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p -adic differential polynomials when poles make a small function

by Alain Escassut, Weiran Lü and Chung Chun Yang

Notation: We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} , by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} , i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions. Throughout the paper, a is a point in \mathbb{K} and R is a strictly positive number and we denote by $d(a, R^-)$ the “open” disk $\{x \in \mathbb{K} : |x - a| < R\}$, by $\mathcal{A}(d(a, R^-))$ the \mathbb{K} -algebra of analytic functions in $d(a, R^-)$ i.e. the \mathbb{K} -algebra of power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converging in $d(a, R^-)$ and we denote by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the \mathbb{K} -subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n \in \mathbb{N}} |a_n| R^n < +\infty$. And we denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

Let \log be a real logarithm function of base $b > 1$ and we set $\log^+(x) = \max(0, \log(x))$. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0. Let $r \in]0, +\infty[$ (resp. $r \in]0, R[$) and let $\gamma \in d(0, r)$. If f has a zero of order n at γ , we put $\omega_\gamma(h) = n$. If f has a pole of order n at γ , we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$.

We must now recall the definition of the Nevanlinna functions in p -adic [1], [3], [4].

We denote by $Z(r, f)$ the *counting function of zeros of f in $d(0, r)$* , counting multiplicities, i.e.

$$Z(r, f) = \max(\omega_0, 0) \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by $\bar{Z}(r, f)$ the *counting function of zeros of f in $d(0, r)$* , ignoring multiplicities, and set

$$\bar{Z}(r, f) = u \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} (\log r - \log |\gamma|)$$

with $u = 1$ when $\omega_0(f) > 0$ and $u = 0$ else.

In the same way, we set $N(r, f) = Z\left(r, \frac{1}{f}\right)$ (resp. $\bar{N}(r, f) = \bar{Z}\left(r, \frac{1}{f}\right)$) to denote the *counting function of poles of f* in $d(0, r)$, counting multiplicities (resp. ignoring multiplicities).

For $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}(d(0, R^-))$, we call *Nevanlinna function of f* the function $T(r, f) = \max\{Z(r, f), N(r, f)\}$.

As usual, given $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) we denote by $S_f(r)$ any function ϕ from $]0, +\infty[$ (resp. from $]0, R[$) to \mathbb{R} such that $\lim_{r \rightarrow +\infty} \frac{\phi(r)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{\phi(r)}{T(r, f)} = 0$).

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$) such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}(d(0, R^-))$) is called a *small function with respect to f* , if it satisfies $\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R^-} \frac{T(r, \alpha)}{T(r, f)} = 0$).

We denote by $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) the set of small meromorphic functions with respect to f in \mathbb{K} (resp. in $d(0, R^-)$). Similarly, we denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(0, R^-))$) the set of small meromorphic functions $w \in \mathcal{A}(\mathbb{K})$ (resp. $w \in \mathcal{A}(d(0, R^-))$) with respect to f in \mathbb{K} (resp. in $d(0, R^-)$).

Theorem: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}_u(d(0, R^-))$) be such that $N(r, f) = S_f(r)$ and let $F(x) = \sum_{j=0}^{n-1} a_j f^j + a_n f^n f^{(k)}$ with $a_0 a_n$ not identically zero, $n \geq 1$, $a_j \in \mathcal{M}_f(\mathbb{K})$, $j = 0, \dots, n$ (resp. $a_j \in \mathcal{M}_f(d(0, R^-))$, $j = 0, \dots, n$). Then F has infinitely many zeros in \mathbb{K} (resp. in $d(0, R^-)$).

Corollary: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}_u(d(0, R^-))$) be such that $N(r, f) = S_f(r)$ and let $a_0, a_1 \in \mathcal{M}_f(\mathbb{K})$ (resp. $a_0, a_1 \in \mathcal{M}_f(d(0, R^-))$) with $a_0 a_1$ non-identically zero. Then f has infinitely many zeros in \mathbb{K} (resp. in $d(0, R^-)$).

Remark: In [5], results were proven with some simlilarity: see Theorems A and B below given by Theorem 2 in [5]. Indeed, the condition $N(r, f) = S_f(r)$ implies that $\limsup_{r \rightarrow +\infty} |f|(r) = +\infty$ (resp. $\limsup_{r \rightarrow R} |f|(r) = +\infty$). But the hypothesis on b was more restrictive.

Theorem A: Let $f \in \mathcal{M}(\mathbb{K}) \setminus K(x)$ satisfy $\liminf_{r \rightarrow \infty} |f|(r) > 0$ and let $b \in \mathbb{K}(x)$ be non-identically zero and such that $\lim_{r \rightarrow +\infty} |b|(r) > 0$. Then $ff' - b$ has infinitely many zeros.

