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Alain Escassut

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**Survey and additional properties  
on the transcendence order over  $\mathbb{Q}_p$  in  $\mathbb{C}_p$ .**

**by Alain Escassut**

**Abstract** The paper is aimed at recalling the notion of transcendence order over  $\mathbb{Q}_p$  and its main properties. Proofs are more detailed than in the paper published in Journal of Number Theory. The main results: the order always is  $\geq 1$  and we construct a number  $b$  that is of order  $1 + \epsilon$  for every  $\epsilon > 0$ . If  $a$  is of order  $\leq t$  and if  $b$  is transcendental over  $\mathbb{Q}_p$  but algebraic over  $\mathbb{Q}_p(a)$ , then  $b$  is of order  $\leq t$  too. Finally, numbers of infinite order are constructed.

**Introduction and results**

We denote by  $\mathbb{C}_p$  the completion of the algebraic closure of the field  $\mathbb{Q}_p$  [1], [4]. The transcendence order of a number  $a$  in  $\mathbb{C}_p$  was introduced in [2]. This notion, which is specific to  $p$ -adic numbers, looks a bit like that of transcendence type [5] but it is quite different because this concerns transcendence over  $\mathbb{Q}_p$ , not over  $\mathbb{Q}$ . In 1978 the notion of transcendence order was defined and we showed the existence of numbers with transcendence order  $\leq 1 + \epsilon$ . This is a result that we will describe and here with some improvements. The existence of numbers with transcendence order 1 was communicated to me and stated in [2] but a mistake in the proof puts in doubt the result even though it seems likely. The order of transcendence is stable through an algebraic extension. Finally, in Theorem 4 we construct numbers with an infinite order of transcendence.

**Definitions and notation:** Given a field  $E$ , we denote by  $E(X)$  the field of rational functions with coefficients in  $E$ . We denote by  $|\cdot|_p$  the  $p$ -adic absolute value defined on  $\mathbb{C}_p$  and by  $\log_p$  the real logarithm function of base  $p$ . We then define  $\Psi_p(x) = \log_p(|x|_p)$ . We denote by  $\|\cdot\|$  the Gauss norm defined on  $\mathbb{C}_p[X]$

$$\text{by } \left\| \sum_{j=0}^n a_j X^j \right\| = \max_{0 \leq j \leq n} |a_j|_p.$$

Let  $\tau \in ]0, +\infty[$ . Let  $F$  be a transcendental extension of  $\mathbb{Q}_p$  provided with an absolute value  $|\cdot|$  extending that of  $\mathbb{Q}_p$ . An element  $a \in F$  will be said *to have transcendence order  $\leq \tau$*  or *order  $\leq \tau$*  in brief, if there exists a constant  $C_a \in ]0, +\infty[$  such that every polynomial  $P \in \mathbb{Q}_p[x]$  satisfies  $\log_p(|P(a)|) \geq \log_p(\|P\|) - C_a(\deg(P))^\tau$ .

We will denote by  $\mathcal{S}(\tau)$  the set of numbers  $x \in \mathbb{C}_p$  having transcendence order  $\leq \tau$ .

We will say that a number  $x \in \mathbb{C}_p$  is of *infinite transcendence order* if it does not belong to  $\mathcal{S}(\tau)$  for all  $\tau \in \mathbb{R}_+^*$ .

Given  $a \in \mathbb{C}_p$  and  $r > 0$  we denote by  $d(a, r)$  the disk  $\{x \in \mathbb{C}_p \mid |x - a| \leq r\}$  and by  $d(a, r^-)$  the disk  $\{x \in \mathbb{C}_p \mid |x - a| < r\}$ . We will denote by  $U$  the disk  $d(0, 1)$ .

**Remark:** By definition, an element  $a \in \mathbb{C}_p$  having transcendence order  $\leq \tau$  is transcendental over  $\mathbb{Q}_p$ .

**Theorem 1:** *Let  $\tau \in ]0, +\infty[$ . If  $\mathcal{S}(\tau) \neq \emptyset$  then  $\tau \geq 1$ .*

**Theorem 2:** *There exists  $b \in \mathbb{C}_p$ , transcendental over  $\mathbb{Q}_p$ , of order  $\leq 1 + \epsilon$  for every  $\epsilon > 0$ .*

**Remark:** In [2] the existence of numbers of transcendence order 1 is stated. Unfortunately, the proof is wrong because it is not possible to compose two  $\mathbb{Q}_p$ -isomorphisms defined in an extension of  $\mathbb{C}_p$ . Therefore, the question of existence of numbers transcendental over  $\mathbb{Q}_p$  of order 1 remains open.

