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Survey and additional properties on p -adic meromorphic functions $f'P'(f)$, $g'P'(g)$ sharing a small function

Kamal Boussaf, Alain Escassut and Jacqueline Ojeda¹

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Abstract

Let \mathbb{K} be a complete algebraically closed p -adic field of characteristic zero. Let f, g be two transcendental meromorphic functions in the whole field \mathbb{K} or meromorphic functions in an open disk that are not quotients of bounded analytic functions. Let P be a polynomial of uniqueness for meromorphic functions in \mathbb{K} or in an open disk and let α be a small meromorphic function with regards to f and g . Here we present the following results: if $f'P'(f)$ and $g'P'(g)$ share α counting multiplicity, then we show that $f = g$ provided that the multiplicity order of zeros of P' satisfy certain inequalities. If α is a Moebius function or a non-zero constant, we can obtain more general results on P . Further, when f, g are entire analytic functions or analytic functions inside an open disk, we can obtain a new result improving that published by the third author

1 Introduction and Main Results

Let f, g be two meromorphic functions in a p -adic field. Here we study polynomials P such that, when $f'P'(f)$ and $g'P'(g)$ share a small function α , then $f = g$. Problems of uniqueness on meromorphic functions were examined first in \mathbb{C} [10], [11], [13], [16], [17], [18], [19], [20], [25], [26], [27] and next in a p -adic field [1], [5], [6], [7], [14], [15], [21], [22], [23], [24]. After examining problems of the form $P(f) = P(g)$, several studies were made on the equality $f'P'(f) = g'P'(g)$, or value sharing questions: if $f'P'(f)$ and $g'P'(g)$ share a value, or a small function, do we have $f = g$? Here we recall results previously obtained no matter what the number of zeros of P' . Results also apply to meromorphic functions inside an open disk. They were published in [2]. Moreover, here we will examine the particular case of analytic functions and will improve results of [23].

Let \mathbb{K} be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by $|\cdot|$. We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} , by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} , i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$ and by $\mathbb{K}(x)$ the field of rational functions.

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Let $a \in \mathbb{K}$ and $R \in]0, +\infty[$. We denote by $d(a, R)$ the closed disk $\{x \in \mathbb{K} : |x - a| \leq R\}$ and by $d(a, R^-)$ the "open" disk $\{x \in \mathbb{K} : |x - a| < R\}$. We denote by $\mathcal{A}(d(a, R^-))$ the set of analytic functions in $d(a, R^-)$, i.e. the \mathbb{K} -algebra of power series $\sum_{n=0}^{\infty} a_n(x - a)^n$ converging in $d(a, R^-)$ and by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions inside $d(a, R^-)$, i.e. the field of fractions of $\mathcal{A}(d(a, R^-))$. Moreover, we denote by $\mathcal{A}_b(d(a, R^-))$ the \mathbb{K} -subalgebra of $\mathcal{A}(d(a, R^-))$ consisting of the bounded analytic functions in $d(a, R^-)$, i.e. which satisfy $\sup_{n \in \mathbb{N}} |a_n|R^n < +\infty$. And we denote by $\mathcal{M}_b(d(a, R^-))$ the field of fractions of $\mathcal{A}_b(d(a, R^-))$. Finally, we denote by $\mathcal{A}_u(d(a, R^-))$ the set of unbounded analytic functions in $d(a, R^-)$, i.e. $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$. Similarly, we set $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$.

The problem of value sharing a small function by functions of the form $f'P'(f)$ was examined first when P was just of the form x^n [10], [21], [27]. More recently it was examined when P was a polynomial such that P' had exactly two distinct zeros [18], [20], [23], both in complex analysis and in p -adic analysis. In [18], [20] the functions where meromorphic on \mathbb{C} , with a small function that was a constant or the identity. In [23], the problem was considered for analytic functions in the field \mathbb{K} : on one hand for entire functions and on the other hand for unbounded analytic functions in an open disk.

Actually solving a value sharing problem involving $f'P'(f)$, $g'P'(g)$ requires to know polynomials of uniqueness P for meromorphic functions.

In [23] the third author studied several problems of uniqueness and particularly the following:

Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental (resp. Let $f, g \in \mathcal{A}_u(d(0, R^-))$) and $\alpha \in \mathcal{A}(\mathbb{K})$ (resp. $\alpha \in \mathcal{A}_u(d(0, R^-))$) be a small function, such that $f^n(f - a)^k f'$ and $g^n(g - a)^k g'$ share α , counting multiplicity, with $n, k \in \mathbb{N}$ and $a \in \mathbb{K} \setminus \{0\}$ (see Theorems D and E below).

Here we consider functions $f, g \in \mathcal{M}(\mathbb{K})$ or $f, g \in \mathcal{M}(d(a, R^-))$ and ordinary polynomials P : we must only assume certain hypotheses on the multiplicity order of the zeros of P' . The method for the various theorems we will show is the following: assuming that $f'P'(f)$ and $g'P'(g)$ share a small function, we first prove that $f'P'(f) = g'P'(g)$. Next, we derive $P(f) = P(g)$. And then, when P is a polynomial of uniqueness for the functions we consider, we can conclude $f = g$.

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field \mathbb{K} and a few specific properties of ultrametric analytic or meromorphic functions [4].

Let \log be a real logarithm function of base $b > 1$ and let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0. Let $r \in]0, +\infty[$ (resp. $r \in]0, R[$) and let $\gamma \in d(0, r)$. If f has a zero of order n at γ , we put $\omega_\gamma(h) = n$. If f has a pole of order n at γ , we put $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$.

