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# Growth of $p$ -adic entire functions and applications

Kamal Boussaf, Abdelbaki Boutabaa and Alain Escassut

## Abstract

Let  $\mathbb{K}$  be an algebraically closed  $p$ -adic complete field of characteristic zero. We define the order of growth  $\rho(f)$  and the type of growth  $\sigma(f)$  of an entire function  $f(x) = \sum_{n=0}^{\infty} a_n x^n$  on  $\mathbb{K}$  as done on  $\mathbb{C}$  and show that  $\rho(f)$  and  $\sigma(f)$  satisfy the same relations as in complex analysis, with regards to the coefficients  $a_n$ . But here another expression  $\psi(f)$  that we call cotype of  $f$ , depending on the number of zeros inside disks is very important and we show under certain wide hypothesis, that  $\psi(f) = \rho(f)\sigma(f)$ , a formula that has no equivalent in complex analysis and suggests that it might hold in the general case. We check that  $\rho(f) = \rho(f')$ ,  $\sigma(f) = \sigma(f')$  and present an asymptotic relation linking the numbers of zeros inside disks for two functions of same order. That applies to a function and its derivative. We show that the derivative of a transcendental entire function  $f$  has infinitely many zeros that are not zeros of  $f$  and particularly we show that  $f'$  cannot divide  $f$  when the  $p$ -adic absolute value of the number of zeros of  $f$  inside disks satisfies certain inequality and particularly when  $f$  is of finite order.

## 1 Growth order

**Definitions and notation:** We denote by  $\mathbb{K}$  an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value  $|\cdot|$ . Analytic functions inside a disk or in the whole field  $\mathbb{K}$  were introduced and studied in many books [1], [5], [6], [7], [8], [9]. Given  $\alpha \in \mathbb{K}$  and  $R \in \mathbb{R}_+^*$ , we denote by  $d(\alpha, R)$  the disk  $\{x \in \mathbb{K} \mid |x - \alpha| \leq R\}$ , by  $d(\alpha, R^-)$  the disk  $\{x \in \mathbb{K} \mid |x - \alpha| < R\}$ , by  $C(\alpha, r)$  the circle  $\{x \in \mathbb{K} \mid |x - \alpha| = r\}$ , by  $\mathcal{A}(\mathbb{K})$  the  $\mathbb{K}$ -algebra of analytic functions in  $\mathbb{K}$  (i.e. the set of power series with an infinite radius of convergence) and by  $\mathcal{M}(\mathbb{K})$  the field of meromorphic functions in  $\mathbb{K}$  (i.e. the field of fractions of  $\mathcal{A}(\mathbb{K})$ ). Given  $f \in \mathcal{M}(\mathbb{K})$ , we will denote by  $q(f, r)$  the number of zeros of  $f$  in  $d(0, r)$ , taking multiplicity into account. Throughout the paper,  $\log$  denotes the Neperian logarithm.

Here we mean to introduce and study the notion of order of growth and type of growth for functions of order  $t$ . We will also introduce a new notion of cotype of growth in relation with the distribution of zeros in disks which plays a major role in processes that are quite different from those in complex analysis. This has an application to the question whether an entire function can be divided by its derivative inside the algebra of entire functions [2], [3].

Similarly to the definition known on complex entire functions [10], given  $f \in \mathcal{A}(\mathbb{K})$ , the superior limit  $\limsup_{r \rightarrow +\infty} \frac{\log(\log(|f|(r)))}{\log(r)}$  is called *the order of growth of  $f$*  or *the order of  $f$*  in brief and is denoted by  $\rho(f)$ . We say that  $f$  has *finite order* if  $\rho(f) < +\infty$ .

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**Theorem 1:** Let  $f, g \in \mathcal{A}(\mathbb{K})$ . Then:

$$\begin{aligned} & \text{if } c(|f|(r))^\alpha \geq |g|(r) \text{ with } \alpha \text{ and } c > 0, \text{ when } r \text{ is big enough, then } \rho(f) \geq \rho(g), \\ & \rho(f+g) \leq \max(\rho(f), \rho(g)), \\ & \rho(fg) = \max(\rho(f), \rho(g)), \end{aligned}$$

**Proof:** Similarly to the complex context we can easily verify that  $\rho(f+g) \leq \max(\rho(f), \rho(g))$ ,  $\rho(fg) \leq \max(\rho(f), \rho(g))$  and if  $c(|f|(r))^\alpha \geq |g|(r)$  with  $\alpha$  and  $c > 0$  when  $r$  is big enough, then  $\rho(f) \geq \rho(g)$ . Let us now show that  $\rho(fg) \geq \max(\rho(f), \rho(g))$ . Since  $\lim_{r \rightarrow +\infty} |g|(r) = +\infty$ , of course we have  $\log(|f.g|(r)) \geq \log(|f|(r))$  when  $r$  is big enough, hence

$$\frac{\log(\log(|f.g|(r)))}{\log(r)} \geq \frac{\log(\log(|f|(r)))}{\log(r)}$$

and therefore  $\rho(f.g) \geq \rho(f)$  and similarly,  $\rho(f.g) \geq \rho(g)$ .

**Corollary 1.1:** Let  $f, g \in \mathcal{A}(\mathbb{K})$ . Then  $\rho(f^n) = \rho(f) \forall n \in \mathbb{N}^*$ . If  $\rho(f) > \rho(g)$ , then  $\rho(f+g) = \rho(f)$ .

**Notation:** Given  $t \in [0, +\infty[$ , we denote by  $\mathcal{A}(\mathbb{K}, t)$  the set of  $f \in \mathcal{A}(\mathbb{K})$  such that  $\rho(f) \leq t$  and we set  $\mathcal{A}^0(\mathbb{K}) = \bigcup_{t \in [0, +\infty[} \mathcal{A}(\mathbb{K}, t)$ .

**Corollary 1.2:** For any  $t \geq 0$ ,  $\mathcal{A}(\mathbb{K}, t)$  is a  $\mathbb{K}$ -subalgebra of  $\mathcal{A}(\mathbb{K})$ . If  $t \leq u$ , then  $\mathcal{A}(\mathbb{K}, t) \subset \mathcal{A}(\mathbb{K}, u)$  and  $\mathcal{A}^0(\mathbb{K})$  is also a  $\mathbb{K}$ -subalgebra of  $\mathcal{A}(\mathbb{K})$ .

In the proofs of various theorems we will use the classical Theorems A, B [6], and Theorem C [4], [7] that we must recall here:

**Theorem A:** Let  $f(x) = \sum_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$ . Then for all  $r > 0$  we have  $|f|(r) = \sup_{n \geq 0} |a_n| r^n = |a_{q(f,r)}| r^{q(f,r)} > |a_n| r^n \forall n > q(f,r)$ . Moreover, if  $f$  is not a constant, the function in  $r$ :  $|f|(r)$  is strictly increasing and tends to  $+\infty$  with  $r$ . If  $f$  is transcendental, the function in  $r$ :  $\frac{|f|(r)}{r^s}$  tends to  $+\infty$  with  $r$ , whenever  $s > 0$ .

If the sequence  $\left(\frac{|a_{n-1}|}{|a_n|}\right)_{n \in \mathbb{N}^*}$  is strictly increasing, then, putting  $\frac{|a_{n-1}|}{|a_n|} = r_n$ ,  $f$  admits in each circle  $C(0, r_n)$  a unique zero taking multiplicity into account and has no other zero in  $\mathbb{K}$ .

**Theorem B:** Let  $f \in \mathcal{A}(\mathbb{K})$  be non-identically zero and let  $r', r'' \in ]0, +\infty[$  with  $r' < r''$ . Then

$$\left(\frac{r''}{r'}\right)^{q(f,r'')} \geq \frac{|f|(r'')}{|f|(r')} \geq \left(\frac{r''}{r'}\right)^{q(f,r')}.$$

**Theorem C:** Let  $f \in \mathcal{A}(\mathbb{K})$ . Then

$$|f'(r)| \leq \frac{|f|(r)}{r} \forall r > 0.$$

**Theorem D:** Let  $f, g \in \mathcal{A}(\mathbb{K})$ . Then  $|f \circ g|(r) = |f|(|g|(r)) \forall r > 0$ .

**Theorem 2:** Let  $f \in \mathcal{A}(\mathbb{K})$  and let  $P \in \mathbb{K}[x]$ . Then  $\rho(P \circ f) = \rho(f)$  and  $\rho(f \circ P) = \deg(P)\rho(f)$ .

