_i - k\ \forall i = 2, 3$.

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

By Theorem B we can state Corollary 4.1.

Corollary 4.1 *Let $a \in \mathbb{K}$ and $R > 0$. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 4$, let $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 4$, $k_i \geq k_{i+1},\ \ 2 \leq i \leq l-1$ and let $k = \sum_{i=2}^{l} k_i$. Suppose P satisfies the following conditions:*

$n \geq 18 + \max(0, 10 - k_2) + \sum_{i=3}^{l} \max(0, 7 - k_i)$,

$n \geq k + 3$,

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(x) = x^{23}(x-1)^{10}(x-2)^4(x-3)^5$. Consider the polynomial function $P(t)$ defined in $[0, +\infty[$. Clearly, $P'(t) < 0\ \forall t \in]0, 1[\cup]1, 2[\cup]2, 3[$ hence $P(3) < P(2) < P(1) < P(0)$. Therefore, $\Phi(P) = 4$ and hence, P is a polynomial of uniqueness for $\mathcal{M}(d(a, R^-))$. On the other hand, we check that P satisfies the hypotheses of Corollary 4.1. Now, consider two functions $f,\ g \in \mathcal{M}_u(d(a, R^-))$ and a constant α. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

And by Theorem C we have Corollary 4.2:

Corollary 4.2 *Let $a \in \mathbb{K}$ and $R > 0$. Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^2$ with $b \in \mathbb{K}^*$. Suppose P satisfies*

$n \geq 18 + \max(0, 10 - k)$.

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_f(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(X) = x^{26}(x-1)^2$. By Theorem A, P is a polynomial of uniqueness for $\mathcal{M}(d(a,R^-))$ and it satisfies all hypotheses of Corollary 4.2. Now, consider two functions $f,\ g \in \mathcal{M}(d(a,R^-))$ and a function $\alpha \in \mathcal{M}_f(d(a,R^-)) \cap \mathcal{M}_g(d(a,R^-))$. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

THEOREM 5. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form $b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$, satisfying $n \geq 10+4l$.*
Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

By Theorem B, we have Corollary 5.1:

Corollary 5.1 *Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$ and be such that P' is of the form $b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$ satisfying:*
$n \geq 10+4l$.
Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(x) = x^{22}(x^2-1)$. Consider the polynomial function $P(t)$ defined in $[-1,+1]$. Clearly, $P'(t) < 0\ \forall t \in]-1,0[\cup]0,1[$ hence $P(-1) < P(0) < P(1)$. Therefore, $\Phi(P) = 3$ and hence, P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$. On the other hand, we check that P satisfies the hypotheses of Corollary 5.1. Now, consider two functions $f,\ g \in \mathcal{M}(\mathbb{K})$ and a function $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

THEOREM 6. *Let $a \in \mathbb{K}$ and $R > 0$. Let P be a polynomial of uniqueness for $\mathcal{M}_u(d(a,R^-))$ such that P' is of the form $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$ satisfying:*
$n \geq 10+4l$.
Let $f, g \in \mathcal{M}_u(d(a,R^-))$ and let $\alpha \in \mathcal{M}_f(d(a,R^-)) \cap \mathcal{M}_g(d(a,R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

By Theorem B, we have Corollary 6.1:

Corollary 6.1 *Let $a \in \mathbb{K}$ and $R > 0$. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 4$ and be such that P' is of the form $P' = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)$ with $l \geq 4$, $b \in \mathbb{K}^*$ and $n \geq 10+4l$.*

Let $f, g \in \mathcal{M}_u(d(a,R^-))$ and let $\alpha \in \mathcal{M}_f(d(a,R^-)) \cap \mathcal{M}_g(d(a,R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P(x) = \dfrac{x^{30}}{30} - \dfrac{2x^{29}}{29} - \dfrac{x^{28}}{28} + \dfrac{2x^{27}}{27}$. Then $P'(x) = x^{29} - 2x^{28} - x^{27} + 2x^{26} = x^{26}(x-1)(x+1)(x-2)$. We check that:
$P(0) = 0$,
$P(1) = \dfrac{1}{30} - \dfrac{2}{29} - \dfrac{1}{28} + \dfrac{2}{27} \neq 0$,
$P(-1) = \dfrac{1}{30} + \dfrac{2}{29} - \dfrac{1}{28} - \dfrac{2}{27} \neq 0, P(1)$, and $P(2) = \dfrac{2^{30}}{30} - \dfrac{2^{30}}{29} - \dfrac{2^{28}}{28} + \dfrac{2^{28}}{27} \neq 0, P(1), P(-1)$.

Then $\Phi(P) = 4$. So, P is a polynomial of uniqueness for both $\mathcal{M}(\mathbb{K})$ and $\mathcal{M}(d(0, R^-))$. Moreover, we have $n = 26, l = 4$, hence we can apply Corollaries 5.1 and 6.1.

Given $f,\ g \in \mathcal{M}(\mathbb{K})$ transcendental or $f,\ g \in \mathcal{M}_u(d(0, R^-))$ such that $f'P'(f)$ and $g'P'(g)$ share a small function α I.M., we have $f = g$.

THEOREM 7. *Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form $P' = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$ satisfying $n \geq 9 + 4l$.*

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function or a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α I.M., then $f = g$.

Example: Let $P'(x) = x^{21}(x^2 - 1)$. Then as in the previous example, P is a uniqueness polynomial for $\mathcal{M}(\mathbb{K})$. Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}(\mathbb{K})$ be a Moebius function or a non-zero constant such that $f'P'(f),\ g'P'(g)$ share α I.M. Then, $f = g$.

THEOREM 8. *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f - a)$ and $g'g^n(g - a)$ share the function α I.M. and if $n \geq 18$, then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{K})$ such that $f = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1} - 1}{h^{n+2} - 1}\Big)h$ and $g = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1} - 1}{h^{n+2} - 1}\Big)$. Moreover, if α is a constant or a Moebius function, then the conclusion holds whenever $n \geq 17$.*

Inside an open disk, we have a version similar to the general case in the whole field.

