



HAL
open science

Survey on Bezout rings of p-adic analytic functions

Bertin Diarra, Alain Escassut

► **To cite this version:**

Bertin Diarra, Alain Escassut. Survey on Bezout rings of p-adic analytic functions. Southeast Asian Bulletin of Mathematics, 2015. hal-01913280

HAL Id: hal-01913280

<https://uca.hal.science/hal-01913280>

Submitted on 6 Nov 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Survey on Bezout rings of p-adic analytic functions

Bertin Diarra and Alain Escassut

Abstract

Let \mathbb{K} be a complete ultrametric algebraically closed field and let $\mathcal{A}(\mathbb{K})$ (resp. $\mathcal{A}(D)$) be the \mathbb{K} -algebra of analytic functions in \mathbb{K} (resp. inside an open disk D). Following results in a paper by M. Lazard, we show that these algebras are Bezout rings, a property that is not showed in that paper. Moreover, the main results leading to the Bezout property is based upon a Mittag-Leffler theorem for meromorphic functions which is not proven in Lazard's paper. Furthermore, that Mittag-Leffler theorem (which is different from Krasner's Mittag-Leffler theorem for analytic elements) lets us find a shorter proof to show that the meromorphic functions admitting primitives are those whose residues are null.

1 Introduction and main results

Let us recall that the ring of analytic functions on a region of the complex number field is well known to be a Bezout ring. The two fundamental theorems necessary for a proof are the Weierstrass factorization theorem and the Mittag-Leffler theorem, (see for instance [6]- Exercice 4 -Chapter 8).

According to results of the paper *Les zéros des fonctions analytiques sur un corps valué complet* by Michel Lazard [7], it appears that in several hypotheses, rings of analytic functions on complete ultrametric algebraically closed fields are Bezout rings. However, that interesting property is not stated. Moreover, it derives from a Mittag-Leffler Theorem referred in general topology whose justification is not relevant. Here we plan to give proofs of all these properties, using results on quasi-invertible analytic elements [3], [4], [5] and on a Mittag-Leffler theorem for meromorphic functions similar to this of complex analysis, but is quite different from Krasner's Mittag-Leffler theorem for analytic elements on an infraconnected subset of \mathbb{K} .

Definitions and notation: Throughout the paper, we denote by \mathbb{K} a complete algebraically closed field of characteristic 0. Given $a \in \mathbb{K}$ and $R > 0$, we denote by $d(a, R)$ the disk $\{x \in \mathbb{K} \mid |x - a| \leq R\}$ and by $d(a, R^-)$ the disk $\{x \in \mathbb{K} \mid |x - a| < R\}$.

We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} and given $a \in \mathbb{K}$ and $R > 0$, we denote by $\mathcal{A}(d(a, R^-))$ the \mathbb{K} -algebra of power series converging in the disk $d(a, R^-)$. The \mathbb{K} -algebra $\mathcal{A}(\mathbb{K})$ is provided with the topology of uniform convergence in all bounded subsets of \mathbb{K} i.e. the topology of uniform convergence in all disks $d(0, R)$. The neighborhoods of a function $f \in \mathcal{A}(\mathbb{K})$ are the sets $\mathcal{W}(f, r, \epsilon) = \{h \in \mathcal{A}(\mathbb{K}) \mid |f - h|(r) \leq \epsilon\}$, with $r > 0$, $\epsilon > 0$. Similarly, given $a \in \mathbb{K}$ and $R > 0$, the \mathbb{K} -algebra $\mathcal{A}(d(a, R^-))$ is provided with the following topology of \mathbb{K} -algebra: given $f \in \mathcal{A}(d(a, R^-))$, the neighborhoods of f are the sets $\mathcal{W}(f, r, \epsilon) = \{h \in \mathcal{A}(d(a, R^-)) \mid |f - h|(r) \leq \epsilon\}$, with $0 < r < R$, $\epsilon > 0$.

