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Survey and additional properties
on the transcendence order over $\mathbb{Q}_p$ in $\mathbb{C}_p$.

by Alain Escassut

Abstract The paper is aimed at recalling the notion of transcendence order over $\mathbb{Q}_p$ and its main properties. Proofs are more detailed than in the paper published in Journal of Number Theory. The main results: the order always is ≥ 1 and we construct a number b that is of order $1+\epsilon$ for every $\epsilon > 0$. If a is of order $\leq t$ and if b is transcendental over $\mathbb{Q}_p$ but algebraic over $\mathbb{Q}_p(a)$, then b is of order $\leq t$ too. Finally, numbers of infinite order are constructed.

Introduction and results

We denote by $\mathbb{C}_p$ the completion of the algebraic closure of the field $\mathbb{Q}_p$ [1], [4]. The transcendence order of a number a in $\mathbb{C}_p$ was introduced in [2]. This notion, which is specific to p-adic numbers, looks a bit like that of transcendence type [5] but it is quite different because this concerns transcendence over $\mathbb{Q}_p$, not over $\mathbb{Q}$. In 1978 the notion of transcendence order was defined and we showed the existence of numbers with transcendence order $\leq 1+\epsilon$. This is a result that we will describe and here with some improvements. The existence of numbers with transcendence order 1 was communicated to me and stated in [2] but a mistake in the proof puts in doubt the result even though it seems likely. The order of transcendence is stable through an algebraic extension. Finally, in Theorem 4 we construct numbers with an infinite order of transcendence.

Definitions and notation: Given a field E, we denote by $E(X)$ the field of rational functions with coefficients in E. We denote by $|\ .\ |_p$ the p-adic absolute value defined on $\mathbb{C}_p$ and by $\log_p$ the real logarithm function of base p. We then define $\Psi_p(x) = \log_p(|x|_p)$. We denote by $\|\ .\ \|$ the Gauss norm defined on $\mathbb{C}_p[X]$ by $\|\sum_{j=0}^{n} a_j X^j\| = \max_{0\leq j\leq n} |a_j|_p$.

Let $\tau \in]0,+\infty[$. Let F be a transcendental extension of $\mathbb{Q}_p$ provided with an absolute value $|\ .\ |$ extending that of $\mathbb{Q}_p$. An element $a \in F$ will be said *to have transcendence order* $\leq \tau$ or *order* $\leq \tau$ in brief, if there exists a constant $C_a \in]0,+\infty[$ such that every polynomial $P \in \mathbb{Q}_p[x]$ satisfies $\log_p(|P(a)|) \geq \log_p(\|P\|) - C_a(\deg(P))^\tau$.

We will denote by $\mathcal{S}(\tau)$ the set of numbers $x \in \mathbb{C}_p$ having transcendence order $\leq \tau$.

We will say that a number $x \in \mathbb{C}_p$ is *of infinite transcendence order* if it does not belong to $\mathcal{S}(\tau)$ for all $\tau \in \mathbb{R}_+^*$

Given $a \in \mathbb{C}_p$ and $r > 0$ we denote by $d(a, r)$ the disk $\{x \in \mathbb{C}_p \mid |x-a| \leq r\}$ and by $d(a, r^-)$ the disk $\{x \in \mathbb{C}_p \mid |x-a| < r\}$. We will denote by U the disk $d(0, 1)$.

Remark: By definition, an element $a \in \mathbb{C}_p$ having transcendence order $\leq \tau$ is transcendental over $\mathbb{Q}_p$.

Theorem 1: *Let $\tau \in]0, +\infty[$. If $\mathcal{S}(\tau) \neq \emptyset$ then $\tau \geq 1$.*

Theorem 2: *There exists $b \in \mathbb{C}_p$, transcendental over $\mathbb{Q}_p$, of order $\leq 1 + \epsilon$ for every $\epsilon > 0$.*

Remark: In [2] the existence of numbers of transcendence order 1 is stated. Unfortunately, the proof is wrong because it is not possible to compose two $\mathbb{Q}_p$-isomorphisms defined in an extension of $\mathbb{C}_p$. Therefore, the question of existence of numbers transcendental over $\mathbb{Q}_p$ of order 1 remains open.

