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p -adic meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function

Kamal Boussaf, Alain Escassut and Jacqueline Ojeda¹

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Abstract

Let \mathbb{K} be a complete algebraically closed p -adic field of characteristic zero. Let f, g be two transcendental meromorphic functions in the whole field \mathbb{K} or meromorphic functions in an open disk that are not quotients of bounded analytic functions. Let P be a polynomial of uniqueness for meromorphic functions in \mathbb{K} or in an open disk and let α be a small meromorphic function with regards to f and g . If $f'P'(f)$ and $g'P'(g)$ share α counting multiplicity, then we show that $f = g$ provided that the multiplicity order of zeroes of P' satisfy certain inequalities. If α is a Moebius function or a non-zero constant, we can obtain more general results on P .

1 Introduction and Main Results

Let f, g be two meromorphic functions in a p -adic field. Here we study polynomials P such that, when $f'P'(f)$ and $g'P'(g)$ share a small function α , then $f = g$. Problems of uniqueness on meromorphic functions were examined first in \mathbb{C} [7], [8], [10], [13], [14], [15], [16], [17], [22], [23] and next in a p -adic field [1], [3], [4], [5], [11], [12], [18], [20], [21]. After examining problems of the form $P(f) = P(g)$, several studies were made on the equality $f'P'(f) = g'P'(g)$, or value sharing questions: if $f'P'(f)$ and $g'P'(g)$ share a value, or a small function, do we have $f = g$? Here we will try to generalize results previously obtained no matter what the number of zeroes of P' . Moreover results also apply to meromorphic functions inside an open disk.

Let \mathbb{K} be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by $|\cdot|$. We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} , by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} , i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions.

Let $a \in \mathbb{K}$ and $R \in]0, +\infty[$. We denote by $d(a, R)$ the closed disk $\{x \in \mathbb{K} : |x - a| \leq R\}$ and by $d(a, R^-)$ the “open” disk $\{x \in \mathbb{K} : |x - a| < R\}$. We denote by $\mathcal{A}(d(a, R^-))$ the set of analytic functions in $d(a, R^-)$, i.e. the \mathbb{K} -algebra of power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converging in $d(a, R^-)$ and by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of

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$\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the \mathbb{K} -subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n \in \mathbb{N}} |a_n| R^n < +\infty$. And we denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

The problem of value sharing a small function by functions of the form $f'P'(f)$ was examined first when P was just of the form x^n [7], [18], [24]. More recently it was examined when P was a polynomial such that P' had exactly two distinct zeroes [15], [17], [20], both in complex analysis and in *p*-adic analysis. In [15], [17] the functions were meromorphic on \mathbb{C} , with a small function that was a constant or the identity. In [20], the problem was considered for analytic functions in the field \mathbb{K} : on one hand for entire functions and on the other hand for unbounded analytic functions in an open disk.

Actually solving a value sharing problem involving $f'P'(f)$, $g'P'(g)$ requires to know polynomials of uniqueness P for meromorphic functions.

In [20] the third author studied several problems of uniqueness and particularly the following:

Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{A}_u(d(0, R^-))$) and $\alpha \in \mathcal{A}(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_u(d(0, R^-))$) be a small function, such that $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share α , counting multiplicity, with $n, k \in \mathbb{N}$ and $a \in \mathbb{K} \setminus \{0\}$ (see Theorems D and E below).

Here we consider functions $f, g \in \mathcal{M}(\mathbb{K})$ or $f, g \in \mathcal{M}(d(a, R^-))$ and ordinary polynomials P : we must only assume certain hypotheses on the multiplicity order of the zeroes of P' . The method for the various theorems we will show is the following: assuming that $f'P'(f)$ and $g'P'(g)$ share a small function, we first prove that $f'P'(f) = g'P'(g)$. Next, we derive $P(f) = P(g)$. And then, when P is a polynomial of uniqueness for the functions we consider, we can conclude $f = g$.

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field \mathbb{K} and a few specific properties of ultrametric analytic or meromorphic functions.

Let \log be a real logarithm function of base $b > 1$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0. Let $r \in]0, +\infty[$ (resp. $r \in]0, R[$) and let $\gamma \in d(0, r)$. If f has a zero of order n at γ , we put $\omega_\gamma(f) = n$. If f has a pole of order n at γ , we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$.

We denote by $Z(r, f)$ the *counting function of zeroes of f in $d(0, r)$* , counting multiplicity, i.e. we set

$$Z(r, f) = \sum_{\omega_\gamma(f) > 0, |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by $\bar{Z}(r, f)$ the *counting function of zeroes of f in $d(0, r)$* , ignoring multiplicity, and set

$$\bar{Z}(r, f) = \sum_{\omega_\gamma(f) > 0, |\gamma| \leq r} (\log r - \log |\gamma|).$$

In the same way, we set $N(r, f) = Z(r, \frac{1}{f})$ (resp. $\bar{N}(r, f) = \bar{Z}(r, \frac{1}{f})$) to denote the *counting function of poles of f in $d(0, r)$* , counting multiplicity (resp. ignoring multiplicity).

For $f \in \mathcal{M}(d(0, R^-))$ having no zero and no pole at 0, the *Nevanlinna function* is defined by $T(r, f) = \max \{Z(r, f) + \log |f(0)|, N(r, f)\}$.

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$) such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0 is called a *small function with respect to f* , if it satisfies $\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R^-} \frac{T(r, \alpha)}{T(r, f)} = 0$).

If 0 is a zero or a pole of f or α , we can make a change of variable such that the new origin is not a zero or a pole for both f and α . Thus it is easily seen that the last relation does not really depend on the origin.

We denote by $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) the set of small meromorphic functions with respect to f in \mathbb{K} (resp. in $d(0, R^-)$).

Remark 1. Thanks to classical properties of the Nevanlinna function $T(r, f)$ with respect to the operations in a field of meromorphic functions, such as $T(r, f + g) \leq T(r, f) + T(r, g)$ and $T(r, fg) \leq T(r, f) + T(r, g)$, for $f, g \in \mathcal{M}(\mathbb{K})$ and $r > 0$, it is easily proved that $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) is a subfield of $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0, R^-))$) and that $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0, R^-))$) is a transcendental extension of $\mathcal{M}_f(\mathbb{K})$ (resp. of $\mathcal{M}_f(d(0, R^-))$) [6].

Let us remember the following definition.

Definition. Let $f, g, \alpha \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g, \alpha \in \mathcal{M}(d(0, R^-))$). We say that f and g share the function α C.M., if $f - \alpha$ and $g - \alpha$ have the same zeroes with the same multiplicity in \mathbb{K} (resp. in $d(0, R^-)$).

Recall that a polynomial $P \in \mathbb{K}[x]$ is called a *polynomial of uniqueness* for a class of functions \mathcal{F} if for any two functions $f, g \in \mathcal{F}$ the property $P(f) = P(g)$ implies $f = g$.

The definition of polynomials of uniqueness was introduced in [16] by P. Li and C. C. Yang and was studied in many papers [9], [10] for complex functions and [1], [4], [5], [11], [12], [14], [21], for *p*-adic functions.

Actually, in a *p*-adic field, we can obtain various results, not only for functions defined in the whole field \mathbb{K} but also for functions defined inside an open disk because the *p*-adic Nevanlinna Theory works inside a disk, for functions of $\mathcal{M}_u(d(0, R^-))$.

Let us recall Theorem A [5], [21]:

Theorem A. Let $P \in \mathbb{K}[x]$ be such that P' has exactly two distinct zeroes γ_1 of order c_1 and γ_2 of order c_2 . Then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. Moreover, if $\min\{c_1, c_2\} \geq 2$, then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.

Theorem A was first proved in [21] with the additional hypothesis $P(c_1) \neq P(c_2)$. Actually this hypothesis is useless because, as showed in Lemma 10 [5], the equality $P(c_1) = P(c_2)$ is impossible since P' only has two distinct zeroes.

Notation: Let $P \in \mathbb{K}[x] \setminus \mathbb{K}$ and let $\Xi(P)$ be the set of zeroes c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c . We denote by $\Phi(P)$ the cardinal of $\Xi(P)$.

Remark 2. If $\deg(P) = q$ then $\Phi(P) \leq q - 1$.

From [5] we have the following results:

Theorem B. Let $d(a, R^-)$ be an open disk in \mathbb{K} and $P \in \mathbb{K}[x]$. If $\Phi(P) \geq 2$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for both

$\mathcal{A}_u(d(a, R^-))$ and $\mathcal{M}(\mathbb{K})$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$.

And from [20] we have:

Theorem C. Let $P \in \mathbb{K}[x]$ be of degree $n \geq 6$ be such that P' only has two distinct zeroes, one of them being of order 2. Then P is a polynomial of uniqueness for $\mathcal{M}_u(d(0, R^-))$.

In [19], the third author proved the following theorems concerning entire functions and analytic functions in a disk:

Theorem D. Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share the function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ C.M. with $n, k \in \mathbb{N}$ and $a \in \mathbb{K} \setminus \{0\}$. If $n \geq \max\{6-k, k+1\}$, then $f = g$. Moreover, if $\alpha \in \mathbb{K} \setminus \{0\}$ and $n \geq \max\{5-k, k+1\}$, then $f = g$.

Theorem E. Let $f, g \in \mathcal{A}_u(d(0, R^-))$, let $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$ and let $a \in \mathbb{K} \setminus \{0\}$. If $f^n(f-a)^2 f'$ and $g^n(g-a)^2 g'$ share the function α C.M. and $n \geq 4$, then $f = g$. Moreover, if $f^n(f-a)f'$ and $g^n(g-a)g'$ share the function α C.M. and $n \geq 5$, then again $f = g$.