Theorem B: Let $f \in \mathcal{M}_u(d(a, R^-))$ satisfy $\limsup_{r \rightarrow R} |f|(r) = +\infty$ and let $b \in \mathbb{K}(x)$ be non-identically zero be such that $|b|(r) > \frac{1}{R}$. Then $ff' - b$ has infinitely many zeros.

The proof:

Lemma 1 is classical [1], [3], [4]:

Lemma 1 : Let $f \in \mathcal{A}(d(0, R^-))$ and suppose that $f(0) \neq 0$. Then $\log(|f|(r)) = \log(|f(0)|) + Z(r, f)$.

The following Lemma 2 is a consequence of Lemma 1.

Lemma 2: Let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}(d(0, R^-))$) be such that $\lim_{r \rightarrow +\infty} Z(r, f) - Z(r, g) = +\infty$ (resp. $\lim_{r \rightarrow R} Z(r, f) - Z(r, g) = +\infty$). Then $|f + g|(r) = |f|(r)$ and $Z(r, f + g) = Z(r, f)$ when r is big enough (resp. when r is close enough to R).

Proof: Without loss of generality, we can suppose that $f(0)$ and $g(0)$ different from 0 and ∞ . When r is big enough (resp. is close enough to R), by Lemma 1 we have $|f|(r) > |g|(r)$, hence $|f + g|(r) = |f|(r)$ and the conclusion follows from Lemma 1 again.

Lemma 3 is useful to deal with meromorphic functions inside a disk: Theorem 25.5 of [2] and Theorem 2.2.3 of [3]:

Definition and notation: Let $a \in \mathbb{K}$, $R > 0$. Let $(a_n, q_n)_{n \in \mathbb{N}}$ be a sequence where $a_n \in (a, R^-)$, $\lim_{n \rightarrow +\infty} |a_n| = R$ and $q_n \in \mathbb{N}^*$. The sequence $(a_n, q_n)_{n \in \mathbb{N}}$ is called a *divisor of $d(a, R^-)$* . Given a divisor $T = (a_n, q_n)_{n \in \mathbb{N}}$ of $d(a, R^-)$, for all $r \in]0, R[$, we set $|T|(r) = \prod_{n=0}^{\infty} (\max(|x|, |a_n|))^{q_n}$.

Lemma 3 : Let $f \in \mathcal{M}(d(0, R^-))$. There exists $b, c \in \mathcal{A}(d(0, R^-))$ such that $Z(r, b) \leq Z(r, f) + O(1)$ and $N(r, b) \leq N(r, f) + O(1)$.

Proof: Let D be the divisor of zeros of f and let E be the divisor of poles of f in $d(0, R^-)$ By Theorem 25.5 of [2] or Theorem 22.3 of [3] we can write f in the form $\frac{b}{c}$

with $b, c \in \mathcal{A}(\mathbb{K})$ such that $|b|(r) \leq |D|(r) + 1$ and $|c|(r) \leq |E|(r) + 1$. Consequently, by Lemma 1 we have $Z(r, b) \leq Z(r, f) + O(1)$ and $N(r, c) \leq N(r, f) + O(1)$.

Lemma 4 is then an easy consequence of Lemma 3:

Lemma 4 : *Let $f \in \mathcal{M}(d(0, R^-))$ and let $w \in \mathcal{M}_f(d(0, R^-))$. We can write w in the form $\frac{b}{c}$ with $b, c \in \mathcal{A}_f(d(0, R^-))$.*

Proof: By Lemma 3, we can write w in the form $\frac{b}{c}$ with $b, c \in \mathcal{A}(d(0, R))$ such that $Z(r, b) \leq Z(r, w) + O(1) \leq T(r, w) + O(1)$ and $Z(r, c) \leq N(r, w) + O(1) \leq T(r, w) + O(1)$. Hence $b, c \in \mathcal{A}_f(d(0, R))$.

Proof of the Theorem: Without loss of generality, we can assume that f has no zero and no pole at 0.

Suppose first $f \in \mathcal{M}(\mathbb{K})$. We can write f in the form $f = \frac{g}{h}$ with $g, h \in \mathcal{A}(\mathbb{K})$ having no common zero and for all $j = 0, \dots, n$, set $a_j = \frac{b_j}{c_j}$ with $b_j, c_j \in \mathcal{A}(\mathbb{K})$ having no common zero. Then by hypotheses, h belongs to $\mathcal{M}_f(\mathbb{K})$ and for each $j = 0, \dots, n$, so do both b_j, c_j .

Let $h(x) = \prod_{j=0}^{\infty} (1 - (\frac{x}{\alpha_j})^{k_j})$, let $\bar{h}(x) = \prod_{j=0}^{\infty} (1 - \frac{x}{\alpha_j})$ and let $h = \bar{h}\tilde{h}$. By classical results [] we know that $f^{(k)}$ is of the form $\frac{g_k}{h(\bar{h})^k}$ with $g_k \in \mathcal{A}(\mathbb{K})$. Consequently we have $N(r, f^{(k)}) \leq (k+1)N(r, f)$ and therefore $N(r, f^{(k)}) = S_f(r)$.

