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Unique range sets of 5 points for unbounded analytic functions inside an open disk

Alain Escassut and Jacqueline Ojeda¹

Abstract

Let \mathbb{K} be a complete algebraically closed p -adic field of characteristic $p \geq 0$ and let $\mathcal{A}_u(d(a, R^-))$ be the set of unbounded analytic functions inside the disk $d(a, R^-) = \{x \in \mathbb{K} \mid |x - a| < R\}$. We recall the definition of urscm and the ultrametric Nevanlinna Theory on 3 small functions in order to find new urscm for $\mathcal{A}_u(d(a, R^-))$. Results depend on the characteristic. In characteristic 0, we can find urscm of 5 points. Some results on bi-urscm are given for meromorphic functions.

1 Introduction and main result

We shall introduce URSCM for p -adic meromorphic functions. Many studies were made in the eighties and the nineties concerning URSCM for functions in \mathbb{C} , [3], [6], [16]. Studies were also made in the non-archimedean context by the late nineties and next [1], [2], [3], [4], [5], [8], [9], [10], [11], [13]. Here, we will only consider the situation in an ultrametric field.

Definitions and notation: Throughout the paper, E is an algebraically closed field of characteristic $p \geq 0$ without any assumption on the existence of an absolute value. A subset S of E is said to be *affinely rigid* if there is no similarity t on E other than the identity, such that $t(S) = S$.

We denote by \mathbb{K} an algebraically closed field complete with respect to an ultrametric absolute value $|\cdot|$ and of characteristic $p \geq 0$. We will denote by q the characteristic exponent of \mathbb{K} : if $p \neq 0$, then $q = p$ and if $p = 0$ then $q = 1$.

Given $\alpha \in \mathbb{K}$ and $R \in \mathbb{R}_+^*$, we denote by $d(\alpha, R)$ the disk $\{x \in \mathbb{K} \mid |x - \alpha| \leq R\}$, by $d(\alpha, R^-)$ the disk $\{x \in \mathbb{K} \mid |x - \alpha| < R\}$, by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of analytic functions in \mathbb{K} (i.e. the set of power series with an infinite radius of convergence) and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} (i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$).

In the same way, given $\alpha \in \mathbb{K}$, $R > 0$ we denote by $\mathcal{A}(d(\alpha, R^-))$ the \mathbb{K} -algebra of analytic functions in $d(\alpha, R^-)$ (i.e. the set of power series with

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an radius of convergence $\geq R$) and by $\mathcal{M}(d(\alpha, R^-))$ the field of fractions of $\mathcal{A}(d(\alpha, R^-))$. We then denote by $\mathcal{A}_b(d(\alpha, R^-))$ the \mathbb{K} -algebra of bounded analytic functions in $d(\alpha, r^-)$ and by $\mathcal{M}_b(d(\alpha, r^-))$ the field of fractions of $\mathcal{A}_b(d(\alpha, r^-))$. And we set $\mathcal{A}_u(d(\alpha, R^-)) = \mathcal{A}(d(\alpha, R^-)) \setminus \mathcal{A}_b(d(\alpha, R^-))$ and $\mathcal{M}_u(d(\alpha, R^-)) = \mathcal{M}(d(\alpha, R^-)) \setminus \mathcal{M}_b(d(\alpha, R^-))$.

Given a family of functions \mathcal{F} defined in \mathbb{K} or in a subset S of \mathbb{K} (resp. in E or in a subset S of E), with values in \mathbb{K} (resp. in E), S is called an *ursim for* \mathcal{F} if for any two non-constant functions $f, g \in \mathcal{F}$ satisfying $f^{-1}(S) = g^{-1}(S)$, these functions are equal.

That definition particularly applies to $\mathcal{A}(\mathbb{K})$, $\mathcal{M}(\mathbb{K})$, $\mathcal{A}_u(d(a, R^-))$, $\mathcal{M}_u(d(a, R^-))$, $\mathbb{K}[x]$, $\mathbb{K}(x)$, $E[x]$, $E(x)$.

We will now recall the definition of URSCM. Given a subset S of E and $f \in E(x)$, we denote by $\mathcal{E}(f, S)$ the set in $E \times \mathbb{N}^*$:

$$\bigcup_{a \in S} \{(z, q) \in E \times \mathbb{N}^* \mid z \text{ is a zero of order } q \text{ of } f(x) - a\}.$$

Similarly, consider now meromorphic functions in the field \mathbb{K} . For a subset S of \mathbb{K} and $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a, R^-))$) we denote by $\mathcal{E}(f, S)$ the set in $\mathbb{K} \times \mathbb{N}^*$: $\bigcup_{a \in S} \{(z, q) \in \mathbb{K} \times \mathbb{N}^* \mid z \text{ is a zero of order } q \text{ of } f(x) - a\}$.

Let \mathcal{F} be a non-empty subset of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{M}(\mathbb{K})$, resp. of $\mathcal{A}(d(a, R^-))$, resp. of $\mathcal{M}(d(a, R^-))$). We say that two non-constant functions $f, g \in \mathcal{F}$ *share* S , *counting multiplicity* if $\mathcal{E}(f, S) = \mathcal{E}(g, S)$; and the set S is called a *unique range set counting multiplicity* (an *URSCM* in brief) *for* \mathcal{F} if for any two non-constant $f, g \in \mathcal{F}$ sharing S counting multiplicity, one has $f = g$. Next, the set S will be called a *bi-URSCM* for \mathcal{F} if for two non-constant functions $f, g \in \mathcal{M}_u(d(a, R^-))$ sharing S counting multiplicity and having the same poles, counting multiplicity, one has $f = g$ [8].