⁰2000 Mathematics Subject Classification: 12J25; 30D35; 30G06.

⁰Keywords: Bezout rings of p-adic analytic functions, meromorphic functions, p-adic value distribution.

We denote by $\mathcal{M}(\mathbb{K})$ the field of fractions of $\mathcal{A}(\mathbb{K})$. The elements of $\mathcal{M}(\mathbb{K})$ are called *meromorphic functions in \mathbb{K}* .

In the same way, given $a \in \mathbb{K}$ and $r > 0$, we denote by $\mathcal{M}(d(a, r^-))$ the field of fractions of $\mathcal{A}(d(a, r^-))$. The elements of $\mathcal{M}(d(a, r^-))$ are called *meromorphic functions in $d(a, r^-)$* . Given a pole b of order q of a meromorphic function $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a, R^-))$), there exists a unique rational function g of the form $\frac{Q(x)}{(x-b)^q}$ with $Q \in \mathbb{K}[x]$, $\deg(Q) \leq q-1$, called *the singular part of f at the pole b* .

The topology defined on $\mathcal{A}(\mathbb{K})$ has expansion to $\mathcal{M}(\mathbb{K})$. The neighborhoods of a function $f \in \mathcal{M}(\mathbb{K})$ are the sets $\mathcal{W}(f, r, \epsilon) = \{h \in \{\mathcal{M}(\mathbb{K}) \mid |f-h|(r) \leq \epsilon\}, \text{ with } r > 0, \epsilon > 0\}$, which implies that f and h have the same singular part at each pole in $d(0, r)$. Similarly, given $a \in \mathbb{K}$ and $R > 0$, the topology defined on $\mathcal{A}(d(a, R^-))$ has expansion to $\mathcal{M}(\mathbb{K})$: given $f \in \mathcal{A}(d(a, R^-))$, the neighborhoods of f are the sets $\mathcal{W}(f, r, \epsilon) = \{h \in \{\mathcal{A}(d(a, R^-)) \mid |f-h|(r) \leq \epsilon\}, \text{ with } 0 < r < R, \epsilon > 0\}$, which implies that f and h have the same singular part at each pole in $d(0, r)$.

We shall define divisors of \mathbb{K} or in a disk $d(a, R^-)$. We then shall define the divisor of an analytic function and of an ideal of an algebra $\mathcal{A}(\mathbb{K})$ or $\mathcal{A}(d(a, r^-))$. Given a divisor T on \mathbb{K} , there is no problem to construct an entire function whose divisor is T . But given a divisor T on a disk $d(a, r^-)$, it is not always possible to find an analytic function (in that disk) whose divisor is T . This is Lazard's problem that we will recall.

Definition: We call a *divisor of \mathbb{K}* (resp. a *divisor of a disk $d(a, R^-)$*) a mapping T from \mathbb{K} (resp. from $d(a, R^-)$) to \mathbb{N} whose support is countable and has a finite intersection with each disk $d(a, r)$, $\forall r > 0$ (resp. $\forall r \in]0, R[$). Thus, a divisor of \mathbb{K} (resp. of $d(a, R^-)$) is characterized by a sequence $(a_n, q_n)_{n \in \mathbb{N}}$ with $a_n \in \mathbb{K}$, $\lim_{n \rightarrow \infty} |a_n| = \infty$, (resp. $a_n \in d(a, R^-)$, $\lim_{n \rightarrow \infty} |a_n - a| = R$), $|a_n| \leq |a_{n+1}|$ and $q_n \in \mathbb{N}^* \forall n \in \mathbb{N}$. So, we will frequently denote a divisor by the sequence $(a_n, q_n)_{n \in \mathbb{N}}$ which characterizes it.

The set of divisors of \mathbb{K} (resp. of $d(a, R^-)$) is provided with a natural multiplicative law that makes it a semi-group. It is also provided with a natural order relation: given two divisors T and T' , we can set $T \leq T'$ when $T(\alpha) \leq T'(\alpha) \forall \alpha \in d(a, R^-)$. Moreover, if T, T' are two divisors such that $T(\alpha) \geq T'(\alpha) \forall \alpha \in d(0, R^-)$, we can define the divisor $\frac{T}{T'}$.