We can now state our main theorems.

Theorem 1. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x-a_1)^n \prod_{i=2}^l (x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4-k_i) + \max(0, 5-k_2), \\ n &\geq k+2, \\ \text{if } l=2, \text{ then } n &\neq 2k, 2k+1, 3k+1, \\ \text{if } l=3, \text{ then } n &\neq 2k+1, 3k_i-k \ \forall i=2,3. \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary 1.1:

Corollary 1.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b(x-a_1)^n \prod_{i=2}^l (x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4-k_i) + \max(0, 5-k_2), \\ n &\geq k+2, \\ \text{if } l=3, \text{ then } n &\neq 2k+1, 3k_i-k \ \forall i=2,3. \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem A we also have Corollary 1.2.

Corollary 1.2 Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x-a_1)^n(x-a_2)^k$ with $\min(k, n) \geq 2$.

Suppose P satisfies the following conditions:

$$n \geq 10 + \max(0, 5 - k),$$

$$n \geq k + 2,$$

$$n \neq 2k, 2k + 1, 3k + 1.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 2. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$n \geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

$$\text{if } l = 2, \text{ then } n \neq 2k, 2k + 1, 3k + 1,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary 2.1.

Corollary 2.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$n \geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem A, we have Corollary 2.2.

Corollary 2.2 Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $\min(k, n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k),$$

$$n \geq k + 2,$$

$$n \neq 2k, 2k + 1, 3k + 1,$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 3. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2, \\ n &\geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary 3.1.

Corollary 3.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2, \\ n &\geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem A, we have Corollary 3.2

Corollary 3.2 Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $k \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq 9 + \max(0, 5 - k), \\ n &\geq k + 2, \\ n &\neq 2k, 2k + 1, 3k + 1. \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 4. Let $a \in K$ and $R > 0$. Let P be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$ and let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ n &\geq k + 3, \\ \text{if } l = 2, \text{ then } n &\neq 2k, 2k + 1, 3k + 1, \\ \text{if } l = 3, \text{ then } n &\neq 2k + 1, 3k_i - k \ \forall i = 2, 3. \end{aligned}$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B we can state Corollary 4.1.

Corollary 4.1 Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 4$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 4$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 3,$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem C we have Corollary 4.2:

Corollary 4.2 Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^2$ with $b \in \mathbb{K}^*$. Suppose P satisfies

$$n \geq 10 + \max(0, 5 - k).$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 5. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form

$$b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^*, \text{ satisfying:}$$

$$n \geq l + 10.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary 5.1:

Corollary 5.1 Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$ and be such that P' is of the form

$$b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying:}$$

$$n \geq l + 10.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem 6. Let $a \in K$ and $R > 0$. Let P be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$ such

$$\text{that } P' \text{ is of the form } P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying:}$$

$$n \geq l + 10.$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary 6.1:

Corollary 6.1 Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 4$ and be such that P' is of

$$\text{the form } P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 4, b \in \mathbb{K}^* \text{ and } n \geq l + 10.$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let $P(x) = \frac{x^{18}}{18} - \frac{2x^{17}}{17} - \frac{x^{16}}{16} + \frac{2x^{15}}{15}$. Then $P'(x) = x^{17} - 2x^{16} - x^{15} + 2x^{14} = x^{14}(x-1)(x+1)(x-2)$. We check that:

$$P(0) = 0,$$

$$P(1) = \frac{1}{18} - \frac{2}{17} - \frac{1}{16} + \frac{2}{15},$$

$$P(-1) = \frac{1}{18} + \frac{2}{17} - \frac{1}{16} - \frac{2}{15} \neq 0, P(1), \text{ and } P(2) = \frac{2^{18}}{18} - \frac{2^{18}}{17} - \frac{2^{16}}{16} + \frac{2^{16}}{15} \neq 0, P(1), P(-1).$$

Then $\Phi(P) = 4$. So, P is a polynomial of uniqueness for both $\mathcal{M}(\mathbb{K})$ and $\mathcal{M}(d(0, R^-))$. Moreover, we have $n = 14, l = 4$, hence we can apply Corollaries 5.1 and 6.1.

Given $f, g \in \mathcal{M}(\mathbb{K})$ transcendental or $f, g \in \mathcal{M}_u(d(0, R^-))$ such that $f'P(f)$ and $g'P(g)$ share a small function α C.M., we have $f = g$.

Theorem 7. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form

$$P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying}$$

$$n \geq l + 9.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P(f)$ and $g'P(g)$ share α C.M., then $f = g$.

Theorem 8. Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form

$$P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying } n \geq l + 9.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P(f)$ and $g'P(g)$ share α C.M., then $f = g$.

Example: Let $P(x) = x^q - ax^{q-2} + b$ with $a \in K^*, b \in K$, with $q \geq 5$ an odd integer. Then q and $q-2$ are relatively prime and hence by Theorem 3.21 [12] P is a uniqueness polynomial for $\mathcal{M}(\mathbb{K})$ and P' admits 0 as a zero of order $n = q - 3$ and two other zeroes of order 1.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}(\mathbb{K})$ be a small function such that $f'P(f)$, $g'P(g)$ share α C.M.

Suppose first $q \geq 17$. By Theorem 5 we have $f = g$. Now suppose $q \geq 15$ and suppose α is a Moebius function or a non-zero constant. Then by Theorem 7 and 8, we have $f = g$.

Theorem 9. Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function α C.M. and if $n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{K})$ such that $f = \frac{a(n+2)}{n+1} \left(\frac{h^{n+1}-1}{h^{n+2}-1} \right) h$ and $g = \frac{a(n+2)}{n+1} \left(\frac{h^{n+1}-1}{h^{n+2}-1} \right)$. Moreover, if α is a constant or a Moebius function, then the conclusion holds whenever $n \geq 11$.

Inside an open disk, we have a version similar to the general case in the whole field.

Theorem 10. Let $f, g \in \mathcal{M}_u(d(0, R^-))$, and let $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function α C.M. and

$n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(d(0, R^-))$ such that $f = \frac{a(n+2)}{n+1} \left(\frac{h^{n+1} - 1}{h^{n+2} - 1} \right) h$ and $g = \frac{a(n+2)}{n+1} \left(\frac{h^{n+1} - 1}{h^{n+2} - 1} \right)$.

Remark 3. In Theorems 9 and 10, the second conclusion does occur. Indeed, let $h \in \mathcal{M}(\mathbb{K})$ (resp. let $h \in \mathcal{M}_u(d(0, R^-))$). Now, let us precisely define f and g as: $g = \left(\frac{n+2}{n+1} \right) \left(\frac{h^{n+1} - 1}{h^{n+2} - 1} \right)$ and $f = hg$. Then we can see that the polynomial $P(y) = \frac{1}{n+2}y^{n+2} - \frac{1}{n+1}y^{n+1}$ satisfies $P(f) = P(g)$, hence $f'P'(f) = g'P'(g)$, therefore $f'P'(f)$ and $g'P'(g)$ trivially share any function.

2 Basic Results

Let us recall a few classical lemmas [3], [4], [12]:

Lemma 1. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$), let $a \in \mathbb{K}$, let $Q(x) \in \mathbb{K}[x]$ of degree s . Then $T(r, Q(f)) = sT(r, f) + O(1)$ and $T(r, f'Q(f)) \geq sT(r, f) + O(1)$.

Lemma 2. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$). Then $N(r, f') = N(r, f) + \bar{N}(r, f)$, $Z(r, f') \leq Z(r, f) + \bar{N}(r, f) - \log r + O(1)$. Moreover, $T(r, f) - Z(r, f) \leq T(r, f') - Z(r, f') + O(1)$.

Notation: Given two meromorphic functions $f, g \in \mathcal{M}(K)$ (resp. $f, g \in \mathcal{M}(d(0, R^-))$), we will denote by $\Psi_{f,g}$ the function

$$\frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}.$$

Lemma 3. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$) and let $\psi = \frac{f'}{f}$. Then $Z(r, \psi) \leq N(r, \psi) - \log r + O(1)$.

The following Lemma 4 is an immediate consequence of Lemma 3:

Lemma 4. The function $\Psi_{f,g}$ satisfies $Z(r, \Psi_{f,g}) \leq N(r, \Psi_{f,g}) - \log r$.

Lemma 5. Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. Let $f, g \in \mathcal{M}(d(0, R^-))$). If a is a simple zero of $f-1$ and $g-1$, it is a zero of $\Psi_{f,g}$.

In order to state the following lemma, we must recall the definition of quasi-exceptional values [19].

- (i) Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$. (resp. let $f \in \mathcal{M}_u(d(0, R^-))$). Then b will be said to be a *Picard exceptional value* of f (or just an *exceptional value*) if $f(x) \neq b \forall x \in \mathbb{K}$ (resp $f(x) \neq b \forall x \in d(0, R^-)$).

- (ii) Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}(x)$. (resp. let $f \in \mathcal{M}_u(d(0, R^-))$) and let $b \in \mathbb{K}$. Then b will be said to be a *quasi-exceptional value* of f if the function $f - b$ has a finite number of zeroes in \mathbb{K} (resp. in $d(0, R^-)$).