Particularly, if we consider a family $\mathcal{F} \subset \mathcal{A}(K)$ or $\mathcal{F} \subset \mathcal{A}_u(d(a, R^-))$ and a set $S = \{a_1, \dots, a_t\} \subset \mathbb{K}$ (resp. a set $S = \{a_1, \dots, a_t\} \subset E$) with $a_i \neq a_j \forall i \neq j$, we can set $P(X) = \prod_{j=1}^t (X - a_j)$ and then the set $S = \{a_1, \dots, a_t\}$ is an URSCM for \mathcal{F} if for any two functions $f, g \in \mathcal{F}$ such that $P \circ f$ and $P \circ g$ have the same zeros with the same multiplicity, then $f = g$.

Similarly, if we consider a family $\mathcal{F} \subset \mathcal{M}(K)$ or $\mathcal{F} \subset \mathcal{M}_u(d(a, R^-))$ and a set $S = \{a_1, \dots, a_t\} \subset \mathbb{K}$ (resp. a set $S = \{a_1, \dots, a_t\} \subset E$) with $a_i \neq a_j \forall i \neq j$, we can set $P(X) = \prod_{j=1}^t (X - a_j)$ and then the set $S = \{a_1, \dots, a_t\}$ is a bi-URSCM for \mathcal{F} if for any two functions $f, g \in \mathcal{F}$ having the same poles (counting multiplicity) such that $P \circ f$ and $P \circ g$ have the same zeros with the same multiplicity, then $f = g$.

Remark: An URSCM S for a family of functions $\mathcal{F} = \mathcal{M}(\mathbb{K})$, $\mathcal{A}(\mathbb{K})$, $\mathcal{M}_u(d(a, R^-))$, $\mathcal{A}_u(d(a, R^-))$ must obviously be affinely rigid. Indeed suppose that S is not affinely rigid and let t be a similarity of \mathbb{K} such that $t(S) = S$. Then, if f belongs to \mathcal{F} , so does $f \circ t$ and therefore we can check that $\mathcal{E}(f, S) =$

$\mathcal{E}(f \circ t, S)$. And it is a bi-URSCM if for any two functions $f, g \in \mathcal{F}$ such that $P \circ f$ and $P \circ g$ have the same zeros and the same poles, counting multiplicity, then $f = g$.

Similar definitions were given for meromorphic functions on \mathbb{C} before these questions were examined on the field \mathbb{K} . URSCM of only 11 points for complex meromorphic functions in the whole field \mathbb{C} were found in [16] and the same method showed the existence of URSCM of only 7 points for complex entire functions. So far, they are the smallest known in \mathbb{C} .

In the field \mathbb{K} , the same method lets us find URSCM of 11 points for $\mathcal{M}_u(d(a, R^-))$ and URSCM of 10 points for $\mathcal{M}(\mathbb{K})$.

In 1996, URSCM for polynomials on a field such as E were characterized: they are just the affinely rigid subsets of E [9]. Particularly, the smallest URSCM for polynomials are the affinely rigid sets of 3 points. Concerning entire functions on the field \mathbb{K} , URSCM of 3 points were found: they also are the affinely rigid sets of 3 points [9]. Next, URSCM of 7 points were found for unbounded analytic functions in a disk $d(a, R^-)$ [10]. Here we will show the existence of another family of URSCM for $\mathcal{A}_u(d(a, R^-))$, looking for sets of less than 7 points.

The notion of URSCM is closely linked to that of strong uniqueness polynomial.

Definition: A polynomial $P \in \mathbb{K}[x]$ is called a *strong uniqueness polynomial* for a subset $\mathcal{F} \subset E(x)$ (resp. $\mathcal{F} \subset \mathcal{M}(\mathbb{K})$, resp. $\mathcal{F} \subset \mathcal{M}(d(a, R^-))$) if, given $f, g \in \mathcal{F}$, the equality $P(f) = P(g)$ implies $f = g$.

The following basic result is immediate and useful to understand the role of URSCM:

Proposition A: Let $S = \{a_1, \dots, a_n\} \subset E$, (resp. $S = \{a_1, \dots, a_n\} \subset \mathbb{K}$), let $a \in \mathbb{K}$, let $R \in \mathbb{R}_+^*$ and let $P(x) = \prod_{i=1}^n (x - a_i)$. Given any two functions $f, g \in E[x]$ (resp. $f, g \in \mathcal{A}(\mathbb{K})$, resp. $f, g \in \mathcal{A}(d(a, R^-))$) then $\mathcal{E}(f, S) = \mathcal{E}(g, S)$ if and only if $\frac{P(f)}{P(g)}$ is a constant in E^* (resp. is a constant in \mathbb{K}^* , resp. is an invertible function in $\mathcal{A}(d(a, R^-))$). Given any two functions $f, g \in E(x)$ (resp. $f, g \in \mathcal{M}(\mathbb{K})$, resp. $f, g \in \mathcal{M}(d(a, R^-))$) having the same poles counting multiplicity, then $\mathcal{E}(f, S) = \mathcal{E}(g, S)$ if and only if $\frac{P(f)}{P(g)}$ is a constant in E^* (resp. is a constant in \mathbb{K}^* , resp. is an invertible function in $\mathcal{A}(d(a, R^-))$).

Corollary A1 Let $S = \{a_1, \dots, a_n\} \subset \mathbb{K}$ (resp. let $S = \{a_1, \dots, a_n\} \subset E$) and let $P(x) = \prod_{i=1}^n (x - a_i)$. Then P is a polynomial of strong uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $E[x]$) if and only if $S = \{a_1, \dots, a_n\}$ is an URSCM for $\mathcal{A}(\mathbb{K})$ (resp. for $E[x]$).