Theorem 3: *Let $x \in \mathbb{C}_p$ belong to $\mathcal{S}(\tau)$ ($\tau \geq 1$) and let $y \in \mathbb{C}_p$ be transcendental over $\mathbb{Q}_p$ but algebraic over $\mathbb{Q}_p(x)$. Then y also belongs to $\mathcal{S}(\tau)$.*

Corollary 3.1: *Let $a \in \mathcal{S}(\tau)$ and let E be an algebraic extension of $\mathbb{Q}_p(a)$ and let $h(X) \in E(X) \setminus E$. Then $h(a)$ belongs to $\mathcal{S}(\tau)$.*

Particularly, we notice that if $a \in \mathcal{S}(\tau)$ then $\dfrac{1}{a}$ and a^n belong to $\mathcal{S}(\tau)$.

Theorem 4: *There exists numbers in $\mathbb{C}_p$ having infinite order.*

The proofs

Notation: We denote by $|\ .\ |_\infty$ the Archimedean absolute value defined on $\mathbb{R}$.

Proof of Theorem 1: Let $a \in \mathbb{C}_p$, $a \neq 0$, be trenscendental over $\mathbb{Q}_p$ and have transcendence order $\leq \tau$. We can find $b \in \Omega_p$, $(b \neq 0)$ such that $|a - b|_p < 1$. Consider the minimal polynomial Q of b over $\mathbb{Q}_p$. Let $b_2, ..., b_q$ be the conjugates of b over $\mathbb{Q}_p$ and set $b_1 = b$. Since $\mathbb{Q}_p$ is complete, we notice that all conjugates b_j of b over $\mathbb{Q}_p$ satisfy $|b_j|_p = |b|_p = |a|_p$.

Suppose first that $|a|_p \leq 1$. Since $|b_j|_p = |b|_p = |a|_p \leq 1$, all coefficients of Q belong to $\mathbb{Z}_p$. Obviously Q is monic, hence $\|Q\| = 1$. By hypothesis, there exists $C_a \in]0, +\infty[$ such that $\Psi_p(P(a)) \geq \log_p(\|P\|) - C_a(\deg(P))^\tau \ \forall P \in \mathbb{Q}_p[x]$. Consequently, $-n\Psi_p(Q(a)) = -\Psi_p((Q(a))^n) \leq C_a(n\deg(Q))^\tau \ \forall n \in \mathbb{N}^*$. Since $Q(b) = 0$ and since, by Q is clearly 1-Lipschitzian in the disk U, we have $-\Psi_p(Q(a)) > 0$ and therefore, if $\tau < 1$, the inequality $-n\Psi_p(Q(a)) \leq C_a(n\deg(Q))^\tau \ \forall n \in \mathbb{N}^*$ is impossible when n tends to $+\infty$.

Suppose now $|a|_p > 1$. Set $Q(X) = \sum_{k=0}^{q} c_k X^k$. Since the b_j satisfy $|b_j|_p = |a|_p$, $(1 \leq j \leq q)$, we have $|c_k|_p \leq (|a|_p)^{q-k}$ and particularly $|c_0|_p = \prod_{j=1}^{q} |b_j|_p = |a|^p$. Consequently, $\|Q\| = (|a|_p)^q$ and therefore, considering the sequence $(Q^n)_{n\in\mathbb{N}}$, for every $n \in \mathbb{N}^*$ we have,

(1) $-n\log_p(|Q(a)|_p) \leq -nq\log_p(|a|_p) + C_a(nq)^\tau$.

On the other hand, $Q(a) = Q(a) - Q(b) = (a-b)\sum_{k=1}^{q} c_k\Big(\sum_{j=0}^{k} a_j b^{k-j-1}\Big)$ and hence $|Q(a)|_p \leq |a-b|_p(|a|_p)^{q-1}$. Consequently, we obtain
$-n\log_p(|a-b|_p) - n(q-1)\log_p(|a|_p) \leq -n\log_p(|Q(a)|_p)$ and hence, by (1):
$-n\log_p(|a-b|_p) - n(q-1)\log_p(|a|_p) \leq -nq\log_p(|a|_p) + C_a(nq)^\tau$. Finally,
$n(\log_p(|a|_p) - \log_p(|a-b|_p)) \leq C_a(nq)^\tau$. Since $|a|_p > 1$ and $|a-b|_p < 1$, this inequality is impossible again when n tends to $+\infty$, which ends the proof.