THEOREM 9. *Let $f,\ g \in \mathcal{M}_u(d(0, R^-))$, and let $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f - a)$ and $g'g^n(g - a)$ share the function α C.M. and $n \geq 18$, then either $f = g$ or there exists $h \in \mathcal{M}(d(0, R^-))$ such that $f = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1} - 1}{h^{n+2} - 1}\Big)h$ and $g = \frac{a(n+2)}{n+1}\Big(\frac{h^{n+1} - 1}{h^{n+2} - 1}\Big)$.*

REMARK 3. *As already noticed in [2], in Theorems 8 and 9, the second conclusion does occur. Indeed, let $h \in \mathcal{M}(\mathbb{K})$ (resp. let $h \in \mathcal{M}_u(d(0, R^-))$). Now, let us precisely define f and g as: $g = (\frac{n+2}{n+1})\Big(\frac{h^{n+1)} - 1}{h^{n+2} - 1}\Big)$ and $f = hg$. Then we can see that the polynomial $P(y) = \frac{1}{n+2}y^{n+2} - \frac{1}{n+1}y^{n+1}$ satisfies $P(f) = P(g)$, hence $f'P'(f) = g'P'(g)$, therefore $f'P'(f)$ and $g'P'(g)$ trivially share any function.*

Consider now the situation with analytic functions.

THEOREM 10. *Let $P(x) \in \mathbb{K}[x]$ be a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}(d(a, R^-))$) and let $P'(x) = b(x - a_1)^n \prod_{i=2}^{l}(x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $f,\ g \in \mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}(d(0, R^-))$) and be such that $f'P'(f)$ and $g'P'(g)$ share a function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ I.M. (resp. $\alpha \in$*

$\mathcal{A}_f(d(a,R^-)) \cap \mathcal{A}_g(d(a,R^-))$ *I.M.). If* $n \geq \sum_{i=2}^{l} k_i + 2$ *(resp.* $n \geq \sum_{i=2}^{l} k_i + 3$*) and* $n \geq 8 + \max(0, 10 - k_2) + \sum_{i=3}^{l} \max(0, 7 - k_i)$*, then* $f = g$.

Example: Let $P'(x) = x^{20}(x-1)^{10}(x-2)^7$. Consider the polynomial function $P(t)$ defined in $[0, +\infty[$, vanishing at 0. Clearly, $P'(t) > 0 \ \forall t \in]0,1[\cup]1,2[$ hence $P(2) > P(1) > P(0) = 0$. Therefore, $\Phi(P) = 3$ and hence, P is a polynomial of uniqueness for $\mathcal{A}(d(a,R^-))$. On the other hand, P satisfies the hypotheses of Theorem 8. Now, consider two functions $f,\ g \in \mathcal{A}(d(a,R^-))$ such that $f'P'(f)$ and $g'P'(g)$ share a function $\alpha \in \mathcal{A}_f(d(a,R^-)) \cap \mathcal{A}_g(d(a,R^-))$ I.M. Then $f = g$.

By Theorem A, we can derive this corollary:

Corollary 10.1 *Let* $P(x) \in \mathbb{K}[x]$ *and let* $P'(x) = x^n(x-a)^k$. *Let* $f,\ g \in \mathcal{A}(\mathbb{K})$ *be such that* $f'P'(f)$ *and* $g'P'(g)$ *share a function* $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ *I.M. If* $n \geq k+2$ *and* $n \geq 8 + \max(0, 10-k)$*, then* $f = g$.

THEOREM 11. *Let* $P(x) \in \mathbb{K}[x]$ *and let* $P'(x) = b(x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i}$ *with* $b \in \mathbb{K}^*$, $f,\ g \in \mathcal{A}(\mathbb{K})$ *be a polynomial of uniqueness for* $\mathcal{A}(\mathbb{K})$ *and be such that* $f'P'(f)$ *and* $g'P'(g)$ *share an affine function* α *or a non-zero constant* α *I.M. If* $n \geq \sum_{i=2}^{l} k_i + 1$ *and* $n \geq 7 + \max(0, 10 - k_2) + \sum_{i=3}^{l} \max(0, 7 - k_i)$*, then* $f = g$.

2. Basic Results

All lemmas used in [2] apply again here, even those whose hypotheses involve sharing values because they only concern sharing values IM, except Lemma 8 that we will thoroughly check.

Notation: Given two meromorphic functions $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{M}(d(0,R^-))$), we will denote by $\Theta_{f,g}$ the function

$$\frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}.$$

LEMMA 1. *Let* $f \in \mathcal{M}(\mathbb{K})$ *(resp. Let* $f \in \mathcal{M}(d(0,R^-))$*), let* $a \in \mathbb{K}$, *let* $Q(x) \in \mathbb{K}[x]$ *of degree* s. *Then* $T(r, Q(f)) = sT(r,f) + O(1)$ *and* $T(r, f'Q(f)) \geq sT(r,f) + O(1)$.

LEMMA 2. *Let* $f \in \mathcal{M}(\mathbb{K})$ *(resp. Let* $f \in \mathcal{M}(d(0,R^-))$*). Then* $N(r,f') = N(r,f) + \overline{N}(r,f)$, $Z(r,f') \leq Z(r,f) + \overline{N}(r,f) - \log r + O(1)$. *Moreover,* $T(r,f) - Z(r,f) \leq T(r,f') - Z(r,f') + O(1)$.