**Theorem 3:** *Let  $x \in \mathbb{C}_p$  belong to  $\mathcal{S}(\tau)$  ( $\tau \geq 1$ ) and let  $y \in \mathbb{C}_p$  be transcendental over  $\mathbb{Q}_p$  but algebraic over  $\mathbb{Q}_p(x)$ . Then  $y$  also belongs to  $\mathcal{S}(\tau)$ .*

**Corollary 3.1:** *Let  $a \in \mathcal{S}(\tau)$  and let  $E$  be an algebraic extension of  $\mathbb{Q}_p(a)$  and let  $h(X) \in E(X) \setminus E$ . Then  $h(a)$  belongs to  $\mathcal{S}(\tau)$ .*

Particularly, we notice that if  $a \in \mathcal{S}(\tau)$  then  $\frac{1}{a}$  and  $a^n$  belong to  $\mathcal{S}(\tau)$ .

**Theorem 4:** *There exists numbers in  $\mathbb{C}_p$  having infinite order.*

## The proofs

**Notation:** We denote by  $|\cdot|_\infty$  the Archimedean absolute value defined on  $\mathbb{R}$ .

**Proof of Theorem 1:** Let  $a \in \mathbb{C}_p$ ,  $a \neq 0$ , be transcendental over  $\mathbb{Q}_p$  and have transcendence order  $\leq \tau$ . We can find  $b \in \mathbb{Q}_p$ , ( $b \neq 0$ ) such that  $|a - b|_p < 1$ . Consider the minimal polynomial  $Q$  of  $b$  over  $\mathbb{Q}_p$ . Let  $b_2, \dots, b_q$  be the conjugates of  $b$  over  $\mathbb{Q}_p$  and set  $b_1 = b$ . Since  $\mathbb{Q}_p$  is complete, we notice that all conjugates  $b_j$  of  $b$  over  $\mathbb{Q}_p$  satisfy  $|b_j|_p = |b|_p = |a|_p$ .

Suppose first that  $|a|_p \leq 1$ . Since  $|b_j|_p = |b|_p = |a|_p \leq 1$ , all coefficients of  $Q$  belong to  $\mathbb{Z}_p$ . Obviously  $Q$  is monic, hence  $\|Q\| = 1$ . By hypothesis, there exists  $C_a \in ]0, +\infty[$  such that  $\Psi_p(P(a)) \geq \log_p(\|P\|) - C_a(\deg(P))^\tau \forall P \in \mathbb{Q}_p[x]$ . Consequently,  $-n\Psi_p(Q(a)) = -\Psi_p((Q(a))^n) \leq C_a(n \deg(Q))^\tau \forall n \in \mathbb{N}^*$ . Since  $Q(b) = 0$  and since, by  $Q$  is clearly 1-Lipschitzian in the disk  $U$ , we have  $-\Psi_p(Q(a)) > 0$  and therefore, if  $\tau < 1$ , the inequality  $-n\Psi_p(Q(a)) \leq C_a(n \deg(Q))^\tau \forall n \in \mathbb{N}^*$  is impossible when  $n$  tends to  $+\infty$ .

Suppose now  $|a|_p > 1$ . Set  $Q(X) = \sum_{k=0}^q c_k X^k$ . Since the  $b_j$  satisfy  $|b_j|_p = |a|_p$ , ( $1 \leq j \leq q$ ), we have  $|c_k|_p \leq (|a|_p)^{q-k}$  and particularly  $|c_0|_p = \prod_{j=1}^q |b_j|_p = |a|^q$ . Consequently,  $\|Q\| = (|a|_p)^q$  and therefore, considering the sequence  $(Q^n)_{n \in \mathbb{N}}$ , for every  $n \in \mathbb{N}^*$  we have,

$$(1) \quad -n \log_p(|Q(a)|_p) \leq -nq \log_p(|a|_p) + C_a(nq)^\tau.$$

On the other hand,  $Q(a) = Q(a) - Q(b) = (a - b) \sum_{k=1}^q c_k \left( \sum_{j=0}^k a_j b^{k-j-1} \right)$  and

hence  $|Q(a)|_p \leq |a - b|_p (|a|_p)^{q-1}$ . Consequently, we obtain

$$-n \log_p(|a - b|_p) - n(q - 1) \log_p(|a|_p) \leq -n \log_p(|Q(a)|_p) \text{ and hence, by (1):}$$

$$-n \log_p(|a - b|_p) - n(q - 1) \log_p(|a|_p) \leq -nq \log_p(|a|_p) + C_a(nq)^\tau. \text{ Finally,}$$