Given $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(a, R^-))$), we can define *the divisor of f* , denoted by $\mathcal{D}(f)$ on \mathbb{K} (resp. of $d(a, R^-)$) as $\mathcal{D}(f)(\alpha) = 0$ whenever $f(\alpha) \neq 0$ and $\mathcal{D}(f)(\alpha) = s$ when f has a zero of order s at α .

Similarly, given an ideal I of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{A}(d(a, R^-))$) we will denote by $\mathcal{D}(I)$ the lower bound of the the $\mathcal{D}(f) \ f \in I$ and $\mathcal{D}(I)$ will be called *the divisor of I* .

Remark: Let $f \in \mathcal{A}(d(a, R^-))$ and let $(a_n, q_n)_{n \in \mathbb{N}} = \mathcal{D}(f)$. Then $\omega_{a_n}(f) = q_n \forall n \in \mathbb{N}$ and $\omega_\alpha(f) = 0 \forall \alpha \in d(a, R^-) \setminus \{a_n \mid n \in \mathbb{N}\}$.

Theorem 1 Let $a \in \mathbb{K}$, $R > 0$. Let $f, g \in \mathcal{A}(\mathbb{K})$ (resp. $f, g \in \text{calA}(d(a, R^-))$) be such that $\mathcal{D}(f) \geq \mathcal{D}(g)$. Then there exists $h \in \mathcal{A}(\mathbb{K})$ (resp. $h \in \mathcal{A}(d(a, R^-))$) such that $f = gh$.

Corollary 1.1 Let $a \in \mathbb{K}$, $R > 0$. Let I be an ideal of $\mathcal{A}(\mathbb{K})$ (resp. an ideal of $\mathcal{A}(d(a, R^-))$) and suppose that there exists $g \in I$ such that $\mathcal{D}(g) = \mathcal{D}(I)$. Then $I = g\mathcal{A}(\mathbb{K})$ (resp. $I = g\mathcal{A}(d(a, R^-))$).

Theorem 2: Let $T = (\alpha_n, q_n)_{n \in \mathbb{N}}$ be a divisor of \mathbb{K} . The infinite product $\prod_{n=1}^{\infty} (1 - \frac{x}{\alpha_n})^{q_n}$ is

uniformly convergent in all bounded subsets of \mathbb{K} and defines an entire function $f \in \mathcal{A}(\mathbb{K})$ such that $\mathcal{D}(f) = T$. Moreover, given $g \in \mathcal{A}(\mathbb{K})$ such that $\mathcal{D}(g) = T$, then g is of the form λf .

Recall the following theorem [5]:

Theorem 3: *Suppose that \mathbb{K} is spherically complete, let $a \in \mathbb{K}$, let $R > 0$ and let T be a divisor of $d(a, R^-)$. There exists $f \in \mathcal{A}(d(a, R^-))$ such that $\mathcal{D}(f) = T$.*

Notation: Given a divisor E of \mathbb{K} , we will denote by $\mathcal{T}(E)$ the ideal of the $f \in \mathcal{A}(\mathbb{K})$ such that $\mathcal{D}(f) \geq E$. Similarly, given $a \in \mathbb{K}$ and $R > 0$ and a divisor E of $d(a, R^-)$, we will denote by $\mathcal{T}_{a,R}(E)$ the ideal of the $f \in \mathcal{A}(d(a, R^-))$ such that $\mathcal{D}(f) \geq E$.