The following results are then immediate [19]:

Lemma 6. *Let $f \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}$ (resp. let $f \in \mathcal{A}_u(d(0, R^-))$). Then f has no exceptional value. If f is transcendental, it has no quasi-exceptional value. Let $f \in \mathcal{M}(\mathbb{K}) \setminus \mathbb{K}$ (resp. let $f \in \mathcal{M}_u(d(0, R^-))$). Then f has at most one exceptional value in \mathbb{K} . Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. let $f \in \mathcal{M}_u(d(0, R^-))$). Then f has at most one quasi-exceptional value in \mathbb{K} .*

We now have to recall the *ultrametric Nevanlinna Second Main Theorem* in a basic form which we will frequently use.

Let $f \in \mathcal{M}\mathbb{K}$ (resp. $f \in \mathcal{M}(d(0, R^-))$) satisfy $f'(0) \neq 0, \infty$. Let S be a finite subset of \mathbb{K} and $r \in]0, +\infty[$ (resp. $r \in]0, R[$). We denote by $Z_0^S(r, f')$ the counting function of zeroes of f' in $d(0, r)$ which are not zeroes of any $f - s$ for $s \in S$. This is, if $(\gamma_n)_{n \in \mathbb{N}}$ is the finite or infinite sequence of zeroes of f' in $d(0, r)$ that are not zeroes of $f - s$ for $s \in S$, with multiplicity order q_n respectively, we set

$$Z_0^S(r, f') = \sum_{|\gamma_n| \leq r} q_n (\log r - \log |\gamma_n|).$$

Theorem N. [2], [3] *Let $a_1, \dots, a_q \in \mathbb{K}$ with $q \geq 2, q \in \mathbb{N}$, and let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$). Let $S = \{a_1, \dots, a_q\}$. Assume that none of f , f' and $f - a_j$ with $1 \leq j \leq q$, equals 0 or ∞ at the origin. Then, for $r > 0$ (resp. for $r \in]0, R[$), we have*

$$(q-1)T(r, f) \leq \sum_{j=1}^q \bar{Z}(r, f - a_j) + \bar{N}(r, f) - Z_0^S(r, f') - \log r + O(1).$$

3 Specific Lemmas

We will need the following Lemma 7:

Lemma 7. *Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(d(0, R^-))$). Let $P(x) = x^{n+1}Q(x)$ be a polynomial such that $n \geq \deg(Q) + 2$ (resp. $n \geq \deg(Q) + 3$). If $P'(f)f' = P'(g)g'$ then $P(f) = P(g)$.*

Proof : Put $k = \deg(Q)$. Since $P'(f)f' = P'(g)g'$ there exists $c \in \mathbb{K}$ such that $P(f) = P(g) + c$. Suppose that $c \neq 0$. Then by Theorem N, we have

$$T(r, P(f)) \leq \bar{Z}(r, P(f)) + \bar{Z}(r, P(f) - c) + \bar{N}(r, P(f)) - \log r + O(1). \quad (1)$$

Obviously we see that $\bar{Z}(r, P(f)) = \bar{Z}(r, f^{n+1}Q(f)) = \bar{Z}(r, fQ(f)) \leq T(r, fQ(f))$. By Lemma 1 we have $T(r, fQ(f)) = (k+1)T(r, f) + O(1)$ and then $\bar{Z}(r, P(f)) \leq (k+1)T(r, f) + O(1)$. We also have $\bar{Z}(r, P(f) - c) = \bar{Z}(r, P(g)) \leq \bar{Z}(r, g) + \bar{Z}(r, Q(g)) \leq T(r, g) + T(r, Q(g))$. Then by Lemma

1, $\bar{Z}(r, P(f) - c) \leq (k+1)T(r, g) + O(1)$. Notice that $\bar{N}(r, P(f)) = \bar{N}(r, f) \leq T(r, f) + O(1)$ then by (1) we obtain

$$T(r, P(f)) \leq (k+2)T(r, f) + (k+1)T(r, g) - \log r + O(1). \quad (2)$$

According to Lemma 1 we have $T(r, P(f)) = (n+k+1)T(r, f) + O(1)$. Then by (2) we have

$$nT(r, f) \leq T(r, f) + (k+1)T(r, g) - \log r + O(1). \quad (3)$$

We similarly we obtain

$$nT(r, g) \leq T(r, g) + (k+1)T(r, f) - \log r + O(1). \quad (4)$$

Hence adding (3) and (4) we have

$$n(T(r, f) + T(r, g)) \leq (k+2)(T(r, f) + T(r, g)) - 2\log r + O(1)$$

and then

$$0 \leq (k+2-n)(T(r, f) + T(r, g)) - 2\log r + O(1).$$

That leads to a contradiction because $n \geq k+2$ (resp. $n \geq k+3$) and $\lim_{r \rightarrow +\infty} (T(r, f) + T(r, g)) = +\infty$ (resp. $\lim_{r \rightarrow R} (T(r, f) + T(r, g)) = +\infty$). Thus $c = 0$ and consequently $P(f) = P(g)$.

Lemma 8. Let $F, G \in \mathcal{M}(\mathbb{K})$ (resp. Let $F, G \in \mathcal{M}(d(0, R^-))$) be non-constant, having no zero and no pole at 0 and sharing the value 1 C.M.

If $\Psi_{F,G} = 0$ and if

$$\lim_{r \rightarrow +\infty} \left(T(r, F) - [\bar{Z}(r, F) + \bar{N}(r, F) + \bar{Z}(r, G) + \bar{N}(r, G)] \right) = +\infty$$

(resp.

$$\lim_{r \rightarrow R^-} \left(T(r, F) - [\bar{Z}(r, F) + \bar{N}(r, F) + \bar{Z}(r, G) + \bar{N}(r, G)] \right) = +\infty)$$

then either $F = G$ or $FG = 1$.

Proof. Suppose $\Psi_{F,G} = 0$ and that the above limit is $+\infty$ in the situation we consider: $F, G \in \mathcal{M}(\mathbb{K})$ or $F, G \in \mathcal{M}(d(0, R^-))$.

Since $\Psi_{F,G} = \frac{\phi'}{\phi}$ with $\phi = \left(\frac{F'}{(F-1)^2} \right) \left(\frac{(G-1)^2}{G'} \right)$, there exist $a, b \in \mathbb{K}$ with $a \neq 0$, such that $\frac{1}{F-1} = \frac{a}{G-1} + b$, this is, $F = \frac{(1+b)G + a - (1+b)}{bG + (a-b)}$. Hence

$$F = \frac{AG + B}{CG + D} \quad (1)$$

with $A, B, C, D \in \mathbb{K}$.

Let $r > 0$ (resp. Let $r \in]0, R[$). Consider the following three cases:

- **Case 1:** $A \neq 0$ and $C = 0$.

By (1), we have $F - \frac{B}{D} = \frac{A}{D}G$. Suppose $B \neq 0$. Then $\bar{Z}\left(r, F - \frac{B}{D}\right) = \bar{Z}(r, G)$. So, applying Theorem N to F , we obtain

$$\begin{aligned} T(r, F) &\leq \bar{Z}(r, F) + \bar{Z}\left(r, F - \frac{B}{D}\right) + \bar{N}(r, F) - \log r + O(1) \\ &= \bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) - \log r + O(1) \\ &< \bar{Z}(r, F) + \bar{N}(r, F) + \bar{Z}(r, G) + \bar{N}(r, G) + O(1), \end{aligned}$$

a contradiction to our hypothesis. Thus $B = 0$ and, so $F = \frac{A}{D}G$.

Suppose $\frac{A}{D} \neq 1$. Since F and G share 1 C.M. and $F = \frac{A}{D}G$, we have $(F(x), G(x)) \neq (1, 1)$ $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(0, R^-)$), because if $F(x) = 1$, then $G(x) = 1$ and hence $\frac{A}{D} = 1$, a contradiction. But $G(x) = 1$ if and only if $F(x) = \frac{A}{D}$. Thus F cannot take values 1 and $\frac{A}{D}$ and hence F has two exceptional values. Consequently, by Lemma 6, F is a constant, a contradiction. Thereby $\frac{A}{D} = 1$, and hence $F = G$.

- **Case 2:** $A = 0$ and $C \neq 0$.

By (2), we have $G = \frac{B}{CF} - \frac{D}{C}$. Suppose $D \neq 0$. Since $T(r, F) = T\left(r, \frac{1}{F}\right) + O(1)$ and $\bar{Z}\left(r, \frac{1}{F} - \frac{D}{B}\right) = \bar{Z}(r, G)$, applying Theorem N to F , we have

$$\begin{aligned} T(r, F) &\leq \bar{Z}\left(r, \frac{1}{F}\right) + \bar{Z}\left(r, \frac{1}{F} - \frac{D}{B}\right) + \bar{N}\left(r, \frac{1}{F}\right) - \log r + O(1) \\ &= \bar{Z}(r, F) + \bar{Z}(r, G) + \bar{N}(r, F) - \log r + O(1) \\ &< \bar{Z}(r, F) + \bar{N}(r, F) + \bar{Z}(r, G) + \bar{N}(r, G) + O(1), \end{aligned}$$

a contradiction to our hypothesis, again. Thus $D = 0$ and, so $F = \frac{B}{CG}$.

Now, suppose $\frac{B}{C} \neq 1$. Using the same argument as in Case 1, we conclude that $F(x) - 1 \neq 0$ and $G(x) - 1 \neq 0$ $\forall x \in \mathbb{K}$ (resp. $\forall x \in d(0, R^-)$). Moreover, $G(x) = 1$ if and only if $F(x) = \frac{B}{C}$. Then, it is necessary that $F(x) \neq \frac{B}{C}$. Hence, as in Case 1, F omits two values in \mathbb{K} which is impossible (Lemma 6), F is a constant, a contradiction again. Consequently $\frac{B}{C} = 1$ and, hence $FG = 1$.