LEMMA 3. *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0,R^-))$) and let $\phi = \dfrac{f'}{f}$. Then $Z(r,\phi) \leq N(r,\phi) - \log r + O(1)$.*

LEMMA 4. *The function $\Theta_{f,g}$ satisfies $Z(r,\Theta_{f,g}) \leq N(r,\Theta_{f,g}) - \log r + O(1)$.*

LEMMA 5. *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ (resp. Let $f,\ g \in \mathcal{M}(d(0,R^-))$). If a is a simple zero of $f-1$ and $g-1$, it is a zero of $\Theta_{f,g}$.*

In order to state the next lemma, we must recall the definition of quasi-exceptional values.

(i) Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$. (resp. let $f \in \mathcal{M}_u(d(0,R^-))$). Then b will be said to be *a Picard exceptional value* of f (or just *an exceptional value*) if $f(x) \neq b\ \forall x \in \mathbb{K}$ (resp $f(x) \neq b\ \forall x \in d(0,R^-)$).

(ii) Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. (resp. let $f \in \mathcal{M}_u(d(0,R^-))$ and let $b \in \mathbb{K}$. Then b will be said to be *a quasi-exceptional value* of f if the function $f-b$ has a finite number of zeros in $\mathbb{K}$ (resp. in $d(0,R^-)$).

LEMMA 6. *Let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}$ $\big($resp. let $f \in \mathcal{A}_u(d(0,R^-))\big)$. Then f has no exceptional value. If f is transcendental, it has no quasi-exceptional value. Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$ $\big($resp. let $f \in \mathcal{M}_u(d(0,R^-))\big)$. Then f has at most one exceptional value in $\mathbb{K}$. Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental $\big($resp. let $f \in \mathcal{M}_u(d(0,R^-))\big)$. Then f has at most one quasi-exceptional value in $\mathbb{K}$.*

We now have to recall the *ultrametric Nevanlinna Second Main Theorem* in a basic form which we will frequently use.

Let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. $f \in \mathcal{M}(d(0,R^-))\big)$ satisfy $f'(0) \neq 0, \infty$. Let S be a finite subset of $\mathbb{K}$ and $r \in]0,+\infty[$ $\big($resp. $r \in]0,R[\big)$. We denote by $Z_0^S(r,f')$ *the counting function of zeros of f' in $d(0,r)$ which are not zeros of any $f-s$ for $s \in S$*. This is, if $(\gamma_n)_{n\in\mathbb{N}}$ is the finite or infinite sequence of zeros of f' in $d(0,r)$ that are not zeros of $f-s$ for $s \in S$, with multiplicy order q_n respectively, we set

$$Z_0^S(r,f') = \sum_{|\gamma_n| \leq r} q_n(\log r - \log|\gamma_n|).$$

Theorem N. [4], [5] *Let $a_1,...,a_q \in \mathbb{K}$ with $q \geq 2, q \in \mathbb{N}$, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0,R^-))$). Let $S = \{a_1,...,a_q\}$. Assume that none of $f,\ f'$ and $f-a_j$ with $1 \leq j \leq q$, equals 0 or ∞ at the origin. Then, for $r > 0$ $\big($resp. for $r \in]0,R[\big)$, we have*

$$(q-1)T(r,f) \leq \sum_{j=1}^{q} \overline{Z}(r,f-a_j) + \overline{N}(r,f) - Z_0^S(r,f') - \log r + O(1).$$

3. Specific Lemmas

The following Lemma is proven in [2].

LEMMA 7. *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f,\ g \in \mathcal{M}_u(d(0,R^-))$). Let $P(x) = x^{n+1}Q(x)$ be a polynomial such that $n \geq deg(Q)+2$ (resp. $n \geq deg(Q)+3$). If $P'(f)f' = P'(g)g'$ then $P(f) = P(g)$.*

Lemma 8 is proven in [2] under the assumption that $F,\ G$ share 1 value C.M. But actually, the hypothesis IM is sufficient to conclude.

LEMMA 8. *Let $F, G \in \mathcal{M}(\mathbb{K})$ $\big($resp. let $F, G \in \mathcal{M}(d(0,R^-))\big)$ be non-constant, having no zero and no pole at 0 and sharing the value 1 I.M.*

If $\Theta_{F,G} = 0$ and if

$$\lim_{r\to+\infty}\left(T(r,F) - [\overline{Z}(r,F) + \overline{N}(r,F) + \overline{Z}(r,G) + \overline{N}(r,G)]\right) = +\infty$$

(resp.

$$\lim_{r\to R^-}\left(T(r,F) - [\overline{Z}(r,F) + \overline{N}(r,F) + \overline{Z}(r,G) + \overline{N}(r,G)]\right) = +\infty)$$

then either $F = G$ or $FG = 1$.

The following Lemma is proven in [2] and in [9], (Theorem 56.3). Here, we just have to add the last sentence which actually is immediate: *$f,\ g$ can not belong to $\mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}_u(d(0,R^-))$).*

LEMMA 9. *Let $Q(x) = (x-a_1)^n \prod_{i=2}^{l}(x-a_i)^{k_i} \in \mathbb{K}[x]$ $(a_i \neq a_j,\ \forall i \neq j)$ with $l \geq 2$ and $n \geq \max\{k_2,..,k_l\}$ and let $k = \sum_{i=2}^{l} k_i$. Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f,\ g \in \mathcal{M}_u(d(0,R^-))$) such that $\theta = Q(f)f'Q(g)g'$ is a small function with respect to f and g. We have the following :*

If $l = 2$ then n belongs to $\{k,\ k+1,\ 2k,\ 2k+1,\ 3k+1\}$.

If $l = 3$ then n belongs to $\{\frac{k}{2}, k+1, 2k+1, 3k_2-k, .., 3k_l-k\}$.

If $l \geq 4$ then $n = k+1$.

If θ is a constant and $f,\ g \in \mathcal{M}(\mathbb{K})$ then $n = k+1$.

Moreover, $f,\ g$ can not belong to $\mathcal{A}(\mathbb{K})$ (resp. $f,\ g \in \mathcal{A}_u(d(0,R^-))$).