Lemma A: *Let $P(X) \in \mathbb{Q}[x]$ and let $a \in U$, let $t = X - a$ and let $Q(t) = P(X)$. Then $\|P\| = \|Q\|$.*

Lemma B: *Let $b \in \mathbb{C}_p$ be transcendental over $\mathbb{Q}_p$ and let $q \in \mathbb{N}^*$. There exists a real constant $m > 0$ such that $|Q(b)|_p \geq m$ for every polynomial $Q \in \mathbb{Q}_p[X]$ such that $\|Q\| = 1$ and $\deg(Q) \leq q$.*

Proof: Suppose that Lemma B is wrong and let $(Q_n)_{n\in\mathbb{N}^*}$ be a sequence of $\mathbb{Q}_p[X]$ such that $\|Q_n\| = 1$, $\deg(Q_n) \leq q \ \forall n \in \mathbb{N}^*$ and such that $\lim_{n\to+\infty} Q_n(b) = 0$. We then can extract from the sequence (Q_n) a subsequence converging to a polynomial $Q \in \mathbb{Q}_p[X]$ such that $\|Q\| = 1$ and $Q(b) = 0$, a contradiction since b is transcendental over $\mathbb{Q}_p$.

Lemma C: *Let $a \in \mathbb{C}_p$ be algebraic over $\mathbb{Q}_p$, $\log_p(|a|)$ is of the form $\dfrac{\lambda}{t}$, with*

λ and t in $\mathbb{N}$. Take $m,\ n \in \mathbb{N}$ and $b \in \mathbb{C}_p$ such that $\log_p(|b|)$ is of the form $\frac{u}{w}$ with $u \in \mathbb{N}$ and $w \in \mathbb{N}$ prime, prime with u and such that $w > \max(m, n, t)$.

Let $f,\ g \in \mathbb{Q}_p[a]$ be such that $|fb^m|_p = |gb^n|_p$. Then $m = n$.

Proof: We notice that for every $x \in \mathbb{Q}_p[a]$, $\log_p(|x|)$ is of the form $\frac{\ell}{t}$ with $\ell \in \mathbb{N}$. Consequently, $\log_p(|f|)$ is of the form $\frac{h}{t}$ and $\log_p(|g|)$ is of the form $\frac{k}{t}$ with h and $k \in \mathbb{N}^*$. Consequently, $\log_p(|fb^m|) = \frac{h}{t} + \frac{mu}{w}$ and $\log_p(|gb^n|) = \frac{k}{t} + \frac{nu}{w}$ and therefore, due to the equality $|fb^m|_p = |gb^n|_p$, we have $(h-k)w = ut(n-m)$. But since $w > t$, it is prime with ut, hence it must divide $n - m$, which is impossible because $\max(m, n) < w$, except if $m = n$.

Proof of Theorem 2: Consider first a strictly decreasing sequence $(\epsilon_n)_{n\in}$ such that $\lim_{n\to+\infty} \epsilon_n = 0$ and $\lim_{n\to+\infty} \epsilon_n \log_p(n) = +\infty$.

We can always divide any polynomial $P \in \mathbb{Q}_p[x]$ by some $\lambda \in \mathbb{Q}_p$ such that $|\lambda|_p = \|P\|$ and hence we go back to the hypothesis $\|P\| = 1$. So, if we can find some $b \in \mathbb{C}_p$ and, for every $\varepsilon > 0$, a constant $C(\varepsilon) > 0$ and show that for every $P \in \mathbb{Q}[X]$ such that $\|P\| = 1$, we have $-\log_p(|P(b)|_p) \leq C(\varepsilon)(\deg(P))^{1+\varepsilon}$, Theorem 2 will be proven.

By induction we can define a strictly increasing sequence $(r_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$ and a sequence $(a_n)_{n\in\mathbb{N}}$ of $\mathbb{C}_p$ with $r_n = \frac{u_n}{v_n}$, irreducible and $(v_n)_{n\in\mathbb{N}}$ a strictly increasing sequence of prime numbers and a sequence $(l_n)_{n\in\mathbb{N}^*}$ satisfying further the following properties:

i) $\lim\limits_{n\to+\infty} r_n = +\infty$,

ii) for every $n \in \mathbb{N}$, $n^{\epsilon_n} < r_n < (n+1)^{\epsilon_n}$,

iii) $v_n > \prod_{j=1}^{n-1} v_j$

iv) $(a_n)^{v_n} = p^{u_n}$

v) $(a_n)^{l_n} = a_{n-1}$

By construction, the sequence $(|a_n|_p)_{n\in\mathbb{N}}$ is strictly decreasing and tends to 0 and all terms belong to U. Set $b = \sum_{n=1}^{\infty} a_n$. Now, let us fix $\varepsilon > 0$. We will show that b is transcendental over $\mathbb{Q}_p$ and has a transcendence order $\leq 1 + \varepsilon$.

