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Complex and p -adic meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function

Alain Escassut, Kamal Boussaf and Jacqueline Ojeda¹

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Abstract

Let \mathbb{K} be a complete algebraically closed p -adic field of characteristic zero. We apply results in algebraic geometry and a new Nevanlinna theorem for p -adic meromorphic functions in order to prove results of uniqueness in value sharing problems, both on \mathbb{K} and on \mathbb{C} . Let P be a polynomial of uniqueness for meromorphic functions in \mathbb{K} or \mathbb{C} or in an open disk. Let f , g be two transcendental meromorphic functions in the whole field \mathbb{K} or in \mathbb{C} or meromorphic functions in an open disk of \mathbb{K} that are not quotients of bounded analytic functions. We show that if $f'P'(f)$ and $g'P'(g)$ share a small function α counting multiplicity, then $f = g$, provided that the multiplicity order of zeros of P' satisfy certain inequalities. A breakthrough in this paper consists of replacing inequalities $n \geq k + 2$ or $n \geq k + 3$ used in previous papers by Hypothesis (G). In the p -adic context, another consists of giving a lower bound for a sum of q counting functions of zeros with $(q - 1)$ times the characteristic function of the considered meromorphic function.

Notation and definition: Let \mathbb{K} be an algebraically closed field of characteristic zero, complete with respect to an ultrametric absolute value $|\cdot|$. We will denote by \mathbb{E} a field that is either \mathbb{K} or \mathbb{C} . Throughout the paper we denote by a a point in \mathbb{K} . Given $R \in]0, +\infty[$ we define disks $d(a, R) = \{x \in \mathbb{K} \mid |x - a| \leq R\}$ and disks $d(a, R^-) = \{x \in \mathbb{K} \mid |x - a| < R\}$.

A polynomial $Q(X) \in \mathbb{E}[X]$ is called a *polynomial of uniqueness for a family of functions \mathcal{F} defined in a subset of \mathbb{E}* if $Q(f) = Q(g)$ implies $f = g$. The definition of polynomials of uniqueness was introduced in [19] by P. Li and C. C. Yang and was studied in many papers [11], [13], [20] for complex functions and in [1], [2], [9], [10], [17], [18], for p -adic functions.

Throughout the paper we will denote by $P(X)$ a polynomial in $\mathbb{E}[X]$ such that $P'(X)$ is of the form $b \prod_{i=1}^l (X - a_i)^{k_i}$ with $l \geq 2$ and $k_1 \geq 2$. The polynomial P will be said to *satisfy Hypothesis (G)* if $P(a_i) + P(a_j) \neq 0 \forall i \neq j$.

We will improve the main theorems obtained in [5] and [6] with the help of a new hypothesis denoted by Hypothesis (G) and by thoroughly examining the situation with p -adic and complex analytic and meromorphic functions in order to avoid a lot of exclusions. Moreover, we will prove a new theorem completing the 2nd Main Theorem for p -adic meromorphic functions. Thanks to this new theorem we will give more precisions in results on value-sharing problems.

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Notation: Let L be an algebraically closed field, let $P \in L[x] \setminus L$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c . We denote by $\Phi(P)$ its cardinal.

We denote by $\mathcal{A}(\mathbb{E})$ the \mathbb{E} -algebra of entire functions in \mathbb{E} , by $\mathcal{M}(\mathbb{E})$ the field of meromorphic functions in \mathbb{E} , i.e. the field of fractions of $\mathcal{A}(\mathbb{E})$ and by $\mathbb{E}(x)$ the field of rational functions. Throughout the paper, we denote by $\mathcal{A}(d(a, R^-))$ the \mathbb{K} -algebra of analytic functions in $d(a, R^-)$ i.e. the \mathbb{K} -algebra of power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converging in $d(a, R^-)$ and we denote

by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the \mathbb{K} -subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n \in \mathbb{N}} |a_n|R^n < +\infty$. We denote

by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$ and finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

Theorem O1 [9]: Let $P(X) \in \mathbb{K}[X]$. If $\Phi(P) \geq 2$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ and for $\mathcal{A}_u(d(a, R^-))$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$.

Let $P(X) \in \mathbb{C}[X]$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.

Concerning polynomials such that P' has exactly two distinct zeros, we know other results:

Theorem O2 [1], [2], [18]: Let $P \in \mathbb{K}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 with $\min\{c_1, c_2\} \geq 2$. Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.

Theorem O3 [9], [17]: Let $P \in \mathbb{K}[x]$ be of degree $n \geq 6$ be such that P' only has two distinct zeros, one of them being of order 2. Then P is a polynomial of uniqueness for $\mathcal{M}_u(d(0, R^-))$.

Theorem O4 [18]: Let $P \in \mathbb{C}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 with $\min\{c_1, c_2\} \geq 2$ and $\max\{c_1, c_2\} \geq 3$. Then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$.

In order to state theorems and recall the definition of a small function, we must recall the definition of the classical Nevanlinna functions both on a p -adic field and on the field \mathbb{C} together with a few specific properties of ultrametric analytic or meromorphic functions [7], [11], [13].

Notation: Let \log be a real logarithm function of base $b > 1$ and let $\log^+(x) = \max(0, \log(x))$. Let $f \in \mathcal{M}(\mathbb{E})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0. Let $r \in]0, +\infty[$ (resp. $r \in]0, R[$) and let $\gamma \in d(0, r)$. If f has a zero of order n at γ , we put $\omega_\gamma(h) = n$. If f has a pole of order n at γ , we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$. These definitions of Nevanlinna's functions are equivalent to those defined in [7].

We denote by $Z(r, f)$ the counting function of zeros of f in $d(0, r)$, counting multiplicities, i.e.

$$Z(r, f) = \max(\omega_0, 0) \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by $\bar{Z}(r, f)$ the counting function of zeros of f in $d(0, r)$, ignoring multiplicities, and set

$$\bar{Z}(r, f) = u \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} (\log r - \log |\gamma|)$$

with $u = 1$ when $\omega_0(f) > 0$ and $u = 0$ else.

In the same way, we set $N(r, f) = Z\left(r, \frac{1}{f}\right)$ (resp. $\bar{N}(r, f) = \bar{Z}\left(r, \frac{1}{f}\right)$) to denote the *counting function of poles of f* in $d(0, r)$, counting multiplicities (resp. ignoring multiplicities).

For $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}(d(0, R^-))$, we call *Nevanlinna function of f* the function $T(r, f) = \max\{Z(r, f), N(r, f)\}$.

Consider now a function $f \in \mathcal{M}(\mathbb{C})$. We can define a function $m(r, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ and we call *Nevanlinna function of f* the function $T(r, f) = m(r, f) + N(r, f)$.

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f \in \mathcal{M}(\mathbb{E})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$) such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{E})$ (resp. $\alpha \in \mathcal{M}(d(0, R^-))$) is called a *small function with respect to f* , if it satisfies

$$\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0 \quad \left(\text{resp. } \lim_{r \rightarrow R^-} \frac{T(r, \alpha)}{T(r, f)} = 0\right).$$

We denote by $\mathcal{M}_f(\mathbb{E})$ (resp. $\mathcal{M}_f(d(0, R^-))$) the set of small meromorphic functions with respect to f in \mathbb{E} (resp. in $d(0, R^-)$).

Remark 1. Thanks to classical properties of the Nevanlinna function $T(r, f)$ with respect to the operations in a field of meromorphic functions, such as $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$ and $T(r, fg) \leq T(r, f) + T(r, g) + O(1)$, for $f, g \in \mathcal{M}(\mathbb{K})$ and $r > 0$, it is easily proven that $\mathcal{M}_f(\mathbb{E})$ (resp. $\mathcal{M}_f(d(0, R^-))$) is a subfield of $\mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0, R^-))$) and that $\mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0, R^-))$) is a transcendental extension of $\mathcal{M}_f(\mathbb{E})$ (resp. of $\mathcal{M}_f(d(0, R^-))$) [10].

Let us remember the following definition.

Definitions. Let $f, g, \alpha \in \mathcal{M}(\mathbb{E})$ (resp. let $f, g, \alpha \in \mathcal{M}(d(0, R^-))$). We say that f and g share the function α C.M., if $f - \alpha$ and $g - \alpha$ have the same zeros with the same multiplicities in \mathbb{E} (resp. in $d(0, R^-)$).

In [5] and [6] we have obtained this general Theorem (where results of [5] and [6] here are gathered):

Theorem T: Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, (resp for $\mathcal{M}(\mathbb{C})$, resp. for $\mathcal{M}(d(0, R^-))$) with $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$k_1 \geq k + 2 \quad (\text{resp. } k_1 \geq k + 3, \text{ resp. } k_1 \geq k + 3)$$

$$\text{if } l = 2, \text{ then } k_1 \neq k + 1, 2k, 2k + 1, 3k + 1,$$

$$\text{if } l = 3, \text{ then } k_1 \neq k + 1, 2k + 1, 3k_i - k \quad \forall i = 2, 3.$$

If $l \geq 4$, then $k_1 \neq k + 1$

Let $f, g \in \mathcal{M}(\mathbb{E})$ (resp. $f, g \in \mathcal{M}_u((d(a, R^-))$) be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{E}) \cap \mathcal{M}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

In the field \mathbb{K} , several particular applications were given when the small function is a constant or a Moebius function. On \mathbb{C} , we can't get similar refinements because the complex Nevanlinna Theory is less accurate than the p -adic Nevanlinna Theory.