Lemma 10 is known and easily checked [6], [24]:

LEMMA 10. *Let $f,\ g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. let $f,\ g \in \mathcal{M}_u(d(0,R^-))$) satisfy $(f-a)f^n = (g-a)g^n$ with $a \in \mathbb{K}$ and let $h = \dfrac{f}{g}$. If h is not identically 1, then*

$$g = \frac{h^n-1}{h^{n+1}-1},\ f = \frac{h^{n+1}-h}{h^{n+1}-1}.$$

Notation: Let $f, g \in \mathcal{M}(\mathbb{K})$ $\big($resp. Let $f \in \mathcal{M}(d(0,R^-))\big)$ be such that $f(0) \neq 0, \infty$ We denote by $Z_{1)}(r,f)$ the counting function of simple zeros of f, we denote by $Z_{(2}(r,f)$ the counting function of multiple zeros of f, each counted with multiplcity and we denote by $\overline{Z}_{(2}(r,f)$ the counting function of multiple zeros of f, each counted without multiplcity.

In the same way, we set $N_{1)}(r,f)=Z_{1)}(r,\frac{1}{f}),\ N_{(2}(r,f)=Z_{(2}(r,\frac{1}{f}),\overline{N}_{(2}(r,f)=\overline{Z}_{(2}(r,\frac{1}{f})$. Consequently, by definition, one has $\overline{Z}(r,f)=Z_{1)}(r,f)+\overline{Z}_{(2}(r,f)$, $\overline{N}(r,f)=N_{1)}(r,f)+\overline{N}_{(2}(r,f)$.

Finally we denote by $Z_{[2]}(r,f)$ the counting fiunction of the zeros of f each counted with multiplicity when it is at most 2 and with multiplicity 2 when it is bigger and we set $N_{[2]}(r,f)=Z_{[2]}(r,\frac{1}{f})$.

Moreover, here we denote by $Z_0(r,f')$ the counting function of the zeros of f' that are not zeros of $f(f-1)$.

Given $f,g\in\mathcal{M}(\mathbb{K})$ (resp. Let $f,g\in\mathcal{M}(d(0,R^-))$), we denote by $\Theta_{f,g}$ the function

$$\frac{f''}{f'}-\frac{2f'}{f-1}-\frac{g''}{g'}+\frac{2g'}{g-1}.$$

The following Lemma 11 is proven in [24] for complex functions. Here we check that it holds for p-adic meromorphic functions. Moreover, we enjoy an additional negative function in $-\log r$. This is a generalization published in [16].

LEMMA 11. *Let $F,G\in\mathcal{M}(\mathbb{K})$ $\big($resp. Let $F,\ G\in\mathcal{M}(d(0,R^-))\big)$ be distinct and share the value 1 I.M. If $\Theta_{F,G}$ is not identically zero, then,*

$$T(r,F)\leq N_{[2]}(r,F))+Z_{[2]}(r,F)+N_{[2]}(r,G)+Z_{[2]}(r,G)$$
$$+2\big(\overline{Z}(r,F)+2\overline{N}(r,F)\big)+\overline{Z}(r,G)+\overline{N}(r,G)-4\log(r)+O(1).$$

PROOF. By hypotheses, we have $\overline{Z}(r,F-1)=Z_{1)}(r,F-1)+\overline{Z}_{(2}(r,F-1)$.

Next, we can easilly check that each simple zero of $F-1$ and $G-1$ is a zero of $\Theta_{F,G}$. Therefore
(2) $Z_{1)}(r,F-1)\leq Z(r,\Theta_{F,G})+\overline{Z}_{(2}(r,G-1)$ and similarly
(3) $Z_{1)}(r,G-1)\leq Z(r,\Theta_{F,G})+\overline{Z}_{(2}(r,F-1)$.

On the other hand, by Lemma 4, we have $Z(r,\Theta_{F,G})\leq N(r,\Theta_{F,G})-\log r+O(1)$. Consequently, by (2) we obtain
(4) $Z_{1)}(r,F-1)\leq N(r,\Theta_{F,G})+\overline{Z}_{(2}(r,G-1)-\log r+O(1)$
and similarly by (3),
(5) $Z_{1)}(r,G-1)\leq N(r,\Theta_{F,G})+\overline{Z}_{(2}(r,F-1)-\log r+O(1)$.

Let $Z_0(r,F')$ denote the counting function of zeros of F' when $F(x)\neq 0$ and 1. Of course, each pole of $\Theta_{F,G}$ is of order 1. Next, if F has a pole of order 1, it is not a pole of $\frac{F''}{F'}-\frac{2F'}{F-1}$ and if F has a zero of order ≥ 2 it is not a pole of $\frac{F''}{F'}-\frac{2F'}{F-1}$, and similarly for $\frac{G''}{G'}-\frac{2G'}{G-1}$.

Consequently, a pole of $\Theta_{F,G}$ can only occur either when $F-1$ or $G-1$ has a zero of order ≥ 2 or when F' or G' has a zero that is not a zero of $F-1$ or when F or G has a pole of order ≥ 2 or when $F-1$ or $G-1$ has a zero of order ≥ 2. On the other hand, we can dissociate zeros of F' when $F(x)\neq 0$ and 1 and other zeros of F' and similarly for G'. Therefore

$$(6)\ N(r,\Theta_{F,G})\leq\overline{Z}_0(r,F')+\overline{Z}_0(r,G')+Z_{(2}(r,F)+Z_{(2}(r,G)+\overline{N}_{(2}(r,F)$$
$$+\overline{N}_{(2}(r,G)+\big(\overline{Z}_{(2}(r,F-1)+\overline{Z}_{(2}(r,G-1)\big)$$