Since the sequence (ϵ_n) tends to 0, we can find an integer $t(\varepsilon)$ such that $\epsilon_m < \varepsilon\ \forall m \geq t(\varepsilon)$. Thus, as a first step, let us take $q \geq t(\varepsilon)$ and let us find a constant $C(\varepsilon) > 0$, not depending on b, such that for every $P \in \mathbb{Q}[X]$ satisfying $\|P\| = 1$ and $\deg(P) = q$, we have $-\log_p(|P(b)|_p) \leq C(\varepsilon)q^{1+\varepsilon}$.

For each $n \in \mathbb{N}^*$, set $b_n = \sum_{m=1}^{n} a_m$. Since the sequence $(|a_m|_p)_{m\in\mathbb{N}}$ is strictly decreasing, we have $|b - b_n|_p = |a_{n+1}|_p$ and since P is obviously 1-Lipschitzian in the disk U, we have $|P(b) - P(b_n)|_p \leq |a_{n+1}|_p$ hence

(1) $\log_p(|P(b) - P(b_n)|_p) \leq \log_p(|a_{n+1}|_p) = -r_{n+1}$.

Now, since the sequence $\epsilon_n \log_p(n)$ tends to $+\infty$, we can choose $n(q)$ such that $(n(q)+1)^{\epsilon_{n(q)+1}} > (q+1)^{1+\varepsilon}$. Then by (1) we have

(2) $\log_p(|P(b)-P(b_{n(q)})|_p) < \log_p(|a_{n(q)+1}|_p) = -(r_{n(q)+1}) < -(n(q)+1)^{\epsilon_{n(q)+1}} < -(q+1)^{1+\varepsilon}$.

We will show the following inequality (3)

(3) $-\log_p(|P(b_{n(q)})|_p) \leq (q+1)^{1+\varepsilon}$.

Thus, suppose (3) is wrong. Set $h_q = \sum_{m=q}^{n(q)} a_m$. Then $b_{n(q)} = b_{q-1} + h_q$. Now, developping P at the point b_{q-1}, we have

(4) $\log_p(|P(b_{n(q)})|_p) = \log_p\left(\Big|\sum_{m=0}^{q} \frac{P^{(m)}(b_{q-1})}{m!}(h_q)^m\Big|_p\right) < -(q+1)^{1+\varepsilon}$

Consider now the sum $\sum_{m=0}^{q} \frac{P^{(m)}(b_{q-1})}{m!}(h_q)^m$. Since the sequence $|a_m|_p$ is strictly decreasing, we have $|h_q|_p = |a_q|_p$, hence $\log_p(|h_q|_p) = -r_q$. On the other hand, due to the hypothesis $r_q = \frac{u_q}{v_q}$, it appears that v_q is a prime integer, prime to u_q and bigger than q and than $\prod_{j=1}^{q-1} v_j$. And thanks to the hypothesis v), each $P(b_m)$ is a polynomial in a_{q-1}. Consequently, we can apply Lemma C and we can see that for each $m = 0, ..., q-1$, all the $\Big|\frac{P^{(m)}(b_{q-1})}{m!}(h_q)^m\Big|_p$ are pairwise distinct. Consequently we have

(5) $\Big|\sum_{m=0}^{q} \frac{P^{(m)}(b_{q-1})}{m!}(h_q)^m\Big|_p = \max_{1\leq m\leq q}\Big|\frac{P^{(m)}(b_{q-1})}{m!}(h_q)^m\Big|_p$.

Next, since $-\log_p(|h_q|_p) = r_q < (q+1)^{\varepsilon}$, for each integer $m = 1, ..., q$, we have $\log_p(|(h_q)^m|_p) = -mr_q > -m(q+1)^{\varepsilon} \geq -q(q+1)^{\varepsilon}$, hence

(6) $\log_p(|(h_q)^m|_p) \geq -q(q+1)^\varepsilon > -(q+1)^{1+\varepsilon}\ \forall m \leq q$.