**Proof:** Let  $n = \deg(P)$ . For  $r$  big enough, we have  $\log(\log(|f|(r))) \leq \log(\log(|P \circ f|(r))) \leq \log((n+1)\log(|f|(r))) = \log(n+1) + \log(\log(|f|(r)))$ . Consequently,

$$\begin{aligned} \limsup_{r \rightarrow +\infty} \left( \frac{\log(\log(|f|(r)))}{\log(r)} \right) &\leq \limsup_{r \rightarrow +\infty} \left( \frac{\log(\log(|P \circ f|(r)))}{\log(r)} \right) \\ &\leq \limsup_{r \rightarrow +\infty} \left( \frac{\log(n+1) + \log(\log(|f|(r)))}{\log(r)} \right) \end{aligned}$$

and therefore  $\rho(P \circ f) = \rho(f)$ .

Next, for  $r$  big enough, we have

$$\frac{\log(\log(|f|(r)))}{\log(r)} \leq \frac{\log(\log(|f \circ P|(r)))}{\log(r)} = \left( \frac{\log(\log(|f \circ P|(r)))}{\log(|P|(r))} \right) \left( \frac{\log(|P|(r))}{\log(r)} \right)$$

Now,

$$\limsup_{r \rightarrow +\infty} \left( \frac{\log(\log(|f \circ P|(r)))}{\log(|P|(r))} \right) = \limsup_{r \rightarrow +\infty} \left( \frac{\log(\log(|f|(r)))}{\log(r)} \right)$$

because the function  $h$  defined in  $[0, +\infty[$  as  $h(r) = |P|(r)$  is obviously an increasing continuous bijection from  $[0, +\infty[$  onto  $[|P(0)|, +\infty[$ . On the other hand, it is obviously seen that

$\limsup_{r \rightarrow +\infty} \left( \frac{\log(|P|(r))}{\log(r)} \right) = n$ . Consequently,

$$\limsup_{r \rightarrow +\infty} \left( \frac{\log(\log(|f \circ P|(r)))}{\log(|P|(r))} \right) = n \limsup_{r \rightarrow +\infty} \left( \frac{\log(\log(|f|(r)))}{\log(r)} \right)$$

and hence  $\rho(f \circ P) = n\rho(f)$ .

**Theorem 3:** Let  $f, g \in \mathcal{A}(\mathbb{K})$  be transcendental. If  $\rho(f) \neq 0$ , then  $\rho(f \circ g) = +\infty$ . If  $\rho(f) = 0$ , then  $\rho(f \circ g) \geq \rho(g)$ .

**Proof:** Let us fix an integer  $n \in \mathbb{N}$ . Let  $f(x) = \sum_{j=0}^{\infty} a_n x^n$  and  $g(x) = \sum_{j=0}^{\infty} b_n x^n$ . Since  $g$  is transcendental, for every  $n \in \mathbb{N}$ , there exists  $r_n$  such that  $q(g, r_n) \geq n$ . Then  $|g|(r) \geq |b_n| r^n \forall r \geq r_n$  and hence, by Theorem 2, we have

$$(1) \quad \rho(f \circ g) \geq n\rho(f).$$

Relation (1) is true for every  $n \in \mathbb{N}$ . Suppose first that  $\rho(f) \neq 0$ . Then by (1) we have  $\rho(f \circ g) = +\infty$ .

Now, suppose  $\rho(f) = 0$ . Let  $k \in \mathbb{N}$  be such that  $a_k \neq 0$ . Let  $s_0$  be such that  $q(f, s_0) \geq k$ . Then  $|f|(r) \geq |a_k| r^k \forall r \geq s_0$ , hence  $|f \circ g|(r) \geq |a_k| (|g|(r))^k \forall r \geq s_0$ , hence by Theorems 1 and 2 we have  $\rho(f \circ g) \geq \rho(g)$ .

**Theorem 4:** Let  $f \in \mathcal{A}(\mathbb{K})$  be not identically zero. If there exists  $s \geq 0$  such that  $\limsup_{r \rightarrow +\infty} \left( \frac{q(f, r)}{r^s} \right) < +\infty$  then  $\rho(f)$  is the lowest bound of the set of  $s \in [0, +\infty[$  such that  $\limsup_{r \rightarrow +\infty} \left( \frac{q(f, r)}{r^s} \right) = 0$ . Moreover, if  $\limsup_{r \rightarrow +\infty} \left( \frac{q(f, r)}{r^t} \right)$  is a number  $b \in ]0, +\infty[$ , then  $\rho(f) = t$ . If there exists no  $s$  such that  $\limsup_{r \rightarrow +\infty} \left( \frac{q(f, r)}{r^s} \right) < +\infty$ , then  $\rho(f) = +\infty$ .

**Proof of Theorem 4:** The proof holds in two statements. First we will prove that given  $f \in \mathcal{A}(\mathbb{K})$  nonconstant and such that for some  $t \geq 0$ ,  $\limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^t}$  is finite, then  $\rho(f) \leq t$ .

Set  $\limsup_{r \rightarrow +\infty} \left( \frac{q(f, r)}{r^t} \right) = b \in [0, +\infty[$ . Let us fix  $\epsilon > 0$ . We can find  $R > 1$  such that  $|f|(R) > e^2$  and  $\frac{q(f, r)}{r^t} \leq b + \epsilon \forall r \geq R$  and hence, by Theorem B, we have  $\frac{|f|(r)}{|f|(R)} \leq \left(\frac{r}{R}\right)^{q(f, r)} \leq \left(\frac{r}{R}\right)^{r^t(b+\epsilon)}$ . Therefore, since  $R > 1$ , we have

$$\log(|f|(r)) \leq \log(|f|(R)) + r^t(b + \epsilon)(\log(r)).$$

Now, when  $u > 2$ ,  $v > 2$ , we check that  $\log(u + v) \leq \log(u) + \log(v)$ . Applying that inequality with  $u = \log(|f|(R))$  and  $v = r^t(b + \epsilon)(\log(r))$  when  $r^t(b + \epsilon)(\log(r)) > 2$ , that yields

$$\log(\log(|f|(r))) \leq \log(\log(|f|(R))) + t \log(r) + \log(b + \epsilon) + \log(\log(r)).$$

Consequently,

$$\frac{\log(\log(|f|(r)))}{\log(r)} \leq \frac{\log(\log(|f|(R))) + t \log(r) + \log(b + \epsilon) + \log(\log(r))}{\log(r)}$$

and hence we can check that

$$\limsup_{r \rightarrow +\infty} \frac{\log(\log(|f|(r)))}{\log(r)} \leq t$$

which proves the first claim.

Second, we will prove that given  $f \in \mathcal{A}(\mathbb{K})$  not identically zero and such that for some  $t \geq 0$ , we have  $\limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^t} > 0$ , then  $\rho(f) \geq t$ .

By hypotheses, there exists a sequence  $(r_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} r_n = +\infty$  and such that  $\lim_{n \rightarrow +\infty} \frac{q(f, r_n)}{r_n^t} > 0$ . Thus there exists  $b > 0$  such that  $\lim_{n \rightarrow +\infty} \frac{q(f, r_n)}{r_n^t} \geq b$ . We can assume that  $|f|(r_0) \geq 1$ , hence by Theorem A,  $|f|(r_n) \geq 1 \forall n$ . Let  $\lambda \in ]1, +\infty[$ . By Theorem B we have

$$\frac{|f|(\lambda r_n)}{|f|(r_n)} \geq (\lambda)^{q(f, r_n)} \geq (\lambda)^{[b(r_n)^t]}$$

hence

$$\log(|f|(\lambda r_n)) \geq \log(|f|(r_n)) + b(r_n)^t \log(\lambda).$$

Since  $|f|(r_n) \geq 1$ , we have  $\log(\log(|f|(\lambda r_n))) \geq \log(b \log(\lambda)) + t \log(r_n)$  therefore

$$\frac{\log(\log(|f|(\lambda r_n)))}{\log(r_n)} \geq t + \frac{\log(b \log(\lambda))}{\log(r_n)} \quad \forall n \in \mathbb{N}$$

and hence

$$\limsup_{r \rightarrow +\infty} \frac{\log(\log(|f|(r)))}{\log(r)} \geq t$$

which ends the proof the second claim.