- **Case 3:** $AC \neq 0$.

By (1), we have $F - \frac{A}{C} = \frac{B - \frac{AD}{C}}{CG + D}$ and hence $\bar{Z}\left(r, F - \frac{A}{C}\right) = \bar{N}(r, G)$. Applying Theorem N

to F , we have

$$\begin{aligned} T(r, F) &\leq \overline{Z}(r, F) + \overline{Z}\left(r, F - \frac{A}{C}\right) + \overline{N}(r, F) - \log r + O(1) \\ &= \overline{Z}(r, F) + \overline{N}(r, G) + \overline{N}(r, F) - \log r + O(1) \\ &< \overline{Z}(r, F) + \overline{N}(r, F) + \overline{Z}(r, G) + \overline{N}(r, G) + O(1), \end{aligned}$$

a contradiction to our hypothesis, again.

□

Lemma 9. Let $Q(x) = (x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i} \in \mathbb{K}[x]$ ($a_i \neq a_j$, $\forall i \neq j$) with $l \geq 2$ and $n \geq \max\{k_2, \dots, k_l\}$ and let $k = \sum_{i=2}^l k_i$. Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. $f, g \in \mathcal{M}_u(d(0, R^-))$) such that $\theta = Q(f)f'Q(g)g'$ is a small function with respect to f and g . We have the following :

If $l = 2$ then n belongs to $\{k, k+1, 2k, 2k+1, 3k+1\}$.

If $l = 3$ then n belongs to $\{\frac{k}{2}, k+1, 2k+1, 3k_2 - k, \dots, 3k_l - k\}$.

If $l \geq 4$ then $n = k+1$.

If θ is a constant and $f, g \in \mathcal{M}(\mathbb{K})$ then $n = k+1$.

Proof. Without loss of generality, we can assume $a_1 = 0$. Suppose $f, g \in \mathcal{M}(\mathbb{K})$ (resp. $f, g \in \mathcal{M}_u(d(0, R^-))$) satisfy

$$f^n \left(\prod_{i=2}^l (f - a_i)^{k_i} \right) f' g^n \left(\prod_{i=2}^l (g - a_i)^{k_i} \right) g' = \theta \quad (1)$$

Let Σ be the set of zeroes and poles of θ . We will denote by $S(r)$ any function φ defined in $]0, +\infty[$ (resp. in $]0, R[$) such that $\lim_{r \rightarrow +\infty} \frac{\varphi(r)}{T(r, f)} = 0$ and $\lim_{r \rightarrow +\infty} \frac{\varphi(r)}{T(r, g)} = 0$.

Let $\gamma \in \mathbb{K} \setminus \Sigma$ (resp. $\gamma \in d(0, R^-) \setminus \Sigma$) be a zero of g of order s . Clearly, by (1), γ is a pole of f of order, for example, t . And since γ is neither a zero nor a pole of θ we can derive the following relation

$$s(n+1) = t(n+k+1) + 2 \quad (2)$$

Now, suppose that for $i \in \{2, \dots, l\}$, $g - a_i$ has a zero $\gamma \in \mathbb{K} \setminus \Sigma$ (resp. $\gamma \in d(0, R^-) \setminus \Sigma$) of order s_i . It is a pole of f of order t_i . So, by (2), we obtain

$$s_i(k_i+1) = t_i(n+k+1) + 2 \quad (3)$$

By (2) and (3) it is obvious that $s > t$ and $s_i > t_i$ and hence, $s \geq 2$, $s_i \geq 2$.

Consider now a pole $\gamma \in \mathbb{K} \setminus \Sigma$ (resp. $\gamma \in d(0, R^-) \setminus \Sigma$) of f . Either it is a zero of g , or it is a zero of $g - a_i$ for some $i \in \{2, \dots, l\}$, or it is a zero of g' that is neither a zero of g nor a zero of $g - a_i$ ($\forall i \in \{2, \dots, l\}$). Let $Z_0(r, g')$ be the counting function of zeroes of g' that are neither a zero of g nor a zero of $g - a_i$ for all $i \in \{2, \dots, l\}$ (counting multiplicity) and let $\overline{Z}_0(r, g')$ be the counting function of zeroes of g' that are neither a zero of g nor a zero of $g - a_i$ for all $i \in \{2, \dots, l\}$, ignoring multiplicity. Since $T(r, \theta) = S(r)$, we have

$$\overline{N}(r, f) \leq \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}_0(r, g') + S(r) \quad (4)$$

And if θ is a constant, we have

$$\overline{N}(r, f) \leq \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}_0(r, g') \quad (5)$$

Now, by Theorem N, we have $(l-1)T(r, f) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{N}(r, f) - Z_0(r, f') - \log r + O(1)$, hence by (4), we obtain

$$(l-1)T(r, f) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}_0(r, g') - Z_0(r, f') + S(r) \quad (6)$$

and if θ is a constant, by (5) we have

$$(l-1)T(r, f) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}_0(r, g') - Z_0(r, f') - \log r + O(1) \quad (7)$$

And similarly, with f , in the general case we have

$$(l-1)T(r, g) \leq \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}_0(r, f') - Z_0(r, g') + S(r) \quad (8)$$

and if θ is a constant, we have

$$(l-1)T(r, g) \leq \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}_0(r, f') - Z_0(r, g') - \log r + O(1) \quad (9)$$

Hence, adding (6) and (8), in the general case we obtain

$$(l-1)(T(r, f) + T(r, g)) \leq 2(\overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i)) + S(r) \quad (10)$$

and if θ is a constant, by (7) and (9) we have

$$(l-1)(T(r, f) + T(r, g)) \leq 2(\overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i)) - 2 \log r + O(1) \quad (11)$$

Case $l=2$:

Without loss of generality, we can assume $a_2 = 1$. Relation (10) now becomes

$$T(r, f) + T(r, g) \leq 2(\overline{Z}(r, g) + \overline{Z}(r, g - 1) + \overline{Z}(r, f) + \overline{Z}(r, f - 1)) + S(r) \quad (10 - a)$$

Suppose now that all zeroes of f , $f - 1$, g , $g - 1$ are at least of order 5, except maybe those lying in Σ : then

$$\begin{aligned} \overline{Z}(r, f) &\leq \frac{1}{5}T(r, f) + S(r), \quad \overline{Z}(r, f - 1) \leq \frac{1}{5}T(r, f) + S(r), \\ \overline{Z}(r, g) &\leq \frac{1}{5}T(r, g) + S(r), \quad \overline{Z}(r, g - 1) \leq \frac{1}{5}T(r, g - 1) + S(r) \end{aligned}$$

a contradiction to (10-a), proving the statement of the Lemma.

Consequently, we will examine all situations leading to zeroes of order ≤ 4 for f , $f - 1$, g , $g - 1$ out of Σ . Actually, since f and g play the same role with respect to n and k , it is sufficient to examine the situation, for instance, when g or $g - 1$ has a zero out of Σ of order $s \leq 4$. In each case we denote by t the order of the pole of f which is a zero of g or $g - 1$. Since $s > t$, we only have to examine zeroes of g or $g - 1$ that are poles of f of order 1, 2, 3.

Suppose first g has a zero $\gamma \notin \Sigma$ of order $s = 2$. Then

$$2(n + 1) = t(k + n + 1) + 2 \quad (12)$$

By (12) if $t = 1$ we find a solution:

$$n = k + 1 \quad (13)$$

Next, if $t \geq 2$, we check that $2n + 2 < t(k + n + 1) + 2$, hence (13) is the only solution.

Suppose now g has a zero $\gamma \notin \Sigma$ of order $s = 3$. Then

$$3(n + 1) = t(k + n + 1) + 2 \quad (14)$$

By (14) if $t = 1$ we find no solution because $k \leq n$.

If $t = 2$ we find

$$n = 2k + 1 \quad (15)$$

If $t \geq 3$ we have $3(n + 3) < 3(k + n + 1) + 2$ hence (15) is the only solution.

Suppose now g has a zero of order $s = 4$. Then

$$4(n + 1) = t(k + n + 1) + 2 \quad (16)$$

If $t = 1$, since $k \leq n$, we have $4(n + 1) > t(k + n + 1) + 2$.

If $t = 2$, by (16) we have a solution

$$n = k \quad (17)$$

If $t = 3$, we have another solution

$$n = 3k + 1 \quad (18)$$

Consequently, by (13), (15), (16), (18), all possibilities for g to have a zero of order $s \leq 4$ are as follows:

$$n = k + 1, s = 2, \quad n = 2k + 1, s = 3, \quad n = k, s = 4, \quad n = 3k + 1, s = 4. \quad (19)$$

Now, we will examine zeroes of $g - 1$ out of Σ of order ≤ 4 . So, the order s' of $g - 1$ satisfies

$$s'(k + 1) = t(k + n + 1) + 2 \quad (20)$$

Suppose first $g - 1$ has a zero $\gamma \notin \Sigma$ of order $s' = 2$. Then by (20), we have

$$2(k + 1) = t(k + n + 1) + 2 \quad (21)$$

Since $k \leq n$, we find no solution neither when $t = 1$ that would lead to $k = n + 1$, nor when $t \geq 2$ because $2(k + 1) < t(k + n + 1) + 2$.

Suppose now that $s' = 3$.

