



HAL
open science

p-adic differential polynomials when poles make a small function

Alain Escassut, Weiran Lü, Chung Chun Yang

► **To cite this version:**

Alain Escassut, Weiran Lü, Chung Chun Yang. p-adic differential polynomials when poles make a small function. p-Adic Numbers, Ultrametric Analysis and Applications, 2014. hal-01920257

HAL Id: hal-01920257

<https://uca.hal.science/hal-01920257v1>

Submitted on 13 Nov 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

p -adic differential polynomials when poles make a small function

by **Alain Escassut, Weiran Lü and Chung Chun Yang**

Notation: We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} , by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} , i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions. Throughout the paper, a is a point in \mathbb{K} and R is a strictly positive number and we denote by $d(a, R^-)$ the “open” disk $\{x \in \mathbb{K} : |x - a| < R\}$, by $\mathcal{A}(d(a, R^-))$ the \mathbb{K} -algebra of analytic functions in $d(a, R^-)$ i.e. the \mathbb{K} -algebra of power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converging in $d(a, R^-)$ and we denote by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the \mathbb{K} -subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n \in \mathbb{N}} |a_n|R^n < +\infty$. And we denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

Let \log be a real logarithm function of base $b > 1$ and we set $\log^+(x) = \max(0, \log(x))$. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0. Let $r \in]0, +\infty[$ (resp. $r \in]0, R[$) and let $\gamma \in d(0, r)$. If f has a zero of order n at γ , we put $\omega_\gamma(h) = n$. If f has a pole of order n at γ , we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$.

We must now recall the definition of the Nevanlinna functions in p -adic [1], [3], [4].

We denote by $Z(r, f)$ the *counting function of zeros of f in $d(0, r)$* , counting multiplicities, i.e.

$$Z(r, f) = \max(\omega_0, 0) \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by $\bar{Z}(r, f)$ the *counting function of zeros of f in $d(0, r)$* , ignoring multiplicities, and set

$$\bar{Z}(r, f) = u \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} (\log r - \log |\gamma|)$$

with $u = 1$ when $\omega_0(f) > 0$ and $u = 0$ else.

In the same way, we set $N(r, f) = Z\left(r, \frac{1}{f}\right)$ (resp. $\bar{N}(r, f) = \bar{Z}\left(r, \frac{1}{f}\right)$) to denote the *counting function of poles of f* in $d(0, r)$, counting multiplicities (resp. ignoring multiplicities).

For $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}(d(0, R^-))$, we call *Nevanlinna function of f* the function $T(r, f) = \max\{Z(r, f), N(r, f)\}$.

As usual, given $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) we denote by $S_f(r)$ any function ϕ from $]0, +\infty[$ (resp. from $]0, R[$) to \mathbb{R} such that $\lim_{r \rightarrow +\infty} \frac{\phi(r)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R} \frac{\phi(r)}{T(r, f)} = 0$).

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$) such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}(d(0, R^-))$) is called a *small function with respect to f* , if it satisfies $\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R^-} \frac{T(r, \alpha)}{T(r, f)} = 0$).

We denote by $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) the set of small meromorphic functions with respect to f in \mathbb{K} (resp. in $d(0, R^-)$). Similarly, we denote by $\mathcal{A}_f(\mathbb{K})$ (resp. $\mathcal{A}_f(d(0, R^-))$) the set of small meromorphic functions $w \in \mathcal{A}(\mathbb{K})$ (resp. $w \in \mathcal{A}(d(0, R^-))$) with respect to f in \mathbb{K} (resp. in $d(0, R^-)$).

Theorem: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}_u(d(0, R^-))$) be such that $N(r, f) = S_f(r)$ and let $F(x) = \sum_{j=0}^{n-1} a_j f^j + a_n f^n f^{(k)}$ with $a_0 a_n$ not identically zero, $n \geq 1$, $a_j \in \mathcal{M}_f(\mathbb{K})$, $j = 0, \dots, n$ (resp. $a_j \in \mathcal{M}_f(d(0, R^-))$, $j = 0, \dots, n$). Then F has infinitely many zeros in \mathbb{K} (resp. in $d(0, R^-)$).

Corollary: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}_u(d(0, R^-))$) be such that $N(r, f) = S_f(r)$ and let $a_0, a_1 \in \mathcal{M}_f(\mathbb{K})$ (resp. $a_0, a_1 \in \mathcal{M}_f(d(0, R^-))$) with $a_0 a_1$ non-identically zero. Then f has infinitely many zeros in \mathbb{K} (resp. in $d(0, R^-)$).

Remark: In [5], results were proven with some simlilarity: see Theorems A and B below given by Theorem 2 in [5]. Indeed, the condition $N(r, f) = S_f(r)$ implies that $\limsup_{r \rightarrow +\infty} |f|(r) = +\infty$ (resp. $\limsup_{r \rightarrow R} |f|(r) = +\infty$). But the hypothesis on b was more restrictive.