**Example:** Suppose that for each  $r > 0$ , we have  $q(f, r) \in [r^t \log r, r^t \log r + 1]$ . Then of course, for every  $s > t$ , we have  $\limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^s} = 0$  and  $\limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^t} = +\infty$ , so there exists no  $t > 0$  such that  $\frac{q(f, r)}{r^t}$  have non-zero superior limit  $b < +\infty$ .

**Definition and notation:** Let  $t \in [0, +\infty[$  and let  $f \in \mathcal{A}(\mathbb{K})$  of order  $t$ . We set  $\psi(f) = \limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^t}$  and call  $\psi(f)$  the *cotype* of  $f$ .

**Theorem 5:** Let  $f, g \in \mathcal{A}^0(\mathbb{K})$  be such that  $\rho(f) = \rho(g)$ . Then  $\max(\psi(f), \psi(g)) \leq \psi(fg) \leq \psi(f) + \psi(g)$ .

**Proof:** By Theorem 1, we have  $\rho(f.g) = \rho(f)$ . For each  $r > 0$ , we have  $q(f.g, r) = q(f, r) + q(g, r)$ , so the conclusion is immediate.

Theorem 6 is similar to a well known statement in complex analysis and its proof also is similar when  $\rho(f) < +\infty$  [10] but is different when  $\rho(f) = +\infty$ .

**Theorem 6:** Let  $f(x) = \sum_{n=0}^{+\infty} a_n x^n \in \mathcal{A}(\mathbb{K})$ . Then  $\rho(f) = \limsup_{n \rightarrow +\infty} \left( \frac{n \log(n)}{-\log |a_n|} \right)$ .

**Proof:** If  $\rho(f) < +\infty$ , the proof is identical to the one made in the complex context, replacing  $M(f, r)$  by the multiplicative norm  $|f|(r)$  (see [10], Proposition 11.4).

Suppose now that  $t = +\infty$ . Suppose that  $\limsup_{n \rightarrow +\infty} \frac{n \log n}{(-\log |a_n|)} < +\infty$ . Let us take  $s \in \mathbb{N}$  such that

$$(2) \quad \frac{n \log n}{(-\log |a_n|)} < s \quad \forall n \in \mathbb{N}.$$

By Theorem 4, we have  $\limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^s} = +\infty$ . So, we can take a sequence  $(r_m)_{m \in \mathbb{N}}$  such that

$$(3) \quad \lim_{m \rightarrow +\infty} \frac{q(f, r_m)}{(r_m)^s} = +\infty.$$

For simplicity, set  $u_m = q(f, r_m)$ ,  $m \in \mathbb{N}$ . By (2), for  $m$  big enough we have

$$u_m \log(u_m) < s(-\log(|a_{u_m}|)) = s \log \left( \frac{1}{|a_{u_m}|} \right)$$

hence

$$\frac{1}{(u_m)^{u_m}} > |a_{u_m}|^s,$$

therefore

$$|a_{u_m}|^s (r_m)^{s u_m} < \frac{(r_m)^{s u_m}}{(u_m)^{u_m}}$$

i.e.

$$(|f|(r_m))^s < \left(\frac{r_m}{u_m}\right)^{u_m}$$

But by Theorem A, we have  $\lim_{r \rightarrow +\infty} |f|(r) = +\infty$ , hence  $(r_m)^s > u_m$  when  $m$  is big enough and therefore  $\limsup_{m \rightarrow +\infty} \frac{q(f, r_m)}{(r_m)^s} \leq 1$ , a contradiction to (3). Consequently, (2) is impossible and therefore

$$\limsup_{n \rightarrow +\infty} \left( \frac{n \log(n)}{-\log |a_n|} \right) = +\infty = \rho(f).$$

**Remark:** Of course, polynomials have a growth order equal to 0. On  $\mathbb{K}$  as on  $\mathbb{C}$  we can easily construct transcendental entire functions of order 0 or of order  $\infty$ .

**Example 1:** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{K}$  such that  $-\log |a_n| \in [n(\log n)^2, n(\log n)^2 + 1]$ . Then clearly,

$$\lim_{n \rightarrow +\infty} \frac{\log |a_n|}{n} = -\infty$$

hence the function  $\sum_{n=0}^{\infty} a_n x^n$  has radius of convergence equal to  $+\infty$ . On the other hand,

$$\lim_{n \rightarrow +\infty} \frac{n \log n}{-\log |a_n|} = 0$$

hence  $\rho(f) = 0$ .

**Example 2:** Let  $(a_n)_{n \in \mathbb{N}}$  be a sequence in  $\mathbb{K}$  such that  $-\log |a_n| \in [n\sqrt{\log n}, n\sqrt{\log n} + 1]$ . Then

$$\lim_{n \rightarrow +\infty} \frac{\log |a_n|}{n} = -\infty$$

again and hence the function  $\sum_{n=0}^{\infty} a_n x^n$  has radius of convergence equal to  $+\infty$ . On the other hand,

$$\lim_{n \rightarrow +\infty} \left( \frac{n \log n}{-\log |a_n|} \right) = +\infty$$

hence  $\rho(f) = +\infty$ .

**Definition and notation:** In complex analysis, the type of growth is defined for an entire function of order  $t$  as  $\sigma(f) = \limsup_{r \rightarrow +\infty} \frac{\log(M_f(r))}{r^t}$ , with  $t < +\infty$ . Of course the same notion may

be defined for  $f \in \mathcal{A}(\mathbb{K})$ . Given  $f \in \mathcal{A}^0(\mathbb{K})$  of order  $t$ , we set  $\sigma(f) = \limsup_{r \rightarrow +\infty} \frac{\log(|f|(r))}{r^t}$  and  $\sigma(f)$  is called *the type of growth of  $f$* .

**Theorem 7:** Let  $f, g \in \mathcal{A}^0(\mathbb{K})$ . Then  $\sigma(fg) \leq \sigma(f) + \sigma(g)$  and  $\sigma(f+g) \leq \max(\sigma(f), \sigma(g))$ . If  $\rho(f) = \rho(g)$ , then  $\max(\sigma(f), \sigma(g)) \leq \sigma(fg)$  and if  $c|f|(r) \geq |g|(r)$  with  $c > 0$  when  $r$  is big enough, then  $\sigma(f) \geq \sigma(g)$ .

**Proof:** Let  $s = \rho(g)$  and  $t = \rho(f)$  and suppose  $s \leq t$ . When  $r$  is big enough, we have  $\max(\log(|f|(r)), \log(|g|(r))) \leq \log(|f.g|(r)) = \log(|f|(r)) + \log(|g|(r))$ . By Theorem 1, we have  $\rho(fg) = t$ . Therefore

$$\begin{aligned} \limsup_{r \rightarrow +\infty} \left( \frac{\log(|f.g|(r))}{r^t} \right) &\leq \limsup_{r \rightarrow +\infty} \left( \frac{\log(|f|(r))}{r^t} \right) + \limsup_{r \rightarrow +\infty} \left( \frac{\log(|g|(r))}{r^t} \right) \\ &\leq \limsup_{r \rightarrow +\infty} \left( \frac{\log(|f|(r))}{r^t} \right) + \limsup_{r \rightarrow +\infty} \left( \frac{\log(|g|(r))}{r^s} \right) = \sigma(f) + \sigma(g). \end{aligned}$$

Similarly,

$$\begin{aligned} \sigma(f+g) &= \limsup_{r \rightarrow +\infty} \left( \frac{|f+g|(r)}{r^t} \right) \leq \limsup_{r \rightarrow +\infty} \left( \frac{\max(|f|(r), |g|(r))}{r^t} \right) \\ &\leq \max \left( \limsup_{r \rightarrow +\infty} \left( \frac{|f|(r)}{r^t} \right), \limsup_{r \rightarrow +\infty} \left( \frac{|g|(r)}{r^s} \right) \right) = \max(\sigma(f), \sigma(g)). \end{aligned}$$

Now, suppose  $s = t$ . Then

$$\max \left( \limsup_{r \rightarrow +\infty} \left( \frac{\log(|f|(r))}{r^t} \right), \limsup_{r \rightarrow +\infty} \left( \frac{\log(|g|(r))}{r^t} \right) \right) \leq \limsup_{r \rightarrow +\infty} \left( \frac{\log(|f.g|(r))}{r^t} \right)$$

hence  $\sigma(fg) \geq \max(\sigma(f), \sigma(g))$ . Suppose now  $c|f|(r) \geq |g|(r)$  when  $r$  is big enough, then, assuming again that  $s = t$ , it is obvious that  $\sigma(f) \geq \sigma(g)$ .