Theorem 4: *For every divisor E of \mathbb{K} , $\mathcal{T}(E)$ is a closed ideal of $\mathcal{A}(\mathbb{K})$. Moreover, \mathcal{T} is a bijection from the set of divisors of \mathbb{K} onto the set of closed ideals of $\mathcal{A}(\mathbb{K})$. Further, given a closed ideal I of $\mathcal{A}(\mathbb{K})$, then $I = \mathcal{T}(\mathcal{D}(I))$.*

Corollary 4.1: *Every closed ideal of $\mathcal{A}(\mathbb{K})$ is principal.*

Proof. Indeed, consider a closed ideal I and let $E = \mathcal{D}(I)$. By Theorem 4 I is of the form $\mathcal{T}(E)$ with $E = \mathcal{D}(I)$. By Theorem 2 there exists $g \in \mathcal{A}(\mathbb{K})$ such that $\mathcal{D}(g) = E$ and of course, g belongs to I . Hence $g\mathcal{A}(\mathbb{K}) \subset I$. Now, let $f \in I$. Then $\mathcal{D}(f) \geq E$, hence by Theorem 1, f factorizes in the form gh with $h \in \mathcal{A}(\mathbb{K})$, hence $I = g\mathcal{A}(\mathbb{K})$. \square

Similarly, we have Theorem 5:

Theorem 5: *Let $a \in \mathbb{K}$ and take $R > 0$. For every divisor E of $d(a, R^-)$, $\mathcal{T}_{a,R}(E)$ is a closed ideal of $\mathcal{A}(d(a, R^-))$. Moreover $\mathcal{T}_{a,R}$ is a bijection from the set of divisors of $d(a, R^-)$ onto the set of closed ideals of $\mathcal{A}(d(a, R^-))$. Further, given a closed ideal I of $\mathcal{A}(d(a, R^-))$, then $I = \mathcal{T}_{a,R}(\mathcal{D}(I))$.*

Corollary 5.1: *Suppose \mathbb{K} is spherically complete. Then all closed ideals of $\mathcal{A}(d(a, R^-))$ are principal.*

Proof. Let I be a closed ideal of $\mathcal{A}(d(a, R^-))$ and let $E = \mathcal{D}(I)$. By Theorem 5 we have $I = \mathcal{T}_{a,R}(E)$. Now, by Theorem 3 there exists $g \in \mathcal{A}(\mathbb{K})$ such that $\mathcal{D}(g) = E$ and of course g belongs to $\mathcal{T}_{a,R}(E)$ hence to I . Consequently, $g\mathcal{A}(d(a, R^-)) \subset I$. Conversely, by Corollary 1.1, we have $I = g\mathcal{A}(d(a, R^-))$. \square

The following Mittag-Leffler for meromorphic functions is similar to the classical Mittag-Leffler theorem for complex meromorphic functions [2] and is quite different from the Mittag-Leffler theorem for analytic elements due to Mark Krasner [5], [3], [4]. The justification given in [6] does not seem relevant.

Theorem 6 (Mittag-Leffler theorem for meromorphic functions): *Let $(a_m, q_m)_{m \in \mathbb{N}}$ be a divisor of \mathbb{K} (resp. of $d(a, R^-)$), $a \in \mathbb{K}$, $R > 0$ and for every $n \in \mathbb{N}$, let $Q_m \in \mathbb{K}[x]$ be of degree $< q_m$. There exists $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a, R^-))$) admitting for poles each a_m of order q_m and no other pole and such that its singular part at a_m is $\frac{Q_m}{(x - a_m)^{q_m}}$.*

Thanks to Theorem 6, we can derive an easy proof of the characterization of meromorphic functions admitting primitives, a property proven in [1] with a longer proof.

Theorem 7: \mathbb{K} is supposed to have characteristic 0. A function $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(a, R^-))$, $a \in \mathbb{K}$, $R > 0$) admits primitives in $\mathcal{M}(\mathbb{K})$ (resp. in $\mathcal{M}(d(a, R^-))$) if and only if all residues of f are null.

Theorem 8: Every ideal of finite type of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{A}(d(a, R^-))$, $a \in \mathbb{K}$, $R > 0$) is closed and is of the form $\mathcal{T}(E)$ (resp. $\mathcal{T}_{a,R}(E)$) with E a divisor of \mathbb{K} (resp. of $d(a, R^-)$).