$n(\log_p(|a|_p) - \log_p(|a - b|_p)) \leq C_a(nq)^\tau$ . Since  $|a|_p > 1$  and  $|a - b|_p < 1$ , this inequality is impossible again when  $n$  tends to  $+\infty$ , which ends the proof.

**Lemma A:** *Let  $P(X) \in \mathbb{Q}[x]$  and let  $a \in U$ , let  $t = X - a$  and let  $Q(t) = P(X)$ . Then  $\|P\| = \|Q\|$ .*

**Lemma B:** *Let  $b \in \mathbb{C}_p$  be transcendental over  $\mathbb{Q}_p$  and let  $q \in \mathbb{N}^*$ . There exists a real constant  $m > 0$  such that  $|Q(b)|_p \geq m$  for every polynomial  $Q \in \mathbb{Q}_p[X]$  such that  $\|Q\| = 1$  and  $\deg(Q) \leq q$ .*

**Proof:** Suppose that Lemma B is wrong and let  $(Q_n)_{n \in \mathbb{N}^*}$  be a sequence of  $\mathbb{Q}_p[X]$  such that  $\|Q_n\| = 1$ ,  $\deg(Q_n) \leq q \forall n \in \mathbb{N}^*$  and such that  $\lim_{n \rightarrow +\infty} Q_n(b) = 0$ .

We then can extract from the sequence  $(Q_n)$  a subsequence converging to a polynomial  $Q \in \mathbb{Q}_p[X]$  such that  $\|Q\| = 1$  and  $Q(b) = 0$ , a contradiction since  $b$  is transcendental over  $\mathbb{Q}_p$ .

**Lemma C:** *Let  $a \in \mathbb{C}_p$  be algebraic over  $\mathbb{Q}_p$ ,  $\log_p(|a|)$  is of the form  $\frac{\lambda}{t}$ , with*

$\lambda$  and  $t$  in  $\mathbb{N}$ . Take  $m, n \in \mathbb{N}$  and  $b \in \mathbb{C}_p$  such that  $\log_p(|b|)$  is of the form  $\frac{u}{w}$  with  $u \in \mathbb{N}$  and  $w \in \mathbb{N}$  prime, prime with  $u$  and such that  $w > \max(m, n, t)$ .

Let  $f, g \in \mathbb{Q}_p[a]$  be such that  $|fb^m|_p = |gb^n|_p$ . Then  $m = n$ .

**Proof:** We notice that for every  $x \in \mathbb{Q}_p[a]$ ,  $\log_p(|x|)$  is of the form  $\frac{\ell}{t}$  with  $\ell \in \mathbb{N}$ . Consequently,  $\log_p(|f|)$  is of the form  $\frac{h}{t}$  and  $\log_p(|g|)$  is of the form  $\frac{k}{t}$  with  $h$  and  $k \in \mathbb{N}^*$ . Consequently,  $\log_p(|fb^m|) = \frac{h}{t} + \frac{mu}{w}$  and  $\log_p(|gb^n|) = \frac{k}{t} + \frac{nu}{w}$  and therefore, due to the equality  $|fb^m|_p = |gb^n|_p$ , we have  $(h - k)w = ut(n - m)$ . But since  $w > t$ , it is prime with  $ut$ , hence it must divide  $n - m$ , which is impossible because  $\max(m, n) < w$ , except if  $m = n$ .

**Proof of Theorem 2:** Consider first a strictly decreasing sequence  $(\epsilon_n)_{n \in \mathbb{N}}$  such that  $\lim_{n \rightarrow +\infty} \epsilon_n = 0$  and  $\lim_{n \rightarrow +\infty} \epsilon_n \log_p(n) = +\infty$ .