**Corollary 7.1:** *Let  $f, g \in \mathcal{A}^0(\mathbb{K})$  be such that  $\rho(f) = \rho(g)$  and  $\sigma(f) > \sigma(g)$ . Then  $\sigma(f+g) = \sigma(f)$ .*

Now, we notice that  $\sigma(f)$  may be computed by the same formula as on  $\mathbb{C}$ . Since the proof is the same we will not reproduce it (10], Proposition 11.5).

**Theorem 8:** *Let  $f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}^0(\mathbb{K})$  such that  $\rho(f) \in ]0, +\infty[$ . Then  $\sigma(f)\rho(f)e = \limsup_{n \rightarrow +\infty} (n \sqrt[n]{|a_n|^t})$ .*

**Definition:** Let us say that an entire function  $\sum_{n=1}^{+\infty} c_n x^n \in \mathcal{A}(\mathbb{K})$  satisfies Hypothesis  $L$  when the sequence  $(\frac{|c_{n-1}|}{|c_n|})_{n \in \mathbb{N}}$  is strictly increasing.

**Theorem 9:** *Let  $f \in \mathcal{A}(\mathbb{K})$  such that  $\rho(f) \in ]0, +\infty[$ .*

- i) If  $\sigma(f) = \lim_{r \rightarrow +\infty} \frac{\log(|f|(r))}{r^{\rho(f)}}$ , then  $\psi(f) \geq \rho(f)\sigma(f)$ .*
- ii) If  $\psi(f) = \lim_{r \rightarrow +\infty} \frac{q(f, r)}{r^{\rho(f)}}$ , then  $\psi(f) = \rho(f)\sigma(f)$ .*

**Proof:** Let  $f(x) = \sum_{m=0}^{+\infty} a_m x^m$ . Without loss of generality, we may assume that  $f(0) = 1$ . Set  $t = \rho(f)$  and  $\ell = \frac{1}{\rho(f)}$  and let us denote by  $(C(0, s_m))_{m \in \mathbb{N}}$  the sequence of circles containing at least one zero of  $f$ , with  $s_m < s_{m+1}$ .



Suppose first that  $f$  satisfies Hypothesis  $L$ . By Theorem A, actually each circle  $C(0, s_m)$ ,  $m \in \mathbb{N}$  contains a unique zero of  $f$  and  $f$  has no other zero in  $\mathbb{K}$ . Moreover, for each  $m \in \mathbb{N}$ , we have  $s_m = \frac{|a_{m-1}|}{|a_m|}$ . Consequently,  $q(f, s_m) = m$ ,  $m \in \mathbb{N}$ . Now, when  $r$  belongs to the interval  $[s_m, s_{m+1}[$ ,  $f$  admits exactly  $m$  zeros in  $d(0, r)$  and hence  $\frac{q(f, r)}{r^t}$  is maximum in  $[s_m, s_{m+1}[$  when  $r = s_m$ . Consequently, we have

$$\psi(f) = \limsup_{m \rightarrow +\infty} \frac{m}{(s_m)^t}.$$

So we can write  $\frac{m}{(s_m)^t} = \psi(f) + \epsilon_m$  with  $\limsup_{m \rightarrow +\infty} \epsilon_m = 0$  hence

$$(1) \quad s_m = \left( \frac{m}{\psi(f) + \epsilon_m} \right)^\ell.$$

Now, let  $(\phi(m))_{m \in \mathbb{N}}$  be a strictly increasing sequence of integers and consider the expression  $E(m) = \log(|f|(s_{\phi(m)})) = \sum_{k=1}^{\phi(m)} \log(s_{\phi(m)}) - \log(s_k)$ . By (1) we have

$$\begin{aligned} E(m) &= \ell \left( \phi(m) \log(\phi(m)) - \phi(m) \log(\psi(f) + \epsilon_{\phi(m)}) - \sum_{k=1}^{\phi(m)} \log(k) + \sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k) \right) \\ &= \ell \left( \phi(m) \log(\phi(m)) - \phi(m) \log(\phi(m) + \phi(m) + O(1) - \phi(m) \log(\psi(f) + \epsilon_{\phi(m)}) + \sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k) \right) \end{aligned}$$

hence

$$(2) \quad E(m) = \ell \left( \phi(m) + O(1) - \phi(m) (\log(\psi(f) + \epsilon_{\phi(m)}) + \sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k)) \right)$$

Suppose first that  $\sigma(f) = \lim_{r \rightarrow +\infty} \frac{\log(|f|(r))}{r^t}$  and let us choose for the sequence  $(\phi(m))_{m \in \mathbb{N}}$  a sequence such that  $\lim_{m \rightarrow +\infty} \frac{q(f, s_{\phi(m)})}{(s_{\phi(m)})^t} = \psi(f)$  i.e.  $\lim_{m \rightarrow +\infty} \frac{\phi(m)}{(s_{\phi(m)})^t} = \psi(f)$ . Obviously, we can check that

$$\sigma(f) = \lim_{m \rightarrow +\infty} \frac{\log(|f|(s_{\phi(m)}))}{(s_{\phi(m)})^t} = \lim_{m \rightarrow +\infty} \frac{E(m)}{(s_{\phi(m)})^t}$$

hence by (2)

$$(3) \quad \sigma(f) = \lim_{m \rightarrow +\infty} \frac{\ell \left( \phi(m) + O(1) - \phi(m) (\log(\psi(f) + \epsilon_{\phi(m)}) + \sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k)) \right)}{(s_{\phi(m)})^t}.$$

Here we notice that  $\lim_{m \rightarrow +\infty} \ell \left( \frac{\phi(m) + O(1) - \phi(m) (\log(\psi(f) + \epsilon_{\phi(m)}) + \sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k))}{(s_{\phi(m)})^t} \right) = \psi(f) (1 - \log(\psi(f)))$ , therefore by (3),  $\frac{\sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k)}{(s_{\phi(m)})^t}$  admits a limit when  $m$  tends to  $+\infty$ , which is  $\sigma(f) - \psi(f) (1 - \log(\psi(f)))$ .

$\log(\psi(f))$ ). Next, since  $\limsup_{m \rightarrow +\infty} (\epsilon_{\phi(m)}) = 0$ , we can check that

$$(4) \quad \lim_{m \rightarrow +\infty} \left( \frac{\sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k)}{(s_{\phi(m)})^t} \right) \leq \lim_{m \rightarrow +\infty} \left( \frac{\phi(m) \log(\psi(f))}{(s_{\phi(m)})^t} \right) = \psi(f) \log(\psi(f))$$

Indeed, let us fix  $\omega > 0$  and let  $M \in \mathbb{N}$  be such that  $\epsilon_k \leq \omega \forall k > M$  and

$$(5) \quad \frac{\sum_{k=1}^M \log(\psi(f) + \epsilon_k)}{(s_{\phi(m)})^t} \leq \omega \quad \forall \phi(m) > M.$$

Then

$$\frac{\sum_{k=M+1}^{\phi(m)} \log(\psi(f) + \epsilon_k)}{(s_{\phi(m)})^t} \leq \log(\psi(f) + \omega) \frac{\phi(m)}{(s_{\phi(m)})^t} \leq \log(\psi(f) + \omega) \psi(f).$$

Consequently, by (5), we have

$$\limsup_{m \rightarrow +\infty} \left( \frac{\sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k)}{(s_{\phi(m)})^t} \right) = \lim_{m \rightarrow +\infty} \left( \frac{\sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k)}{(s_{\phi(m)})^t} \right) \leq \omega + \log(\psi(f) + \omega) \psi(f).$$

This is true for each  $\omega > 0$  and hence finishes proving (4). Now, by (3) we have

$$\sigma(f) = \ell \left( \psi(f) - \psi(f) \log(\psi(f)) + \lim_{m \rightarrow +\infty} \frac{\sum_{k=1}^{\phi(m)} \log(\psi(f) + \epsilon_k)}{(s_{\phi(m)})^t} \right)$$

hence by (4), we obtain  $\sigma(f) \leq \ell \psi(f)$  and therefore  $\rho(f) \sigma(f) \leq \psi(f)$ .