Corollary 8.1: $\mathcal{A}(\mathbb{K})$ is a Bezout ring.

Proof. Indeed, consider an ideal of finite type I . By Theorem 8, it is closed and hence, by Corollary 4.1 it is principal. \square

And by Corollary 5.1, we have Corollary 8.2

Corollary 8.2: Let $a \in \mathbb{K}$ and let $R > 0$. If \mathbb{K} is spherically complete, $\mathcal{A}(d(a, R^-))$ is a Bezout ring.

Proof. Indeed, consider an ideal of finite type I . By Theorem 8, it is of the form $\mathcal{T}_{a,R}(E)$ with E a divisor of $d(a, R^-)$ and hence, by Theorem 5, it is closed. But then, by Corollary 5.1, it is principal. \square

2 The proofs

Recall that given a closed bounded subset D of \mathbb{K} , we denote by $R(D)$ the \mathbb{K} -algebra of rational function $h \in \mathbb{K}(x)$ with no pole in D . That algebra is provided with the norm of uniform convergence $\| \cdot \|_D$ on D that makes it a normed \mathbb{K} -algebra. We then denote by $H(D)$ the Banach K -algebra completion of $R(D)$ with respect to that norm, whose elements are called *analytic elements on D* .

By Theorem 24.3 in [3], we can derive the following (well known) lemma:

Lemma 1: Let $\alpha \in \overset{\circ}{D}$. Let $q \in \mathbb{N}^*$ and let (g_n) be a sequence of $H(D)$ such that the sequence $(x - \alpha)^q g_n$ converges in $H(D)$. Then the sequence $(g_n)_{n \in \mathbb{N}}$ also converges in $H(D)$.

Now, the proof of Theorem 1 is immediate:

Proof. Let $T = \mathcal{D}(g) = (a_n, q_n)_{n \in \mathbb{N}}$. Let us fix $r > 0$ (resp. $r \in]0, R[$), let $s \in \mathbb{N}$ be such that $|a_n| \leq r \forall n \leq s$ and $|a_n| > r \forall n > s$. Let $P_r(x) = \prod_{n=0}^s (1 - \frac{x^{q_n}}{a_n})$. We can factorise f in the form $P_r \hat{f}$ and similarly, we can factorize g in the form $P_r \hat{g}$, hence $\frac{f}{g} = \frac{\hat{f}}{\hat{g}}$. Since \hat{g} has no zero in $d(0, r)$ it is invertible in $H(d(0, r))$, hence $\frac{f}{g}$ belongs to $H(d(0, r))$. This is true for all $r > 0$ (resp. for all $r \in]0, R[$) and hence $\frac{f}{g}$ belongs to $\mathcal{A}(\mathbb{K})$ (resp. to $\mathcal{A}(d(a, R^-))$). \square

We can now prove Theorem 4 and 5:

Proof. (Theorems 4 and 5) Let E be a divisor of \mathbb{K} (resp. of $d(a, R^-)$). First, let us check that $\mathcal{T}(E)$ (resp. $\mathcal{T}_{a,R}(E)$) is a closed ideal of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{A}(d(a, R^-))$). Let $E = (a_n, q_n)_{n \in \mathbb{N}}$ and let $(f_m)_{m \in \mathbb{N}}$ be a sequence of elements of $\mathcal{T}(E)$ (resp. of $\mathcal{T}_{a,R}(E)$) converging to a limit f in $\mathcal{A}(\mathbb{K})$ (resp. in $\mathcal{A}(d(a, R^-))$). For every $n \in \mathbb{N}$, each f_m admits a_n as a zero of order at least q_n , hence by Lemma 1, so does f . Consequently, f belongs to $\mathcal{T}(E)$ (resp. f belongs to $\mathcal{T}_{a,R}(E)$).