We can always divide any polynomial  $P \in \mathbb{Q}_p[x]$  by some  $\lambda \in \mathbb{Q}_p$  such that  $|\lambda|_p = \|P\|$  and hence we go back to the hypothesis  $\|P\| = 1$ . So, if we can find some  $b \in \mathbb{C}_p$  and, for every  $\epsilon > 0$ , a constant  $C(\epsilon) > 0$  and show that for every  $P \in \mathbb{Q}[X]$  such that  $\|P\| = 1$ , we have  $-\log_p(|P(b)|_p) \leq C(\epsilon)(\deg(P))^{1+\epsilon}$ , Theorem 2 will be proven.

By induction we can define a strictly increasing sequence  $(r_n)_{n \in \mathbb{N}}$  of  $\mathbb{Q}$  and a sequence  $(a_n)_{n \in \mathbb{N}}$  of  $\mathbb{C}_p$  with  $r_n = \frac{u_n}{v_n}$ , irreducible and  $(v_n)_{n \in \mathbb{N}}$  a strictly increasing sequence of prime numbers and a sequence  $(l_n)_{n \in \mathbb{N}^*}$  satisfying further the following properties:

- i)  $\lim_{n \rightarrow +\infty} r_n = +\infty$ ,
- ii) for every  $n \in \mathbb{N}$ ,  $n^{\epsilon_n} < r_n < (n+1)^{\epsilon_n}$ ,
- iii)  $v_n > \prod_{j=1}^{n-1} v_j$
- iv)  $(a_n)^{v_n} = p^{u_n}$
- v)  $(a_n)^{l_n} = a_{n-1}$

By construction, the sequence  $(|a_n|_p)_{n \in \mathbb{N}}$  is strictly decreasing and tends to 0 and all terms belong to  $U$ . Set  $b = \sum_{n=1}^{\infty} a_n$ . Now, let us fix  $\epsilon > 0$ . We will show that  $b$  is transcendental over  $\mathbb{Q}_p$  and has a transcendence order  $\leq 1 + \epsilon$ .

Since the sequence  $(\epsilon_n)$  tends to 0, we can find an integer  $t(\epsilon)$  such that  $\epsilon_m < \epsilon \forall m \geq t(\epsilon)$ . Thus, as a first step, let us take  $q \geq t(\epsilon)$  and let us find a constant  $C(\epsilon) > 0$ , not depending on  $b$ , such that for every  $P \in \mathbb{Q}[X]$  satisfying  $\|P\| = 1$  and  $\deg(P) = q$ , we have  $-\log_p(|P(b)|_p) \leq C(\epsilon)q^{1+\epsilon}$ .

For each  $n \in \mathbb{N}^*$ , set  $b_n = \sum_{m=1}^n a_m$ . Since the sequence  $(|a_m|_p)_{m \in \mathbb{N}}$  is strictly decreasing, we have  $|b - b_n|_p = |a_{n+1}|_p$  and since  $P$  is obviously 1-Lipschitzian in the disk  $U$ , we have  $|P(b) - P(b_n)|_p \leq |a_{n+1}|_p$  hence

$$(1) \log_p(|P(b) - P(b_n)|_p) \leq \log_p(|a_{n+1}|_p) = -r_{n+1}.$$

Now, since the sequence  $\epsilon_n \log_p(n)$  tends to  $+\infty$ , we can choose  $n(q)$  such that  $(n(q) + 1)^{\epsilon_{n(q)+1}} > (q + 1)^{1+\epsilon}$ . Then by (1) we have

$$(2) \log_p(|P(b) - P(b_{n(q)})|_p) < \log_p(|a_{n(q)+1}|_p) = -(r_{n(q)+1}) < -(n(q)+1)^{\epsilon_{n(q)+1}} < -(q+1)^{1+\epsilon}.$$

We will show the following inequality (3)

$$(3) -\log_p(|P(b_{n(q)})|_p) \leq (q+1)^{1+\epsilon}.$$

Thus, suppose (3) is wrong. Set  $h_q = \sum_{m=q}^{n(q)} a_m$ . Then  $b_{n(q)} = b_{q-1} + h_q$ . Now, developping  $P$  at the point  $b_{q-1}$ , we have

$$(4) \log_p(|P(b_{n(q)})|_p) = \log_p \left( \left| \sum_{m=0}^q \frac{P^{(m)}(b_{q-1})}{m!} (h_q)^m \right|_p \right) < -(q+1)^{1+\epsilon}$$