In the present paper, thanks to the new Hypothesis (G) introduced below, we mean to avoid the hypothesis $k_1 \geq k + 2$ for $\mathcal{M}(\mathbb{K})$ and $k_1 \geq k + 3$ for $\mathcal{M}(\mathbb{C})$ and for $\mathcal{M}(d(a, R^-))$.

But first, we have a new theorem for p -adic analytic functions:

First we can improve results of [5] concerning p -adic analytic functions.

Theorem 1. Let $P(X) \in \mathbb{K}[X]$ be a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}_u(d(a, R^-))$) and let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$. Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental (resp. let $f, g \in \mathcal{A}_u(d(a, R^-))$), be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$). If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$. Moreover, if f, g belong to $\mathcal{A}(\mathbb{K})$, if α is a constant and if $\sum_{i=1}^l k_i \geq 2l + 1$ then $f = g$.

Corollary 1.1: Let $P(X) \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 2$ and let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$. Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$. If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$. Moreover, if α is a constant and if $\sum_{i=1}^l k_i \geq 2l + 1$ then $f = g$.

Example: Let $c \in \mathbb{K}$ be a solution of the algebraic equation:

$$X^{11} \left(\frac{1}{11} - \frac{1}{10} \right) - X^9 \left(\frac{1}{9} - \frac{1}{8} \right) + X \left(\frac{1}{10} - \frac{1}{8} \right) - \frac{1}{11} + \frac{1}{9} = 0.$$

Let

$$P(X) = \frac{X^{11}}{11} - \frac{cX^{10}}{10} - \frac{X^9}{9} + \frac{cX^8}{8}$$

Then we can check that $P'(X) = X^7(X - 1)(X + 1)(X - c)$,

$P(1) = P(c) \neq 0$ and that $P(1) \neq 0$, $P(-1) \neq 0$, $P(1) + P(-1) = c \left(\frac{1}{4} - \frac{1}{5} \right)$ and $P(-1) - P(1) = 2 \left(\frac{1}{11} - \frac{1}{9} \right)$, hence $P(-1) \neq P(c)$. Consequently, we can apply Corollary 1.1 and show that if $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$, then $f = g$.

Remark 2. Recall Hypothesis (F) due to H. Fujimoto [12]. A polynomial Q is said to satisfy Hypothesis (F) if the restriction of Q to the set of zeros of Q' is injective. In the last example, we may notice that Hypothesis (F) is not satisfied by P .

Corollary 1.2: Let $P(X) \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 3$ and let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$. Let $f, g \in \mathcal{A}_u(d(a, R^-))$ be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$. If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$.

Corollary 1.3: Let $P(X) \in \mathbb{K}[X]$ be such that $\Phi(P) \geq 2$ (resp. $\Phi(P) \geq 3$) and let $P'(X) = bX^n \prod_{i=2}^l (X - a_i)$ with $l \geq 3$ and let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \mathcal{A}_u(d(a, R^-))$) be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$). If $n \geq l + 3$ then $f = g$. Moreover, if f, g belong to $\mathcal{A}(\mathbb{K})$, if α is a constant and if $n \geq l + 2$ then $f = g$.

In order to improve results of [5] on p -adic meromorphic functions and of [6] on complex meromorphic functions, we have to state Propositions P1 and P2 derived from results of [3] and [4].

Notation and definition: Henceforth we assume that $P(a_1) = 0$ and that $P'(X)$ is of the form $b \prod_{i=1}^l (X - a_i)^{k_i}$ with $l \geq 2$. The polynomial P will be said to satisfy Hypothesis (G) if $P(a_i) + P(a_j) \neq 0 \forall i \neq j$

Proposition P1: Let $P \in \mathbb{K}[X]$ satisfy Hypothesis (G) and $n \geq 2$ (resp. $n \geq 3$). If meromorphic functions $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) satisfy $P(f(x)) = P(g(x)) + C$ ($C \in \mathbb{K}^*$, $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(a, R^-)$)) then both f and g are constant (resp. f and g belong to $\mathcal{M}_b(d(a, R^-))$).

Proposition P2: Let $P \in \mathbb{C}[X]$ satisfy Hypothesis (G) and $n \geq 3$. If meromorphic functions $f, g \in \mathcal{M}(\mathbb{C})$ satisfy $P(f(x)) = P(g(x)) + C$ ($C \in \mathbb{C}^*$, $\forall x \in \mathbb{C}$) then both f and g are constant.

From [5] and thanks to Propositions P1 we can now derive the following Theorems 2, 3, 4, 5:

Theorem 2. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, (resp for $\mathcal{M}(d(0, R^-))$) with $l \geq 2$, let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, let u_5 be the biggest of the i such that $k_i > 4$ and let $s_5 = \max(0, u_5 - 3)$ and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > m$ and let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{m=5}^{\infty} s_m),$$

either $k_1 \geq k + 2$ (resp. $k_1 \geq k + 3$, resp. $k_1 \geq k + 3$) or P satisfies Hypothesis (G),
if $l = 2$, then $k_1 \neq k + 1, 2k, 2k + 1, 3k + 1$,

if $l = 3$, then $k_1 \neq \frac{k}{2}$, $k_1 \neq k + 1$, $2k + 1$, $3k_i - k \forall i = 2, 3$.

If $l \geq 4$, then $k_1 \neq k + 1$

Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark 3. the sum $\sum_{m=5}^{\infty} s_m$ is obviously finite.

Corollary 2.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$ and hypothesis (G), let $P' = b \prod_{i=1}^l (X - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

if $l = 3$, then $k_1 \neq \frac{k}{2}$, $k + 1$, $2k + 1$, $3k_i - k \forall i = 2, 3$,

if $l \geq 4$, then $n \neq k + 1$. Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{20}}{20} - \frac{X^{19}}{19} - \frac{4X^{18}}{18} + \frac{4X^{17}}{17} \\ + \frac{6X^{16}}{16} - \frac{6X^{15}}{15} - \frac{4X^{14}}{14} + \frac{4X^{13}}{13} + \frac{X^{12}}{12} - \frac{X^{11}}{11}$$

We can check that $P'(X) = X^{10}(X - 1)^5(X + 1)^4$ and

$$P(0) = 0, \quad P(1) = \sum_{j=0}^4 C_4^j (-1)^j \left(\frac{1}{10 + 2j} - \frac{1}{9 + 2j} \right),$$

$$P(-1) = - \sum_{j=0}^4 C_4^j \left(\frac{1}{10 + 2j} + \frac{1}{9 + 2j} \right)$$

Consequently, we have $\Phi(P) = 3$ and we check that Hypothesis (G) is satisfied. Now, let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Remark: In that example, we have $k_1 = 10$, $k = 9$. Applying our previous work, a conclusion would have required $n \geq k + 2 = 11$.

Theorem 3. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{C})$, with $l \geq 2$, let $P'(X) = b \prod_{i=1}^l (X - a_i)^{k_i}$

with $b \in \mathbb{K}^*$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, let u_5 be the biggest of the i such that $k_i > 4$ and let $s_5 = u_5 - 3$ and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > m$ and let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{m=5}^{\infty} s_m),$$

either $k_1 \geq k + 3$ or P satisfies Hypothesis (G),

if $l = 2$, then $k_1 \neq k + 1, 2k, 2k + 1, 3k + 1$,

if $l = 3$, then $k_1 \neq \frac{k}{2}, k_1 \neq k + 1, 2k + 1, 3k_i - k \forall i = 2, 3$.

If $l \geq 4$, then $k_1 \neq k + 1$

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Corollary 3.1 Let $P \in \mathbb{C}[X]$ satisfy $\Phi(P) \geq 4$ and Hypothesis (G), let $P' = b \prod_{i=1}^l (X - a_i)^{k_i}$,

$k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$$k_1 \geq 10 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{m=5}^{\infty} s_m)$$

$$k_1 \neq k + 1.$$

Let $f, g \in \mathcal{M}(\mathbb{C})$ and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

As noticed in [5], if f, g belong to $\mathcal{M}(\mathbb{K})$ and if α is a constant or a Moebius function, we can get a more accurate statement:

Theorem 4. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b \prod_{i=1}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$,

$l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$.

Suppose P satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

either $k_1 \geq k + 2$ or P satisfies (G)

if $l = 2$, then $k_1 \neq k + 1, 2k, 2k + 1, 3k + 1$,

if $l = 3$, then $k_1 \neq \frac{k}{2}, k + 1, 2k + 1, 3k_i - k \forall i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem O1, we can derive Corollary 4.1.

Corollary 4.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b \prod_{i=1}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$,

$k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$.

Suppose P satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

either $n \geq k + 2$ or P satisfies (G),

if $l = 3$, then $k_1 \neq \frac{k}{2}, k + 1, 2k + 1, 3k_i - k \forall i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem O2 we have Corollary 4.2.

Corollary 4.2 Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $k \leq n$, $\min(k, n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k),$$

either $n \geq k + 2$ or P satisfies (G),

$$n \neq k + 1, 2k, 2k + 1, 3k + 1,$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 5. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b \prod_{i=1}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$,

$l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 4)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 3)$. Suppose P satisfies the following conditions:

either $k_1 \geq k + 2$ or P satisfies (G)

$$k_1 \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

$$k_1 \neq k + 1$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem O1, we can derive Corollary 5.1.