Remark: Let $P(x) = x^4 - 4x^3$ and let j be a primitive 3-rd root of 1. Clearly, $P(jf) = jP(f) \forall f \in \mathcal{M}(\mathbb{K})$, hence P is not a polynomial of strong uniqueness for $\mathcal{A}(\mathbb{K})$ or for $E[x]$.

As usual, given $a \in \mathbb{K}$ and $n \in \mathbb{N}$, we denote by $\sqrt[n]{a}$ the unique $b \in \mathbb{K}$ such that $b^{(p^n)} = a$.

Given $m, n \in \mathbb{N}$ we set $m \prec n$ if m divides n and $m \not\prec n$ if m does not divide n . When $p \neq 0$, we denote by \mathcal{S} the \mathbb{F}_p -automorphism of \mathbb{K} defined by $\mathcal{S}(x) = \sqrt[p]{x}$. More generally this mapping has continuation to a \mathbb{K} -algebra automorphism of $\mathbb{K}[X]$ as $\mathcal{S}(c \prod_{j=1}^n (X - a_j)) = \mathcal{S}(c) \prod_{j=1}^n (X - \mathcal{S}(a_j))$, $c \in \mathbb{K}$.

Proposition B: Suppose $p \neq 0$. Let $r > 0$ and let $f \in \mathcal{M}(d(a, r^-))$. Then $\sqrt[p]{f}$ belongs to $\mathcal{M}(d(a, r^-))$ if and only if $f' = 0$. Moreover, there exists a unique $t \in \mathbb{N}$ such that $\sqrt[p^t]{f} \in \mathcal{M}(d(a, r^-))$ and $(\sqrt[p^t]{f})' \neq 0$.

Proof: If f is of the form l^p with $l \in \mathcal{M}(d(a, r^-))$, then of course we have $f' = 0$. Now, suppose that $f' = 0$. If $f \in \mathcal{A}(d(a, r^-))$, then obviously all non-zero coefficients have an index multiple of p , hence f is of the form l^p , with $l \in \mathcal{A}(d(a, r^-))$. We now consider the general case when $f \in \mathcal{M}(d(a, r^-))$. Let $(b_n, t_n)_{n \in \mathbb{N}}$ be the sequence of poles of f inside $d(a, r^-)$ where t_n is the multiplicity order of b_n . By Theorem 25.5 [14] we can find $h \in \mathcal{A}(d(a, r^-))$ such that $\omega_{b_n}(h) \geq t_n \forall n \in \mathbb{N}$. Clearly fh^p belongs to $\mathcal{A}(d(a, r^-))$ and satisfies $(fh^p)' = 0$. Consequently, fh^p is of the form g^p , with $g \in \mathcal{A}(d(a, r^-))$, therefore $f = \left(\frac{g}{h}\right)^p$. On the other hand, the set of integers s such that $\sqrt[p^s]{f}$ belongs to $\mathcal{M}(d(a, r^-))$ is obviously bounded and therefore admits a biggest element, which ends the proof.

Definition and notation: Suppose $p \neq 0$. Given, $f \in \mathcal{M}(d(a, r^-))$, we will call *ramification index of f* the integer t such that $\sqrt[p^t]{f} \in \mathcal{M}(d(a, r^-))$ and $(\sqrt[p^t]{f})' \neq 0$.

In the same way, given an algebraically closed field B of characteristic $p \neq 0$ and $P(x) \in B[x]$, we call *ramification index of P* the unique integer t such that $\sqrt[p^t]{P} \in B[x]$ and $(\sqrt[p^t]{P})' \neq 0$. This ramification index will be denoted by $\text{ram}(f)$ for any $f \in \mathcal{M}(d(a, r^-))$ or $f \in \mathcal{M}(\mathbb{K})$ and similarly it will be denoted by $\text{ram}(P)$ for any $P \in B[x]$.

Henceforth, given $t \in \mathbb{N}^*$, we will denote by $\mathcal{A}_t(d(a, R^-))$ the subset of the functions $f \in \mathcal{A}(d(a, R^-))$ having a ramification index $\leq t$ and similarly, we put $\mathcal{A}_{u,t}(d(a, R^-)) = \mathcal{A}_t(d(a, R^-)) \cap \mathcal{A}_u(d(a, R^-))$.

Given $k \in \mathbb{K}^*$ and $n, m \in \mathbb{N}^*$ with $m < n$, we set $Q_{n,m,k}(x) = x^n - x^m + k$ and we denote by $Y_{n,m,k}$ the set of zeros of $Q_{n,m,k}$. In the same way, we set $Q_{n,k}(x) = x^n - x^{n-1} + k$ and we denote by $Y_{n,k}$ the set of zeros of $Q_{n,k}$.

Remark: Suppose $p \neq 0$ and let $f \in \mathcal{M}(d(a, r^-))$ have ramification index t as an element of $\mathcal{M}(d(a, r^-))$. For every $r' \in]0, r[$, f has the same ramification index as an element of $\mathcal{M}(d(a, r'^-))$ because of course, on one hand, $\sqrt[p^t]{f} \in \mathcal{M}(d(a, r'^-))$ and on the other hand, by properties of analytic functions, $(\sqrt[p^t]{f})'$ is not identically zero inside $d(a, r')$.

As recalled above, in [9] the smallest urscm for $\mathcal{A}_u(d(a, R^-))$ have 7 points. By Corollary 2.2 we can find a new family of urscm for $\mathcal{A}_u(d(a, R^-))$, with particularly urscm of 5 points.