We can write $f(x)$ in the form $\frac{\sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k}{(h)^{n+1} (\bar{h})^k \prod_{j=0}^n c_j}$ whereas each w_j is a product of the form $b_j \prod_{m \neq j, 0 \leq m \leq n} c_m h^{n-j}$. Thus, we can check that $Z(r, b_j \prod_{m \neq j, 0 \leq m \leq n} c_m h^{n-j}) = S_f(r)$. Consequently,

$$Z(r, F) = Z(r, \sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k) + S_f(r).$$

Now, since $N(r, f) = S_f(r)$, we have $T(r, f) = Z(r, f) = Z(r, g)$. Then for each $j = 0, \dots, n-1$, we have $Z(r, w_n g^n g_k) \geq Z(r, w_n g^n)$ and hence $\lim_{r \rightarrow +\infty} Z(r, w_n g^n g_k) - Z(r, g^j) = +\infty$. Consequently, by Lemma 2, we have

$$Z(r, \sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k) + S_f(r) = nZ(r, g) + S_f(r) \text{ when } r \text{ is big enough. That}$$

implies $Z(r, F) = nZ(r, g) + S_f(r) = nT(r, f) + S_f(r)$ when r is big enough, proving that F has infinitely many zeros.

Suppose now that f belongs to $\mathcal{M}_u(d(0, R^-))$. By Lemma 3 we can write f in the form $\frac{g}{h}$ with $g, h \in \mathcal{A}_f(d(0, R^-))$ and by Lemma 4, for each $j = 0, \dots, n$, we can write a_j in the form $\frac{b_j}{c_j}$ with $b_j, c_j \in \mathcal{A}_f(d(0, R^-))$. The proof then goes on like when f belongs to $\mathcal{M}(\mathbb{K})$.

Let $h(x) = \prod_{j=0}^{\infty} (1 - (\frac{x}{\alpha_j})^{k_j})$, let $\bar{h}(x) = \prod_{j=0}^{\infty} (1 - \frac{x}{\alpha_j})$ and let $h = \bar{h}\tilde{h}$. Then $f^{(k)}$ is of the form $\frac{g_k}{h(\bar{h})^k}$ with $g_k \in \mathcal{A}(d(0, R^-))$. Consequently we have $N(r, f^{(k)}) \leq (k+1)N(r, f)$ and therefore $N(r, f^{(k)}) = S_f(r)$.

We can write $f(x)$ in the form $\frac{\sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k}{(h)^{n+1} (\bar{h})^k \prod_{j=0}^n c_j}$ whereas each w_j is a product of the form $b_j \prod_{m \neq j, 0 \leq m \leq n} c_m h^{n-j}$. Thus, we can check that $Z(r, b_j \prod_{m \neq j, 0 \leq m \leq n} c_m h^{n-j}) = S_f(r)$. Consequently,

$$Z(r, F) = Z(r, \sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k) + S_f(r).$$

Now, since $N(r, f) = S_f(r)$, we have $T(r, f) = Z(r, f) = Z(r, g)$. Then for each $j = 0, \dots, n-1$, we have $Z(r, w_n g^n g_k) \geq Z(r, w_n g^n)$ and hence $\lim_{r \rightarrow R} Z(r, w_n g^n g_k) - Z(r, g^j) = +\infty$. Consequently, by Lemma 2, we have $Z(r, \sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k) + S_f(r) = nZ(r, g) + S_f(r)$ when r is close enough to R . That implies $Z(r, F) = nZ(r, g) + S_f(r) = nT(r, f) + S_f(r)$ when r is close enough to R , proving that F has infinitely many zeros.

Reference:

- [1] **Boutabaa, A.** *Théorie de Nevanlinna p-adique*, Manuscripta Math. 67, p. 251-269 (1990).
- [2] **Escassut, A.** *Analytic Elements in p-adic Analysis*. World Scientific Publishing Co. Pte. Ltd. Singapore, (1995).
- [3] **Escassut, A.** *p-adic Value Distribution*. Some Topics on Value Distribution and Differentiability in Complex and P-adic Analysis, p. 42- 138. Mathematics Monograph, Series 11. Science Press.(Beijing 2008).

- [4] **Hu, P.C. and Yang, C.C.** *Meromorphic Functions over non-Archimedean Fields*, Kluwer Academic Publishers, (2000).
- [5] **Ojeda, J.** *On Hayman's Conjecture over a p -adic field*, Taiwanese Journal of Mathematics 12 (9), p. 2295-2313, (2008).