Consider now the sum  $\sum_{m=0}^q \frac{P^{(m)}(b_{q-1})}{m!} (h_q)^m$ . Since the sequence  $|a_m|_p$  is strictly decreasing, we have  $|h_q|_p = |a_q|_p$ , hence  $\log_p(|h_q|_p) = -r_q$ . On the other hand, due to the hypothesis  $r_q = \frac{u_q}{v_q}$ , it appears that  $v_q$  is a prime integer, prime to  $u_q$

and bigger than  $q$  and than  $\prod_{j=1}^{q-1} v_j$ . And thanks to the hypothesis v), each  $P(b_m)$

is a polynomial in  $a_{q-1}$ . Consequently, we can apply Lemma C and we can see that for each  $m = 0, \dots, q-1$ , all the  $\left| \frac{P^{(m)}(b_{q-1})}{m!} (h_q)^m \right|_p$  are pairwise distinct. Consequently we have

$$(5) \left| \sum_{m=0}^q \frac{P^{(m)}(b_{q-1})}{m!} (h_q)^m \right|_p = \max_{1 \leq m \leq q} \left| \frac{P^{(m)}(b_{q-1})}{m!} (h_q)^m \right|_p.$$

Next, since  $-\log_p(|h_q|_p) = r_q < (q+1)^\epsilon$ , for each integer  $m = 1, \dots, q$ , we have  $\log_p(|(h_q)^m|_p) = -mr_q > -m(q+1)^\epsilon \geq -q(q+1)^\epsilon$ , hence

$$(6) \log_p(|(h_q)^m|_p) \geq -q(q+1)^\varepsilon > -(q+1)^{1+\varepsilon} \quad \forall m \leq q.$$

Consequently, by (4), (5) and (6), the polynomial  $Q(X) = \sum_{m=0}^q \frac{P^{(m)}(b_{q-1})}{m!} (X)^m$

has all coefficients in  $d(0, 1^-)$  and hence we have  $\|Q\| < 1$ . But since  $|b_{q-1}|_p < 1$ , by Lemma A, we have  $\|P\| = \|Q\| < 1$ , a contradiction to the hypothesis  $\|P\| = 1$ . Therefore, Relation (3) is proven for every polynomial  $P \in \mathbb{Q}_p[X]$  of degree  $q \geq t(\varepsilon)$ , such that  $\|P\| = 1$ . Consequently, by (3) we obviously have a constant  $C > 0$ , not depending on  $b$ , such that  $-\log_p(|P(b)|_p) \leq C(\deg(P))^{1+\varepsilon}$  for every  $P \in \mathbb{Q}_p[X]$  such that  $\deg(P) \geq t(\varepsilon)$  and  $\|P\| = 1$ .

Particularly  $b$  is transcendental over  $\mathbb{Q}_p$  because if it were algebraic, the degrees of polynomials  $P \in \mathbb{Q}_p[X]$  such that  $P(b) = 0$  wouldn't be bounded. Finally, by Lemma B there exists a constant  $m > 0$  such that  $|Q(b)|_p \geq m$  for every polynomial  $Q \in \mathbb{Q}_p[X]$  of degree  $q \leq t(\varepsilon)$  and  $\|Q\| = 1$ . Therefore  $b$  is clearly of order  $\leq 1 + \varepsilon$ .

**Lemma D:** *Let  $F$  be an extension of  $\mathbb{Q}_p$  of transcendence degree 1 containing  $\Omega_p$  and provided with an absolute value  $|\cdot|_F$  that extends the absolute value of  $\mathbb{Q}_p$ . Let  $\mathcal{H}$  be the residue class field of  $F$ . If  $\mathcal{H}$  is transcendental over  $\mathbb{F}_p$ , then any  $t \in F$  transcendental over  $\mathbb{Q}_p$  has transcendence order 1 over  $\mathbb{Q}_p$ .*

**Proof:** Suppose  $\mathcal{H}$  is transcendental over  $\mathbb{F}_p$  and let  $t \in F$  be transcendental over  $\mathbb{Q}_p$ . Since  $F$  has a transcendence degree 1 over  $\mathbb{Q}_p$ ,  $F$  is an algebraic extension of  $\mathbb{Q}_p(t)$ . Let us show that  $t$  has transcendence order 1.