Theorem 1: *Let $t \in \mathbb{N}^*$ and let $f, g \in \mathcal{M}_u(d(a, R^-))$ be such that the function $\phi = \frac{f^n - f^m + k}{g^n - g^m + k}$ is invertible in $\mathcal{A}(d(a, R^-))$. Let t be the ramification index of $\frac{f^n - f^m - k(\phi - 1)}{f^n - f^m}$. If $2mq^t > n(2q^t - 1) + 3q^t$ then $f = g$.*

Corollary 1.1 : *Suppose \mathbb{K} is of characteristic 0. If $2m > n + 3$ then $Y(n, m, k)$ is a bi-urscm for $\mathcal{M}_u(d(a, R^-))$.*

Corollary 1.2 : *Suppose \mathbb{K} is of characteristic 0. If $n \geq 6$, then $Y(n, k)$ is a bi-urscm for $\mathcal{M}_u(d(a, R^-))$.*

Theorem 2: *Let $t \in \mathbb{N}^*$ and let $f, g \in \mathcal{A}_u(d(a, R^-))$ be such that the function $\phi = \frac{f^n - f^m + k}{g^n - g^m + k}$ is invertible in $\mathcal{A}(d(a, R^-))$. Let t be the ramification index of $\frac{f^n - f^m - k(\phi - 1)}{f^n - f^m}$. If $2mq^t > n(2q^t - 1) + 2q^t$ then $f = g$.*

Corollary 2.1 : *Suppose \mathbb{K} is of characteristic 0. If $2m \geq n + 3$, then $Y(n, m, k)$, is an urscm for $\mathcal{A}_u(d(a, R^-))$.*

Corollary 2.2 : *Suppose \mathbb{K} is of characteristic 0. If $n \geq 5$, then $Y(n, k)$, is an urscm for $\mathcal{A}_u(d(a, R^-))$.*

Remark: We don't know whether there exists an urscm for $\mathcal{A}_u(d(a, R^-))$ of 4 points or 3 points.

2 The Proof

We must recall the definition of the counting functions in the Nevanlinna Theory.

Definitions and notation: Let $f \in \mathcal{M}(d(a, R^-))$ and let $\alpha \in d(a, R^-)$. If f admits α as a zero of order q , we set $\omega_\alpha(f) = q$; if f admits α as a pole of order q , we set $\omega_\alpha(f) = -q$; and if α is neither a zero nor a pole for f , we set $\omega_\alpha(f) = 0$.

We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$ in the following way:

Let (a_n) , $1 \leq n \leq \sigma(r)$ be the finite sequence of zeros of f such that $0 < |a_n| \leq |a_{n+1}| \leq |a_{\sigma(r)}| \leq r$, of respective order s_n .

We set $Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{n=1}^{\sigma(r)} s_n (\log r - \log |a_n|)$.

Similarly, we set $N(r, f) = Z(r, \frac{1}{f})$.

In order to define the counting function of zeros of f without multiplicity, we put $\overline{\omega}_0(f) = 0$ if $\omega_0(f) \leq 0$ and $\overline{\omega}_0(f) = 1$ if $\omega_0(f) \geq 1$.

In the sequel, I will denote an interval of the form $[\rho, +\infty[$, with $\rho > 0$, and J will denote an interval of the form $[\rho, R]$.

Next, denoting by $E(r, f)$ the set $\{a \in d(0, r) \mid \omega_a(f) > 0, p^{\text{ram}(f)+1} \nmid \omega_a(f)\}$,

if $0 \notin E(r, f)$ we set $\tilde{Z}(r, f) = \sum_{\alpha \in E(r, f)} \log \frac{r}{|\alpha|}$

and if $0 \in E(r, f)$ we set $\tilde{Z}(r, f) = \log r + \sum_{\alpha \in E(r, f), \alpha \neq 0} \log \frac{r}{|\alpha|}$.

Similarly we define $\tilde{N}(r, f) = \tilde{Z}(r, \frac{1}{f})$.

We can now define the Nevanlinna characteristic function of f : $T(r, f) = \max(Z(r, f), T(r, f))$.

Assume that f' is not identically 0.

Let $V(r, f) = \{a \in d(0, r) \mid \omega_a(f) < 0, p^{\text{ram}(f)+1} \prec \omega_a(f)\}$. We put

$$N_0(r, f') = \sum_{\alpha \in V(r, f)} [\omega_\alpha(f') - \omega_\alpha(f)] \log \frac{r}{|\alpha|}.$$

Given a finite subset S of \mathbb{K} , we put $\Lambda'(r, f, S) = \{a \in d(0, r) \mid f'(a) = 0, f(a) \notin S\}$ and $\Lambda''(r, f, S) = \{a \in d(0, r) \mid p^{\text{ram}(f)+1} \prec \omega_a(f - f(a)), f(a) \in S\}$. Then we can define

$$Z_0^S(r, f') = \sum_{\alpha \in \Lambda'(r, f, S)} \omega_\alpha(f') \log \frac{r}{|\alpha|} + \sum_{\alpha \in \Lambda''(r, f, S)} [\omega_\alpha(f') - \omega_\alpha(f - f(\alpha))] \log \frac{r}{|\alpha|}.$$

Remarks: 1) It is easily verified that all the above functions are positive.

2) If $p = 0$, we have $\overline{Z}(r, f) = \tilde{Z}(r, f)$ and $\overline{N}(r, f) = \tilde{N}(r, f)$.

Lemma 1: Let $f \in \mathcal{M}(d(0, R^-))$, let $t = r(f)$ and let $g = \sqrt[t]{f}$. Then $\tilde{Z}(r, f) = \tilde{Z}(r, g)$ and $\tilde{N}(r, f) = \tilde{N}(r, g)$.