Now, suppose that  $\psi(f) = \lim_{r \rightarrow +\infty} \frac{q(f, r)}{r^t}$ . Then we can take  $\phi(m) = m \forall m \in \mathbb{N}$  hence we have

$$\psi(f) = \lim_{m \rightarrow +\infty} \frac{m}{(s_m)^t} \quad \text{and} \quad \lim_{m \rightarrow +\infty} \epsilon_m = 0.$$

Set  $f(x) = \sum_{n=0}^{\infty} a_n x^n$ . By Theorem A, for each  $m \in \mathbb{N}$ , we have  $s_m = \frac{|a_{m-1}|}{|a_m|}$ . Consequently,  $q(f, s_m) = m$ . Hence

$$(6) \quad \psi(f) = \lim_{m \rightarrow +\infty} \frac{m}{(s_m)^t} = \lim_{m \rightarrow +\infty} \left( m \left( \frac{|a_m|}{|a_{m-1}|} \right)^t \right).$$

For every  $m \in \mathbb{N}$ , set  $b_m = m! |a_m|^t$ . By hypothesis and by (6) we have  $\lim_{m \rightarrow +\infty} \left( \frac{b_m}{b_{m-1}} \right) = \psi(f)$ .

But by d'Alembert-Cauchy's Theorem,  $\lim_{m \rightarrow +\infty} \left( \frac{b_m}{b_{m-1}} \right) = \lim_{m \rightarrow +\infty} \sqrt[m]{b_m} = \psi(f)$ , hence

$\lim_{m \rightarrow +\infty} \sqrt[m]{m! |a_m|^t} = \psi(f)$ . On the other hand, by Stirling's formula, we have  $\lim_{m \rightarrow +\infty} \frac{\sqrt[m]{m!}}{m} = \frac{1}{e}$ , hence

$\lim_{m \rightarrow +\infty} m \left( \sqrt[m]{|a_m|} \right)^t = e \psi(f)$ . But by Theorem 8, we have  $\lim_{m \rightarrow +\infty} \left( m \left( \sqrt[m]{|a_m|} \right)^t \right) = (e t \sigma(f))$ .

Consequently,  $\psi(f) = t \sigma(f)$ , which proves the theorem when  $f$  satisfies Hypothesis  $L$ .

Consider now the general case when  $f$  is no longer supposed to satisfy Hypothesis  $L$ . For each  $m \in \mathbb{N}$ , let  $u_m$  be the total number of zeros of  $f$  in  $C(0, s_m)$ , taking multiplicity into account.

For each  $m \in \mathbb{N}$ , we can put  $l_m = \sum_{k=1}^m u_k$ , hence  $l_m$  is the total number of zeros of  $f$  the disk  $d(0, s_m)$ , taking multiplicity into account. As previously remarked,

$$(7) \quad \psi(f) = \limsup_{m \rightarrow +\infty} \frac{l_m}{(s_m)^t}.$$

We will construct a new function  $g$  satisfying  $\rho(g) = \rho(f)$ ,  $\psi(g) = \psi(f)$ ,  $\sigma(g) = \sigma(f)$ .

For each  $m \in \mathbb{N}$ , let us set  $s'_m = \max(s_{m_1}, s_m - \frac{1}{u_m m})$ , let us take  $u_m$  points  $\beta_{m,j}$ ,  $j = 1, \dots, u_m$  in  $\mathbb{K}$  satisfying

$$s'_m < |\beta_{m,1}| < \dots < |\beta_{m,u_m}| = s_m$$

and let

$$g(x) = \prod_{m=1}^{+\infty} \left( \prod_{j=1}^{u_m} \left( 1 - \frac{x}{\beta_{m,j}} \right) \right).$$

Obviously, we have  $q(g, s_m) = q(f, s_m)$ . On the other hand, we can check that when  $m$  is big enough, we have  $\frac{q(g, s_m) - 1}{(s'_m)^t} < q(g, s_m)$  hence

$$\sup_{r \in [s_{m-1}, s_m[} \frac{q(g, r)}{r^t} = \frac{q(g, s_m)}{(s_m)^t} = \frac{q(f, s_m)}{(s_m)^t}$$

Therefore, by (7) we have

$$(8) \quad \psi(g) = \psi(f) = \limsup_{m \rightarrow +\infty} \frac{q(f, s_m)}{(s_m)^t}.$$

Particularly, if  $\psi(f) = \lim_{r \rightarrow +\infty} \frac{q(f, r)}{r^t}$ , then  $\psi(f) = \lim_{m \rightarrow +\infty} \frac{q(f, s_m)}{(s_m)^t} = \lim_{m \rightarrow +\infty} \frac{q(g, s_m)}{(s_m)^t} = \psi(g)$ .

Now consider  $\log(|g|(r)) - \log(|f|(r))$  when  $r \in [s_{m-1}, s_m[$ . On one hand, we check that

$$\log(|f|(r)) = \sum_{j=1}^{m-1} u_j (\log(r) - \log(s_j))$$

and

$$\log(|f|(r)) \geq \log(|g|(r)) \geq \log(|f|(r)) - \sum_{j=1}^m u_j (\log(s_j) - \log(s'_j)).$$

On the other hand we notice that  $0 \leq \sum_{j=1}^m u_j (\log(s_j) - \log(s'_j)) \leq \frac{1}{j(j-1)}$ . Therefore  $\log(|f|(r)) - \log(|g|(r))$  is positive and bounded when  $r$  tends to  $+\infty$ . Consequently, we have

$$\lim_{r \rightarrow +\infty} \frac{\log(\log(|f|(r)) - \log(\log(|g|(r)))}{\log(r)} = 0$$

and hence  $\rho(f) = \rho(g) = t$ . Further,

$$\lim_{r \rightarrow +\infty} \left( \frac{\log(|f|(r))}{r^t} - \frac{\log(|g|(r))}{r^t} \right) = 0,$$

hence  $\sigma(f) = \sigma(g)$ . Moreover, by (8)  $\psi(g) = \psi(f) = \lim_{r \rightarrow +\infty} \frac{q(g, r)}{r^t}$ .

Thus, as announced,  $g$  satisfies  $\rho(g) = \rho(f)$ ,  $\sigma(g) = \sigma(f)$ ,  $\psi(g) = \psi(f)$  and by construction,  $g$  satisfies the Hypothesis  $L$ . Consequently, we can apply Theorem 9 proven when  $f$  satisfies Hypothesis  $L$ . Therefore, assuming  $i$ ) we have  $\psi(g) \leq \rho(g)\sigma(g)$  and assuming  $ii$ ) then  $\psi(g) = \rho(g)\sigma(g)$ . That ends the proof of Theorem 9.

**Remark:** The conclusions of Theorem 9 hold for  $\psi(f) = \sigma(f) = +\infty$ .

We will now present Example 3 where neither  $\psi(f)$  nor  $\sigma(f)$  are obtained as limits but only as superior limits: we will show that the equality  $\psi(f) = \rho(f)\sigma(f)$  holds again.

**Example 3:** Let  $r_n = 2^n$ ,  $n \in \mathbb{N}$  and let  $f \in \mathcal{A}(\mathbb{K})$  have exactly  $2^n$  zeros in  $C(0, r_n)$  and satisfy  $f(0) = 1$ . Then  $q(f, r_n) = 2^{n+1} - 1 \forall n \in \mathbb{N}$ . We can see that the function  $h(r)$  defined in  $[r_n, r_{n+1}[$  by  $h(r) = \frac{q(f, r)}{r}$  is decreasing and satisfies  $h(r_n) = \frac{2^{n+1} - 2}{2^n}$  and  $\lim_{r \rightarrow r_{n+1}} \frac{h(r)}{r} = \frac{2^{n+1} - 2}{2^{n+1}}$ . Consequently,  $\limsup_{r \rightarrow +\infty} h(r) = 2$  and  $\liminf_{r \rightarrow +\infty} h(r) = 1$ . Particularly, by Theorem 4, we have  $\rho(f) = 1$  and of course  $\psi(f) = 2$ .