And hence by (4) we have

$$(7)\ Z_{1)}(r,F-1)\leq\overline{Z}_0(r,F')+\overline{Z}_0(r,G')+\overline{N}_{(2}(r,F)+\overline{N}_{(2}(r,G)$$
$$+\overline{Z}_{(2}(r,F-1)+2\overline{Z}_{(2}(r,G-1)-\log(r)+O(1)$$

and similarly by (5),

$$(8)\ Z_{1)}(r,G-1) \le \overline{Z}_0(r,F') + \overline{Z}_0(r,G') + \overline{N}_{(2}(r,F) + \overline{N}_{(2}(r,G) + \overline{Z}_{(2}(r,G-1) + 2\overline{Z}_{(2}(r,F-1) - \log(r) + O(1).$$

Moreover, we have

$$(9)\ Z_{1)}(r,F-1) \le \overline{Z}_0(r,F') + \overline{Z}_0(r,G') + \overline{N}_{(2}(r,F) + \overline{N}_{(2}(r,G) + \overline{Z}_{(2}(r,F-1) + \overline{Z}_{(2}(r,G-1) - \log(r) + O(1)$$

and similarly

$$(10)\ Z_{1)}(r,G-1) \le \overline{Z}_0(r,F') + \overline{Z}_0(r,G') + \overline{N}_{(2}(r,F) + \overline{N}_{(2}(r,G) + \overline{Z}_{(2}(r,G-1) + \overline{Z}_{(2}(r,F-1) - \log(r) + O(1).$$

Now, since each zero of $F-1$ of order ≥ 2 is a zero of F' that is not a zero of F, we have

$$Z(r,F') = Z_0(r,F') + Z_{(2}(r,F-1) - \overline{Z}_{(2}(r,F-1) + Z_{(2}(r,F) - \overline{Z}_{(2}(r,F)$$

hence

$$(11)\ \overline{Z}_0(r,F') + \overline{Z}_{(2}(r,F-1) + Z_{(2}(r,F) - \overline{Z}_{(2}(r,F) \le Z(r,F')$$

and similarly

$$(12)\ \overline{Z}_0(r,G') + \overline{Z}_{(2}(r,G-1) + Z_{(2}(r,G) - \overline{Z}_{(2}(r,G) \le Z(r,G').$$

Hence by Lemma 3 and by (11) and (12) we have

$$(13)\ \overline{Z}_{(2}(r,F-1) \le -\overline{Z}_0(r,F') + \overline{Z}_{(2}(r,F) - Z_{(2}(r,F) + Z(r,F) + \overline{N}(r,F) - \log r + O(1)$$

and similarly

$$(14)\ \overline{Z}_{(2}(r,G-1) \le -\overline{Z}_0(r,G') + \overline{Z}_{(2}(r,G) - Z_{(2}(r,G) + Z(r,G) + \overline{N}(r,G) - \log r + O(1).$$

Now $Z(r,F) + \overline{Z}_{(2}(r,F) - Z_{(2}(r,F)$ is just $\overline{Z}(r,F)$. Consequently, by (13) we obtain

$$(15)\ \overline{Z}_{(2}(r,F-1) \le \overline{N}(r,F) + \overline{Z}(r,F) - \overline{Z}_0(r,F') - \log r + O(1)$$

and similarly by (14),

$$(16)\ \overline{Z}_{(2}(r,G-1) \le \overline{N}(r,G) + \overline{Z}(r,G) - \overline{Z}_0(r,G') - \log r + O(1).$$

Consequently, by (7), (8), (15) and (16) we obtain

$$Z_{1)}(r,F-1) \le \overline{Z}_0(r,F') + \overline{Z}_0(r,G') + \overline{N}_{(2}(r,F) + \overline{N}_{(2}(r,G) + \overline{Z}_{(2}(r,F) + \overline{Z}_{(2}(r,G) + (\overline{N}(r,F) + \overline{Z}(r,F) - \overline{Z}_0(r,F')) + 2((\overline{N}(r,G) + \overline{Z}(r,G) - \overline{Z}_0(r,G') - \overline{Z}_0(r,F') - 2\log(r) + O(1)$$

i.e.

$$(17)\ Z_{1)}(r,F-1) \le N_{[2]}(r,F) + N_{[2]}(r,G) + Z_{[2]}(r,F) + Z_{[2]}(r,G) + \overline{N}(r,G) + \overline{Z}(r,G) - 2\log(r) + O(1)$$

and similarly

$$(18)\ Z_{1)}(r,G-1) \le N_{[2]}(r,F) + N_{[2]}(r,G) + Z_{[2]}(r,F) + Z_{[2]}(r,G) + \overline{N}(r,F) + \overline{Z}(r,F) - 2\log(r) + O(1)$$

Morever, by (9) and (10) we have

$$(19)\ Z_{1)}(r,F-1) \le N_{[2]}(r,F)+N_{[2]}(r,G)+Z_{[2]}(r,F)+Z_{[2]}(r,G) - \log(r)+O(1)$$

and similarly

$$(20)\ Z_{1)}(r,G-1) \le N_{[2]}(r,F)+N_{[2]}(r,G)+Z_{[2]}(r,F)+Z_{[2]}(r,G) - \log(r)+O(1)$$

Now, Theorem N lets us write

$$T(r,F) \le \overline{N}(r,F)+\overline{Z}(r,F)+\overline{Z}(r,F-1)-Z_0(r,F')-\log r+O(1)$$

hence

$$T(r,F) \le \overline{N}(r,F)+\overline{Z}(r,F)+Z_{1)}(r,F-1)+\overline{Z}_{(2}(r,F-1)-\log r+O(1)$$

and hence, by (14) and (16), we can derive

$$T(r,F) \le N_{[2]}(r,F)+Z_{[2]}(r,F)+N_{[2]}](r,G)+Z_{[2]}(r,G)$$
$$+2\big(\overline{Z}(r,F)+\overline{N}(r,F)\big)+\overline{N}(r,G)+\overline{Z}(r,G)-4\log r+O(1)$$

and similarly

$$T(r,G) \le N_{[2]}(r,F)+Z_{[2]}(r,F)+N_{[2]}](r,G)+Z_{[2]}(r,G)$$
$$+2\big(\overline{Z}(r,G)+\overline{N}(r,G)\big)+\overline{N}(r,F)+\overline{Z}(r,F)-4\log r+O(1).$$