Proof: Let a be a zero of f and let $s = \omega_a(f)$. Then s is of the form nt with $n \in \mathbb{N}^*$. If $n = 1$, then a belongs to both $E(r, f)$ and $E(r, g)$; and if $n > 1$, then $a \notin E(r, f)$. But then a is a zero of order n of g and hence, a does not belong to $E(r, g)$.

The following Lemmas 2 and 3 are easily checked [12]:

Lemma 2: Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ be pairwise distinct, let $P(u) = \prod_{i=1}^n (u - \alpha_i)$

and let $f \in \mathcal{M}(d(0, R^-))$. Then $Z(r, P(f)) = \sum_{i=1}^n Z(r, f - \alpha_i)$ and $\tilde{Z}(r, P(f)) =$

$$\sum_{i=1}^n \tilde{Z}(r, f - \alpha_i).$$

Lemma 3: Let $f \in \mathcal{M}(d(0, R^-))$ be such that f' is not identically zero and let $\alpha \in d(0, R^-)$. We have $\omega_\alpha(f') = \omega_\alpha(f) - 1$ if $p \nmid \omega_\alpha(f)$ and $\omega_\alpha(f') \geq \omega_\alpha(f)$ if $p \mid \omega_\alpha(f)$.

Lemmas 4 and 5 are consequences of Lemma 3.

Lemma 4: *Let $f \in \mathcal{M}(d(0, R^-))$ be such that $f' \neq 0$ and let S be a finite subset of \mathbb{K} . Then:*

$$\sum_{b \in S} \left(Z(r, f - b) - \tilde{Z}(r, f - b) \right) = Z(r, f') - Z_0^S(r, f').$$

Lemma 5: *Let $f \in \mathcal{M}(d(0, R^-))$ be such that $f' \neq 0$ and let $0 < r < R$. Then $N(r, f') = N(r, f) + \tilde{N}(r, f) - N_0(r, f')$.*

Lemma 6: *Let $f \in \mathcal{M}(d(0, R^-))$ be such that $f' \neq 0$ and let $0 < r < R$. Then:*

$$Z(r, f') \leq Z(r, f) + \tilde{N}(r, f) - N_0(r, f') - \log r + O(1), (r \in J).$$

Proof: Without loss of generality, up to change of variable, we can assume that both f and f' have no zero and no pole at 0. Let $|f|(r)$ denote the circular value of f defined as $|f|(r) = \lim_{|x| \rightarrow r, |x| \neq r} |f(x)|$

By classical results such as Theorem 23.13 [14], we have $Z(r, f) - N(r, f) = \log(|f|(r)) - \log(|f(0)|)$, and $Z(r, f') - N(r, f') = \log(|f'|(r)) - \log(|f'(0)|)$. But, it is well-known that $|f'|(r) \leq \frac{|f|(r)}{r}$ (Theorem 1.5.10 [15]); hence we obtain

$$Z(r, f') \leq N(r, f') - N(r, f) + Z(r, f) - \log r + O(1).$$

Moreover, by Lemma 4 we have $N(r, f') - N(r, f) = \tilde{N}(r, f) - N_0(r, f')$, which completes the proof.

We know Proposition C [9].

Proposition C : *Let $f \in \mathcal{M}(d(0, R^-))$. Then f belongs to $\mathcal{M}_b(d(0, R^-))$ if and only if $T(r, f)$ is bounded when r tends to R .*

Corollary C1: *Let $f \in \mathcal{M}(d(0, R^-))$. Then $\mathcal{M}_b(d(0, R^-))$ is a subset of $\mathcal{M}_f(d(0, R^-))$ and $\mathcal{A}_b(d(0, R^-))$ is a subset of $\mathcal{A}_f(d(0, R^-))$.*

Remark: Particularly, an invertible function $f \in \mathcal{A}(d(a, R^-))$ has a constant absolute value and therefore lies in $\mathcal{A}_b(d(a, R^-))$.

The following Theorem D1 is known as Second Main Theorem on Three Small Functions [17]. It holds in p-adic analysis as well as in complex analysis, where it was shown first [17]. Notice that this theorem was generalized to any finite set of small functions by Yamanoy in complex analysis [18], through methods that have no equivalent on a p-adic field.

Remark: Let $f \in \mathcal{M}_u(d(0, R^-))$ and let $w \in \mathcal{M}_b(d(0, R^-))$. Then of course, $w \in \mathcal{M}_f(d(0, R^-))$.

The previous results enable us to prove the ultrametric Nevanlinna Main Theorem in a basic form:

Theorem D1: Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, with $n \geq 2$, and let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(\mathbb{K})$) of ramification index t . Let $S = \{\sqrt[t]{\alpha_1}, \dots, \sqrt[t]{\alpha_n}\}$. Then we have:

$$\frac{(n-1)T(r, f)}{q^t} \leq \sum_{i=1}^n \tilde{Z}(r, f - \alpha_i) + Z(r, (\sqrt[t]{f})') - Z_0^S(r, (\sqrt[t]{f})') + O(1) \quad \forall r \in J$$

(resp. $\forall r \in I$).

Moreover, if f belongs to $\mathcal{A}(d(0, R^-))$ (resp. $f \in \mathcal{A}(\mathbb{K})$), then

$$\frac{nT(r, f)}{q^t} \leq \sum_{i=1}^n \tilde{Z}(r, f - \alpha_i) + Z(r, (\sqrt[t]{f})') - Z_0^S(r, (\sqrt[t]{f})') + O(1) \quad \forall r \in J$$

(resp. $\forall r \in I$).