Suppose first that  $|t|_F \leq 1$ . Then, by a classical result (Corollary 6.8 in [3]) the field  $\mathcal{H}$  is algebraic over the residue class field  $\mathcal{Q}_t$  of the field  $\mathbb{Q}_p(t)$  with the absolute value of  $F$ , since  $F$  is algebraic over  $\mathbb{Q}_p(t)$ . Now by hypothesis  $\mathcal{H}$  is transcendental over  $\mathbb{F}_p$  hence  $\mathcal{Q}_t$  must also be transcendental over  $\mathbb{F}_p$ . Suppose now that  $t$  belongs to the closure of  $\Omega_p$  in  $F$  with respect to the topology of  $F$ . The natural isomorphism from  $\mathbb{Q}_p(t)$  into  $\Omega_p$  makes an isomorphism from  $\mathcal{Q}_t$  into the residue class field of  $\Omega_p$ . But the residue class field of  $\Omega_p$  is algebraic over  $\mathbb{F}_p$ , hence so is  $\mathcal{Q}_t$ , a contradiction. So, we have proven that  $t$  does not belong to the closure of  $\Omega_p$  in  $F$ . Therefore, there exists a disk  $d(t, r)$  in  $F$  (with  $0 < r \leq 1$ ) such that  $d(t, r) \cap \Omega_p = \emptyset$ .

Let  $Q(X) \in \mathbb{Q}_p[X]$  be of degree  $n$ , such that  $\|Q\| = 1$  and consider its factorization in  $\Omega_p[X]$ :  $\prod_{k=1}^n (a_k X - b_k)$  with  $a_k, b_k \in \Omega_p$  and  $\|a_k X - b_k\| = 1 \quad \forall k = 1, \dots, n$ .

Let  $\Xi$  be the mapping defined on  $F$  as  $\Xi(z) = \log_p(|z|_F)$ . Since  $F \supset \Omega_p$ , we know that the absolute value  $|\cdot|_F$  of  $F$  induces  $|\cdot|_p$  on  $\Omega_p$  because the extension of the absolute value  $|\cdot|_p$  of  $\mathbb{Q}_p$  to  $\Omega_p$  is unique.

If  $|a_k|_p = 1$ , then  $|a_k t - b_k|_F = |t - \frac{b_k}{a_k}|_F \geq r$  and if  $|a_k|_p < 1$ , then  $|a_k t - b_k|_F = |b_k|_p = 1 \geq r$ . Thus, in all cases, we have  $|Q(t)|_F \geq r^{\deg(Q)}$  hence  $\Xi(Q(t)) \geq (\log_p r) \deg(Q)$ . Consequently, we have proven that  $t$  has transcendence order  $\leq 1$  whenever  $|t|_F \leq 1$ .

Suppose now that  $|t|_F > 1$ . We can set  $Q(t) = t^n G(\frac{1}{t})$  with  $G \in \mathbb{Q}_p[X]$  and  $\deg(G) \leq \deg(Q)$ . Since  $|t|_F > 1$ , we have  $\Xi(G(\frac{1}{t})) \geq (\log_p r) \deg(G) \geq (\log_p r) \deg(Q)$  and therefore  $\Xi(Q(t)) \geq (\log_p r) \deg(Q) + \deg(Q) \log_p(|t|_F)$ , which finishes proving that  $t$  has transcendence order  $\leq 1$ .

**Proof of Theorem 3:** Let  $E = \mathbb{Q}_p(x, y)$  and let  $m = [E : \mathbb{Q}_p(x)]$ . Let  $y_1 = y$  and  $y_2, \dots, y_m$  be the conjugates of  $y$  in  $\mathbb{C}_p$ .

We set  $(t - y_1) \cdots (t - y_m) = \sum_{i=0}^m S_i t^{m-i}$  in order to define the  $i$ -th symmetric function  $S_i(y_1, \dots, y_m)$ , with respect to the extension  $E$  over  $\mathbb{Q}_p(x)$ .

We know that the  $S_i$  belong to  $\mathbb{Q}_p(x)$  and hence are of the form  $\frac{f_i(x)}{G(x)}$  with  $f_i(x), G(x) \in \mathbb{Q}_p[x]$  and  $\|G\| = 1$ . Let  $t_i$  be the degree of  $f_i$  and set  $A = \max(t_i \mid 1 \leq i \leq m)$ . Let  $\mathcal{N}$  denote the  $\mathbb{Q}_p(x)$ -algebraic norm defined on  $E$ .