Corollary 5.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b \prod_{i=1}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$,

$k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$, let $k = \sum_{i=2}^l k_i$, and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > 4$, let $s_5 = \max(0, u_5 - 3)$ and for every $m \geq 6$, let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

$k_1 \geq k + 2$ or P satisfies Hypothesis (G),

$$n \geq 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 1, \sum_{m=5}^{\infty} s_m)$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem O2, we have Corollary 5.2

Corollary 5.2 Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $\min(k, n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

$n \geq 9 + \max(0, 5 - k)$,
either $n \geq k + 2$ or P satisfies (G)
 $n \neq k + 1$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{24}}{24} - \frac{10X^{23}}{23} + \frac{36X^{22}}{22} - \frac{40X^{21}}{21} - \frac{74X^{20}}{20} + \frac{226X^{19}}{19} \\ - \frac{84X^{18}}{18} - \frac{312X^{17}}{17} + \frac{321X^{16}}{16} + \frac{88X^{15}}{15} \\ - \frac{280X^{14}}{14} + \frac{48X^{13}}{13} + \frac{80X^{12}}{12} - \frac{32X^{11}}{11}$$

We can check that $P'(X) = X^{10}(X - 2)^5(X + 1)^4(X - 1)^4$. Next, we have $P(2) < -134378$, $P(1) \in]-2, 11[$, $P(-1) \in]2, 18; 2, 19[$. Therefore, $P(0)$, $P(1)$, $P(-1)$, $P(2)$ are all distinct, hence $\Phi(P) = 4$. Moreover, Hypothesis (G) is satisfied.

Now, let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g \in \mathcal{M}_u(d(a, R^-))$), resp. let $f, g \in \mathcal{M}(\mathbb{C})$ and let $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. let $\alpha \in \mathcal{M}(d(a, R^-))$), resp. let $\alpha \in \mathcal{M}(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Particularly, when f, g are entire functions in \mathbb{C} we can simplify the hypothesis:

Theorem 6. Let P be a polynomial of uniqueness for $\mathcal{A}(\mathbb{C})$ with $l \geq 2$ and $k_i \geq k_{i+1}$, $1 \leq i \leq l - 1$ when $l > 2$ and let $k = \sum_{i=2}^l k_i$, let u_5 be the biggest of the i such that $k_i > 4$ and let $s_5 = \max(0, u_5 - 3)$ and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > m$ and let $s_m = \max(0, u_m - 2)$. Suppose P satisfies the following conditions:

either $k_1 \geq k + 2$ or P satisfies hypothesis (G)

$$k_1 \geq 5 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 3, \sum_{m=5}^{\infty} s_m).$$

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Proposition P2, we have Corollaries 6.1 and 6.2:

Corollary 6.1 Let $P \in \mathbb{C}[X]$, let $P' = b \prod_{i=1}^l (X - a_i)^{k_i}$ with $b \in \mathbb{C}^*$, $k_i \geq k_{i+1}$, $1 \leq i \leq l - 1$ and

let $k = \sum_{i=2}^l k_i$, let u_5 be the biggest of the i such that $k_i > 4$ and let $t_5 = u_5 - 3$ and for each $m \in \mathbb{N}$, let u_m be the biggest of the i such that $k_i > m$ and let $t_m = \max(0, u_m - 2)$.

Suppose P satisfies the following conditions

either $k_1 \geq k + 2$ or P satisfies hypothesis (G)

$$k_1 \geq 5 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l - 3, \sum_{m=5}^{\infty} s_m)$$

Let $f, g \in \mathcal{M}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{C}) \cap \mathcal{M}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Corollary 6.2 *Let $P \in \mathbb{C}[X]$ and let $P' = b(X - a_1)^n(X - a)^k$ with $\min(k, n) \geq 2$ and $\max(n, k) \geq 3$. Suppose that P satisfies $n \geq 5 + \max(0, 5 - k)$,*

Let $f, g \in \mathcal{A}(\mathbb{C})$ be transcendental and let $\alpha \in \mathcal{A}_f(\mathbb{C}) \cap \mathcal{A}_g(\mathbb{C})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let

$$P(X) = \frac{X^{11}}{11} + \frac{5X^{10}}{10} + \frac{10X^9}{9} + \frac{10X^8}{8} + \frac{5X^7}{7} + \frac{X^6}{6}.$$

Then $P'(X) = X^5(X + 1)^5$. We can apply Corollary 6.2: given $f, g \in \mathcal{A}(\mathbb{C})$ transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{M}(\mathbb{C})$ C.M., we have $f = g$.

1 The proofs:

Notation: As usual, given a function $f \in \mathcal{M}(\mathbb{E})$ (resp. $\mathcal{M}(d(0, R^-))$), we denote by $S_f(r)$ a function of r defined in $]0, +\infty[$ (resp. in $]0, R[$) such that $\lim_{r \rightarrow +\infty} \frac{S_f(r)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{S_f(r)}{T(r, f)} = 0$)

We must recall the classical Nevanlinna Main Theorem:

Theorem N1: [7], [12] *Let $a_1, \dots, a_n \in \mathbb{K}$ (resp. $a_1, \dots, a_n \in \mathbb{K}$, resp. $a_1, \dots, a_n \in \mathbb{C}$) with $n \geq 2, n \in \mathbb{N}$, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$, resp. let $f \in \mathcal{M}(\mathbb{C})$). Let $S = \{a_1, \dots, a_n\}$. Then, for $r > 0$ we have*

$$(n - 1)T(r, f) \leq \sum_{j=1}^n \bar{Z}(r, f - a_j) + \bar{N}(r, f) - \log r + O(1),$$

resp.

$$(n - 1)T(r, f) \leq \sum_{j=1}^n \bar{Z}(r, f - a_j) + \bar{N}(r, f) + O(1),$$

resp.

$$(n - 1)T(r, f) \leq \sum_{j=1}^n \bar{Z}(r, f - a_j) + \bar{N}(r, f) + S_f(r).$$

Let us recall the following corollary of the Nevanlinna Second Main Theorem on three small function:

Theorem N2: *Let $f \in \mathcal{A}(\mathbb{K})$ (resp. let $f \in \mathcal{A}(d(0, R^-))$, resp. let $f \in \mathcal{A}(\mathbb{C})$) and let $u \in f \in \mathcal{A}_f(\mathbb{K})$ (resp. let $u \in \mathcal{A}_f(d(0, R^-))$, resp $u \in f \in \mathcal{A}_f(\mathbb{C})$). Then $T(r, f) \leq \bar{Z}(r, f) + \bar{Z}(r, f - u) + S_f(r)$.*

In order to prove Theorem N3 we need additional lemmas:

Notation: Let $f \in \mathcal{M}(d(a, R^-))$, and let $r \in]0, R[$. By classical results [8], [10] we know that $|f(x)|$ has a limit when $|x|$ tends to r , while being different from r .

We set $|f|(r) = \lim_{|x| \rightarrow r, |x| \neq r} |f(x)|$.

Lemma 1. For every $r \in]0, R[$, the mapping $|\cdot|(r)$ is an ultrametric multiplicative norm on $\mathcal{M}(d(0, R^-))$.

The following Lemma 2 is the p -adic Schwarz formula:

Lemma 2. Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(0, R^-))$) and let $r', r'' \in]0, +\infty[$ (resp. let $r', r'' \in]0, R[$) satisfy $r' < r''$. Then $\log(|f|(r'')) - \log(|f|(r')) = Z(r'', f) - Z(r', f)$.

Lemma 3. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$). Suppose that there exists $a \in \mathbb{K}$ and a sequence of intervals $I_n = [u_n, v_n]$ such that $u_n < v_n < u_{n+1}$, $\lim_{n \rightarrow +\infty} u_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} u_n = R$) and $\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} qT(r, f) - Z(r, f - a) \right) = +\infty$. Set $L = \bigcup_{n=0}^{+\infty} I_n$. Let $b \in \mathbb{K}$, $b \neq a$. Then $Z(r, f - b) = T(r, f) + O(1) \forall r \in L$.

Proof: We know that the Nevanlinna functions of a meromorphic function f are the same in \mathbb{K} and in an algebraically closed complete extension of \mathbb{K} whose absolute value extends that of \mathbb{K} . Consequently, without loss of generality, we can suppose that \mathbb{K} is spherically complete because we know that such a field does admit a spherically complete algebraically closed extension whose absolute value expands that of \mathbb{K} . If f belongs to $\mathcal{M}(\mathbb{K})$, we can obviously set it in the form $\frac{g}{h}$ where g, h belong to $\mathcal{A}(\mathbb{K})$ and have no common zero. Next, since \mathbb{K} is supposed to be spherically complete, if f belongs to $\mathcal{M}(d(0, R^-))$ we can also set it in the form $\frac{g}{h}$ where g, h belong to $\mathcal{A}(d(0, R^-))$ and have no common zero [8], [10]. Consequently, we have $T(r, f) = \max(Z(r, g), Z(r, h))$.

By hypothesis we have $\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} T(r, f) - Z(r, f - a) \right) = +\infty$ i.e.

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} T(r, f) - Z(r, f - a) \right) = +\infty \text{ i.e.}$$

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} \max(Z(r, g), Z(r, h)) - Z(r, g - ah) \right) = +\infty.$$

Set $B_n = \inf_{r \in I_n} \max(Z(r, g), Z(r, h)) - Z(r, g - ah)$. Since the sequence B_n tends to $+\infty$, clearly, by Lemma 2, the sequence (D_n) defined as $D_n = \sup_{r \in I_n} \left(\frac{|g - ah|(r)}{\max(|g|(r), |h|(r))} \right)$ tends to zero.