□

The following Lemma is immediate:

LEMMA 12. *Let $f \in \mathcal{M}(\mathbb{K})$ $\big($resp. Let $f \in \mathcal{M}(d(0,R^-))\big)$ be non-constant. Let $P(x)=\prod_{i=1}^{l}(x-a_i)^{k_i} \in \mathbb{K}[x]$ be of degree q. Then $\overline{Z}(r,P(f)) \le \sum_{i=1}^{l}\overline{Z}(r,f-a_i) \le lT(r,f)$.*

4. Proof of Theorems

Proof of the Theorems. The polynomial P is considered in theorems 1, 2, 3, 4, 5, 6, 7, 10, 11 and we can assume $a_1=0$. In Theorems 8, and 9, we call P the polynomial such that $P'(x)=x^n(x-a)^k$ and $P(0)=0$. Set $F=\dfrac{f'P'(f)}{\alpha}$ and $G=\dfrac{g'P'(g)}{\alpha}$. Clearly F and G share the value 1 C.M. Since $f,\ g$ are transcendental (resp. unbounded), we notice that so are F and G. Recall that

$$\Theta_{F,G}=\frac{F''}{F'}-\frac{2F'}{F-1}-\frac{G''}{G'}+\frac{2G'}{G-1}$$

We will prove that under the hypotheses of each theorem, $\Theta_{F,G}$ is identically zero.

Set $\widehat{F}=P(f),\ \widehat{G}=P(g)$. We notice that $P(x)$ is of the form $x^{n+1}Q(x)$ with $Q \in K[x]$ of degree k. Now, with help of Lemma 2, we can check that we have

$$T(r,\widehat{F})-Z(r,\widehat{F}) \le T(r,\widehat{F}')-Z(r,\widehat{F}')+O(1)$$

Consequently, since $(\widehat{F})'=\alpha F$, we have

$$(1)\qquad T(r,\widehat{F}) \le T(r,F)+Z(r,\widehat{F})-Z(r,F)+T(r,\alpha)+O(1),$$

hence, by (1), we obtain

$$T(r,\widehat{F}) \leq T(r,F)+(n+1)Z(r,f)+Z\big(r,Q(f)\big)-nZ(r,f)-\sum_{i=2}^{l} k_i Z(r,f-a_i)-Z(r,f')+T(r,\alpha)+O(1).$$

i.e.

$$(2)\quad T(r,\widehat{F}) \leq T(r,F)+Z(r,f)+Z\big(r,Q(f)\big)-\sum_{i=2}^{l} k_i Z(r,f-a_i)-Z(r,f')+T(r,\alpha)+O(1).$$

and similarly,

$$(3)\quad T(r,\widehat{G}) \leq T(r,G)+Z(r,g)+Z\big(r,Q(g)\big)-\sum_{i=2}^{l} k_i Z(r,g-a_i)-Z(r,g')+T(r,\alpha)+O(1).$$

Now, using Lemma 13, it follows from the definition of F and G that

$$(4)\quad Z_{[2]}(r,F)+N_{[2]}(r,F) \leq 2Z(r,f)+2\sum_{i=2}^{l} Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+T(r,\alpha)+O(1)$$

and similarly

$$(5)\quad Z_{[2]}(r,G)+N_{[2]}(r,G) \leq 2Z(r,g)+2\sum_{i=2}^{l} Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+T(r,\alpha)+O(1)$$

And particularly, if $k_i = 1,\ \forall i \in \{2,..,l\}$, then

$$(6)\quad Z_{[2]}(r,F)+N_{[2]}(r,F) \leq 2Z(r,f)+\sum_{i=2}^{l} Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+T(r,\alpha)+O(1)$$

and similarly

$$(7)\quad Z_{[2]}(r,G)+N_{[2]}(r,G) \leq 2Z(r,g)+\sum_{i=2}^{l} Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+T(r,\alpha)+O(1).$$

Suppose now that $\Theta_{F,G}$ is not identically zero. Now, by Lemma 11, we have

$$\begin{aligned} T(r,F) \leq\ & Z_{[2]}(r,F)+N_{[2]}(r,F)+Z_{[2]}(r,G)+N_{[2]}(r,G)\\ &+2\big(\overline{N}(r,F)+\overline{Z}(r,F)\big)+\overline{N}(r,G)+\overline{Z}(r,G)-4\log r+O(1)\end{aligned}$$

hence by (2), we obtain

$$\begin{aligned} T(r,\widehat{F}) \leq\ & Z_{[2]}(r,F)+N_{[2]}(r,F)+Z_{[2]}(r,G)+N_{[2]}(r,G)+Z(r,f)+Z(r,Q(f))\\ &+2\big(\overline{N}(r,F)+\overline{Z}(r,F)\big)+\overline{N}(r,G)+\overline{Z}(r,G)\\ &-\sum_{i=2}^{l} k_i Z(r,f-a_i)-Z(r,f')+T(r,\alpha)-4\log r+O(1)\end{aligned}$$

and hence by (4) and (5):

$$\begin{aligned} T(r,\widehat{F}) \leq\ & 2Z(r,f)+2\sum_{i=2}^{l} Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+2Z(r,g)+2\sum_{i=2}^{l} Z(r,g-a_i)+Z(r,g')+\\ &+2\big(\overline{N}(r,F)+\overline{Z}(r,F)\big)+\overline{N}(r,G)+\overline{Z}(r,G)\end{aligned}$$

$$+2\overline{N}(r,g)+Z(r,f)+Z(r,Q(f))-\sum_{i=2}^{l}k_iZ(r,f-a_i)-Z(r,f')+2T(r,\alpha)-4\log r+O(1) \tag{8}$$

and similarly,

$$T(r,\widehat{G})\leq 2Z(r,g)+2\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+2Z(r,f)+2\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')$$
$$+2\big(\overline{N}(r,G)+\overline{Z}(r,G)\big)+\overline{N}(r,F)+\overline{Z}(r,F)$$

$$+2\overline{N}(r,f)+Z(r,g)+Z(r,Q(g))-\sum_{i=2}^{l}k_iZ(r,g-a_i)-Z(r,g')+2T(r,\alpha)-4\log r+O(1). \tag{9}$$