Now, let us show that \mathcal{T} (resp. $\mathcal{T}_{a,R}$) is injective. Let E, F be two distinct divisors of \mathbb{K} (resp. of $d(a, R^-)$). Without loss of generality, we can suppose that E admits a pair (b, s) with $s > 0$ and that F either does not admit any pair (b, m) or admits a pair (b, m) with $m < s$. Let $f \in \mathcal{T}(F)$ (resp. let $f \in \mathcal{T}_{a,R}(F)$) and suppose that $\omega_b(f) \geq s$. Then by Lemma 1 f factorizes in the form $(x - b)^{s-m}g$ with $g \in \mathcal{A}(\mathbb{K})$ (resp. $g \in \mathcal{A}(d(a, R^-))$) and of course g belongs to $\mathcal{T}(F)$ (resp. to $\mathcal{T}_{a,R}(F)$). But by construction, g does not belong to $\mathcal{T}(E)$ (resp. to $\mathcal{T}_{a,R}(E)$) because $\omega_b(g) < s$. Therefore $\mathcal{T}(E) \neq \mathcal{T}(F)$ (resp. $\mathcal{T}_{a,R}(E) \neq \mathcal{T}_{a,R}(F)$). So, \mathcal{T} (resp. $\mathcal{T}_{a,R}$) is injective.

Let us show that it is also surjective. Let I be a closed ideal of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{A}(d(a, R^-))$) and let $E = \mathcal{D}(I)$. Then E is of the form $(a_n, q_n)_{n \in \mathbb{N}}$ with $|a_n| \leq |a_{n+1}|$ and $\lim_{n \rightarrow +\infty} |a_n| = +\infty$ (resp. $\lim_{n \rightarrow +\infty} |a_n| = R$), hence there is a unique $s \in \mathbb{N}$ such that $a_n \in d(0, r) \forall n \leq s$ and $a_n \notin d(0, r) \forall n > s$.

Let $J = \mathcal{T}(E)$ (resp. $J = \mathcal{T}_{a,R}(E)$). Then of course, $I \subset J$. Let us show that $J \subset I$. Let $f \in J$ and take $r > 0$. Denoting by P_r the polynomial $\prod_{i=0}^s (X - a_i)^{q_i}$ by Theorem 27.2, $I \cap H(d(0, r)) = P_r(x)H(d(0, r))$. But now all functions $g \in J \cap H(d(0, r))$ also are of the form $P_r(x)h(x)$ with $h \in H(d(0, r))$. Consequently, in $H(d(0, r))$ we can write f in the form $f = \sum_{j=1}^m g_j h_j$

with $g_j \in I$ and $h_j \in H(d(0, r))$. Let $\epsilon > 0$ be fixed.

For each $j = 1, \dots, m$, narrowing each h_j by a polynomial ℓ_j in $H(d(0, r))$, we can find $\ell_j \in \mathbb{K}[x]$ such that $|g_j(h_j - \ell_j)|(r) \leq \epsilon$. Now, let $\phi_r = \sum_{j=1}^m g_j \ell_j$. Then ϕ_r belongs to I and satisfies $|\phi_r - f|(r) \leq \epsilon$. This is true for each $r > 0$ and for every $\epsilon > 0$. Consequently, since I is closed, f belongs to I . This finishes proving that \mathcal{T} (resp. $\mathcal{T}_{a,R}$) is surjective. Further, we have proven that $I = \mathcal{T}(\mathcal{D}(I))$ (resp. $I = \mathcal{T}_{a,R}(\mathcal{D}(I))$). \square

We will now prove Theorem 6:

Proof. The proof is similar to that in the complex case [2]. Without loss of generality, we can suppose that $|a_m| \leq |a_{m+1}|$. Let $(t(n))_{n \in \mathbb{N}}$ be the strictly increasing sequence such that $|a_{t(n)}| < |a_{t(n+1)}|$ and let $r_n = |a_{t(n)}|$ $n \in \mathbb{N}$.