Theorem A: Let $f \in \mathcal{M}(\mathbb{K}) \setminus K(x)$ satisfy $\liminf_{r \rightarrow \infty} |f|(r) > 0$ and let $b \in \mathbb{K}(x)$ be non-identically zero and such that $\lim_{r \rightarrow +\infty} |b|(r) > 0$. Then $ff' - b$ has infinitely many zeros.

Theorem B: Let $f \in \mathcal{M}_u(d(a, R^-))$ satisfy $\limsup_{r \rightarrow R} |f|(r) = +\infty$ and let $b \in \mathbb{K}(x)$ be non-identically zero be such that $|b|(r) > \frac{1}{R}$. Then $ff' - b$ has infinitely many zeros.

The proof:

Lemma 1 is classical [1], [3], [4]:

Lemma 1 : Let $f \in \mathcal{A}(d(0, R^-))$ and suppose that $f(0) \neq 0$. Then $\log(|f|(r)) = \log(|f(0)|) + Z(r, f)$.

The following Lemma 2 is a consequence of Lemma 1.

Lemma 2: Let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. let $f, g \in \mathcal{A}(d(0, R^-))$) be such that $\lim_{r \rightarrow +\infty} Z(r, f) - Z(r, g) = +\infty$ (resp. $\lim_{r \rightarrow R} Z(r, f) - Z(r, g) = +\infty$). Then $|f + g|(r) = |f|(r)$ and $Z(r, f + g) = Z(r, f)$ when r is big enough (resp. when r is close enough to R).

Proof: Without loss of generality, we can suppose that $f(0)$ and $g(0)$ different from 0 and ∞ . When r is big enough (resp. is close enough to R), by Lemma 1 we have $|f|(r) > |g|(r)$, hence $|f + g|(r) = |f|(r)$ and the conclusion follows from Lemma 1 again.

Lemma 3 is useful to deal with meromorphic functions inside a disk: Theorem 25.5 of [2] and Theorem 2.2.3 of [3]:

Definition and notation: Let $a \in \mathbb{K}$, $R > 0$. Let $(a_n, q_n)_{n \in \mathbb{N}}$ be a sequence where $a_n \in (a, R^-)$, $\lim_{n \rightarrow +\infty} |a_n| = R$ and $q_n \in \mathbb{N}^*$. The sequence $(a_n, q_n)_{n \in \mathbb{N}}$ is called a *divisor of $d(a, R^-)$* . Given a divisor $T = (a_n, q_n)_{n \in \mathbb{N}}$ of $d(a, R^-)$, for all $r \in]0, R[$, we set $|T|(r) = \prod_{n=0}^{\infty} (\max(|x|, |a_n|))^{q_n}$.

Lemma 3 : Let $f \in \mathcal{M}(d(0, R^-))$. There exists $b, c \in \mathcal{A}(d(0, R^-))$ such that $Z(r, b) \leq Z(r, f) + O(1)$ and $N(r, b) \leq N(r, f) + O(1)$.

Proof: Let D be the divisor of zeros of f and let E be the divisor of poles of f in $d(0, R^-)$ By Theorem 25.5 of [2] or Theorem 22.3 of [3] we can write f in the form $\frac{b}{c}$

with $b, c \in \mathcal{A}(\mathbb{K})$ such that $|b|(r) \leq |D|(r) + 1$ and $|c|(r) \leq |E|(r) + 1$. Consequently, by Lemma 1 we have $Z(r, b) \leq Z(r, f) + O(1)$ and $N(r, c) \leq N(r, f) + O(1)$.

Lemma 4 is then an easy consequence of Lemma 3:

Lemma 4 : *Let $f \in \mathcal{M}(d(0, R^-))$ and let $w \in \mathcal{M}_f(d(0, R^-))$. We can write w in the form $\frac{b}{c}$ with $b, c \in \mathcal{A}_f(d(0, R^-))$.*

Proof: By Lemma 3, we can write w in the form $\frac{b}{c}$ with $b, c \in \mathcal{A}(d(0, R))$ such that $Z(r, b) \leq Z(r, w) + O(1) \leq T(r, w) + O(1)$ and $Z(r, c) \leq N(r, w) + O(1) \leq T(r, w) + O(1)$. Hence $b, c \in \mathcal{A}_f(d(0, R))$.

Proof of the Theorem: Without loss of generality, we can assume that f has no zero and no pole at 0.

Suppose first $f \in \mathcal{M}(\mathbb{K})$. We can write f in the form $f = \frac{g}{h}$ with $g, h \in \mathcal{A}(\mathbb{K})$ having no common zero and for all $j = 0, \dots, n$, set $a_j = \frac{b_j}{c_j}$ with $b_j, c_j \in \mathcal{A}(\mathbb{K})$ having no common zero. Then by hypotheses, h belongs to $\mathcal{M}_f(\mathbb{K})$ and for each $j = 0, \dots, n$, so do both b_j, c_j .