Consequently,

$$T(r,\widehat{F})+T(r,\widehat{G})\leq 7\big(Z(r,f)+Z(r,g)\big)+7\sum_{i=2}^{l}\big(Z(r,f-a_i)+Z(r,f-a_i)\big)+Z(r,g')+Z(r,f')$$
$$+4\big(\overline{N}(r,f)+2\overline{N}(r,g)\big)+Z(r,Q(f))+Z(r,Q(g))$$
$$-\sum_{i=2}^{l}k_iZ(r,f-a_i)-Z(r,f')+2T(r,\alpha)-4\log r+O(1) \tag{10}$$

Particularly, if $k_i=1\ \forall i=2,...,l$, then we have

$$T(r,\widehat{F})\leq 2Z(r,f)+\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')+2\overline{N}(r,f)+2Z(r,g)+\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')$$
$$+2\big(\overline{N}(r,F)+\overline{Z}(r,F)\big)+\overline{N}(r,G)+\overline{Z}(r,G)$$

$$+2\overline{N}(r,g)+Z(r,f)+Z(r,Q(f))-\sum_{i=2}^{l}Z(r,f-a_i)-Z(r,f')+2T(r,\alpha)-4\log r+O(1) \tag{11}$$

and similarly,

$$T(r,\widehat{G})\leq 2Z(r,g)+\sum_{i=2}^{l}Z(r,g-a_i)+Z(r,g')+2\overline{N}(r,g)+2Z(r,f)+\sum_{i=2}^{l}Z(r,f-a_i)+Z(r,f')$$
$$+2\big(\overline{N}(r,G)+\overline{Z}(r,G)\big)+\overline{N}(r,F)+\overline{Z}(r,F)$$

$$+2\overline{N}(r,f)+Z(r,g)+Z(r,Q(g))-\sum_{i=2}^{l}Z(r,g-a_i)-Z(r,g')+2T(r,\alpha)-4\log r+O(1). \tag{12}$$

therefore by (11) and (12) we can adapt (10) and obtain

$$T(r,\widehat{F})+T(r,\widehat{G})\leq 7\big(Z(r,f)+Z(r,g)\big)+4\sum_{i=2}^{l}\big(Z(r,f-a_i)+Z(r,f-a_i)\big)+Z(r,g')+Z(r,f')$$
$$+4\big(\overline{N}(r,f)+\overline{N}(r,g)\big)+Z(r,Q(f))+Z(r,Q(g))+4T(r,\alpha)-8\log r+O(1) \tag{13}$$

Now, let us go back to the general case. By (10) we have

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq 5(Z(r,f)+Z(r,g))+\sum_{i=2}^{l}(4-k_i)(Z(r,f-a_i)+Z(r,g-a_i))+(Z(r,f')+Z(r,g'))+$$
$$+3\big(\overline{Z}(r,F)+\overline{Z}(r,G)\big)+7(\overline{N}(r,f)+\overline{N}(r,g))+(Z(r,Q(f))+Z(r,Q(g)))+4T(r,\alpha)$$
(14)
$$+3\Big(Z(r,f')+\sum_{i=2}^{l}(Z(r,f-a_i)+Z(r,g-a_i))+Z(r,f)+Z(r,g)+Z(r,g')\Big)+4T(r,\alpha)-8\log r+O(1).$$

Here, by Lemma 2, we may notice that we can give $Z(r,f')$ the following upper bound $Z(r,f') \leq Z(r,f-a_2)+\overline{N}(r,f)-\log(r)$ and similarly $Z(r,g') \leq Z(r,g-a_2)+\overline{N}(r,g-\log(r))$. Consequently, by (14) we obtain

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq 8\big(Z(r,f)+Z(r,g)\big)+10\big(\overline{N}(r,f)+\overline{N}(r,g)\big)+(10-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big)$$
$$+\sum_{i=3}^{l}(7-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)+Z(r,Q(f))+Z(r;Q(g))+4T(r\alpha)-10\log(r)+O(1)$$

therefore

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq 8\big(Z(r,f)+Z(r,g)\big)+10\big(\overline{N}(r,f)+\overline{N}(r,g)\big)+(10-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big)$$
(15) $$+\sum_{i=3}^{l}(7-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)+Z(r,Q(f))+Z(r;Q(g))+4T(r,\alpha)-10\log(r)+O(1).$$

Now, we notice that gven $m \in \mathbb{Z}$ and $h \in \mathcal{M}(\mathbb{K})$ (resp. $h \in \mathcal{M}(d(0,R^-))$ we have $mZ(r,h) \leq \max(0,mT(r,h)$. Consequently, by (15) we obtain

$$T(r,\widehat{F})+T(r,\widehat{G}) \leq 8\big(Z(r,f)+Z(r,g)\big)+10\big(\overline{N}(r,f)+\overline{N}(r,g)\big)+\max\big(0,(10-k_2)\big)\big(T(r,f)+T(r,g)\big)\big)$$
(16) $$+\sum_{i=3}^{l}\max\big(0,(7-k_i)\big)\big(T(r,f)+T(r,g)\big)+Z(r,Q(f))+Z(r,Q(g))+4T(r,\alpha)-10\log(r)+O(1).$$