Now, following the same method as in Theorem 2.5.9 [15], we can obtain that classical form of the Nevalinna inequality where \bar{Z} and \bar{N} are replaced by \tilde{Z} and \tilde{N} .

Theorem D2: Let $\alpha_1, \dots, \alpha_n \in \mathbb{K}$, with $n \geq 2$, and let $f \in \mathcal{M}(d(0, R^-))$ (resp. $f \in \mathcal{M}(\mathbb{K})$) of ramification index t . Let $S = \{\sqrt[t]{\alpha_1}, \dots, \sqrt[t]{\alpha_n}\}$. Then we have:

$$\frac{(n-1)T(r, f)}{q^t} \leq \sum_{i=1}^n \tilde{Z}(r, f - \alpha_i) + \tilde{N}(r, f) - Z_0^S(r, (\sqrt[t]{f})') - N_0(r, (\sqrt[t]{f})') - \log r + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I).$$

Proof of Theorems D1 and D2: The proof of Theorems D1 and D2 was given in [12]. We will recall it. For convenience, we put $g = \sqrt[t]{f}$, and $\beta_i = \sqrt[t]{\alpha_i}$ for every $i = 1, \dots, n$. So $S = \{\beta_1, \dots, \beta_n\}$.

Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(a, R^-))$) and let $(a_n, s_n)_{n \in \mathbb{N}}$ be the set of zeros of f in \mathbb{K} (resp. in $d(a, R^-)$) with $|a_n| \leq |a_{n+1}|$ whereas s_n is the order of multiplicity of a_n . We denote by $\mathcal{D}(f)$ the sequence $(a_n, s_n)_{n \in \mathbb{N}}$.

By Theorem 25.5 [14] there exist $\phi, \psi \in \mathcal{A}(d(0, R^-))$ such that $g = \frac{\phi}{\psi}$, and

$$(1) \quad Z(r, \phi) \leq Z(r, g) + 1,$$

$$(2) \quad Z(r, \psi) \leq N(r, g) + 1.$$

By Lemma 2.5.5 [15], there exists $A \in \mathbb{R}$ and for any $r \in J$ (resp. $r \in I$), there exists $l(r) \in \{1, \dots, n\}$ such that $Z(r, \phi - \beta_j \psi) \geq \max(Z(r, \phi), Z(r, \psi)) + A \quad \forall j \neq l(r)$, therefore there exists $B \in \mathbb{R}$ such that

$$(3) \quad Z(r, \phi - \beta_i \psi) \geq T(r, g) + B \quad \forall i \neq l(r), \quad \forall r \in J \text{ (resp. } \forall r \in I).$$

We check that $\mathcal{D}(\phi) - \mathcal{D}(\frac{\phi}{\psi}) = \mathcal{D}(\psi) - \mathcal{D}(\frac{\psi}{g})$, therefore

$$\mathcal{D}(\phi - \beta_i \psi) = \mathcal{D}(g - \beta_i) + \mathcal{D}(\psi) - \mathcal{D}(\frac{1}{g - \beta_i}) = \mathcal{D}(g - \beta_i) + \mathcal{D}(\psi) - \mathcal{D}(\frac{1}{g}).$$

Then, applying counting functions, we have $Z(r, \phi - \beta_i \psi) = Z(r, g - \beta_i) + Z(r, \psi) - N(r, g)$, and therefore, by (2), we obtain

$$(4) \quad Z(r, \phi - \beta_i \psi) \leq Z(r, g - \beta_i) + 1.$$

Then, by (3) and (4) we obtain $(n-1)(T(r, g) + B)$

$$\leq \sum_{\substack{1 \leq i \leq n, \\ i \neq l(r)}} Z(r, \phi - \beta_i \psi) \leq \sum_{\substack{1 \leq i \leq n, \\ i \neq l(r)}} Z(r, g - \beta_i) + n - 1 \quad \forall r \in J \text{ (resp. } \forall r \in I).$$

Putting $M = (n-1)(1-B)$, we obtain:

$$(5) \quad (n-1)T(r, g) \leq \sum_{i=1}^n Z(r, g - \beta_i) + M - Z(r, g - \beta_{l(r)}) \quad \forall r \in J \text{ (resp. } \forall r \in I).$$

By Lemma 4, we have

$$\sum_{i=1}^n Z(r, g - \beta_i) = \sum_{i=1}^n \tilde{Z}(r, g - \beta_i) + Z(r, g') - Z_0^S(r, g'), \text{ hence by (5) we obtain,}$$

$$(6) \quad (n-1)T(r, g) \leq \sum_{i=1}^n \tilde{Z}(r, g - \beta_i) + Z(r, g') - Z_0^S(r, g') - Z(r, g - \beta_{l(r)}) + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I).$$

Now, since $T(r, g) = \frac{T(r, f)}{q^t}$ and since $\tilde{Z}(r, g - \beta_i) = \tilde{Z}(r, f - \alpha_j) \quad \forall j = i, \dots, n$, we obtain

$$\frac{(n-1)T(r, f)}{q^t} \leq \sum_{i=1}^n \tilde{Z}(r, f - \alpha_i) + Z(r, (\sqrt[t]{f})') - Z_0^S(r, (\sqrt[t]{f})') + O(1) \quad \forall r \in J$$

(resp. $\forall r \in I$).