If $t = 1$ we find a solution:

$$n = 2k \quad (22)$$

If $t \geq 2$, we have no solution with $k \leq n$ because $3(k + 1) < t(k + n + 1) + 2$

Suppose now that $s' = 4$.

If $t = 1$ we find a solution:

$$n = 3k + 1 \quad (23)$$

If $t = 2$ we find another solution:

$$n = k \quad (24)$$

If $t \geq 3$, we find no solution with $k \leq n$ because $4(k + 1) < t(k + n + 1) + 2$

Consequently, by (22), (23), (24), all possibilities for $g - 1$ to have a zero $\gamma \notin \Sigma$ of order $s \leq 4$ are as follows:

$$n = 2k, s' = 3, \quad n = 3k + 1, s' = 4, \quad n = k, s' = 4. \quad (25)$$

Thus, we have proved that when $n \neq k, k + 1, 2k, 2k + 1, 3k + 1$, none of the zeroes of $f, f - 1, g, g - 1$ out of Σ is of order ≤ 4 and therefore the general statement of the Lemma is proved in the case $l = 2$.

Now, suppose that θ is a constant and f, g belong to $\mathcal{M}(K)$ and suppose that $n \neq k + 1$. We notice that Σ is now empty. Now (11) gets

$$T(r, f) + T(r, g) \leq 2(\overline{Z}(r, g) + \overline{Z}(r, g - 1) + \overline{Z}(r, f) + \overline{Z}(r, f - 1)) - 2 \log r + O(1) \quad (11 - a)$$

First, we have seen that zeroes of order 2 for g or $g - 1$ (hence also for f and $f - 1$) only occur when $n = k + 1$. Consequently, excluding this case, all zeroes of $f, f - 1, g, g - 1$ are of order ≥ 3 . We will examine each case.

Suppose first that all zeroes of $f, f - 1, g, g - 1$ are at least of order 4. Then $\overline{Z}(r, f) \leq \frac{1}{4}T(r, f)$, $\overline{Z}(r, f - 1) \leq \frac{1}{4}T(r, f)$, $\overline{Z}(r, g) \leq \frac{1}{4}T(r, g)$, $\overline{Z}(r, g - 1) \leq \frac{1}{4}T(r, g - 1)$ a contradiction to (11-a).

And finally, suppose that all zeroes of f and g are at least of order 3 and all zeroes of $f - 1$ and $g - 1$ are at least of order 6. Then we have $\overline{Z}(r, f) \leq \frac{1}{3}T(r, f)$, $\overline{Z}(r, f - 1) \leq \frac{1}{6}T(r, f)$, $\overline{Z}(r, g) \leq \frac{1}{3}T(r, g)$, $\overline{Z}(r, g - 1) \leq \frac{1}{6}T(r, g - 1)$, a contradiction to (11-a) again.

Recall that when f has a pole of order 4, g or $g - 1$, if it has a zero, must have a zero of order ≥ 5 . Consequently, we only have to examine zeroes of g or $g - 1$ that are poles of f of order 1, 2, 3.

For each pair (n, k) leading to an order $s > 2$ of zero of g , we will precisely examine the possible order of zeroes of $g - 1$ and vice-versa.

First we have to consider the case $n = 2k + 1$. We know that if g has a zero, it is at least of order 3. If $g - 1$ has a zero, by (3) its order s satisfies

$$s(k + 1) = t(3k + 2) + 2 \quad (26)$$

We can check that no solution (s, t) exists with $s \leq 4$.

Suppose now $s = 5$.

If $t = 1$, we check that $5(k + 1) > 3k + 4$

If $t \geq 2$, we have $5k + 5 < t(3k + 2) + 2$.

Hence, if $n = 2k + 1$, a zero of $g - 1$ has order ≥ 6 . Indeed, such a possibility exists with $s = 6$ and $t = 2$. Consequently, we have

$$\overline{Z}(r, g) + \overline{Z}(r, g - 1) \leq \frac{1}{2}T(r, g) \quad (27)$$

Suppose now $n = 3k + 1$. We have seen that all zeroes of g and $g - 1$ have order at least 4. Consequently, we have

$$\overline{Z}(r, g) + \overline{Z}(r, g - 1) \leq \frac{1}{2}T(r, g) + O(1) \quad (28)$$

Suppose now $n = k$, then g and $g - 1$ play the same role. All zeroes of g and $g - 1$ are at least of order ≥ 4 hence we have again:

$$\overline{Z}(r, g) + \overline{Z}(r, g - 1) \leq \frac{1}{2}T(r, g) + O(1). \quad (29)$$

Finally, suppose $n = 2k$. We have seen that g admits no zero of order $s < 5$. So we must examine the case $s = 5$. By (2) we have $5(2k + 1) = t(2k + k + 1) + 2$. Then $t = 1$ is impossible.

If $t = 2$, we have $10k + 5 = 6k + 4$, impossible.

If $t = 3$, we have $10k + 5 = 9k + 5$.

And if $t > 3$, then $5(2k + 1) < t(3k + 1) + 2$ for all $k \geq 2$.

Consequently, all zeroes of g have order at least 6 and hence we have again

$$\overline{Z}(r, g) + \overline{Z}(r, g - 1) \leq \frac{1}{2}T(r, g). \quad (30)$$

Thus, since $n \neq k + 1$, by (27), (28), (29), (30) and the symmetric inequalities for f instead of g , we have proved a contradiction to (11-a).

Case 1=3 : Suppose that all zeroes of f , g , $f - a_i$, $g - a_i$ $\forall i \in \{2, 3\}$ are at least of order 4, except maybe those lying in Σ : then

$$\overline{Z}(r, f) \leq \frac{1}{4}T(r, f) + S(r) \quad \text{and} \quad \forall i \in \{2, 3\}, \quad \overline{Z}(r, f - a_i) \leq \frac{1}{4}T(r, f) + S(r),$$

$$\overline{Z}(r, g) \leq \frac{1}{4}T(r, g) + S(r) \quad \text{and} \quad \forall i \in \{2, 3\}, \quad \overline{Z}(r, g - a_i) \leq \frac{1}{4}T(r, g) + S(r)$$

Then using (10) we obtain $l \leq 2$ a contradiction.

Consequently, we will examine all n and k_i ($i \in \{2, 3\}$) leading to zeroes out of Σ of order ≤ 3 for f , g , $f - a_i$, $g - a_i$ for all $i \in \{2, 3\}$. And since f and g play the same role, it is sufficient to examine the situation, for instance, when g or some $g - a_i$ has a zero of order less than 3. In each case we denote by t the order of the pole of f which is a zero of g or $g - a_i$ for some i . Recall that when f has a pole of order 3, g or $g - a_i$, if it has a zero, must have a zero of order ≥ 4 . Consequently, we only have to examine zeroes of g or $g - a_i$ ($\forall i \in \{2, 3\}$) that are poles of f of order 1, 2.

Suppose first g has a zero $\gamma \notin \Sigma$ of order $s = 2$. Then by (2) we have

$$2(n + 1) = t(k + n + 1) + 2 \quad (31)$$

By (31) if $t = 1$ we find a solution:

$$n = k + 1 \quad (32)$$

Next, if $t = 2$, we check that $2n + 2 < 2(k + n + 1) + 2$, hence (32) is the only solution.

Suppose now g has a zero $\gamma \notin \Sigma$ of order $s = 3$. Then

$$3(n + 1) = t(k + n + 1) + 2 \quad (33)$$

By (33) if $t = 1$ we find a solution :

$$n = \frac{k}{2} \quad (34)$$

If $t = 2$ we find

$$n = 2k + 1 \quad (35)$$

Consequently, by (32), (34), (35) all possibilities for g to have a zero of order $s \leq 3$ are as follows:

$$n = k + 1, s = 2, \quad n = \frac{k}{2}, s = 3, \quad n = 2k + 1, s = 3.$$

Now, let $i \in \{2, 3\}$ and examine zeroes of $g - a_i$, $\gamma \notin \Sigma$ of order $s_i \leq 3$. So, by (3), the order s_i of $g - a_i$ satisfies

$$s_i(k_i + 1) = t(k + n + 1) + 2 \quad (36)$$

Suppose first $g - a_i$ has a zero $\gamma \notin \Sigma$ of order $s_i = 2$. Then by (36), we have

$$2(k_i + 1) = t(k + n + 1) + 2 \quad (37)$$

Since $k_i \leq n$ and $k_i \leq k$ we have $2(k_i + 1) < t(k + n + 1) + 2$. Hence we find no solution for (37).

Suppose now $s_i = 3$.

If $t = 1$ we find a solution:

$$3k_i = n + k \quad (38)$$

If $t = 2$, we have no solution because $3k_i < 2(n + k)$.

Consequently, the unique possibility for $g - a_i$ to have a zero $\gamma \notin \Sigma$ of order $s_i \leq 3$ is :

$$n + k = 3k_i, \quad s_i = 3.$$

Thus, we have proved that when $n \neq k + 1, \frac{k}{2}, 2k + 1, 3k_i - k$ none of the zeroes of $f, g, f - a_i, g - a_i$ ($\forall i \in \{2, 3\}$) out of Σ is of order ≤ 3 and therefore the statement of the Lemma is proved in the case $l = 3$.