Therefore, by Lemma 1 we have $|g|(r) = |ah|(r)$ in I_n when n is big enough. Consequently, by Lemma 2 we have $Z(r, g) = Z(r, ah) + O(1) \forall r \in L$ and hence $T(r, f) = Z(r, h) + O(1) = Z(r, g) + O(1) \forall r \in L$.

Now, consider $g - bh = g - ah + (a - b)h$. By hypothesis we have

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} Z(r, h) - Z(r, g - ah) \right) = +\infty.$$

On the other hand, of course $Z(r, (a - b)h) = Z(r, h) + O(1)$. Consequently, since

$$Z(r, g - bh) = Z(r, g - ah + (a - b)h) \text{ we have } \lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} (Z(r, (a - b)h) - Z(r, g - ah)) \right) = +\infty.$$

Consider now the sequence (E_n) defined as $E_n = \sup_{r \in I_n} \left(\frac{|g - ah|(r)}{|(a - b)h|(r)} \right)$. By Lemma 2, that sequence tends to zero and hence, when r is big enough in L , by Lemma 1 we have $|g - bh|(r) = |a - bh|(r)$. Consequently, when r is big enough in L we have $Z(r, g - bh) = Z(r, bh) = Z(r, h) + O(1)$.

Moreover, we have seen that $Z(r, g) = Z(r, h) + O(1)$ in L , hence $\max(Z(r, g), Z(r, h)) = Z(r, g - bh) + O(1) = \max(Z(r, g - bh), Z(r, h) + O(1))$ i.e. $T(r, f) = T(r, f - b) + O(1)$ in L .

The second Main Theorem is well known in complex and p -adic analysis and is recalled below. But first, we can give here a new theorem of that kind which will be efficient in Theorem 3, 4, 5.

Theorem N3: *Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct. Then $(q-1)T(r, f) \leq \sum_{j=1}^q Z(r, f - a_j) + O(1)$.*

Proof: Suppose the theorem is wrong. There exists $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) and a_1, \dots, a_q such that $(q-1)T(r, f) - \sum_{j=1}^q Z(r, f - a_j)$ admits no superior bound in $]0, +\infty[$. So, there exists a sequence of intervals $J_s = [w_s, y_s]$ such that $w_s < y_s < w_{s+1}$, $\lim_{s \rightarrow +\infty} w_s = +\infty$ (resp. $\lim_{s \rightarrow +\infty} w_s = R$) and

$$\lim_{s \rightarrow +\infty} \left(\inf_{r \in J_s} (q-1)T(r, f) - \sum_{j=1}^q Z(r, f - a_j) \right) = +\infty. \quad (1)$$

Let $M = \bigcup_{s=0}^{\infty} J_s$. For each $j = 1, \dots, q$, we have $Z(r, f - a_j) \leq T(r, f) + O(1)$ in \mathbb{R}_+ and hence (1) implies that there exists an index t and a sequence of intervals $I_n = [u_n, v_n]$ included in M , such that $u_n < v_n < u_{n+1}$, $\lim_{n \rightarrow +\infty} u_n = +\infty$ (resp. $\lim_{n \rightarrow +\infty} u_n = R$) and

$$\lim_{n \rightarrow +\infty} \left(\inf_{r \in I_n} (T(r, f) - Z(r, f - a_t)) \right) = +\infty. \quad (2)$$

Let $L = \bigcup_{n=1}^{\infty} I_n$. Then by Lemma 3, in L we have $Z(r, g - a_k h) = T(r, f) + O(1) \forall k \neq t$. Therefore $\sum_{j=1}^q Z(r, f - a_j) \geq (q-1)T(r, f) + O(1)$ in L , a contradiction to (1). Consequently, the Theorem is not wrong.

Remark 4. *Theorem N3 is trivial for analytic functions since by definition, for a function $f \in \mathcal{A}(\mathbb{K})$ or $\mathcal{A}(d(0, R^-))$ we have $T(r, f) = Z(r, f)$. On the other hand, the theorem does not apply to meromorphic functions in \mathbb{C} . Indeed, consider a meromorphic function f on \mathbb{C} omitting two values a and b . We have $Z(r, f - a) + Z(r, f - b) = 0$.*

In the proof of Theorems 2- 6 will need the following Lemmas:

Lemma 4. *Let $Q \in \mathbb{K}[x]$ (resp. $Q \in \mathbb{K}[X]$, resp. $Q \in \mathbb{C}[x]$) be of degree n and let $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(d(0, R^-))$), resp. $f \in \mathcal{M}(\mathbb{C})$ be transcendental. Then $N(r, f') = N(r, f) + \bar{N}(r, f)$, $Z(r, f') \leq Z(r, f) + \bar{N}(r, f) + O(1)$, $nT(r, f) \leq T(r, f'Q(f)) \leq (n+2)T(r, f) - \log r + O(1)$ (resp. $nT(r, f) \leq T(r, f'Q(f)) \leq (n+2)T(r, f) + O(1)$, resp. $nT(r, f) \leq T(r, f'Q(f)) + m(r, \frac{1}{f'}) \leq (n+2)T(r, f) + S_f(r)$).*

Particularly, if $f \in \mathcal{A}(\mathbb{K})$, (resp. $f \in \mathcal{A}(d(0, R^-))$), then $nT(r, f) \leq T(r, f'Q(f)) \leq (n+1)T(r, f) - \log r + O(1)$ (resp. $nT(r, f) \leq T(r, f'Q(f)) \leq (n+1)T(r, f) + O(1)$)

Let us recall the following corollary of the Nevanlinna Second Main Theorem on three small function:

Lemma 5. *Let $Q(X) \in \mathbb{K}[X]$ and let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}_u(d(0, R^-))$) be such that $Q(f) - Q(g)$ is bounded. Then $f = g$.*

Proof: The polynomial $Q(X) - Q(Y)$ factorizes in the form $(X - Y)F(X, Y)$ with $F(X, Y) \in \mathbb{K}[X, Y]$. Since $Q(f) - Q(g)$ is bounded, so are both factors because the semi-norm $|\cdot|(r)$ is multiplicative on $\mathcal{A}(\mathbb{K})$ (resp. on $\mathcal{A}_u(d(0, R^-))$). Consequently, $f - g$ is a constant c (resp. is a bounded function $u \in \mathcal{A}_b(d(0, R^-))$). Therefore $F(f, g) = F(f, f+c)$ (resp. $F(f, g) = F(f; f+u)$). Let $n = \deg(Q)$. Then we can check that $F(X, X+c)$ is a polynomial in X of degree $n-1$. Consequently, if $f \in (\mathbb{K})$, $F(f, f+c)$ is a non-constant entire function and therefore is unbounded in \mathbb{K} . Similarly, $f \in (d(0, R^-))$, $F(X, X+u)$ is a polynomial in X of degree $n-1$ with coefficients in $\mathcal{A}(d(0, R^-))$ and therefore $F(f, f+u)$ is unbounded in $d(0, R^-)$, which ends the proof.

Proof of Theorem 1. Without loss of generality, we may assume that $b = 1$. Put $F = f' \prod_{j=1}^l (f - a_j)^{k_j}$ and $G = g' \prod_{j=1}^l (g - a_j)^{k_j}$. Since $f, g \in \mathcal{A}(\mathbb{K})$ and since F and G share α C.M., then $\frac{F - \alpha}{G - \alpha}$ is a meromorphic function having no zeros and no poles in \mathbb{K} (resp. in $d(0, R^-)$), hence it is a constant u in $\mathbb{K} \setminus \{0\}$ (resp. it is an invertible function $u \in \mathcal{A}(d(0, R^-))$).

Suppose $u \neq 1$. Then,

$$F = uG + \alpha(1 - c). \quad (1)$$

Let $r > 0$. Since $\alpha(1 - u) \in \mathcal{A}_f(\mathbb{K})$ (resp. $\alpha(1 - u) \in \mathcal{A}_f(d(0, R^-))$), $\alpha(1 - u)$ obviously belongs to $\mathcal{A}_F(\mathbb{K})$ (resp. to $\mathcal{A}_F(d(0, R^-))$). So, applying Theorem N2 to F , we obtain

$$\begin{aligned} T(r, F) &\leq \bar{Z}(r, F) + \bar{Z}(r, F - \alpha(1 - c)) + S_F(r) = \bar{Z}(r, F) + \bar{Z}(G) + S_F(r) \\ &= \sum_{j=1}^l \bar{Z}(r, (f - a_j)^{k_j}) + \bar{Z}(r, f') + \sum_{j=1}^l \bar{Z}(r, (g - a_j)^{k_j}) + \bar{Z}(r, g') + S_f(r) \\ &\leq l(T(r, f) + T(r, g)) + Z(r, f') + Z(r, g') + S_f(r). \end{aligned}$$

We also notice that if $f, g \in \mathcal{A}(\mathbb{K})$ and if $\alpha \in \mathbb{K}$, we have $T(r, F) \leq \bar{Z}(r, F) + \bar{Z}(r, F - \alpha(1 - c)) - \log r + O(1)$ and therefore we obtain

$$T(r, F) \leq l(T(r, f) + T(r, g)) + Z(r, f') + Z(r, g') - \log r + O(1).$$

Now, let us go back to the general case. Since f is entire, by Lemma 4 we have $T(r, F) = (\sum_{j=1}^l k_j)T(r, f) + Z(r, f') + O(1)$. Consequently, $(\sum_{j=1}^l k_j)T(r, f) \leq l(T(r, f) + T(r, g)) + Z(r, g') + S_f(r)$.