Suppose now that f belongs to $\mathcal{A}(d(a, R^-))$ or to $\mathcal{A}(\mathbb{K})$. Then so does g . By Lemma 2.5.5 [15] we have $Z(r, g - \beta_{l(r)}) = T(r, g) + O(1) \quad \forall r \in J$ (resp. $\forall r \in I$) so, by (6) we obtain

$$nT(r, g) \leq \sum_{i=1}^n \tilde{Z}(r, g - \beta_i) + Z(r, g') - Z_0^S(r, g') + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I),$$

and consequently,

$$\frac{nT(r, f)}{q^t} \leq \sum_{i=1}^n \tilde{Z}(r, f - \alpha_i) + Z(r, (\sqrt[t]{f})') - Z_0^S(r, (\sqrt[t]{f})') + O(1) \quad \forall r \in J \text{ (resp. } \forall r \in I).$$

Now, returning to the general case, we have $g' = (g - \beta_{l(r)})'$ and $\tilde{N}(r, g) = \tilde{N}(r, g - \beta_{l(r)})$. So, by Lemma 6, we have:

$$(7) \quad Z(r, g') - Z(r, g - \beta_{l(r)}) \leq \tilde{N}(r, g) - N_0(r, g') - \log r + O(1).$$

Finally, by (6), (7) we obtain

$$\frac{(n-1)T(r, f)}{q^t} \leq \sum_{i=1}^n \tilde{Z}(r, f - \alpha_i) + \tilde{N}(r, f) - Z_0^S(r, (\sqrt[t]{f})') - N_0(r, (\sqrt[t]{f})') -$$

$\log r \quad \forall r \in J$ (resp. $\forall r \in I$).

That completes the proof.

Theorem D3: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$) and let $u_1, u_2, u_3 \in \mathcal{M}_f(\mathbb{K})$ (resp. $u_1, u_2, u_3 \in \mathcal{M}_f(d(0, R^-))$) be pairwise distinct. Let*

$$\phi(x) = \frac{(f(x) - u_1(x))(u_2(x) - u_3(x))}{(f(x) - u_3(x))(u_2(x) - u_1(x))} \text{ and let } t \text{ be the ramification index of } \phi.$$

$$\text{Then } \frac{T(r, f)}{q^t} \leq \sum_{j=1}^3 \tilde{Z}(r, f - u_j) + o(T(r, f)).$$

Proof: By Theorem D2, we have

$$(1) \quad \frac{T(r, \phi)}{q^t} \leq \tilde{Z}(r, \phi) + \tilde{Z}(r, \phi - 1) + \tilde{N}(r, \phi) + O(1).$$

Next, we have $T(r, f) \leq T(r, f - u_j) + T(r, u_j)$ ($j = 1, 2, 3$), hence $T(r, f) \leq T(r, \frac{u_3 - u_1}{f - u_3}) + o(T(r, f))$, thereby $T(r, f) \leq T(r, \frac{u_3 - u_1}{f - u_3} + 1) + o(T(r, f)) = T(r, \frac{f - u_1}{f - u_3}) + o(T(r, f))$.

Now, $T(r, \frac{u_2 - u_1}{u_2 - u_3}) = o(T(r, f))$. Consequently, by writing $\frac{f - u_1}{f - u_3} = \phi \left(\frac{u_2 - u_1}{u_2 - u_3} \right)$ we have $T(r, \frac{f - u_1}{f - u_3}) \leq T(r, \phi) + T(r, \frac{u_2 - u_1}{u_2 - u_3}) \leq T(r, \phi) + o(T(r, f))$ and finally $T(r, f) \leq T(r, \phi) + o(T(r, f))$. Thus, by (1) we obtain

$$(2) \quad \frac{T(r, f)}{q^t} \leq \tilde{Z}(r, \phi) + \tilde{Z}(r, \phi - 1) + \tilde{N}(r, \phi) + o(T(r, f)).$$

Now, we can check that

$$\tilde{Z}(r, \phi) + \tilde{Z}(r, \phi - 1) + \tilde{N}(r, \phi) \leq \sum_{j=1}^3 \tilde{Z}(r, f - u_j) + \sum_{1 \leq j < k \leq 3} \tilde{Z}(r, u_k - u_j) \leq \sum_{j=1}^3 \tilde{Z}(r, f - u_j) + o(T(r, f))$$

which, by (2), completes the proof.

We are now ready to state and prove Theorem D4.

Theorem D4: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}_u(d(0, R^-))$), let $u_1, u_2 \in \mathcal{M}_f(\mathbb{K})$ (resp. $u_1, u_2 \in \mathcal{M}_f(d(0, R^-))$) be distinct and let t be the ramification index of $\frac{f(x) - u_1(x)}{f(x) - u_2(x)}$. Then*

$$\frac{T(r, f)}{q^t} \leq \tilde{Z}(r, f - u_1) + \tilde{Z}(r, f - u_2) + \tilde{N}(r, f) + o(T(r, f)).$$

Proof: Let $g = \frac{1}{f}$, $w_j = \frac{1}{u_j}$, $j = 1, 2$, $w_3 = 0$. Clearly, $T(r, g) = T(r, f) + O(1)$, $T(r, w_j) = T(r, u_j)$, $j = 1, 2$, so we can apply Theorem D3 to g, w_1, w_2, w_3 . On the other hand,

$$\frac{(g(x) - w_1(x))w_2(x)}{(g(x) - w_2(x))w_1(x)} = \frac{f(x) - u_1(x)}{f(x) - u_2(x)}.$$

Thus by Theorem D3 we have: $\frac{T(r, g)}{q^t} \leq \tilde{Z}(r, g - w_1) + \tilde{Z}(r, g - w_2) + \tilde{Z}(r, g) + o(T(r, g))$.

But we notice that $\tilde{Z}(r, g - w_j) = \tilde{Z}(r, f - u_j)$ for $j = 1, 2$ and $\tilde{Z}(r, g) = \tilde{N}(r, f)$. Moreover, we know that $o(T(r, g)) = o(T(r, f))$. Consequently, the claim is proven when $u_1 u_2$ is not identically zero.