Case $l \geq 4$:

Suppose now that all zeroes of $f, g, f - a_i, g - a_i$ $\forall i \in \{2, \dots, l\}$ are at least of order 3, except maybe those lying in Σ : then

$$\begin{aligned} \overline{Z}(r, f) &\leq \frac{1}{3}T(r, f) + S(r) \quad \text{and } \forall i \in \{2, \dots, l\}, \quad \overline{Z}(r, f - a_i) \leq \frac{1}{3}T(r, f) + S(r), \\ \overline{Z}(r, g) &\leq \frac{1}{3}T(r, g) + S(r) \quad \text{and } \forall i \in \{2, \dots, l\}, \quad \overline{Z}(r, g - a_i) \leq \frac{1}{3}T(r, g) + S(r). \end{aligned}$$

Then using (10) we obtain $l \leq 3$, a contradiction.

Consequently, we will examine all n and k_i ($i \in \{2, \dots, l\}$) leading to zeroes out of Σ of order ≤ 2 for $f, g, f - a_i, g - a_i$ for all $i \in \{2, \dots, l\}$. Actually, since f and g play the same role, it is sufficient to examine the situation, for instance, when g or some $g - a_i$ has a zero of order less than 2. In each case we denote by t the order of the pole of f which is a zero of g or $g - a_i$ for some i . Recall that when f has a pole of order 2, g or $g - a_i$, if it has a zero, must have a zero of order ≥ 3 . And then, we only have to examine zeroes of g or $g - a_i$ ($\forall i \in \{2, \dots, l\}$) that are poles of f of order 1.

Suppose first g has a zero $\gamma \notin \Sigma$ of order $s = 2$. Then γ is a pole of f of order $t = 1$. Then by (2) we have

$$2(n + 1) = (k + n + 1) + 2 \quad (39)$$

We find a solution :

$$n = k + 1. \quad (40)$$

Now, let $i \in \{2, \dots, l\}$ and suppose $g - a_i$ has a zero $\gamma \notin \Sigma$ of order $s_i = 2$. Then γ is a pole of f of order $t = 1$. So by (3) we have :

$$2(k_i + 1) = (n + k + 1) + 2$$

That means $2k_i = n + k + 1$. Since $k_i \leq n$ and $k_i \leq k$, we find no solution when $s_i = 2$ and $t = 1$.

Consequently, by (40), the only possibility for g or some $g - a_i$ to have a zero $\gamma \notin \Sigma$ of order ≤ 2 is :

$$n = k + 1.$$

As in the case $l = 2$, we now have to consider the situation $l \geq 3$ when $\theta \in K$ and f, g belong to $\mathcal{M}(K)$.

Obviously $\Sigma = \emptyset$ and we have seen that zeroes of order 2 for $f, g, f - a_i, g - a_i$ ($\forall i \in \{2, \dots, l\}$) only occur when $n = k + 1$. Consequently, excluding this case, all zeroes of $f, g, f - a_i, g - a_i$ ($\forall i \in \{2, \dots, l\}$) are of order ≥ 3 .

Thus suppose $n \neq k + 1$. Then,

$$\bar{Z}(r, f) \leq \frac{1}{3}T(r, f) \quad \text{and } \forall i \in \{2, \dots, l\}, \quad \bar{Z}(r, f - a_i) \leq \frac{1}{3}T(r, f) + O(1),$$

$$\bar{Z}(r, g) \leq \frac{1}{3}T(r, g) \quad \text{and } \forall i \in \{2, \dots, l\}, \quad \bar{Z}(r, g - a_i) \leq \frac{1}{3}T(r, g) + O(1)$$

By (11), we obtain $(l - 1)(T(r, f) + T(r, g)) \leq \frac{2l}{3}(T(r, f) + T(r, g)) - 2 \log(r) + O(1)$. Hence $l < 3$ a contradiction. This finishes the proof of the Lemma. \square

Lemma 10 is known and easily checked [4], [23]:

Lemma 10. *Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental (resp. let $f, g \in \mathcal{M}_u(d(0, R^-))$) satisfy $(f - a)f^n = (g - a)g^n$ with $a \in \mathbb{K}$ and let $h = \frac{f}{g}$. If h is not identically 1, then*

$$g = \frac{h^n - 1}{h^{n+1} - 1}, \quad f = \frac{h^{n+1} - h}{h^{n+1} - 1}.$$

Notation: Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$) be such that $f(0) \neq 0, \infty$

We denote by $Z_1(r, f)$ the counting function of simple zeroes of f and by $N_1(r, f)$ the counting function of simple poles of f .

We denote by $\bar{Z}_{(2)}(r, f)$ the counting function of multiple zeroes of f , each counted without multiplicity, and we denote by $N_{(2)}(r, f)$ the counting function of multiple poles of f , each counted without multiplicity, Consequently, by definition, one has $\bar{Z}(r, f) = Z_1(r, f) + \bar{Z}_{(2)}(r, f)$, $\bar{N}(r, f) = N_1(r, f) + \bar{N}_{(2)}(r, f)$

Finally we denote by $Z_{[2]}(r, f)$ the counting function of the zeroes of f each counted multiplicity when it is at most 2 and with multiplicity 2 when it is bigger.

And here we denote by $Z_0(r, f')$ the counting function of the zeroes of f' that are not zeroes of $f(f - 1)$.

We will now prove the following Lemma 11 in a similar way as in [13], with however some special properties due to *p*-adic analytic functions:

Lemma 11. *Let $f, g \in \mathcal{M}(\mathbb{K})$ (resp. Let $f \in \mathcal{M}(d(0, R^-))$) be such that $f(0) \neq 0, \infty$, and share the value 1 C.M. If $\Psi_{f,g}$ is not identically zero, then,*

$$\max(T(r, f), T(r, g)) \leq N_{[2]}(r, f) + Z_{[2]}(r, f) + N_{[2]}(r, g) + Z_{[2]}(r, g) - 3 \log r$$

Proof. Since f and g share 1 CM, each simple zero of $f - 1$ is a simple zero of $g - 1$ and is a zero of $\Psi_{f,g}$. Consequently, we have

$$\overline{Z}_{(2)}(r, f - 1) = \overline{Z}_{(2)}(r, g - 1) \quad (1)$$

and

$$Z_{(1)}(r, f - 1) = Z_{(1)}(r, g - 1) \leq Z(r, \Psi_{f,g}) \quad (2)$$

Now, by Lemma 4 we have $Z(r, \Psi_{f,g}) \leq N(r, \Psi_{f,g}) - \log r$. Hence by (2) we obtain

$$Z_{(1)}(r, f - 1) = Z_{(1)}(r, g - 1) \leq N(r, \Psi_{f,g}) - \log r \quad (3)$$

On the other hand, all poles of $\Psi_{f,g}$ are simple and only occur at zeroes of f' and g' and at multiple poles of f and g . Consequently, we have

$$N(r, \Psi_{f,g}) \leq \overline{Z}_0(r, f') + \overline{Z}_0(r, g') + \overline{Z}_{(2)}(r, f) + \overline{Z}_{(2)}(r, g) \quad (4)$$

By (3) and (4), we have

$$Z_{(1)}(r, f - 1) \leq \overline{Z}_0(r, f') + \overline{Z}_0(r, g') + \overline{Z}_{(2)}(r, f) + \overline{Z}_{(2)}(r, g) - \log r,$$

therefore

$$Z_{(1)}(r, f - 1) \leq \overline{N}_{(2)}(r, f) + \overline{N}_{(2)}(r, g) + \overline{Z}_0(r, f') + \overline{Z}_0(r, g') + \overline{Z}_{(2)}(r, f) + \overline{Z}_{(2)}(r, g) - \log r \quad (5)$$

Recall that by Lemma 2 we have

$$Z(r, g') \leq Z(r, g) + \overline{N}(r, g) - \log r \quad (6)$$

and from the definition of $Z_0(r, g')$ we have

$$\overline{Z}_0(r, g') + \overline{Z}_{(2)}(r, g - 1) + Z_{(2)}(r, g) - \overline{Z}_{(2)}(r, g) \leq Z(r, g'),$$

consequently, we obtain

$$\overline{Z}_0(r, g') + \overline{Z}_{(2)}(r, g - 1) \leq Z(r, g') + \overline{Z}_{(2)}(r, g) - Z_{(2)}(r, g) \quad (7)$$

But by (6) and (7) we have

$$\overline{Z}_0(r, g') + \overline{Z}_{(2)}(r, g - 1) \leq Z(r, g) + \overline{Z}_{(2)}(r, g) - Z_{(2)}(r, g) + \overline{N}(r, g) - \log r$$

and $Z(r, g) + \overline{Z}_{(2)}(r, g) - Z_{(2)}(r, g)$ is just $\overline{Z}(r, g)$. Consequently, by the last inequality, we have

$$\overline{Z}_0(r, g') + \overline{Z}_{(2)}(r, g - 1) \leq \overline{N}(r, g) + \overline{Z}(r, g) - \log r \quad (8)$$

Now, Theorem N lets us write

$$T(r, f) \leq \overline{N}(r, f) + \overline{Z}(r, f) + \overline{Z}(r, f - 1) - Z_0(r, f') - \log r \quad (9)$$

By (2) we notice that $\overline{Z}(r, f - 1) = Z_{(1)}(r, f - 1) + \overline{Z}_{(2)}(r, f - 1) = Z_{(1)}(r, f - 1) + \overline{Z}_{(2)}(r, g - 1)$. So, by (9) we have

$$T(r, f) \leq \overline{N}(r, f) + \overline{Z}(r, f) + Z_{(1)}(r, f - 1) + \overline{Z}_{(2)}(r, g - 1) - Z_0(r, f') - \log r$$

and hence, by (8), we derive

$$T(r, f) \leq \overline{N}(r, f) + \overline{Z}(r, f) + Z_1(r, f - 1) + \overline{N}(r, g) + \overline{Z}(r, g) - \overline{Z}_0(r, g') - Z_0(r, f') - 2 \log r.$$