For each $m \in \mathbb{N}$, set $S_m(x) = \frac{Q_m(x)}{(x - a_m)^{q_m}}$. Now, for each $n \in \mathbb{N}$, we can set $f_n = \sum_{m \in L_n} S_m(x)$.

So, by construction, f_n belongs to $H(d(0, r_{n-1}))$ hence there exists $P_n \in \mathbb{K}[x]$ such that

$$\|f_n - P_n\|_{d(0; r_{n-1})} \leq \left(\frac{1}{2}\right)^n.$$

Consequently, the sequence $(f_n - P_n)_{n \in \mathbb{N}}$ converges to 0 with respect to the topology of $\mathcal{M}(\mathbb{K})$ (resp. of $\mathcal{M}(d(a, R^-))$). Set $f(x) = f_1(x) + \sum_{n=2}^{\infty} (f_n(x) - P_n(x))$. By construction, f belongs to $\mathcal{M}(d(0, r^-)) \forall r > 0$, hence f belongs to $\mathcal{M}(\mathbb{K})$ (resp. to $\mathcal{M}(d(a, R^-))$). Moreover, the poles of f are the points a_m , $m \in \mathbb{N}$. Let us take $q \geq 1$ and $\rho > 0$ such that $|a_m - a_q| \geq \rho \forall m \neq q$. Then $f - S_q$ belongs to $H(d(a, \rho))$, so S_q is the singular part of f at S_q . \square

Proof of Theorem 7:

Proof. Let a be a pole of f . According to the Laurent series of f at a , if f admits primitives then f has no residue different from zero at a because the function $\frac{1}{x-a}$ has no primitive in $\mathcal{M}(d(a, r))$ (whenever $r > 0$). Now let $(a_m)_{m \in \mathbb{N}}$ be the sequence of poles of f , each of respective degree q_m and suppose that $\text{res}(f, a_m) = 0$. Since $\text{res}_{a_m}(f) = 0$, the singular part of f at a_m is of the form $\frac{Q_m(x - a_m)}{(x - a_m)^{q_m}}$ with $q_m \geq 2$ and $Q_m(X)$ is a polynomial of degree $\leq q_m - 2$. Consequently, the singular part of f at a_m admits a primitive of the form $\frac{P_m(x - a_m)}{(x - a_m)^{q_m - 1}}$ with $\deg(P_m(X)) \leq q_m - 2$. Then by Theorem 6, there exists $G \in \mathcal{M}(K)$ (resp. $G \in \mathcal{M}(d(a, R^-))$) admitting the a_m for poles with respective singular part $\frac{P_m}{(x - a_m)^{q_m - 1}}$ and no other pole. By construction, for each $m \in \mathbb{N}$, the singular part of G' at a_m is $\frac{Q_m(x - a_m)}{(x - a_m)^{q_m}}$ hence $G' - f$ has no pole at a_m and hence has no pole in \mathbb{K} (resp. in $d(a, R^-)$). Consequently, $G' - f$ belongs to $\mathcal{A}(\mathbb{K})$ (resp. to $\mathcal{A}(d(a, R^-))$). But then $G' - f$ admits a primitive $L \in \mathcal{A}(\mathbb{K})$ (resp. $L \in \mathcal{A}(d(a, R^-))$) and hence the function $F = G - L$ is a primitive of f that belongs to $L \in \mathcal{M}(\mathbb{K})$ (resp. to $L \in \mathcal{M}(d(a, R^-))$). \square

In the proof of Theorem 8 we will need the following lemma

Lemma 2: *Let E be a divisor of \mathbb{K} (resp. a divisor of $d(a, R^-)$, $a \in \mathbb{K}$, $R > 0$) and for each $r > 0$ (resp. $r \in]9, R[$), let $g_r \in H(C(0, r))$. There exists $g \in \mathcal{A}(\mathbb{K})$ (resp. $g \in \mathcal{A}(d(a, R^-))$), not depending on r , such that $\mathcal{D}(g - g_r) \geq E_r$.*