Consequentlly, by (4), (5) and (6), the polynomial $Q(X) = \sum_{m=0}^{q} \frac{P^{(m)}(b_{q-1})}{m!}(X)^m$ has all coefficients in $d(0,1^-)$ and hence we have $\|Q\| < 1$. But since $|b_{q-1}|_p < 1$, by Lemma A, we have $\|P\| = \|Q\| < 1$, a contradiction to the hypothesis $\|P\| = 1$. Therefore, Relation (3) is proven for every polynomial $P \in \mathbb{Q}_p[X]$ of degree $q \geq t(\varepsilon)$, such that $\|P\| = 1$. Consequently, by (3) we obviously have a constant $C > 0$, not depending on b, such that $-\log_p(|P(b)|_p) \leq C(\deg(P))^{1+\varepsilon}$ for every $P \in \mathbb{Q}_p[X]$ such that $\deg(P) \geq t(\varepsilon)$ and $\|P\| = 1$.

Particularly b is transcendental over $\mathbb{Q}_p$ because if it were algebraic, the degrees of polynomials $P \in \mathbb{Q}_p[X]$ such that $P(b) = 0$ wouldn't be bounded. Finally, by Lemma B there exists a constant $m > 0$ such that $|Q(b)|_p \geq m$ for every polynomial $Q \in \mathbb{Q}_p[X]$ of degree $q \leq t(\varepsilon)$ and $\|Q\| = 1$. Therefore b is clearly of order $\leq 1 + \varepsilon$.

Lemma D: *Let F be an extension of $\mathbb{Q}_p$ of transcendence degree 1 containing Ω_p and provided with an absolute value $|\ .\ |_F$ that extends the absolute value of $\mathbb{Q}_p$. Let $\mathcal{H}$ be the residue class field of F. If $\mathcal{H}$ is transcendental over $\mathbb{F}_p$, then any $t \in F$ transcendental over $\mathbb{Q}_p$ has transcendence order 1 over $\mathbb{Q}_p$.*

Proof: Suppose $\mathcal{H}$ is transcendental over $\mathbb{F}_p$ and let $t \in F$ be transcendental over $\mathbb{Q}_p$. Since F has a transcendence degree 1 over $\mathbb{Q}_p$, F is an algebraic extension of $\mathbb{Q}_p(t)$. Let us show that t has transcendence order 1.

Suppose first that $|t|_F \leq 1$. Then, by a classical result (Corollary 6.8 in [3]) the field $\mathcal{H}$ is algebraic over the residue class field $\mathcal{Q}_t$ of the field $\mathbb{Q}_p(t)$ with the absolute value of F, since F is algebraic over $\mathbb{Q}_p(t)$. Now by hypothesis $\mathcal{H}$ is transcendental over $\mathbb{F}_p$ hence $\mathcal{Q}_t$ must also be transcendental over $\mathbb{F}_p$. Suppose now that t belongs to the closure of Ω_p in F with respect to the topology of F. The natural isomorphism from $\mathbb{Q}_p(t)$ into Ω_p makes an isomorphism from $\mathcal{Q}_t$ into the residue class field of Ω_p. But the residue class field of Ω_p is algebraic over $\mathbb{F}_p$, hence so is $\mathcal{Q}_t$, a contradiction. So, we have proven that t does not belong to the closure of Ω_p in F. Therefore, there exists a disk $d(t,r)$ in F (with $0 < r \leq 1$) such that $d(t,r) \cap \Omega_p = \emptyset$.

Let $Q(X) \in \mathbb{Q}_p[X]$ be of degree n, such that $\|Q\| = 1$ and consider its factorization in $\Omega_p[X]$: $\prod_{k=1}^{n}(a_kX - b_k)$ with $a_k,\ b_k \in \Omega_p$ and $\|a_kX - b_k\| = 1\ \forall k = 1, ..., n$.

Let Ξ be the mapping defined on F as $\Xi(z) = \log_p(|z|_F)$. Since $F \supset \Omega_p$, we know that the absolute value $|\ .\ |_F$ of F induces $|\ .\ |_p$ on Ω_p because the extension of the absolute value $|\ .\ |_p$ of $\mathbb{Q}_p$ to Ω_p is unique.

If $|a_k|_p = 1$, then $|a_k t - b_k|_F = |t - \frac{b_k}{a_k}|_F \geq r$ and if $|a_k|_p < 1$, then $|a_k t - b_k|_F = |b_k|_p = 1 \geq r$. Thus, in all cases, we have $|Q(t)|_F \geq r^{\deg(Q)}$ hence $\Xi(Q(t)) \geq (\log_p r)\deg(Q)$. Consequently, we have proven that t has transcendence order ≤ 1 whenever $|t|_F \leq 1$.