And now, by (5) we obtain

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, f) + \overline{Z}(r, f) + \overline{N}_{(2)}(r, f) + \overline{N}_{(2)}(r, g) + \overline{Z}_{(2)}(r, f) + \overline{Z}_{(2)}(r, g) \\ &\quad + \overline{Z}_0(r, g') + \overline{Z}_0(r, f') + \overline{N}(r, g) + \overline{Z}(r, g) - \overline{Z}_0(r, g') - Z_0(r, f') - 3 \log r. \end{aligned}$$

But now, we notice that $\overline{Z}(r, f) + \overline{Z}_{(2)}(r, f) = Z_{[2]}(r, f)$, $\overline{Z}(r, g) + \overline{Z}_{(2)}(r, g) = Z_{[2]}(r, g)$, $\overline{N}(r, f) + \overline{N}_{(2)}(r, f) = N_{[2]}(r, f)$, $\overline{N}(r, g) + \overline{N}_{(2)}(r, g) = N_{[2]}(r, g)$. We then obtain the conclusion of Lemma 11. \square

4 Proof of Theorems

Proof of the Theorems. The polynomial P is considered in theorems 1, 2, 3, 4, 5, 6, 7 and we can assume $a_1 = 0$. In Theorems 8, and 9, we call P the polynomial such that $P'(x) = x^n(x - a)^k$ and $P(0) = 0$. Set $F = \frac{f'P'(f)}{\alpha}$ and $G = \frac{g'P'(g)}{\alpha}$. Clearly F and G share the value 1 C.M. Since f, g are transcendental (resp. unbounded), we notice that so are F and G . Recall that

$$\Psi_{F,G} = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$$

We will prove that under the hypotheses of each theorem, $\Psi_{F,G}$ is identically zero.

Set $\widehat{F} = P(f)$, $\widehat{G} = P(g)$. We notice that $P(x)$ is of the form $x^{n+1}Q(x)$ with $Q \in K[x]$ of degree k . Now, with help of Lemma 2, we can check that we have

$$T(r, \widehat{F}) - Z(r, \widehat{F}) \leq T(r, \widehat{F}') - Z(r, \widehat{F}') + O(1)$$

Consequently, since $(\widehat{F})' = \alpha F$, we have

$$T(r, \widehat{F}) \leq T(r, F) + Z(r, \widehat{F}) - Z(r, F) + T(r, \alpha) + O(1), \quad (1)$$

hence, by (1), we obtain

$$T(r, \widehat{F}) \leq T(r, F) + (n+1)Z(r, f) + Z(r, Q(f)) - nZ(r, f) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + O(1).$$

i.e.

$$T(r, \widehat{F}) \leq T(r, F) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) + O(1). \quad (2)$$

and similarly,

$$T(r, \widehat{G}) \leq T(r, G) + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l k_i Z(r, g - a_i) - Z(r, g') + T(r, \alpha) + O(1). \quad (3)$$

Now, it follows from the definition of F and G that

$$Z_{[2]}(r, F) + N_{[2]}(r, F) \leq 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\bar{N}(r, f) + T(r, \alpha) + O(1) \quad (4)$$

and similarly

$$Z_{[2]}(r, G) + N_{[2]}(r, G) \leq 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\bar{N}(r, g) + T(r, \alpha) + O(1) \quad (5)$$

And particularly, if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, then

$$Z_{[2]}(r, F) + N_{[2]}(r, F) \leq 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\bar{N}(r, f) + T(r, \alpha) + O(1) \quad (6)$$

and similarly

$$Z_{[2]}(r, G) + N_{[2]}(r, G) \leq 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\bar{N}(r, g) + T(r, \alpha) + O(1). \quad (7)$$

Suppose now that $\Psi_{F,G}$ is not identically zero. Now, by Lemma 11, we have

$$T(r, F) \leq Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) - 3 \log r$$

hence by (2), we obtain

$$\begin{aligned} T(r, \hat{F}) &\leq Z_{[2]}(r, F) + N_{[2]}(r, F) + Z_{[2]}(r, G) + N_{[2]}(r, G) + Z(r, f) + Z(r, Q(f)) \\ &\quad - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + T(r, \alpha) - 3 \log r + O(1) \end{aligned}$$

and hence by (4) and (5):

$$\begin{aligned} T(r, \hat{F}) &\leq 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\bar{N}(r, f) + 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + \\ &\quad 2\bar{N}(r, g) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l k_i Z(r, f - a_i) - Z(r, f') + 2T(r, \alpha) - 3 \log r + O(1) \end{aligned} \quad (8)$$

and similarly,

$$\begin{aligned} T(r, \hat{G}) &\leq 2Z(r, g) + 2 \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\bar{N}(r, g) + 2Z(r, f) + 2 \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\bar{N}(r, f) \\ &\quad + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l k_i Z(r, g - a_i) - Z(r, g') + 2T(r, \alpha) - 3 \log r + O(1). \end{aligned} \quad (9)$$

Consequently,

$$T(r, \widehat{F}) + T(r, \widehat{G}) \leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) + (Z(r, f') + Z(r, g')) + 4(\overline{N}(r, f) + \overline{N}(r, g)) + (Z(r, Q(f)) + Z(r, Q(g))) + 4T(r, \alpha) - 6 \log r + O(1). \quad (10)$$

Moreover, if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, then by (6) and (7) we have

$$T(r, \widehat{F}) \leq 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + Z(r, f) + Z(r, Q(f)) - \sum_{i=2}^l Z(r, f - a_i) - Z(r, f') + 2T(r, \alpha) - 3 \log r + O(1)$$

and similarly,

$$T(r, \widehat{G}) \leq 2Z(r, g) + \sum_{i=2}^l Z(r, g - a_i) + Z(r, g') + 2\overline{N}(r, g) + 2Z(r, f) + \sum_{i=2}^l Z(r, f - a_i) + Z(r, f') + 2\overline{N}(r, f) + Z(r, g) + Z(r, Q(g)) - \sum_{i=2}^l Z(r, g - a_i) - Z(r, g') + 2T(r, \alpha) - 3 \log r + O(1).$$

Consequently,

$$T(r, \widehat{F}) + T(r, \widehat{G}) \leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (Z(r, f - a_i) + Z(r, g - a_i)) + Z(r, Q(f)) + Z(r, Q(g)) + (Z(r, f') + Z(r, g')) + 4(\overline{N}(r, f) + \overline{N}(r, g)) + 4T(r, \alpha) - 6 \log r + O(1) \quad (11)$$

Now, let us go back to the general case. By Lemma 2, we can write $Z(r, f') + Z(r, g') \leq Z(r, f - a_2) + Z(r, g - a_2) + \overline{N}(r, f) + \overline{N}(r, g) - 2 \log r$. Hence, in general, by (10) we obtain

$$T(r, \widehat{F}) + T(r, \widehat{G}) \leq 5(Z(r, f) + Z(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2))) + 5(\overline{N}(r, f) + \overline{N}(r, g)) + (Z(r, Q(f)) + Z(r, Q(g))) + 4T(r, \alpha) - 8 \log r + O(1)$$

and hence, since $T(r, Q(f)) = kT(r, f) + O(1)$ and $T(r, Q(g)) = kT(r, g) + O(1)$,

$$T(r, \widehat{F}) + T(r, \widehat{G}) \leq 5(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) + (5 - k_2)((Z(r, f - a_2) + Z(r, g - a_2))) + 5(\overline{N}(r, f) + \overline{N}(r, g)) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1). \quad (12)$$

And if $k_i = 1$, $\forall i \in \{2, \dots, l\}$, by (11) and Lemma 2 we have

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq 5(Z(r, f) + Z(r, g)) + \sum_{i=2}^l (Z(r, f - a_i) + Z(r, g - a_i)) + (l-1)(T(r, f) + T(r, g)) \\ &\quad + (Z(r, f - a_2) + Z(r, g - a_2)) + 5(\overline{N}(r, f) + \overline{N}(r, g)) + 4T(r, \alpha) - 8 \log r + O(1) \end{aligned}$$

hence

$$\begin{aligned} T(r, \widehat{F}) + T(r, \widehat{G}) &\leq \\ 5(T(r, f) + T(r, g)) &+ \sum_{i=2}^l (T(r, f - a_i) + T(r, g - a_i)) + (l-1)(T(r, f) + T(r, g)) + (T(r, f - a_2) + T(r, g - a_2)) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + 4T(r, \alpha) - 8 \log r + O(1) \end{aligned}$$

and hence

$$T(r, \widehat{F}) + T(r, \widehat{G}) \leq (9 + 2l)(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1) \quad (13)$$

Now, let us go back to the general case. Since \widehat{F} is a polynomial in f of degree $n + k + 1$, we have $T(r, \widehat{F}) = (n + k + 1)T(r, f) + O(1)$ and similarly, $T(r, \widehat{G}) = (n + k + 1)T(r, g) + O(1)$, hence by (12) we can derive

$$\begin{aligned} (n + k + 1)(T(r, f) + T(r, g)) &\leq \\ 5(T(r, f) + T(r, g)) &+ (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) \\ &+ 5(\overline{N}(r, f) + \overline{N}(r, g)) + k(T(r, f) + T(r, g)) + 4T(r, \alpha) - 8 \log r + O(1). \end{aligned} \quad (14)$$