Next, by setting $g = f - u_1$ and $u = u_2 - u_1$, we obtain Corollary D5:

Corollary D5: *Let $g \in \mathcal{M}(\mathbb{K})$ (resp. $g \in \mathcal{M}_u(d(0, R^-))$), let $u \in \mathcal{M}_g(\mathbb{K})$ (resp. $u \in \mathcal{M}_g(d(0, R^-))$) and let t be the ramification index of $\frac{g - u}{g}$.*

$$\text{Then } \frac{T(r, g)}{q^t} \leq \tilde{Z}(r, g) + \tilde{Z}(r, g - u) + \tilde{N}(r, g) + o(T(r, g)).$$

Corollary D6: Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$) and let $u_1, u_2 \in \mathcal{A}_f(\mathbb{K})$ (resp. $u_1, u_2 \in \mathcal{A}_f(d(0, R^-))$) be distinct and let t be the ramification index of $\frac{f - u_1}{f - u_2}$. Then

$$\frac{T(r, f)}{q^t} \leq \tilde{Z}(r, f - u_1) + \tilde{Z}(r, f - u_2) + o(T(r, f)).$$

Corollary D7: Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$) and let $u \in \mathcal{A}_f(\mathbb{K})$ (resp. $u \in \mathcal{A}_f(d(0, R^-))$) be non-identically zero and let t be the ramification index of $\frac{f - u}{f}$. Then

$$\frac{T(r, f)}{q^t} \leq \tilde{Z}(r, f) + \tilde{Z}(r, f - u) + o(T(r, f)).$$

In the proof of Theorems 1 and 2 we will need the following lemma:

Lemma 7: Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}_u(d(0, R^-))$) and let t be the ramification index of f . Let $m, n \in \mathbb{N}^*$, $m < n$, be prime to p . Then the ramification index of $f^n - f^m$ is also equal to t .

Proof: Since the lemma is trivial when $p = 0$, we suppose $p \neq 0$, hence $p = q$. Set $h = p^t$ and $F = f^n - f^m$. By hypothesis, since both m, n are prime to p , the ramification index of both f^m, f^n is equal to t and hence so are those of f^{n-m} and $f^{n-m} - 1$. Let $g = \sqrt[h]{f}$. Then g belongs to $\mathcal{A}(\mathbb{K})$ (resp. to $\mathcal{A}_u(d(0, R^-))$) and so does $\sqrt[h]{F}$. Let $G = \sqrt[h]{F}$. Then we can check that $G' = g'g^{m-1}(ng^{n-m} - m)$ hence G' is not identically 0. Consequently, the ramification index of F is t .

Proof of Theorem 1 and Theorem 2: We can obviously suppose $a = 0$. Suppose that f, g are two distinct functions. Let $F = f^n - f^m$. By Corollary D5, we can obtain

$$\frac{T(r, F)}{q^t} \leq \tilde{Z}(r, F) + \tilde{Z}(r, F - k(\phi - 1)) + \tilde{N}(r, F) + o(T(r, f)).$$

Now, clearly, $\tilde{N}(r, F) \leq T(r, f)$ and $\tilde{Z}(r, F) \leq \tilde{Z}(r, f) + \tilde{Z}(r, f^{n-m} - 1) + O(1)$ and $\tilde{Z}(r, f^{n-m} - 1) \leq (n - m)T(r, f)$; hence

$$(1) \quad \tilde{Z}(r, F) \leq (n - m + 1)T(r, f) + o(T(r, f)).$$

Similarly,

$$(2) \quad \tilde{Z}(r, F - k(\phi - 1)) = \tilde{Z}(r, g^n - g^m) \leq \tilde{Z}(r, g) + \tilde{Z}(r, g^{n-m} - 1).$$

Of course, since ϕ is bounded, by Proposition C we have $T(r, f) = T(r, g) + O(1)$; hence, by (1) and (2), we obtain $T(r, F) \leq q^t(2n - 2m + 3)T(r, f) + o(T(r, f))$.

On the other hand, by 2.4.15 [15], we have $T(r, F) = nT(r, f) + O(1)$; hence $nT(r, f) \leq q^t(2n - 2m + 3)T(r, f) + o(T(r, f))$. That yields $2mq^t \leq n(2q^t - 1) + 3q^t$, a contradiction to the hypothesis of Theorem 1.

Next, in the hypotheses of Theorem 2, we have $N(r, f) = N(r, g) = 0$; hence we can get $T(r, F) \leq q^t(2n - 2m + 2)T(r, f) + o(T(r, f))$ and hence $2mq^t \leq n(2q^t - 1) + 2q^t$, a contradiction to the hypotheses of Theorem 2. That ends the proofs of Theorems 1 and 2.

Proof of Corollary 1.1: Suppose $Y(n, m, k)$ is not a bi-ursem for $\mathcal{M}_u(d(a, R^-))$ and let $f, g \in \mathcal{M}_u(d(a, R^-))$ be such that $\mathcal{E}(f, Y(n, m, k)) = \mathcal{E}(g, Y(n, m, k))$. By Proposition A, the function $\phi = \frac{f^n - f^m + k}{g^n - g^m + k}$ is an invertible element of $\mathcal{A}(d(a, R^-))$. And since \mathbb{K} has characteristic zero, we have $nT(r, f) > 2(n - m + 1)T(r, f) + o(T(r, f))$, hence by Theorem 1, $f = g$.

The proof of Corollary 2.1 is similar by applying Theorem 2.

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