Now, let us compute  $\sigma(f)$  and consider the function in  $r$ :  $E(r) = \frac{\log(|f|(r))}{r}$ . When  $r$  belongs to  $[r_n, r_{n+1}]$ , we have

$$E(r) = \frac{(2^{n+1} - 2) \log r - (\log 2)(\sum_{k=1}^n 2^k)}{r}$$

and its derivative is  $E'(r) = \frac{\sum_{k=1}^n 2^k(1 + k \log(2)) - \log(r)}{r^2}$ . We will need to compute

$$(1) \quad \sum_{k=1}^n k 2^k = 2(n 2^{n+1} - (n+1)2^n + 1).$$

Now, the numerator  $U(r)$  of  $E'(r)$  is  $U(r) = \sum_{k=1}^n 2^k(1 + k \log(2)) - \log(r)$  is decreasing in the interval  $[r_n, r_{n+1}]$  and has a unique zero  $\alpha_n$  satisfying, by (1),

$$\log(\alpha_n) = \frac{2^n \left( (\log 2)(n-1 + 2^{-n}) + 2 - 2^{-n+1} \right)}{2^n - 2}$$

thereby  $\log(\alpha_n)$  is of the form  $n \log(2) + \epsilon_n$  with  $\lim_{n \rightarrow +\infty} \epsilon_n = 0$ .

Since  $E'(r)$  is decreasing in  $[r_n, r_{n+1}]$ , we can check that  $E(r)$  passes by a maximum at  $\alpha_n$  and consequently,

$$\sigma(f) = \limsup_{n \rightarrow +\infty} \frac{E(\alpha_n)}{\alpha_n}$$

Therefore  $\sigma(f) = 2 = \psi(f)$ .

Now, we can check that  $\liminf_{r \rightarrow +\infty} \frac{E(r)}{r} < \sigma(f)$ . Indeed consider

$$\frac{E(r_n)}{r_n} = \frac{(2^{n+1} - 2)(\log r_n) - (\log 2) \sum_{k=1}^n k 2^k}{r_n} = \frac{(2^{n+1} - 2)(n \log 2) - (\log 2) \sum_{k=1}^n k 2^k}{2^n}$$

hence by (1), we obtain

$$E(r_n) = \frac{(2^{n+1} - 2)(n \log 2) - 2(\log 2)(n2^{n+1} - (n+1)2^n + 1)}{2^n} = \frac{2(\log 2)(2^n - n - 1)}{2^n}$$

therefore  $\lim_{n \rightarrow \infty} E(r_n) = 2 \log 2$  and hence  $\liminf_{r \rightarrow +\infty} E(r) < \sigma(f)$ .

Now, Theorem 9 and Example 3 suggest the following conjecture:

**Conjecture C1:** *Let  $f \in \mathcal{A}^0(\mathbb{K})$  be such that either  $\sigma(f) < +\infty$  or  $\psi(f) < +\infty$ . Then  $\psi(f) = \rho(f)\sigma(f)$ .*

**Example 4:** infinite type and cotype. Here is an example of  $f \in \mathcal{A}(\mathbb{K}, 1)$  such that  $\sigma(f) = \psi(f) = +\infty$ .

For each  $n \in \mathbb{N}$ , set  $\phi(n) = \sqrt{\log n}$  and let  $u_n$  be defined by  $\log(u_n) = -\frac{n \log n}{1 + \frac{1}{\phi(n)}}$ . For simplicity, suppose first that the set of absolute values of  $|\mathbb{K}|$  is the whole set  $[0, \mathbb{R}]$ . We can take a sequence  $(a_n)$  of  $\mathbb{K}$  such that  $|a_n| = u_n \forall n \in \mathbb{N}^*$ , with  $a_0 = 1$ . Then  $\frac{\log |a_n|}{n} = -\frac{(\log n)\phi(n)}{\phi(n) + 1}$  hence  $\lim_{n \rightarrow +\infty} \frac{\log |a_n|}{n} = -\infty$ , therefore  $f \in \mathcal{A}(\mathbb{K})$ . Next,  $\frac{n \log n}{-\log |a_n|} = \frac{\phi(n)}{\phi(n) + 1}$  hence  $\lim_{n \rightarrow +\infty} \frac{n \log n}{-\log |a_n|} = 1$  therefore  $\rho(f) = 1$ .

Next,  $\log(n|a_n|^{\frac{1}{n}}) = \log n + \frac{\log |a_n|}{n} = \log n - \frac{\phi(n) \log n}{\phi(n) + 1} = \frac{\log n}{\phi(n) + 1}$  and hence  $\sigma(f) = +\infty$ .

Let us now compute  $\psi(f)$ . Now, for each  $n \in \mathbb{N}^*$ , take  $r_n = \frac{u_{n-1}}{u_n}$ . We will first check that the sequence  $(r_n)_{n \in \mathbb{N}^*}$  is strictly increasing when  $n$  is big enough. Indeed, we just have to show that there exists  $M \in \mathbb{N}$  such that

$$(1) \quad \log(u_n) - \log(u_{n+1}) > \log(u_{n-1}) - \log(u_n) \quad \forall n > M.$$

Let  $g$  be the function defined in  $]0, +\infty[$  as  $g(x) = -\frac{x \log x}{1 + \frac{1}{\sqrt{\log x}}}$ . Then we can check that  $g$  is

convex and therefore (1) is proven.

Now, since the sequence  $(r_n)_{n \in \mathbb{N}^*}$  obviously tends to  $+\infty$ , there exists an rang  $N \geq M$  such that  $r_{n+1} > r_n \forall n \geq M$  and  $r_M > r_k \forall k < N$ . Consequently, for each  $n > N$ , we have  $|a_n|r^n > |a_k|r^k \forall k \neq n$  and therefore,  $f$  admits  $n-1$  zeros inside  $d(0, (r_n)^-)$  and a unique zero in  $C(0, r_n)$ , hence  $f$  admits exactly  $n$  zeros in  $d(0, r_n)$ . Consequently, we have

$$(3) \quad q(f, r_n) = n \quad \forall n \geq N.$$

Since  $q(f, r)$  remains equal to  $q(f, r_n)$  for all  $r \in [r_n, r_{n+1}[$ , by (2) we can derive that

$$(3) \quad \limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r} = \limsup_{n \rightarrow +\infty} \frac{q(f, r_n)}{r_n}$$

Now, for  $n \geq N$ , we have

$$\log \frac{q(f, r_n)}{r_n} = \log(n) - \log(u_{n-1}) + \log(u_n) = \log(n) - \frac{n \log n}{1 + \frac{1}{\phi(n)}} + \frac{(n-1) \log(n-1)}{1 + \frac{1}{\phi(n-1)}}$$

Set  $S_n = \frac{n \log n}{1 + \frac{1}{\phi(n)}} - \frac{(n-1) \log(n-1)}{1 + \frac{1}{\phi(n-1)}}$ . Then

$$(4) \quad \log\left(\frac{q(f, r_n)}{r_n}\right) = \log n - S_n.$$

Now, we have

$$S = \frac{\phi(n)\phi(n-1)(n \log(n) - (n-1) \log(n-1)) + n \log(n)\phi(n) - (n-1) \log(n-1)\phi(n-1)}{(\phi(n)+1)(\phi(n-1)+1)}.$$

Set  $A_n = \frac{\phi(n)\phi(n-1)(n \log(n) - (n-1) \log(n-1))}{(\phi(n)+1)(\phi(n-1)+1)}$  and  $B_n = \frac{n \log(n)\phi(n) - (n-1) \log(n-1)\phi(n-1)}{(\phi(n)+1)(\phi(n-1)+1)}$ . Then  $S_n = A_n + B_n$  and the two both  $A_n, B_n$  are positive. By finite increasing theorem applied to the function  $g(x) = x \log x$ , we have

$$(5) \quad A_n \leq \frac{\phi(n)\phi(n-1)(\log n)}{(\phi(n)+1)(\phi(n-1)+1)}.$$

On the other hand, by finite increasing theorem applied to the function  $h(x) = x(\log x)^{\frac{3}{2}}$ , we have

$$(6) \quad B_n \leq \frac{\phi(n)(\log n + \frac{3}{2})}{(\phi(n)+1)(\phi(n-1)+1)}$$

Then by (1), (5), (6) we have

$$\begin{aligned} \log(q(f, r_n)) - A_n - B_n &\geq \frac{\log n \left( (\phi(n)+1)(\phi(n-1)+1) - \phi(n)\phi(n-1) - \phi(n) \right) - \frac{3}{2}\phi(n)}{(\phi(n)+1)(\phi(n-1)+1)} \\ &= \frac{\log n (\phi(n) + \phi(n-1) + 1 - \phi(n)) - \frac{3\phi(n)}{2}}{(\phi(n)+1)(\phi(n-1)+1)} \\ &= \frac{\log n}{\phi(n)+1} - \frac{3\phi(n)}{2(\phi(n)+1)(\phi(n-1)+1)}. \end{aligned}$$

Now, since  $\phi(n) = \sqrt{\log n}$ , it is obvious that

$$\lim_{n \rightarrow +\infty} \log(q(f, r_n)) - S_n = +\infty$$

and therefore by (3) and (4),  $\psi(f) = +\infty$ .