Proof. Let $f \in \mathcal{A}(\mathbb{K})$ (resp. $f \in \mathcal{A}(d(a, R^-))$) be such that $\mathcal{D}(f) \geq E$. By Theorem 6, there exists $F \in \mathcal{M}(\mathbb{K})$ whose principal parts at the poles located in $C(0, r)$ are respectively the same as those of $g_r f^{-1}$ for each $r > 0$. Then fF belongs to $\mathcal{A}(\mathbb{K})$ (resp. to $\mathcal{A}(d(a, R^-))$). Putting $g = fF$, we can see that $\mathcal{D}(g - g_r) \geq E_r$, which ends the proof. \square

We can now prove Theorem 8:

Proof. Let I be an ideal of finite type of $\mathcal{A}(\mathbb{K})$ (resp. of $\mathcal{A}(d(a, R^-))$) generated by f_1, \dots, f_q and let $E = \mathcal{D}(I)$. By Theorem 4 the closure J of I is $\mathcal{T}(E)$ (resp. by Theorem 5 the closure J of I is $\mathcal{T}_{a,R}(E)$). Consequently, we can see that $E = \min(\mathcal{D}(f_1), \dots, \mathcal{D}(f_q))$. Let us fix $r > 0$. In $H(C(0, r))$, there exist $g_{1,r}, \dots, g_{q,r} \in H(C(0, r))$ such that $g = \sum_{j=1}^q g_{j,r} f_j$. For each $j = 2, \dots, q$, let $f_{j,r}$ be the polynomial of the zeros of f_j in $C(0, r)$. By Lemma 2 there exists $g_j \in \mathcal{A}(\mathbb{K})$ (resp. $g_j \in \mathcal{A}(d(a, R^-))$), not depending on r , such that $g_{j,r} - g_j$ be divisible in $H(C(0, r))$ by $\mathcal{D}(f_{1,r})$.

Now, set $h = g - \sum_{j=2}^q g_j f_j$. We have $\mathcal{D}(h) \geq \mathcal{D}(f_1)$ hence h factorizes in the form $g_1 f_1$ with $g_1 \in \mathcal{A}(\mathbb{K})$ (resp. $g_1 \in \mathcal{A}(d(a, R^-))$) and then

$$g = h + \sum_{j=2}^q g_j f_j = \sum_{j=1}^q g_j f_j.$$

□

References

- [1] Boussaf, K., Escassut, A. and Ojeda, J. *Primitives of p -adic meromorphic functions*, Contemporary Mathematics., Vol. 551, p.51-56 (2011).
- [2] Conway, J. B. *Functions of One Complex Variable*. Graduate Text in Mathematics, Springer-Verlag, (1973).
- [3] Escassut, A. *Analytic Elements in p -adic Analysis*, World Scientific Publishing Co. Pte. Ltd. Singapore, (1995).
- [4] Escassut, A. *p -adic Value Distribution*, Some Topics on Value Distribution and Differentiability in Complex and p -adic Analysis, p. 42- 138. Mathematics Monograph, Series 11. Science Press.(Beijing 2008).
- [5] Krasner, M. *Prolongement analytique uniforme et multiforme dans les corps valués complets. Les tendances géométriques en algèbre et théorie des nombres*, Clermont-Ferrand, p.94-141 (1964). Centre National de la Recherche Scientifique (1966), (Colloques internationaux de C.N.R.S. Paris, 143).
- [6] Lazard, M. *Les zéros des fonctions analytiques sur un corps valué complet*, IHES, Publications Mathématiques no. 14, pp. 47-75 (1962).

Laboratoire de Mathématiques, UMR 6620
 Université Blaise Pascal
 24 Avenue des Landais
 63171 AUBIERE-CEDEX
 FRANCE
 bertin.diarra@math.univ-bpclermont.fr
 alain.escassut@math.univ-bpclermont.fr