Now, take  $Q(X) \in \mathbb{Q}_p[X]$  with  $\|Q\| = 1$  and  $\deg(Q) = q$ . We can write

$$Q(X) = a \prod_{k=1}^q (X - c_k) \text{ with } a \in \mathbb{Q}_p \text{ and } c_k \in \Omega_p \text{ and } \|X - c_k\| = 1, k = 1, \dots, q.$$

Set  $R(x) = \mathcal{N}((G(x))^q Q(y)) = (G(x))^{mq} \mathcal{N}(Q(y))$ . We have

$$R(x) = (G(x))^{mq} (a)^m \prod_{\substack{1 \leq k \leq q, \\ 1 \leq i \leq m}} (y_i - c_k) \text{ Hence}$$

$$\begin{aligned} R(x) &= (-1)^{mq} (a)^m (G(x))^{mq} \prod_{1 \leq k \leq q} \left( \sum_{i=0}^m (c_k)^{m-i} S_i(y_1, \dots, y_m) \right) \\ &= (-1)^{mq} (a)^m (G(x))^{(m-1)q} \prod_{1 \leq k \leq q} \left( \sum_{i=0}^m (c_k)^{m-i} f_i(x) \right). \end{aligned}$$

We now set  $J = (m-1) \deg(G) + A$ . Then  $J \geq 1$  and we have



(1)  $\deg(R) \leq J \deg(Q)$ .

Consider the absolute value  $\phi$  on  $\mathbb{Q}_p[x]$  induced by the Gauss norm  $\| \cdot \|$  defined on the algebra of polynomials  $\mathbb{Q}_p[X]$ . Then,  $\phi$  admits an extension  $\widehat{\phi}$  to an algebraic closure of  $\mathbb{Q}_p(x)$ . Particularly,  $\widehat{\phi}$  is defined on  $E$  and obviously satisfies  $\widehat{\phi}(x) = \phi(x) = 1$  since  $\phi$  is induced by the Gauss norm on  $\mathbb{Q}_p[x]$ . Let  $\mathcal{E}$  be the residue class field of  $E$  equipped with  $\widehat{\phi}$  and let  $\bar{x}$  be the residue class of  $x$  in  $\mathcal{E}$ . By construction, the image of  $\mathbb{Z}_p[x]$  in  $\mathcal{E}$  is  $\mathbb{F}_p[\bar{x}]$  and  $\bar{x}$  is obviously transcendental over  $\mathbb{F}_p$ . Let  $\Xi$  be the mapping defined on  $E$  as  $\Xi(z) = \log_p(\widehat{\phi}(z))$ . Then, by definition, for all  $F(x) \in \mathbb{Q}_p[x]$ , we have  $\Xi(F(x)) = \log_p(\|F\|)$ .

On the other hand, by hypothesis,  $y$  is transcendental over  $\mathbb{Q}_p$  and so is  $y_j$ ,  $j = 2, \dots, m$ . We can apply Lemma D because the Gauss norm defined in  $\mathbb{Q}_p(t)$  induces the  $p$ -adic value on  $\mathbb{Q}_p$ . Therefore, by Lemma D there exists a constant  $B > 0$  such that  $\Xi(P(y_j)) \geq -B \deg(P)$  for every polynomial  $P(X) \in \mathbb{Q}_p[X]$  such that  $\|P\| \geq 1$  ( $1 \leq j \leq m$ ).

Particularly, here, we have  $\Xi(Q(y_j)) \geq -B \deg(Q)$  and therefore  $\Xi(\mathcal{N}(Q(y))) \geq -mB \deg(Q)$ . Consequently, we obtain

$$\log_p(\|R\|) = \Xi((G(x))^{mq} \mathcal{N}(Q(y))) \geq -mb \deg(Q) \log_p(\|(G(x))^{mq}\|).$$

But since  $\|G\| = 1$ , we can derive  $\Xi(R(x)) \geq -mB \deg(Q)$  and hence

(2)  $\log_p(\|R(x)\|) \geq -mB \deg(Q)$ .