Similarly, $(\sum_{j=1}^l k_j)T(r, g) \leq l(T(r, g) + T(r, g)) + Z(r, f') + S_f(r)$. Therefore

$$\begin{aligned} (\sum_{j=1}^l k_j)(T(r, f) + T(r, g)) &\leq 2l(T(r, f) + T(r, g)) + Z(r, f') + Z(r, g') + S_f(r) \\ &\leq (2l + 1)(T(r, f) + T(r, g)) + S_f(r). \end{aligned}$$

So, $\sum_{j=1}^l k_j \leq 2l + 1$. Thus, since $\sum_{j=1}^l k_j > 2l + 1$ we have $u = 1$.

And if $\alpha \in \mathbb{K}$, we obtain

$$\begin{aligned} \sum_{j=1}^l k_j (T(r, f) + T(r, g)) &\leq 2l(T(r, f) + T(r, g)) + Z(r, f') + Z(r, g') + S_f(r) \\ &\leq (2l + 1)(T(r, f) + T(r, g)) - 3 \log r + O(1) \end{aligned}$$

because $T(r, f') \leq T(r, f) - \log r + O(1)$, hence $\sum_{j=1}^l k_j \leq 2l$ which also contradicts the hypothesis

$c \neq 1$ whenever $\sum_{j=1}^l k_j > 2l$

Consequently, in the general case, whenever $\sum_{j=1}^l k_j > 2l + 1$, we have $u = 1$ and therefore $f'P(f) = g'P(g)$ hence $P(f) - P(g)$ is a constant D . But then by Lemma 5, we have $P(f) = P(g)$. And since P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$ (resp. for $\mathcal{A}(d(0, R^-))$), we can conclude $f = g$. Similarly, if $f, g \in \mathcal{A}(\mathbb{K})$ and if α is a non-zero constant, we have $u = 1$ whenever $\sum_{j=1}^l k_j > 2l$ and we conclude in the same way. \square

On \mathbb{K} , we have this theorem from results of [4]:

Theorem A: Let $P, Q \in \mathbb{K}[x]$ satisfy one of the following two statements:

$$\begin{aligned} \sum_{a_i \in F'} k_i \geq s - m + 2 \quad (\text{resp. } \sum_{a_i \in \Delta} k_i \geq s - m + 3,) \\ \sum_{b_j \in F''} q_j \geq 2 \quad (\text{resp. } \sum_{b_i \in \Lambda} q_j \geq 3,) \end{aligned}$$

If two meromorphic functions $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) satisfy $P(f(x)) = Q(g(x))$, $x \in \mathbb{K}$, (resp. $x \in d(a, R^-)$) then both f and g are constant (resp. belong to $\mathcal{M}_b(d(a, R^-))$).

and on \mathbb{C} , we have this theorem from results of [3]:

Theorem B: Let $P, Q \in \mathbb{C}[X]$ satisfy one of the two following two conditions:

$$\begin{aligned} \sum_{a_i \in F'} k_j \geq s - m + 3, \\ \sum_{b_j \in F''} q_i \geq 3, \end{aligned}$$

and if the polynomial $P(X) - Q(Y)$ has no factor of degree 1, then there is no non-constant function $f, g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x)) - Q(g(x)) = 0 \forall x \in \mathbb{C}$.

From Theorem B we can derive the following Theorem C:

Theorem C: Let $P, Q \in \mathbb{C}[X]$ satisfy one of the two following conditions:

$$\sum_{a_i \in F'} k_i \geq s - m + 3.$$

$$\sum_{b_j \in F''} q_j \geq 3.$$

Then there is no non-constant function $f, g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x)) - Q(g(x)) = 0 \forall x \in \mathbb{C}$.

Proof of Theorem C: Let $F(X, Y) = P(X) - Q(Y)$. Since \mathbb{C} is algebraically isomorphic to an ultrametric field such as \mathbb{C}_p (with p any prime integer), without loss of generality we can transfer the problem onto the field \mathbb{C}_p . So, the image of the polynomial F in $\mathbb{C}_p[X, Y]$ is a polynomial $\tilde{F}(X, Y)$.

Thus, the hypothesis $\sum_{a_i \in F'} k_i \geq s - m + 3$ still holds in \mathbb{C}_p and similarly, for the hypothesis $\sum_{b_j \in F''} q_j \geq 3$. Suppose for instance $\sum_{a_i \in F'} k_i \geq s - m + 3$. By Theorem B, there is no pair of non-constant functions $f, g \in \mathcal{M}(\mathbb{C}_p)$ such that $P(f(x)) - Q(g(x)) = 0$. Particularly, $\tilde{F}(X, Y)$ admits no factor of degree 1 in $\mathbb{C}_p[X, Y]$. But then, $F(X, Y)$ does not admit a factor of degree 1 in $\mathbb{C}[X, Y]$ either, because the factorization is conserved by a transfer. Now, we can apply Theorem B proving that when two functions $f, g \in \mathcal{M}(\mathbb{C})$ satisfy $P(f(x)) = Q(g(x)) \forall x \in \mathbb{C}$, then they are constant.

Proof of Proposition P1: Suppose that two functions $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}(d(a, R^-))$) satisfy $P(f(x)) = P(g(x)) + C$ ($C \in \mathbb{K}$), $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(a, R^-)$). We can apply Theorem A by putting $Q(X) = P(X) + C$. So, we have $h = l$ and $b_i = a_i$, $i = 1, \dots, l$. Let Γ be the curve of equation $P(X) - P(Y) = C$. By hypothesis we have $n \geq 2$, hence $\deg(P) \geq 3$, so Γ is of degree ≥ 3 . Therefore, if Γ has no singular point, it is of genus ≥ 1 and hence, by Picard-Berkovich Theorem, the conclusion is immediate. Consequently, we can assume that Γ has a singular point (α, β) . But then $P'(\alpha) = P'(\beta) = 0$ and hence (α, β) is of the form (a_h, a_k) . Consequently, $C = P(a_h) - P(a_k)$ and since $C \neq 0$, we have $h \neq k$. We will prove that either $a_1 \in F'$, or $a_1 \in F''$.

Suppose first that $a_1 \notin F' \cup F''$. Since $a_1 \notin F'$, there exists $i \in \{2, \dots, l\}$ such that $P(a_1) = P(a_i) + C$. Now since $1 \notin F''$, there exists $j \in \{2, \dots, l\}$ such that $P(a_1) + C = P(a_j)$. But since $C = -P(a_i)$, we have $P(a_j) = -P(a_i)$, therefore $P(a_i) + P(a_j) = 0$. Since P satisfies (G), we have $i = j$, hence $P(a_i) = 0$. But then $C = 0$, a contradiction. Therefore, we have proven that $a_1 \in F' \cup F''$. Now, by Theorem A, f and g are constant (resp. f and g belong to $\mathcal{M}_b(d(a, R^-))$).

Proof of Proposition P2: Suppose that two functions $f, g \in \mathcal{M}(\mathbb{C})$ satisfy $P(f(x)) = P(g(x)) + C$ ($C \in \mathbb{C}$), $\forall x \in \mathbb{C}$. We will apply Theorem C by putting $Q(X) = P(X) + C$. Since $n \geq 3$, we have $\deg(P) \geq 4$ and hence Γ is of degree ≥ 4 . Consequently, if Γ has no singular point, it has genus ≥ 2 and hence, by Picard's Theorem, there exists no functions $f, g \in \mathcal{M}(\mathbb{C})$ such that $P(f(x)) = P(g(x)) + C$, $x \in \mathbb{C}$. Consequently, we can assume that Γ admits a singular point (a_h, a_k) . The proof is then similar to that of Proposition 1.

Notation: Let $f \in \mathcal{M}(\mathbb{C})$ be such that $f(0) \neq 0, \infty$. We denote by $Z_{[2]}(r, f)$ the counting function of the zeros of f each being counted with multiplicity when it is at most 2 and with multiplicity 2 when it is bigger.

The following basic lemma applies to both complex and meromorphic functions. A proof is given in [5] for p -adic meromorphic functions and in [6] for complex meromorphic functions.

The following Theorem Y is indispensable in the proof of theorems:

Theorem Y: Let $P(x) = (x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i} \in \mathbb{E}[x]$ ($a_i \neq a_j$, $\forall i \neq j$) with $l \geq 2$ and $n \geq \max\{k_2, \dots, k_l\}$ and let $k = \sum_{i=2}^l k_i$. Let $f, g \in \mathcal{M}(\mathbb{E})$ be transcendental (resp. let $f, g \in \mathcal{M}(d(a, R^-))$) and let $\theta = P(f)f'P'(g)g'$. If θ belongs to $\mathcal{M}_f(\mathbb{E}) \cap \mathcal{M}_g(\mathbb{E})$, (resp. if θ belongs to $\mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$) then we have the following :

if $l = 2$ then n belongs to $\{k, k + 1, 2k, 2k + 1, 3k + 1\}$,
 if $l = 3$ then n belongs to $\{\frac{k}{2}, k + 1, 2k + 1, 3k_2 - k, 3k_3 - k\}$,
 if $l \geq 4$ then $n = k + 1$.
 Moreover, if f, g belong to $\mathcal{M}(\mathbb{K})$ and if θ is a constant, then $n = k + 1$.
 Further, if f, g belong to $\mathcal{A}(\mathbb{E})$, then θ does not belong to $\mathcal{A}_f(\mathbb{E})$.