Let $h(x) = \prod_{j=0}^{\infty} (1 - (\frac{x}{\alpha_j})^{k_j})$, let $\bar{h}(x) = \prod_{j=0}^{\infty} (1 - \frac{x}{\alpha_j})$ and let $h = \bar{h}\tilde{h}$. By classical results [] we know that $f^{(k)}$ is of the form $\frac{g_k}{h(\bar{h})^k}$ with $g_k \in \mathcal{A}(\mathbb{K})$. Consequently we have $N(r, f^{(k)}) \leq (k+1)N(r, f)$ and therefore $N(r, f^{(k)}) = S_f(r)$.

We can write $f(x)$ in the form $\frac{\sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k}{(h)^{n+1} (\bar{h})^k \prod_{j=0}^n c_j}$ whereas each w_j is a product of the form $b_j \prod_{m \neq j, 0 \leq m \leq n} c_m h^{n-j}$. Thus, we can check that $Z(r, b_j \prod_{m \neq j, 0 \leq m \leq n} c_m h^{n-j}) = S_f(r)$. Consequently,

$$Z(r, F) = Z(r, \sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k) + S_f(r).$$

Now, since $N(r, f) = S_f(r)$, we have $T(r, f) = Z(r, f) = Z(r, g)$. Then for each $j = 0, \dots, n-1$, we have $Z(r, w_n g^n g_k) \geq Z(r, w_n g^n)$ and hence $\lim_{r \rightarrow +\infty} Z(r, w_n g^n g_k) - Z(r, g^j) = +\infty$. Consequently, by Lemma 2, we have

$$Z(r, \sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k) + S_f(r) = nZ(r, g) + S_f(r) \text{ when } r \text{ is big enough. That}$$

implies $Z(r, F) = nZ(r, g) + S_f(r) = nT(r, f) + S_f(r)$ when r is big enough, proving that F has infinitely many zeros.

Suppose now that f belongs to $\mathcal{M}_u(d(0, R^-))$. By Lemma 3 we can write f in the form $\frac{g}{h}$ with $g, h \in \mathcal{A}_f(d(0, R^-))$ and by Lemma 4, for each $j = 0, \dots, n$, we can write a_j in the form $\frac{b_j}{c_j}$ with $b_j, c_j \in \mathcal{A}_f(d(0, R^-))$. The proof then goes on like when f belongs to $\mathcal{M}(\mathbb{K})$.

Let $h(x) = \prod_{j=0}^{\infty} (1 - (\frac{x}{\alpha_j})^{k_j})$, let $\bar{h}(x) = \prod_{j=0}^{\infty} (1 - \frac{x}{\alpha_j})$ and let $h = \bar{h}\tilde{h}$. Then $f^{(k)}$ is of the form $\frac{g_k}{h(\bar{h})^k}$ with $g_k \in \mathcal{A}(d(0, R^-))$. Consequently we have $N(r, f^{(k)}) \leq (k+1)N(r, f)$ and therefore $N(r, f^{(k)}) = S_f(r)$.

We can write $f(x)$ in the form $\frac{\sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k}{(h)^{n+1} (\bar{h})^k \prod_{j=0}^n c_j}$ whereas each w_j is a product of the form $b_j \prod_{m \neq j, 0 \leq m \leq n} c_m h^{n-j}$. Thus, we can check that $Z(r, b_j \prod_{m \neq j, 0 \leq m \leq n} c_m h^{n-j}) = S_f(r)$. Consequently,

$$Z(r, F) = Z(r, \sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k) + S_f(r).$$

Now, since $N(r, f) = S_f(r)$, we have $T(r, f) = Z(r, f) = Z(r, g)$. Then for each $j = 0, \dots, n-1$, we have $Z(r, w_n g^n g_k) \geq Z(r, w_n g^n)$ and hence $\lim_{r \rightarrow R} Z(r, w_n g^n g_k) - Z(r, g^j) = +\infty$. Consequently, by Lemma 2, we have $Z(r, \sum_{j=0}^{n-1} w_j g^j + w_n g^n g_k) + S_f(r) = nZ(r, g) + S_f(r)$ when r is close enough to R . That implies $Z(r, F) = nZ(r, g) + S_f(r) = nT(r, f) + S_f(r)$ when r is close enough to R , proving that F has infinitely many zeros.

Reference:

- [1] **Boutabaa, A.** *Théorie de Nevanlinna p-adique*, Manuscripta Math. 67, p. 251-269 (1990).
- [2] **Escassut, A.** *Analytic Elements in p-adic Analysis*. World Scientific Publishing Co. Pte. Ltd. Singapore, (1995).
- [3] **Escassut, A.** *p-adic Value Distribution*. Some Topics on Value Distribution and Differentiability in Complex and P-adic Analysis, p. 42- 138. Mathematics Monograph, Series 11. Science Press.(Beijing 2008).

- [4] **Hu, P.C. and Yang, C.C.** *Meromorphic Functions over non-Archimedean Fields*, Kluwer Academic Publishers, (2000).
- [5] **Ojeda, J.** *On Hayman's Conjecture over a p -adic field*, Taiwanese Journal of Mathematics 12 (9), p. 2295-2313, (2008).