Let us consider now the  $p$ -adic absolute value  $|\cdot|_p$  defined on  $\mathbb{C}_p$  together with the function  $\Psi_p$  that is associated. Set  $\ell = \max(0, \Psi_p(y_1), \dots, \Psi_p(y_m))$ . According to the definition of  $R$ , we have

$$\begin{aligned} (3) \quad \Psi_p(Q(y)) &= \Psi_p(R(x)) - m\Psi_p(G(x))(\deg(Q)) - \sum_{j=2}^m \Psi_p(Q(y_j)). \\ &\geq \Psi_p(R(x)) - \deg(Q)(\Psi_p(G(x)) + \ell)(\deg(Q)). \end{aligned}$$

Now, since  $x$  has transcendence order  $\leq \tau$  in  $\mathbb{C}_p$ , with respect to  $|\cdot|_p$ , there exists a constant  $C > 0$  such that  $\Psi_p(P(x)) \geq \log_p(\|P\|) - C(\deg(P))^\tau$  for every  $P \in \mathbb{Q}_p[x]$ , hence  $\Psi_p(R(x)) \geq \log_p(\|R\|) - C(\deg(R))^\tau$  and hence by (2), we can derive  $\Psi_p(R(x)) \geq -mB \deg(Q) - C(\deg R)^\tau$ . Then by (1), since by Theorem 1,  $\tau \geq 1$ , we obtain  $\Psi_p(R(x)) \geq -(mB + C)J^\tau(\deg(Q))^\tau$ .

Now, let  $D = (B + C)J^\tau + m(\Psi_p(G(x)) + \ell)$ . We can check that  $D$  does not depend on  $Q$ . Then by (3) we have  $\Psi_p(Q(y)) \geq -D(\deg(Q))^\tau$  because  $\tau \geq 1$  again. Since we have taken  $Q$  such that  $\|Q\| = 1$ , that ends the proof.

**Proof of Theorem 4:** For each  $n \in \mathbb{N}^*$ , take a primitive root  $u_n$  of order  $n$  of 1, set  $c_n = 1 + \sum_{k=1}^n p^{(k!)} u_k$ ,  $c = \lim_{n \rightarrow +\infty} c_n$  and let  $P_n$  be the minimal polynomial of  $c_n$  over  $\mathbb{Q}_p$ . Clearly, each  $u_n$  is integral over  $\mathbb{Z}_p$ , hence  $c_n$  belongs to the integral closure of  $\mathbb{Z}_p$ . Consequently, for every  $n \in \mathbb{N}^*$ ,  $P_n$  belongs to  $\mathbb{Z}_p[x]$  and therefore  $\|P_n\| = 1$ .

Now, we have  $P_n(c) = P_n(c_n) + (P_n(c) - P_n(c_n)) = P_n(c) - P_n(c_n)$ . But since  $P_n$  is obviously 1-Lipschitzian in  $U$ , we notice that  $|(c^k) - (c_n)^k|_p \leq |c - c_n|_p \leq p^{-(n+1)!}$  and hence  $|P_n(c) - P_n(c_n)|_p \leq p^{-(n+1)!}$ . Consequently,

$$(1) \Psi_p(P_n(c)) \leq -(n+1)!$$

On the other hand, since  $[\mathbb{Q}_p[u_n] : \mathbb{Q}_p] = n$ , each  $c_n$  is at most of degree  $n!$  over  $\mathbb{Q}_p$ , hence  $\deg(P_n) \leq n!$ . Consequently, by (1) we have

$$\lim_{n \rightarrow +\infty} \left( \frac{-\Psi_p(P_n(c))}{\deg(P_n)} \right) = +\infty,$$

which proves that  $c$  has infinite order.

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## References

- [1] **Amice, Y.** *Les nombres p-adiques*, P.U.F. (1975).
- [2] **Escassut, A.** *Transcendence order over  $\mathbb{Q}_p$  in  $\mathbb{C}_p$* , Journal of Number Theory, Vol 16, N. 3, p. 395-402 (1982).
- [3] **Escassut, A.** *Analytic Elements in p-adic Analysis*. World Scientific Publishing Co. Pte. Ltd. Singapore, (1995).
- [4] **Krasner, M.** *Nombre des extensions d'un degré donné d'un corps p-adique. Les tendances géométriques en algèbre et théorie des nombres*, Clermont-Ferrand, p. 143-169 (1964). Centre National de la Recherche Scientifique (1966), (Colloques internationaux de C.N.R.S. Paris, 143).
- [5] **Waldschmidt, M.** *Nombres transcendants*, Lecture Notes in Mathematics 402, (1974).

Alain ESCASSUT  
Laboratoire de Mathématiques UMR 6620  
Université Blaise Pascal  
Les Cézeaux  
63171 AUBIERE  
FRANCE  
alain.escassut@math.univ-bpclermont.fr