Hence

$$\begin{aligned} (n + k + 1)(T(r, f) + T(r, g)) &\leq (10 + k)(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) \\ &+ (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + 4T(r, \alpha) - 8 \log r + O(1). \end{aligned}$$

and hence

$$\begin{aligned} n(T(r, f) + T(r, g)) &\leq 9(T(r, f) + T(r, g)) + \sum_{i=3}^l (4 - k_i)((Z(r, f - a_i) + Z(r, g - a_i))) \\ &+ (5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) + 4T(r, \alpha) - 8 \log r + O(1). \end{aligned} \quad (15)$$

Then at least, for each $i = 3, \dots, l$ we have

$$(4 - k_i)(Z(r, f - a_i) + Z(r, g - a_i)) \leq \max(0, 4 - k_i)(T(r, f) + T(r, g)) + O(1) \text{ and}$$

$$(5 - k_2)(Z(r, f - a_2) + Z(r, g - a_2)) \leq \max(0, 5 - k_2)(T(r, f) + T(r, g)) + O(1).$$

Consequently, by (15) we have

$$n(T(r, f) + T(r, g)) \leq 9(T(r, f) + T(r, g)) + \sum_{i=3}^l \max(0, 4 - k_i)(T(r, f) + T(r, g)) + \max(0, 5 - k_2)(T(r, f) + T(r, g)) + O(1)$$

and hence,

$$n \leq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \quad (16)$$

Moreover, if f, g belong to $\mathcal{M}(\mathbb{K})$ and α is a constant or a Moebius function, then $T(r, \alpha) \leq \log r + O(1)$ and hence by (15) we have

$$n \leq 8 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2) \quad (17)$$

Now, if $k_i = 1, \forall i \in \{2, \dots, l\}$, by (13) we have $n + k + 1 \leq 9 + 2l$, hence

$$n \leq 9 + l. \quad (18)$$

Particularly, if $f, g \in \mathcal{M}(\mathbb{K})$ and if α is a constant or a Moebius function, then

$$n \leq 8 + l \quad (19)$$

Consequently, in the general case, if

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2)$$

we have $\Psi_{F,G} = 0$ which concerns Theorems 1 and 4.

Now, if f, g belong to $\mathcal{M}(\mathbb{K})$ and α is a constant or a Moebius function and if

$$n \geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2)$$

we have $\Psi_{F,G} = 0$ again, which concerns Theorem 2 and 3.

Further, if $k_i = 1, \forall i \in \{2, \dots, l\}$, when $n \geq l + 10$ we have $\Psi_{F,G} = 0$ which concerns Theorems 5, 6, 10 and 9 when α is an ordinary small function.

In the same context, if f, g belong to $\mathcal{M}(\mathbb{K})$, and if α is a constant or a Moebius function, then $\Psi_{F,G} = 0$ as soon as $n \geq l + 9$ Theorems 7, 8 and 9 when α is a constant or a Moebius function.

Thus, henceforth, we can assume that $\Psi_{F,G} = 0$ in each hypothesis of all theorems.

Note that $\Psi_{F,G} = \frac{\phi'}{\phi}$ with $\phi = \left(\frac{F'}{(F-1)^2} \right) \left(\frac{(G-1)^2}{G'} \right)$. Since $\phi = 0$, there exist $A, B \in \mathbb{K}$ such that

$$\frac{1}{G-1} = \frac{A}{F-1} + B \quad (20)$$

and $A \neq 0$.

Now, we will consider the following two cases: $B = 0$ and $B \neq 0$.

Case 1: $B = 0$.

Suppose $A \neq 1$. Then, by (20), we have $F = \frac{1}{A}G + \left(1 - \frac{1}{A}\right)$. Applying Theorem N to F , we obtain

$$\begin{aligned} T(r, F) &\leq \overline{Z}(r, F) + \overline{Z}\left(r, F - \left(1 - \frac{1}{A}\right)\right) + \overline{N}(r, F) - \log r + O(1) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) \\ &\quad + \overline{Z}(r, f') + \overline{Z}(r, g) + \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) + 3T(r, \alpha) - \log r + O(1). \end{aligned} \quad (21)$$

But $\overline{Z}(r, f) \leq T(r, f)$, $\overline{N}(r, f) \leq T(r, f)$, $\overline{Z}(r, f - 1) \leq T(r, f - 1) \leq T(r, f) + O(1)$ and $\overline{Z}(r, f') \leq T(r, f') \leq 2T(r, f) + O(1)$. Moreover, by Lemma 1, $T(r, F) \geq (n + k)T(r, f) - T(r, \alpha)$. Then, considering all the previous inequalities in (12), we deduce that

$$(n + k)T(r, f) \leq (l + 3)T(r, f) + (l + 2)T(r, g) + 4T(r, \alpha) - \log r + O(1). \quad (22)$$

Since f and g satisfy the same hypothesis, we also have

$$(n + k)T(r, g) \leq (l + 3)T(r, g) + (l + 2)T(r, f) + 4T(r, \alpha) - \log r + O(1). \quad (23)$$

Hence, adding (22) and (23), we have

$$(n + k)[T(r, f) + T(r, g)] \leq (2l + 5)[T(r, f) + T(r, g)] + 4T(r, \alpha) + O(1),$$

which leads to a contradiction whenever $n + k \geq (2l + 6)$.

In the hypotheses of all theorems we have

$$n \geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2).$$

That implies

$$\begin{aligned} n + k &\geq 9 + k + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2) = 9 + \sum_{i=2}^l k_i + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2) \\ &= 9 + \sum_{i=3}^l \max(4, k_i) + \max(5, k_2) \geq 9 + 4(l - 2) + 5 \geq 4l + 6. \end{aligned}$$

Consequently, the inequality $n + k \geq (2l + 6)$ is satisfied in Theorems 1, 2, 3, 4.

Next, if all k_i are equal to 1, we assume that $n \geq l + 9$, hence $n + k \geq l + k + 9 = 2l + 8$ (because $l = k + 1$) and hence the inequality $n + k \geq (2l + 6)$ is satisfied again in Theorems 5, 6, 7, 8, 9, 10.

Hence, we have $A = 1$ and this implies that $F = G$. Now, $\alpha F = \alpha G$, i.e. $(\widehat{F})' = (\widehat{G})'$. We assume $n \geq k + 2$ in Theorems 1, 2, 3 and this is automatically satisfied in Theorems 5, 7, 8, 9. And we assume $n \geq k + 3$ in Theorem 4 and this is automatically satisfied in Theorems 6 and 10. Consequently, by Lemma 7, we have $\widehat{F} = \widehat{G}$, i.e. $P(f) = P(g)$. But in Theorems 1, 2, 3, 4, 5, 6, 7,

8 P is a polynomial of uniqueness for the family of meromorphic functions we consider, hence we have $f = g$. And in Theorems 9 and 10, the conclusion comes from Lemma 10.

Case 2: $B \neq 0$.

We have $\overline{Z}(r, F) \leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, f') + T(r, \alpha)$ and $\overline{N}(r, F) \leq \overline{N}(r, f) + T(r, \alpha) + O(1)$ and similarly for G , so we can derive

$$\begin{aligned} \overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) &\leq \overline{Z}(r, f) + \sum_{i=2}^l \overline{Z}(r, f - a_i) + \overline{Z}(r, f') + \overline{Z}(r, g) \\ &+ \sum_{i=2}^l \overline{Z}(r, g - a_i) + \overline{Z}(r, g') + \overline{N}(r, f) + \overline{N}(r, g) + 4T(r, \alpha) + O(1) \\ &\leq 5[T(r, f) + T(r, g)] + 4T(r, \alpha) + O(1). \end{aligned} \quad (24)$$

Moreover, by (20), $T(r, F) = T(r, G) + O(1)$ and, by Lemma 1, we have

$$T(r, f) \leq \frac{1}{n+k} (T(r, F) + T(r, \alpha)) + O(1)$$

and $T(r, g) \leq \frac{1}{n+k} (T(r, G) + T(r, \alpha)) + O(1)$. Consequently,

$$T(r, f) + T(r, g) \leq 2 \left[\frac{1}{n+k} (T(r, F) + T(r, \alpha)) \right] + O(1).$$

Thus, (24) is equivalent to

$$\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \frac{10}{n+k} T(r, F) + \left(\frac{10}{n+k} + 4 \right) T(r, \alpha) + O(1).$$

Now, we can check that $n + k \geq 12$ in all theorems. Consequently, the previous inequality implies

$$\overline{Z}(r, F) + \overline{Z}(r, G) + \overline{N}(r, F) + \overline{N}(r, G) \leq \frac{10}{12} T(r, F) + \left(\frac{10}{12} + 4 \right) T(r, \alpha) + O(1). \quad (25)$$

Consequently, by (25) we can see that the hypotheses of Lemma 8 are satisfied and hence, either $F = G$, or $FG = 1$.

If $FG = 1$, then $f'P'(f)g'P'(g) = \alpha^2$. In Theorems 1, 2, 4 we have assumed that $n \geq k + 2$ and if $l = 2$, then $n \neq 2k, 2k + 1, 3k + 1$ and if $l = 3$ then $n \neq 3k_2 - k, 3k_3 - k$. Moreover, these conditions are automatically satisfied in Theorems 5, 6, 7, 8, 9, 10, so we have a contradiction to Lemma 9. In Theorem 3, we have assumed that $n \geq k + 2$ hence by Lemma 9, we have a contradiction again. Consequently, $F = G$, hence $(\widehat{F})' = (\widehat{G})'$ and therefore we can conclude as in the case $B = 0$.

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