Suppose now that $|t|_F > 1$. We can set $Q(t) = t^n G(\frac{1}{t})$ with $G \in \mathbb{Q}_p[X]$ and $\deg(G) \leq \deg(Q)$. Since $|t|_F > 1$, we have $\Xi(G(\frac{1}{t})) \geq (\log_p r)\deg(G) \geq (\log_p r)\deg(Q)$ and therefore $\Xi(Q(t)) \geq (\log_p r)\deg(Q) + \deg(Q)\log_p(|t|_F)$, which finishes proving that t has transcendence order ≤ 1.

Proof of Theorem 3: Let $E = \mathbb{Q}_p(x, y)$ and let $m = [E : \mathbb{Q}_p(x)]$. Let $y_1 = y$ and $y_2, ..., y_m$ be the conjugates of y in $\mathbb{C}_p$.

We set $(t - y_1)\cdots(t - y_m) = \sum_{i=0}^{m} S_i t^{m-i}$ in order to define the i-th symmetric function $S_i(y_1, ..., y_m)$, with respect to the extension E over $\mathbb{Q}_p(x)$.

We know that the S_i belong to $\mathbb{Q}_p(x)$ and hence are of the form $\frac{f_i(x)}{G(x)}$ with $f_i(x),\ G(x) \in \mathbb{Q}_p[x]$ and $\|G\| = 1$. Let t_i be the degree of f_i and set $A = \max(t_i \mid 1 \leq i \leq m)$. Let $\mathcal{N}$ denote the $\mathbb{Q}_p(x)$-algebraic norm defined on E.

Now, take $Q(X) \in \mathbb{Q}_p[X]$ with $\|Q\| = 1$ and $\deg(Q) = q$. We can write $Q(X) = a\prod_{k=1}^{q}(X - c_k)$ with $a \in \mathbb{Q}_p$ and $c_k \in \Omega_p$ and $\|X - c_k\| = 1,\ k = 1, ..., q$. Set $R(x) = \mathcal{N}((G(x))^q Q(y)) = (G(x))^{mq}\mathcal{N}(Q(y))$. We have $R(x) = (G(x))^{mq}(a)^m \prod_{\substack{1\leq k\leq q,\\ 1\leq i\leq m}} (y_i - c_k)$ Hence

$$R(x) = (-1)^{mq}(a)^m (G(x))^{mq} \prod_{1\leq k\leq q} \Big(\sum_{i=0}^{m} (c_k)^{m-i} S_i(y_1, ..., y_m)\Big)$$

$$= (-1)^{mq}(a)^m (G(x))^{(m-1)q} \prod_{1\leq k\leq q} \Big(\sum_{i=0}^{m} (c_k)^{m-i} f_i(x)\Big).$$

We now set $J = (m-1)\deg(G) + A$. Then $J \geq 1$ and we have

(1) $\deg(R) \leq J \deg(Q)$.

Consider the absolute value ϕ on $\mathbb{Q}_p[x]$ induced by the Gauss norm $\| \ . \ \|$ defined on the algebra of polynomials $\mathbb{Q}_p[X]$. Then, ϕ admits an extension $\widehat{\phi}$ to an algebraic closure of $\mathbb{Q}_p(x)$. Particularly, $\widehat{\phi}$ is defined on E and obviously satisfies $\widehat{\phi}(x) = \phi(x) = 1$ since ϕ is induced by the Gauss norm on $\mathbb{Q}_p[x]$. Let $\mathcal{E}$ be the residue class field of E equiped with $\widehat{\phi}$ and let $\overline{x}$ be the residue class of x in $\mathcal{E}$. By construction, the image of $\mathbb{Z}_p[x]$ in $\mathcal{E}$ is $\mathbb{F}_{\mathrm{p}}[\overline{x}]$ and $\overline{x}$ is obviously transcendental over $\mathbb{F}_{\mathrm{p}}$. Let Ξ be the mapping defined on E as $\Xi(z) = \log_p(\widehat{\phi}(z))$. Then, by definition, for all $F(x) \in \mathbb{Q}_p[x]$, we have $\Xi(F(x)) = \log_p(\|F\|)$.