## 2 Applications to derivatives

**Theorem 10:** Let  $f \in \mathcal{A}(\mathbb{K})$  be not identically zero. Then  $\rho(f) = \rho(f')$ .

**Proof:** By Theorem 6 we have  $\rho(f') = \limsup_{n \rightarrow +\infty} \left( \frac{n \log(n)}{-\log(|(n+1)a_{n+1}|)} \right)$ . But since  $\frac{1}{n} \leq |n| \leq 1$ , we have

$$\limsup_{n \rightarrow +\infty} \left( \frac{n \log(n)}{-\log(|(n+1)a_{n+1}|)} \right) = \limsup_{n \rightarrow +\infty} \left( \frac{n \log(n)}{-\log(|a_{n+1}|)} \right)$$

$$= \limsup_{n \rightarrow +\infty} \left( \frac{(n+1) \log(n+1)}{-\log(|a_{n+1}|)} \right) = \rho(f).$$

**Corollary 10.1:** *The derivation on  $\mathcal{A}(\mathbb{K})$  restricted to the algebra  $\mathcal{A}(\mathbb{K}, t)$  (resp. to  $\mathcal{A}^0(\mathbb{K})$ ) provides that algebra with a derivation.*

In complex analysis, it is known that if an entire function  $f$  has order  $t < +\infty$ , then  $f$  and  $f'$  have same type. We will check that it is the same here.

**Theorem 11:** *Let  $f \in \mathcal{A}(\mathbb{K})$  be not identically zero, of order  $t \in ]0, +\infty[$ . Then  $\sigma(f) = \sigma(f')$ .*

**Proof:** By Theorem 8 we have,

$$\begin{aligned} e\rho(f')\sigma(f') &= \limsup_{n \rightarrow +\infty} \left( n \left( |n+1| |a_{n+1}| \right)^{\frac{t}{n}} \right) = \limsup_{n \rightarrow +\infty} \left( \left( (n+1) \left( |n+1| |a_{n+1}| \right)^{\frac{t}{n}} \right)^{\frac{n}{n+1}} \left( \frac{n}{n+1} \right) \right) \\ &= \limsup_{n \rightarrow +\infty} \left( (n+1) \left( |n+1| |a_{n+1}| \right)^{\frac{t}{n+1}} \right) = e\rho(f)\sigma(f). \end{aligned}$$

But since  $\rho(f) = \rho(f')$  and since  $\rho(f) \neq 0$ , we can see that  $\sigma(f') = \sigma(f)$ .

By Theorems 9 and 10, we can now derive Corollary 11.1:

**Corollary 11.1:** *Let  $f \in \mathcal{A}^0(\mathbb{K})$  be not identically zero, of order  $t < +\infty$ . If  $\psi(f) = \lim_{r \rightarrow +\infty} \frac{q(f, r)}{t}$  and if  $\psi(f') = \lim_{r \rightarrow +\infty} \frac{q(f', r)}{t}$ , then  $\psi(f') = \psi(f)$ .*

Conjecture C1 suggests and implies the following Conjecture C2:

**Conjecture C2**  $\psi(f) = \psi(f') \forall f \in \mathcal{A}^0(\mathbb{K})$ .

**Theorem 12:** *Let  $f, g \in \mathcal{A}(\mathbb{K})$  be transcendental and of same order  $t \in [0, +\infty[$ . Then for every  $\epsilon > 0$ ,*

$$\limsup_{r \rightarrow +\infty} \left( \frac{r^\epsilon q(g, r)}{q(f, r)} \right) = +\infty.$$

**Proof:** Suppose first  $t = 0$ . The proof then is almost trivial. Indeed, for all  $\epsilon > 0$ , we have

$$\lim_{r \rightarrow +\infty} \frac{q(f, r)}{r^\epsilon} = 0 \text{ hence } \lim_{r \rightarrow +\infty} \frac{r^\epsilon}{q(f, r)} = +\infty, \text{ therefore } \lim_{r \rightarrow +\infty} \frac{r^\epsilon q(g, r)}{q(f, r)} = +\infty.$$

Now suppose  $t > 0$ . By Theorem 4, we have  $\limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^t}$  is a finite number  $\ell$  and hence there exists  $\lambda > 0$  such that

$$(1) \quad q(f, r) \leq \lambda r^t \quad \forall r > 1.$$

Now, let us fix  $s \in ]0, t[$ . By hypothesis,  $\rho(g) = \rho(f)$  and hence by Theorem 4, we have

$$\limsup_{r \rightarrow +\infty} \frac{q(g, r)}{r^s} = +\infty \text{ so, there exists an increasing sequence } (r_n)_{n \in \mathbb{N}} \text{ of } \mathbb{R}_+ \text{ such that}$$

$$\lim_{n \rightarrow +\infty} r_n = +\infty \text{ and } \frac{q(g, r_n)}{(r_n)^s} \geq n. \text{ Therefore, by (1), we have}$$

$$\frac{\lambda (r_n)^t q(g, r_n)}{(r_n)^s q(f, r_n)} > \frac{q(g, r_n)}{(r_n)^s} > n$$

and hence

$$\lambda \lim_{n \rightarrow +\infty} \left( \frac{(r_n)^{t-s} q(g, r_n)}{q(f, r_n)} \right) = +\infty.$$

Consequently,

$$(2) \quad \limsup_{r \rightarrow +\infty} \left( \frac{(r)^{t-s} q(g, r)}{q(f, r)} \right) = +\infty.$$

Now, since that holds for all  $s \in ]0, t[$ , the statement derived from (2).

**Remark:** Comparing the number of zeros of  $f'$  to this of  $f$  inside a disk is very uneasy. Now, we can give some precisions. By Theorem 11 we can derive Corollary 12.1:

**Corollary 12.1:** *Let  $f \in \mathcal{A}^0(\mathbb{K})$  be not affine. Then for every  $\epsilon > 0$ , we have*

$$\limsup_{r \rightarrow +\infty} \left( \frac{r^\epsilon q(f', r)}{q(f, r)} \right) = +\infty$$

and

$$\limsup_{r \rightarrow +\infty} \left( \frac{r^\epsilon q(f, r)}{q(f', r)} \right) = +\infty.$$

**Corollary 12.2:** *Let  $f \in \mathcal{A}^0(\mathbb{K})$ . Then  $\psi(f)$  is finite if and only if so is  $\psi(f')$ .*

We can now give a partial solution to a problem that arose in the study of zeros of derivatives of meromorphic functions: given  $f \in \mathcal{A}(\mathbb{K})$ , is it possible that  $f'$  divides  $f$  in the algebra  $\mathcal{A}(\mathbb{K})$ ?

**Theorem 13:** *Let  $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$ . Suppose that for some number  $s > 0$  we have  $\limsup_{r \rightarrow +\infty} |q(f, r)| r^s > 0$  (where  $|q(f, r)|$  is the absolute value of  $q(f, r)$  defined on  $\mathbb{K}$ ). Then  $f'$  has infinitely many zeros that are not zeros of  $f$ .*

**Proof:** Suppose that  $f'$  only has finitely many zeros that are not zeros of  $f$ . Then there exist  $h \in \mathcal{A}(\mathbb{K})$  and  $P \in \mathbb{K}[x]$  such that  $Pf = f'h$ . Without loss of generality, we can assume that  $P$  is monic. Every zero of  $f$  of order  $u \geq 2$  is a zero of  $f'$  of order  $u - 1$  and hence is a zero of  $h$ . And every zero of  $f$  of order 1 is zero of  $h$  of order 1 too. Consequently,  $h$  is not a polynomial.