Lemma 6. Let $f \in \mathcal{M}(\mathbb{K})$, (resp. $f \in \mathcal{M}(d'0, R^-)$), resp. $f \in \mathcal{M}(\mathbb{C})$). Then

$$T(r, f) - Z(r, f) \leq T(r, f') - Z(r, f') + O(1).$$

Now, we can extract the following Lemma 7 from a result that is proven in several papers and particularly in Lemma 3 [14] when $\mathbb{E} = \mathbb{C}$ and, with precisions in Lemma 11 [5] when $\mathbb{E} = \mathbb{K}$.

Lemma 7. Let $f, g \in \mathcal{M}(\mathbb{C})$ (resp. $f, g \in \mathcal{M}(\mathbb{K})$) share the value 1 C.M. If $\Psi_{f,g}$ is not identically zero, then, $\max(T(r, f), T(r, g)) \leq N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) + S_f(r) + S_g(r)$ (resp. $\max(T(r, f), T(r, g)) \leq N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) - 6 \log r$).

We will need the following Lemma 8:

Lemma 8. Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(d(0, R^-)$), resp. $f, g \in \mathcal{M}(\mathbb{C})$). Let $P(x) = x^{n+1}Q(x)$ be a polynomial such that $n \geq \deg(Q) + 2$ (resp. $n \geq \deg(Q) + 3$, resp. $n \geq \deg(Q) + 3$). If $P'(f)f' = P'(g)g'$ then $P(f) = P(g)$.

For simplicity, we can assume $a_1 = 0$. Set $F = \frac{f'P'(f)}{\alpha}$ and $G = \frac{g'P'(g)}{\alpha}$. Clearly F and G share the value 1 C.M. Since f, g are transcendental, we notice that so are F and G . We put

$$\Psi_{F,G} = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$$

We will prove that under the hypotheses of Theorems, $\Psi_{F,G}$ is identically zero.

The following lemma holds in the same way in p -adic analysis and in complex analysis. It is proven in [5] for the p -adic version and in [21] for the complex version. :

Lemma 9. Let $f, g \in \mathcal{M}(\mathbb{E})$ (resp. let $f, g \in \mathcal{M}(d(0, R^-)$) be non-constant and sharing the value 1 C.M. Suppose that $\Psi_{f,g} = 0$ and that

$$\limsup_{r \rightarrow +\infty} \left(\frac{\bar{Z}(r, f) + \bar{Z}(r, g) + \bar{N}(r, f) + \bar{N}(r, g)}{\max(T(r, f), T(r, g))} \right) < 1$$

(resp.

$$\limsup_{r \rightarrow R^-} \left(\frac{\bar{Z}(r, f) + \bar{Z}(r, g) + \bar{N}(r, f) + \bar{N}(r, g)}{\max(T(r, f), T(r, g))} \right) < 1.)$$

Then either $f = g$ or $fg = 1$.

Proofs of Theorem 2, 3, 4, 5, 6, 7

For simplicity, now we set $n = k_1$. Set $F = \frac{f'P'(f)}{\alpha}$, $G = \frac{g'P'(g)}{\alpha}$ and $\widehat{F} = P(f)$, $\widehat{G} = P(g)$. Suppose $F \neq G$. We notice that $P(x)$ is of the form $x^{n+1}Q(x)$ with $Q \in K[x]$ of degree k . Now, with help of Lemma 6, we can check that we have

$$T(r, \widehat{F}) - Z(r, \widehat{F}) \leq T(r, \widehat{F}') - Z(r, \widehat{F}') + O(1)$$

Consequently, since $(\widehat{F})' = \alpha F$, we have

$$T(r, \widehat{F}) \leq T(r, F) + Z(r, \widehat{F}) - Z(r, F) + T(r, \alpha) + O(1), \quad (1)$$

hence, by (1), we obtain

$$T(r, \widehat{F}) \leq T(r, F) + (n+1)Z(r, f) + Z(r, Q(f)) - nZ(r, f) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + O(1).$$

i.e.

$$T(r, \widehat{F}) \leq T(r, F) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + O(1). \quad (2)$$

and similarly,

$$T(r, \widehat{G}) \leq T(r, G) + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l k_i Z(r, g - a_i) - Z(r, g') + T(r, \alpha) + O(1). \quad (3)$$

Now, it follows from the definition of F and G that

$$Z_{[2]}(r, F) + N_{[2]}(r, F) \leq 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + T(r, \alpha) + O(1) \quad (4)$$

and similarly

$$Z_{[2]}(r, G) + N_{[2]}(r, G) \leq 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + T(r, \alpha) + O(1) \quad (5)$$

And particularly, if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, then

$$Z_{[2]}(r, F) + N_{[2]}(r, F) \leq 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + T(r, \alpha) + O(1) \quad (6)$$

and similarly

$$Z_{[2]}(r, G) + N_{[2]}(r, G) \leq 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + T(r, \alpha) + O(1). \quad (7)$$

Suppose now that $\Psi_{F,G}$ is not identically zero.

Let us place us in the p -adic context: $\mathbb{E} = \mathbb{K}$. By Lemma 7, we have

$$T(r, F) \leq Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) - 3 \log r$$

hence by (2), we obtain

$$\begin{aligned} T(r, \widehat{F}) &\leq Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) + Z(r, f) + Z(r, Q(f)) \\ &\quad - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) - 3 \log r + O(1) \end{aligned}$$

and hence by (4) and (5):

$$\begin{aligned} T(r, \widehat{F}) &\leq 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + \\ &\quad 2\overline{N}(r, g) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + 2T(r, \alpha) - 3 \log r + O(1) \quad (8) \end{aligned}$$

and similarly,

$$\begin{aligned} T(r, \widehat{G}) &\leq 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) \\ &\quad + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l k_i Z(r, g - a_i) - Z(r, g') + 2T(r, \alpha) - 3 \log r + O(1). \quad (9) \end{aligned}$$

Consequently,

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) + (Z(r, f') + Z(r, g')) + \\ &\quad 4(\overline{N}(r, f) + \overline{N}(r, g)) + (Z(r, Q(f)) + Z(r, Q(g))) + 4T(r, \alpha) - 6 \log r + O(1). \quad (10) \end{aligned}$$

Moreover, if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, then by (6) and (7) we have

$$\begin{aligned} T(r, \widehat{F}) &\leq 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + \\ &\quad Z(r, g') + 2\overline{N}(r, g) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l Z(r, f - a_i) - Z(r, f') + 2T(r, \alpha) - 3 \log r + O(1) \end{aligned}$$

and similarly,

$$T(r, \widehat{G}) \leq 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) +$$

$$Z(r, f') + 2\bar{N}(r, f) + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l Z(r, g - a_i) - Z(r, g') + 2T(r, \alpha) - 3 \log r + O(1).$$

Consequently,

$$T(r, \hat{F}) + T(r, \hat{G}) \leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (Z(r, f - a_i) + Z(r, g - a_i)) + Z(r, Q(f)) + Z(r, Q(g)) + (Z(r, f') + Z(r, g')) + 4(\bar{N}(r, f) + \bar{N}(r, g)) + 4T(r, \alpha) - 6 \log r + O(1) \quad (11)$$

Now, let us go back to the general case. By Lemma 4 we can write $Z(r, f') + Z(r, g') \leq Z(r, f - a_2) + Z(r, g - a_2) + \bar{N}(r, f) + \bar{N}(r, g) - 2 \log r$. Hence, in general, by (10) we obtain

$$T(r, \hat{F}) + T(r, \hat{G}) \leq 5(Z(r, f) + Z(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2))) + 5(\bar{N}(r, f) + \bar{N}(r, g)) + (Z(r, Q(f)) + Z(r, Q(g))) + 4T(r, \alpha) - 8 \log r + O(1)$$

and hence, since $T(r, Q(f)) = kT(r, f) + O(1)$ and $T(r, Q(g)) = kT(r, g) + O(1)$,

$$T(r, \hat{F}) + T(r, \hat{G}) \leq 5(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2))) + 5(\bar{N}(r, f) + \bar{N}(r, g)) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1). \quad (12)$$

Since \hat{F} is a polynomial in f of degree $n + k + 1$, we have $T(r, \hat{F}) = (n + k + 1)T(r, f) + O(1)$ and similarly, $T(r, \hat{G}) = (n + k + 1)T(r, g) + O(1)$, hence by (12) we can derive

$$(n + k + 1)(T(r, f) + T(r, g)) \leq 5(T(r, f) + T(r, g)) + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + 5(\bar{N}(r, f) + \bar{N}(r, g)) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1). \quad (13)$$

hence

$$(n + k + 1)(Tr, f) + T(r, g) \leq 10(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2))) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1)$$

hence

$$n(Tr, f) + T(r, g) \leq 9(T(r, f) + T(r, g)) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2)))$$

$$(14) \quad + \sum_{i=3}^l (4 - k_i) ((Z(r, f - a_i) + Z(r, g - a_i))) + 4T(r, \alpha) - 8 \log r + O(1)$$

Then $(5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \leq \max(0, 5 - k_2)(T(r, f) + T(r, g)) + O(1)$ and at least, for each $i = 3, \dots, l$ we have $(4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) \leq \max(0, 4 - k_i)(T(r, f) + T(r, g)) + O(1)$.