Now, since $\deg(Q)=k$, we have $T(r,Q(f))=kT(r,f)+O(1)$ and $T(r,Q(g))=kT(r,g)+O(1)$. On the other hand, since $\widehat{F}$ is a polynomial in f of degree $n+k+1$, we have $T(r,\widehat{F})=(n+1+k)T(r,f)+O(1)$ and $T(r,\widehat{G})=(n+1+k)T(r,f)+O(1)$. Consequently, by (16) we obtain

$$n\big((T(r,f)+T(r,g)\big) \leq 17\big(T(r,f)+T(r,g)\big)+\max(0,10-k_2)\big((T(r,f)+T(r,g)\big)$$
(17) $$+\sum_{i=3}^{l}(\max(7-k_i)\big((T(r,f)+T(r,g)\big)+4T(r,\alpha)-10\log(r)+O(1).$$

Thus, in Theorems 1 and 4, if $\Theta_{F,G}$ is not identically zero, Relation (17) is impossible. In Theorem 2, $T(r,\alpha)$ is of the form $\log(r)+O(1)$ and in Theorem 3,$T(r,\alpha)=0$, hence Relation (17) is impossible again if $\Theta_{F,G}$ is not identically zero. So, we have proven that $\Theta_{F,G}$ is identically zero in Theorems 1, 2, 3, 4.

Suppose now that $k_i=1,\ \forall i \in \{2,..,l\}$. Then $k=l$ and then, by (13) we can derive

$$(n+l+1)\big(T(r,f)+T(r,g)\big)\leq(7+4(l-1))\big(T(r,f)+T(r,g)\big)+T(r,f)+T(r,g)$$

$$+(4+1)\big(\overline{N}(r,f)+\overline{N}(r,g)\big)+l\big(T(r,f)+T(r,g)\big)+4T(r,\alpha)-10\log(r)+O(1) \tag{18}$$

therefore $n\leq 9+4l$.

Further, assuming again $k_i=1,\ \forall i\in\{2,..,l\}$, suppose now that α is a Moebius function. Then by (18) we now have $n\leq 8+4l$.

Thus, we can check that in Theorems 5 and 6, if $n\geq 10+4l$, $\Theta_{F,G}$ must be identically zero. And in Theorems 7, if $n\geq 9+4l$, $\Theta_{F,G}$ must be identically zero.

Similarly, in Theorems 8 and 9, if $n\geq 18$, $\Theta_{F,G}$ must be identically zero and in Theorem 8, if α is a constant or a Moebius function, the conclusion holds for $n\geq 17$

Consider now the situation in Theorems 10 and 11. Since $N(r,f)=N(r,g)=0$, by (14) we have

$$T(r,\widehat{F})+T(r,\widehat{G})\leq 8\big(Z(r,f)+Z(r,g)\big)+(10-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big)$$

$$+\sum_{i=3}^{l}(7-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)+Z(r,Q(f))+Z(r,Q(g))+4T(r\alpha)-10\log(r)+O(1)$$

therefore

$$T(r,\widehat{F})+T(r,\widehat{G})\leq 8\big(Z(r,f)+Z(r,g)\big)+(10-k_2)\big(Z(r,f-a_2)+Z(r,g-a_2)\big)$$

$$+\sum_{i=3}^{l}(7-k_i)\big(Z(r,f-a_i)+Z(r,g-a_i)\big)+Z(r,Q(f))+Z(r;Q(g))+4T(r,\alpha)-10\log(r)+O(1). \tag{19}$$

As in Relation (15), given $m\in\mathbb{Z}$ and $h\in\mathcal{M}(\mathbb{K})$ (resp. $h\in\mathcal{M}(d(0,R^-))$ we have $mZ(r,h)\leq\max(0,mT(r,h)$. Consequently, by (19) we obtain

$$T(r,\widehat{F})+T(r,\widehat{G})\leq 8\big(Z(r,f)+Z(r,g)\big)+\max\big(0,(10-k_2)\big)\big(T(r,f)+T(r,g)\big)\big)$$

$$+\sum_{i=3}^{l}\max\big(0,(7-k_i)\big)\big(T(r,f)+T(r,g)\big)+Z(r,Q(f))+Z(r,Q(g))+4T(r,\alpha)-10\log(r)+O(1). \tag{20}$$

Now, since $\deg(Q)=k$, we have $T(r,Q(f))=kT(r,f)+O(1)$ and $T(r,Q(g))=kT(r,g)+O(1)$. On the other hand, since $\widehat{F}$ is a polynomial in f of degree $n+k+1$, we have $T(r,\widehat{F})=(n+1+k)T(r,f)+O(1)$ and $T(r,\widehat{G})=(n+1+k)T(r,f)+O(1)$.

Consequently, by (20) we obtain

$$n\big(T(r,f)+T(r,g)\big)\leq 7\big(T(r,f)+T(r,g)\big)+\max(0,10-k_2)\big((T(r,f)+T(r,g)\big)$$

$$+\sum_{i=3}^{l}(\max(7-k_i)\big((T(r,f)+T(r,g)\big)+4T(r,\alpha)-10\log(r)+O(1). \tag{21}$$

Thus, in Theorems 10, if $\Theta_{F,G}$ is not identically zero, Relation (21) is impossible. In Theorem 11, $T(r,\alpha)$ is of the form $-4\log(r)+O(1)$ hence Relation (21) is impossible again if $\Theta_{F,G}$ is not identically zero. So, we have proven that $\Theta_{F,G}$ is identically zero in Theorems 10 and 11 too.

Consequently, we can assume that $\Theta_{F,G}=0$ in each hypothesis of all theorems. Therefore, we can perform the end of the proof exactly as in [2] or [9] (chapter 59) for Theorems 1, 2, 3, 4, 5, 6,

7, 8, 9, 10, 11 because we check that the reasonning never uses the hypothesis: *f and g share* α *C.M.* That ends the proof of all theorems.

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