Set  $f(x) = \sum_{n=0}^{\infty} a_n x^n$ ,  $f'(x) = \sum_{n=0}^{\infty} c_n x^n$ ,  $h(x) = \sum_{n=0}^{\infty} b_n x^n$  and let  $s = \deg(P)$ . Then  $c_n = (n+1)a_{n+1} \forall n \in \mathbb{N}$ . On the other hand, by Theorem A, given any  $r > 0$  we have  $|f|(r) = |a_{q(f,r)}| r^{q(f,r)}$ ,  $|f'|(r) = |c_{q(f',r)}| r^{q(f',r)} = |(q(f',r) + 1)a_{q(f',r)+1}| r^{q(f',r)}$  and  $|h|(r) = |b_{q(h,r)}| r^{q(h,r)}$ . Since  $h$  has infinitely many zeros, there exists  $r_0 > 0$  such that  $q(h, r) \geq s + 2 \forall r \geq r_0$ , assuming that all zeros of  $P$  belong to  $d(0, s)$ . Then since the norm  $|\cdot|(r)$  is multiplicative, we have  $s + q(f, r) = q(f', r) + q(h, r)$ , hence

$$(1) \quad q(f', r) < q(f, r) - 1 \forall r \geq r_0.$$

Then, by (1) we have  $|c_n| r^n < c_{q(f',r)} r^{q(f',r)} \forall n > q(f', r), \forall r \geq r_0$  and particularly,  $|c_{q(f,r)-1}| r^{q(f,r)-1} < |f'|(r) = |c_{q(f',r)}| r^{q(f',r)}$  i.e.

$$(2) \quad |(q(f, r)a_{q(f,r)}| r^{(q(f,r)-1)} < |f'|(r) = |(q(f', r) + 1)a_{q(f',r)+1}| r^{q(f',r)}$$



On the other hand, since  $Pf = f'h$ , we have  $|P|(r)|f|(r) = |f'|(r)|h|(r)$ , hence since  $P$  is monic,

$$(3) \quad r^s |a_{q(f,r)}| r^{q(f,r)} = |(q(f',r) + 1)a_{q(f',r)+1}| r^{q(f',r)} |b_{q(h,r)}| r^{q(h,r)} \quad \forall r > r_0.$$

By (2) we can derive

$$r^{s-1} |q(f,r)a_{q(f,r)}| r^{q(f,r)} < r^s |(q(f',r) + 1)a_{q(f',r)+1}| r^{q(f',r)}$$

and by (3) we have

$$\left( \frac{|q(f,r)|}{r} \right) |(q(f',r) + 1)a_{q(f',r)+1}| r^{q(f',r)} |b_{q(h,r)}| r^{q(h,r)} < r^s |(q(f',r) + 1)a_{q(f',r)+1}| r^{q(f',r)}$$

therefore we obtain  $|b_{q(h,r)}| r^{q(h,r)-1} |q(f,r)| < r^s$ . Consequently,

$$(4) \quad |h|(r) < \frac{r^{s+1}}{|q(f,r)|}$$

Since  $h$  is transcendental, we have  $\lim_{r \rightarrow +\infty} \frac{|h|(r)}{r^m} = +\infty \quad \forall m > 0$ . Now, suppose that for some integer  $m$  we have  $\limsup_{r \rightarrow +\infty} |q(f,r)| r^m > 0$ , hence there exists a constant  $c$  and an increasing sequence  $(r_n)_{n \in \mathbb{N}^*}$  such that  $r_1 > r_0$ ,  $\lim_{n \rightarrow +\infty} r_n = +\infty$  and  $|q(f,r_n)|(r_n)^m > c \quad \forall n$ . Then  $|h|(r_n) < c(r_n)^{s+1+m} \quad \forall n$ , a contradiction to (4). This finishes proving that  $P$  and  $h$  do not exist.

**Remark:** It is possible to deduce the proof of Theorem 13 by using Lemma 1.4 in [3].

**Corollary 13.1:** *Let  $f \in \mathcal{A}^0(\mathbb{K})$ . Then  $f'$  has infinitely many zeros that are not zeros of  $f$ .*

**Proof:** Indeed, let  $f$  be of order  $t$ . By Theorem 4  $\limsup_{r \rightarrow +\infty} \frac{q(f,r)}{r^t}$  is a finite number and therefore  $\limsup_{r \rightarrow +\infty} |q(f,r)| r^t > 0$ .

**Corollary 13.2:** *Let  $f \in \mathcal{A}^0(\mathbb{K})$ . Then  $f'$  does not divide  $f$  in  $\mathcal{A}(\mathbb{K})$ .*

Corollary 13.3 is a partial solution for the  $p$ -adic Hayman conjecture when  $n = 1$ , which is not solved yet.

**Corollary 13.3:** *Let  $f \in \mathcal{M}(\mathbb{K})$  be such that  $\limsup_{r \rightarrow +\infty} |q(\frac{1}{f}, r)| r^s > 0$  for some  $s > 0$ . Then  $ff'$  has at least one zero.*

**Proof:** Indeed, suppose that  $ff'$  has no zero. Then  $f$  is of the form  $\frac{1}{h}$  with  $h \in \mathcal{A}(\mathbb{K})$  and  $f' = -\frac{h'}{h^2}$  has no zero, hence every zero of  $h'$  is a zero of  $h$ , a contradiction to Theorem 13 since  $\limsup_{r \rightarrow +\infty} |q(h,r)| r^s > 0$ .

**Corollary 13.4:** *Suppose  $\mathbb{K}$  has residue characteristic 0. Then for every  $f \in \mathcal{A}(\mathbb{K})$ ,  $f'$  does not divide  $f$  in  $\mathcal{A}(\mathbb{K})$ .*

**Remarks:** 1) Concerning complex entire functions, we can check that the exponential is of order 1 but is divided by its derivative in the algebra of complex entire functions.

2) It is also possible to derive Corollary 13.4 from Theorem 1 in [2]. Indeed, let  $g = \frac{1}{f}$ . By Theorem 4,  $\limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^t}$  is a finite number. Consequently, there exists  $c > 0$  such that  $q(f, r) \leq cr^t \forall r > 1$  and therefore the number of poles of  $g$  in  $d(0, r)$  is upper bounded by  $cr^t$  whenever  $r > 1$ . Consequently, we can apply Theorem 1 [2] and hence the meromorphic function  $g'$  has infinitely many zeros. Now, suppose that  $f'$  divides  $f$  in  $\mathcal{A}(\mathbb{K})$ . Then every zero of  $f'$  is a zero of  $f$  with an order superior, hence  $\frac{f'}{f^2}$  has no zero, a contradiction.

3) If the residue characteristic of  $\mathbb{K}$  is  $p \neq 0$ , we can easily construct an example of entire function  $f$  of infinite order such that  $f'$  does not divide  $f$  in  $\mathcal{A}(\mathbb{K})$ . Let  $f(x) = \prod_{n=0}^{\infty} (1 - \frac{x}{\alpha_n})^{p^n}$  with  $|\alpha_n| = n + 1$ . We check that  $q(f, n + 1) = \sum_{k=0}^n p^k$  is prime to  $p$  for every  $n \in \mathbb{N}$ . Consequently, Theorem 13 shows that  $f$  is not divided by  $f'$  in  $\mathcal{A}(\mathbb{K})$ . On the other hand, fixing  $t > 0$ , we have

$$\frac{q(f, n + 1)}{(n + 1)^t} \geq \frac{p^n}{(n + 1)^t}$$

hence

$$\limsup_{r \rightarrow +\infty} \frac{q(f, r)}{r^t} = +\infty \forall t > 0$$

therefore,  $f$  is not of finite order.

Theorem 13 suggests the following conjecture:

**Conjecture C3:** *Given  $f \in \mathcal{A}(\mathbb{K})$  (other than  $(x - a)^m$ ,  $a \in \mathbb{K}$ ,  $m \in \mathbb{N}$ ) there exists no  $h \in \mathcal{A}(\mathbb{K})$  such that  $f = f'h$ .*

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