Now suppose $s_5 > 0$. That means that $k_i \geq 5 \forall i = 3, \dots, u_5$ with $l \geq 5$. We notice that the number of indices i superior or equal to 2 such that $k_i \geq 5$ is $u_5 - 2$. Similarly, for each $m > 5$, the number of indices superior or equal to 1 such that $k_i \geq m$ is $u_m - 1$.

Suppose first $\mathbb{E} = \mathbb{K}$. then we can apply Theorem N3 and then we obtain $\sum_{i=3}^{u_5} Z(r, f - a_i) \geq (u_5 - 3)T(r, f) - \log r + O(1)$ and for each $m \geq 6$, $\sum_{i=3}^{u_m} Z(r, g - a_i) \geq (u_m - 2)T(r, g) - \log r + O(1)$, i.e. $\sum_{i=3}^{u_5} Z(r, f - a_i) \geq s_5 T(r, f) - \log r + O(1)$ i.e. $\sum_{i=3}^{u_m} Z(r, g - a_i) \geq s_m T(r, g) - \log r + O(1)$ in Theorems 2, 3, 5.

Consequently, by (14) we obtain

$$(15) \quad n(T(r, f) + T(r, g)) \leq 9(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \\ + \sum_{i=3}^l \max(0, 4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) - \sum_{m=5}^{\infty} s_m(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1)$$

therefore

$$n \leq 9 + \max(5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{j=5}^{\infty} s_j,$$

a contradiction to the hypotheses of Theorems 2.

Consider now the situation in Theorems 4 and 5. In Theorem 4, we have $T(r, \alpha) \leq \log r + O(1)$ and in Theorem 5, $T(r, \alpha) = 0$. Consequently, Relation (15) now implies

$$n(T(r, f) + T(r, g)) \leq 9(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \\ + \sum_{i=3}^l \max(0, 4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) - \sum_{m=5}^{\infty} s_m(T(r, f) + T(r, g)) - 4 \log r + O(1)$$

therefore

$$n < 9 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{m=5}^{\infty} s_m$$

but this is incompatible with the hypothesis

$$n \geq 9 + \max(5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{j=5}^{\infty} s_j).$$

Now, let us consider the complex context: $\mathbb{E} = \mathbb{C}$. All inequalities above hold just by replacing each expression $-q \log r$ by $S_f(r) + S_g(r)$. However, we cannot apply Theorem N3 here but only Theorem N1. Therefore we obtain

$$\sum_{i=3}^{u_5} (Z(r, f - a_i) + Z(r, g - a_i)) \geq (u_5 - 4)(T(r, f) + T(r, g)) = t_5(T(r, f) + T(r, g))$$

and

$$\sum_{i=3}^{U_m} (Z(r, f - a_i) + Z(r, g - a_i)) \geq (u_m - 3)(T(r, f) + T(r, g)) = t_m(T(r, f) + T(r, g)).$$

Therefore we obtain

$$n \leq 9 + \max(5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{m=5}^{\infty} t_m$$

a contradiction to the hypothesis of Theorem 3.

Finally, consider the situation in Theorem 6. Since $N(r, f) = N(r, g) = 0$, Relation (13) gets

$$\begin{aligned} (n + k + 1)(T(r, f) + T(r, g)) &\leq 5(T(r, f) + T(r, g)) + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \\ &+ \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) + S_f(r) + S_g(r). \end{aligned}$$

On the other hand, by applying Theorem N1 to f and g , which now are entire functions, we have

$$\sum_{i=3}^{u_5} Z(r, f - a_i) \geq (u_5 - 3)T(r, f) = s_5 T(r, f), \quad \sum_{i=3}^{u_5} Z(r, g - a_i) \geq (u_5 - 3)T(r, g) = s_5 T(r, g)$$

and

$$\sum_{i=3}^{u_m} Z(r, f - a_i) \geq (u_m - 2)T(r, f) = s_m T(r, f), \quad \sum_{i=3}^{u_m} Z(r, g - a_i) \geq (u_m - 2)T(r, g) = s_m T(r, g).$$

Consequently, $n + k + 1 \leq 5 + k + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{m=1}^{\infty} s_m$ and therefore

$n \leq 4 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \sum_{m=1}^{\infty} s_m$, a contradiction to the hypotheses of Theorem 6. Thus, in the hypotheses of Theorems 2, 3, 4, 5, 6 we have proven that $\Psi_{F,G}$ is

identically zero. Henceforth, we can assume that $\Psi_{F,G} = 0$ in each theorem. Note that we can

write $\Psi_{F,G} = \frac{\phi'}{\phi}$ with $\phi = \left(\frac{F'}{(F-1)^2} \right) \left(\frac{(G-1)^2}{G'} \right)$. Since $\Psi_{F,G} = 0$, there exist $A, B \in \mathbb{E}$ such that

$$(16) \quad \frac{1}{G-1} = \frac{A}{F-1} + B$$

and $A \neq 0$.

We notice that $\bar{Z}(r, f) \leq T(r, f)$,

$$\bar{N}(r, f) \leq T(r, f) \bar{Z}(r, f - a_i) \leq T(r, f - a_i) \leq T(r, f) + O(1), \quad i = 2, \dots, l$$

and $\bar{Z}(r, f') \leq T(r, f') \leq 2T(r, f) + O(1)$. Similarly for g and g' . Moreover, if $\mathbb{E} = \mathbb{K}$ by Lemma 4 we have

$$(17) \quad T(r, F) \geq (n+k)T(r, f).$$

and if $\mathbb{E} = \mathbb{C}$, we have

$$(18) \quad T(r, F) \geq (n+k)T(r, f) - m(r, \frac{1}{f}) + S_f(r).$$

We will show that $F = G$ in each theorem. We first notice that according to all hypotheses in Theorems 2, 3, 4, 5, 6 we have

$$(19) \quad n+k \geq 2l+7$$

and in Theorem 6, we have

$$(20) \quad n+k \geq 2l+5.$$

We will consider the following two cases: $B = 0$ and $B \neq 0$.

Case 1: $B = 0$.

Suppose $A \neq 1$. Then, by (16), we have $F = AG + (1-A)$. Suppose first $\mathbb{E} = \mathbb{K}$. Applying Theorem N1 to F , we obtain

$$(21) \quad \begin{aligned} T(r, F) &\leq \bar{Z}(r, F) + \bar{Z}\left(r, F - (1-A)\right) + \bar{N}(r, F) - \log r + O(1) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) \\ &+ \bar{Z}(r, f') + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) - \log r + O(1). \end{aligned}$$

By (17) and (21) we obtain

$$(22) \quad \begin{aligned} (n+k)T(r, f) &\leq \bar{Z}(r, F) + \bar{Z}\left(r, F - (1-A)\right) + \bar{N}(r, F) - \log r + O(1) \\ &\leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, f') + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) - \log r + O(1). \end{aligned}$$

By (22) we have

$$\begin{aligned} (n+k)T(r, f) &\leq \bar{Z}(r, F) + \bar{Z}\left(r, F - (1-A)\right) + \bar{N}(r, F) - \log r + O(1) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) \\ &+ \bar{Z}(r, f') + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) - \log r + O(1). \end{aligned}$$

hence

$$(n+k)T(r, f) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i)$$

$$(23) \quad +\bar{N}(r, f) + \bar{Z}(r, g') + \bar{Z}(r, f') - \log r + O(1).$$

Then, considering all the previous inequalities, by Lemma 4 we can derive the following from (23):

$$(24) \quad (n+k)T(r, f) \leq (l+3)T(r, f) + (l+2)T(r, g) - 3 \log r + O(1).$$

Since f and g satisfy the same hypothesis, we also have

$$(25) \quad (n+k)T(r, g) \leq (l+3)T(r, g) + (l+2)T(r, f) - 3 \log r + O(1).$$

Hence, adding (24) and (25), we have

$$(n+k)[T(r, f) + T(r, g)] \leq (2l+5)[T(r, f) + T(r, g)] - 6 \log r + O(1)$$

therefore

$$(26) \quad n+k < 2l+5.$$

A contradiction to (20) proving that $A \neq 1$ is impossible whenever $B = 0$, in Theorems 2, 3, 5.

Suppose now $\mathbb{E} = \mathbb{C}$. By (18) we have

$$(n+k)T(r, f) \leq \bar{Z}(r, F) + \bar{Z}\left(r, F - (1-A)\right) + \bar{N}(r, F) + m\left(r, \frac{1}{f'}\right) + S_F(r) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) \\ + \bar{Z}(r, f') + m\left(r, \frac{1}{f'}\right) + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) + S_f(r) + S_g(r).$$

Here we notice that $\bar{Z}(r, f') + m\left(r, \frac{1}{f'}\right) \leq T\left(r, \frac{1}{f'}\right) = T(r, f') + O(1)$, hence $(n+k)T(r, f) \leq$

$$(27) \quad \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, g) + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{N}(r, f) + \bar{Z}(r, g') + T(r, f') + S_f(r) + S_g(r).$$

Then, considering all the previous inequalities in (27), similarly we can derive

$$(28) \quad (n+k)T(r, f) \leq (l+3)T(r, f) + (l+2)T(r, g) + S_f(r) + S_g(r).$$

Since f and g satisfy the same hypothesis, we also have

$$(29) \quad (n+k)T(r, g) \leq (l+3)T(r, g) + (l+2)T(r, f) + S_f(r) + S_g(r).$$

Hence, adding (28) and (29), we have

$$(n+k)[T(r, f) + T(r, g)] \leq (2l+5)[T(r, f) + T(r, g)] + S_f(r) + S_g(r)$$

therefore $n+k \leq 2l+5$, a contradiction to (20) proving that $A \neq 1$ is impossible whenever $B = 0$, in Theorem 3.