On the other hand, by hypothesis, y is transcendental over $\mathbb{Q}_p$ and so is $y_j,\ j = 2, ..., m$. We can apply Lemma D because the Gauss norm defined in $\mathbb{Q}_p(t)$ induces the p-adic value on $\mathbb{Q}_p$. Therefore, by Lemma D there exists a constant $B > 0$ such that $\Xi(P(y_j))) \geq -B \deg(P)$ for every polynomial $P(X) \in \mathbb{Q}_p[X]$ such that $\|P\| \geq 1$ $(1 \leq j \leq m)$.

Particularly, here, we have $\Xi(Q(y_j)) \geq -B \deg(Q)$ and therefore $\Xi(\mathcal{N}(Q(y))) \geq -mB \deg(Q)$. Consequently, we obtain

$$\log_p(\|R\|) = \Xi((G(x))^{mq}\mathcal{N}(Q(y))) \geq -mb \deg(Q) \log_p(\|(G(x))^{mq}\|).$$

But since $\|G\| = 1$, we can derive $\Xi(R(x)) \geq -mB \deg(Q)$ and hence

(2) $\log_p(\|R(x)\|) \geq -mB \deg(Q)$.

Let us consider now the p-adic absolute value $| \ . \ |_p$ defined on $\mathbb{C}_p$ together with the function Ψ_p that is associated. Set $\ell = \max(0, \Psi_p(y_1), ..., \Psi_p(y_m))$. According to the definition of R, we have

(3) $\Psi_p(Q(y)) = \Psi_p(R(x)) - m\Psi_p(G(x))(\deg(Q)) - \sum_{j=2}^{m} \Psi_p(Q(y_j))$.

$$\geq \Psi_p(R(x)) - \deg(Q)(\Psi_p(G(x)) + \ell)(\deg(Q)).$$

Now, since x has transcendence order $\leq \tau$ in $\mathbb{C}_p$, with respect to $| \ . \ |_p$, there exists a constant $C > 0$ such that $\Psi_p(P(x)) \geq \log_p(\|P\|) - C(\deg(P))^\tau$ for every $P \in \mathbb{Q}_p[x]$, hence $\Psi_p(R(x)) \geq \log_p(\|R\|) - C(\deg(R))^\tau$ and hence by (2), we can derive $\Psi_p(R(x)) \geq -mB \deg(Q) - C(\deg R)^\tau$. Then by (1), since by Theorem 1, $\tau \geq 1$, we obtain $\Psi_p(R(x)) \geq -(mB + C)J^\tau(\deg(Q))^\tau$.

Now, let $D = (B + C)J^\tau + m(\Psi_p(G(x)) + \ell)$. We can check that D does not depend on Q. Then by (3) we have $\Psi_p(Q(y)) \geq -D(\deg(Q))^\tau$ because $\tau \geq 1$ again. Since we have taken Q such that $\|Q\| = 1$, that ends the proof.

Proof of Theorem 4: For each $n \in \mathbb{N}^*$, take a primitive root u_n of order n of 1, set $c_n = 1 + \sum_{k=1}^{n} p^{(k!)} u_k$, $c = \lim_{n \to +\infty} c_n$ and let P_n be the minimal polinomial of c_n over $\mathbb{Q}_p$. Clearly, each u_n is integral over $\mathbb{Z}_p$, hence c_n belongs to the integral closure of $\mathbb{Z}_p$. Consequently, for every $n \in \mathbb{N}^*$, P_n belongs to $\mathbb{Z}_p[x]$ and therefore $\|P_n\| = 1$.

Now, we have $P_n(c) = P_n(c_n) + (P_n(c) - P_n(c_n)) = P_n(c) - P_n(c_n)$. But since P_n is obviously 1-Lipschitzian in U, we notice that $|(c^k) - (c_n)^k|_p \leq |c - c_n|_p \leq p^{-(n+1)!}$ and hence $|P_n(c) - P_n(c_n)|_p \leq p^{-(n+1)!}$. Consequently,

(1) $\Psi_p(P_n(c)) \leq -(n+1)!$.

On the other hand, since $[\mathbb{Q}_p[u_n] : \mathbb{Q}_p] = n$, each c_n is at most of degree $n!$ over $\mathbb{Q}_p$, hence $\deg(P_n) \leq n!$. Consequently, by (1) we have

$$\lim_{n \to +\infty} \Big(\frac{-\Psi_p(P_n(c))}{\deg(P_n)} \Big) = +\infty,$$

which proves that c has infinite order.

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