Consider now the situation in Theorem 6. By hypothesis we have $k_1 \geq 5 + \max(0, 5 - k_2) + \sum_{i=3}^l \max(0, 4 - k_i) - \min(2l, \sum_{m=5}^{\infty} s_m)$ hence $n+k \geq 10 + 4(l-2) - \sum_{m=5}^{\infty} s_m = 4l+2 - \sum_{m=5}^{\infty} s_m$

Since $N(r, f) = N(r, g) = 0$, we can use Theorem N1 for entire functions and we obtain

$$\sum_{i=3}^{u_5} Z(r, f - a_i) \geq (u_5 - 3)T(r, f) + S_f(r) + S_g(r)$$

and for each $m \geq 6$,

$$\sum_{i=3}^{u_m} Z(r, g - a_i) \geq (u_m - 2)T(r, g) + S_f(r) + S_g(r)$$

i.e.

$$\sum_{i=3}^{u_5} Z(r, f - a_i) \geq s_5 T(r, f) + S_f(r) + S_g(r)$$

and

$$\sum_{i=3}^{u_m} Z(r, g - a_i) \geq s_m T(r, g) + S_f(r) + S_g(r).$$

Now, Relation (13) now gets

$$\begin{aligned} (n + k + 1)(T(r, f) + T(r, g)) &\leq 5(T(r, f) + T(r, g)) + (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \\ &\quad + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + k(T(r, f) + T(r, g)) + S_f(r) + S_g(r) \end{aligned}$$

therefore $n + k \leq 9 + 4(l - 2) - \sum_{j=5}^{\infty} s_j = 2l + 1 - \sum_{m=5}^{\infty} s_m$ a contradiction to the hypothesis $n + k \geq 2l + 5$ of Theorem 6. Consequently, the hypothesis $A \neq 1$ does not hold when $B = 0$.

Henceforth we suppose $B \neq 0$.

Case 2: $B \neq 0$.

Consider first the situation when $\mathbb{E} = \mathbb{K}$, i.e. in Theorems 2 and in Theorems 4 and 5. By (17) we have Immediately,

$$\begin{aligned} \overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) &\leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, f') \\ &\quad + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) + \overline{N}(r, g) + 4T(r, \alpha) + O(1) \\ &\leq (l+1)[T(r, f) + T(r, g)] + T(r, f') + T(r, g') + 4T(r, \alpha) + O(1) \leq (l+3)(T(r, f) + T(r, g)) + 4T(r, \alpha) - 2 \log r \end{aligned}$$

hence by Lemma 4,

$$(30) \quad \overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq (l + 3)(T(r, f) + T(r, g)) + 4T(r, \alpha) - 2 \log r + O(1)$$

Moreover, by (16), $T(r, F) = T(r, G) + O(1)$ and by Lemma 4, we have

$$T(r, f) \leq \frac{1}{n+k}(T(r, F) + T(r, \alpha)) + O(1)$$

and $T(r, g) \leq \frac{1}{n+k}(T(r, G) + T(r, \alpha)) + O(1)$. Consequently,

$$T(r, f) + T(r, g) \leq 2 \left[\frac{1}{n+k}(T(r, F) + T(r, \alpha)) \right] + O(1).$$

$$(31) \quad \bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) \leq \frac{2l+6}{n+k}T(r, F) + \left(\frac{2l+6}{n+k} + 4\right)T(r, \alpha) - 2\log r + O(1).$$

Now, by Hypotheses, in Theorems 2, 3, 5 by (19) we have $n+k \geq 2l+7$. Consequently, by Relation (31) we obtain

$$(32) \quad \bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) \leq \frac{2l+6}{2l+7}T(r, F) + \left(\frac{2l+6}{2l+7} + 4\right)T(r, \alpha) + O(1).$$

and similiary

$$(33) \quad \bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) \leq \frac{2l+6}{2l+7}T(r, G) + \left(\frac{2l+6}{2l+7} + 4\right)T(r, \alpha) + O(1).$$

hence

$$\limsup_{r \rightarrow +\infty} \left(\frac{\bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G)}{\max(T(r, F), T(r, G))} \right) < 1$$

Therefore, by Lemma 9, in Theorems 2, 3, 5 we have either $F = G$, or $FG = 1$.

Suppose $FG = 1$. Then $f'P'(f)g'P'(g) = \alpha^2$. But in Theorems 2, 3, 5 we have assumed that $n \neq k+1$ and if $l = 2$, then $n \neq 2k, 2k+1, 3k+1$ and if $l = 3$ then $n \neq k, 3k_2 - k, 3k_3 - k$. Consequently, we have a contradiction to Theorem Y. Thus, the hypothesis $FG = 1$ is impossible and therefore we have $F = G$.

Consider now the situation when $\mathbb{E} = \mathbb{C}$, i.e. in Theorems 3 and 6. The proof is very similar to that in the case when $\mathbb{E} = \mathbb{K}$. We have

$$\bar{Z}(r, F) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, f') + S_f(r)$$

and

$$\bar{N}(r, F) \leq \bar{N}(r, f) + S_f(r)$$

and similarly for G , so we can derive

$$\bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) \leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, f') + \bar{Z}(r, g)$$

$$+ \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + \bar{N}(r, f) + \bar{N}(r, g) + S_f(r) + S_g(r)$$

$$(34) \quad \leq (l+2)[T(r, f) + T(r, g)] + S_f(r) + S_g(r).$$

Moreover, by (16), $T(r, F) = T(r, G) + O(1)$ and, by Lemma 4, we have

$$T(r, f) \leq \frac{1}{n+k}T(r, F) + S_f(r)$$

and $T(r, g) \leq \frac{1}{n+k}T(r, G) + S_g(r)$. Consequently,

$$T(r, f) + T(r, g) \leq \frac{2}{n+k}T(r, F) + S_f(r) + S_g(r).$$

Thus, (34) implies

$$\bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) \leq \frac{2l+6}{n+k}T(r, F) + S_f(r) + S_g(r).$$

Now, as in Theorems 2, 3, 5, we can check that $n+k \geq 2l+7$ in theorem 3. Consequently, the previous inequality implies

$$\bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) \leq \frac{2l+6}{2l+7}T(r, F) + S_f(r) + S_g(r)$$

and similarly,

$$\bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) + \bar{N}(r, G) \leq \frac{2l+6}{2l+7}T(r, G) + S_f(r) + S_g(r)$$

hence by Lemma 9 again, we have $F = G$ or $FG = 1$. Then, by Theorem Y as in Theorems 2, 3, 5, the hypotheses of Theorem 3 prevent the case $FG = 1$ and therefore $F = G$.

Consider now the situation in Theorem 6. Relation (34) implies

$$(35) \quad \bar{Z}(r, F) + \bar{Z}(r, G) \leq (l+2)[T(r, f) + T(r, g)] + S_f(r) + S_g(r).$$

Moreover, by (16), $T(r, F) = T(r, G) + O(1)$ and, by Lemma 4, we have

$T(r, f) \leq \frac{1}{n+k}T(r, F) + S_f(r)$ and $T(r, g) \leq \frac{1}{n+k}T(r, G) + S_g(r)$. Consequently,

$$T(r, f) + T(r, g) \leq \frac{2}{n+k}T(r, F) + S_f(r) + S_g(r).$$

Thus, (220) implies

$$\begin{aligned} \bar{Z}(r, F) + \bar{Z}(r, G) &\leq \bar{Z}(r, f) + \sum_{i=2}^l \bar{Z}(r, f - a_i) + \bar{Z}(r, f') + \bar{Z}(r, g) \\ &\quad + \sum_{i=2}^l \bar{Z}(r, g - a_i) + \bar{Z}(r, g') + S_f(r) + S_g(r) \\ &\leq 4[T(r, f) + T(r, g)] + S_f(r) + S_g(r). \end{aligned}$$

Therefore

$$\bar{Z}(r, F) + \bar{Z}(r, G) \leq \frac{2l+4}{n+k}T(r, F) + S_f(r) + S_g(r)$$

hence by (20) we have

$$\bar{Z}(r, F) + \bar{Z}(r, G) \leq \frac{2l+4}{2l+5}T(r, F) + S_f(r) + S_g(r).$$

In the same way, this proves that either $F = G$ or $FG = 1$. But by Theorem Y, $FG = 1$ is impossible. Hence $F = G$.

Thus, in Theorems 2, 3, 4, 5, 6 we have proven that $F = G$ i.e. $f'P'(f) = g'P'(g)$. Consequently, $P(f) - P(g)$ is a constant C . Then, by Lemma 8 and Proposition P1, in Theorems 2, 3, 5 we have $P(f) = P(g)$ and by Lemma 8 and Proposition P2, we have $P(f) = P(g)$ in Theorems 3 and 6. Finally, in each theorem, P is a polynomial of uniqueness for the family of functions we consider. Consequently, $f = g$.

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