We denote by $Z(r, f)$ the *counting function of zeros of f in $d(0, r)$* , counting multiplicity, i.e.

$$Z(r, f) = \max(\omega_0, 0) \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by $\overline{Z}(r, f)$ the *counting function of zeros of f in $d(0, r)$* , ignoring multiplicity, and set

$$\overline{Z}(r, f) = u \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} (\log r - \log |\gamma|)$$

with $u = 1$ when $\omega_0(f) > 0$ and $u = 0$ else.

In the same way, we set $N(r, f) = Z\left(r, \frac{1}{f}\right)$ (resp. $\overline{N}(r, f) = \overline{Z}\left(r, \frac{1}{f}\right)$) to denote the *counting function of poles of f* in $d(0, r)$, counting multiplicity (resp. ignoring multiplicity).

For $f \in \mathcal{M}(d(0, R^-))$ having no zero and no pole at 0, the *Nevanlinna function* is defined by $T(r, f) = \max\{Z(r, f), N(r, f)\}$.

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

Definition. Let $f \in \mathcal{M}(\mathbb{K})$ (resp. let $f \in \mathcal{M}(d(0, R^-))$) such that $f(0) \neq 0, \infty$. A function $\alpha \in \mathcal{M}(\mathbb{K})$ (resp. $\alpha \in \mathcal{M}(d(0, R^-))$) having no zero and no pole at 0 is called a *small function with respect to f* , if it satisfies $\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0$ (resp. $\lim_{r \rightarrow R^-} \frac{T(r, \alpha)}{T(r, f)} = 0$).

If 0 is a zero or a pole of f or α , we can make a change of variable such that the new origin is not a zero or a pole for both f and α . Thus it is easily seen that the last relation does not really depend on the origin.

We denote by $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) the set of small meromorphic functions with respect to f in \mathbb{K} (resp. in $d(0, R^-)$).

Remark 1. Thanks to classical properties of the Nevanlinna function $T(r, f)$ with respect to the operations in a field of meromorphic functions, such as $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$ and $T(r, fg) \leq T(r, f) + T(r, g) + O(1)$, for $f, g \in \mathcal{M}(\mathbb{K})$ and $r > 0$, it is easily proved that $\mathcal{M}_f(\mathbb{K})$ (resp. $\mathcal{M}_f(d(0, R^-))$) is a subfield of $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0, R^-))$) and that $\mathcal{M}(\mathbb{K})$ (resp. $\mathcal{M}(d(0, R^-))$) is a transcendental extension of $\mathcal{M}_f(\mathbb{K})$ (resp. of $\mathcal{M}_f(d(0, R^-))$) [9].

Let us remember the following definition.

Definition. Let $f, g, \alpha \in \mathcal{M}(\mathbb{K})$ (resp. let $f, g, \alpha \in \mathcal{M}(d(0, R^-))$). We say that f and g share the function α C.M., if $f - \alpha$ and $g - \alpha$ have the same zeros with the same multiplicity in \mathbb{K} (resp. in $d(0, R^-)$).

Recall that a polynomial $P \in \mathbb{K}[x]$ is called a *polynomial of uniqueness* for a class of functions \mathcal{F} if for any two functions $f, g \in \mathcal{F}$ the property $P(f) = P(g)$ implies $f = g$.

The definition of polynomials of uniqueness was introduced in [19] by P. Li and C. C. Yang and was studied in many papers [12], [13] for complex functions and [1], [6], [7], [14], [15], [17], [24], for p -adic functions.

Actually, in a p -adic field, we can obtain various results, not only for functions defined in the whole field \mathbb{K} but also for functions defined inside an open disk because the p -adic Nevanlinna Theory works inside a disk, for functions of $\mathcal{M}_u(d(0, R^-))$.

The following theorem is classical for analytic functions [26] and was proved for meromorphic functions in [7] and in [14] but it is also a consequence of Theorem 1 in [24].

Theorem A: Let $P \in \mathbb{K}[x]$ be such that P' has exactly two distinct zeros γ_1 of order c_1 and γ_2 of order c_2 . Then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. Moreover, if $\min\{c_1, c_2\} \geq 2$, then P is a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$.

Another way to obtain polynomials of uniqueness was given in [7]. It also applies to meromorphic functions inside an open disk.

Notation: Let $P \in \mathbb{K}[x] \setminus \mathbb{K}$ and let $\Xi(P)$ be the set of zeros c of P' such that $P(c) \neq P(d)$ for every zero d of P' other than c . We denote by $\Phi(P)$ the cardinal of $\Xi(P)$.

Remark 2. If $\deg(P) = q$ then $\Phi(P) \leq q - 1$.

From [7] we have the following results:

Theorem B: Let $d(a, R^-)$ be an open disk in \mathbb{K} and $P \in \mathbb{K}[x]$. If $\Phi(P) \geq 2$ then P is a polynomial of uniqueness for $\mathcal{A}(\mathbb{K})$. If $\Phi(P) \geq 3$ then P is a polynomial of uniqueness for both $\mathcal{A}_u(d(a, R^-))$ and $\mathcal{M}(\mathbb{K})$. If $\Phi(P) \geq 4$ then P is a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$.

We can find the statements of Theorem C in [23]:

Theorem C: Let $P \in K[x]$ such that P' admits exactly 2 distinct zeros of respective order n and k .

- i) Suppose $k = 2$, $n \geq 3$. Then P is a polynomial of uniqueness for $\mathcal{M}_u(d(0, R^-))$.
- ii) Suppose $k = 1$, $n \geq 1$. Then P is a polynomial of uniqueness for $\mathcal{A}_u(d(0, R^-))$.

In [23], the third author proved the following theorems D and E concerning entire functions and analytic functions in a disk when the polynomial P has only two distinct zeros:

Theorem D: Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $f^n(f-a)^k f'$ and $g^n(g-a)^k g'$ share the function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ C.M. with $n, k \in \mathbb{N}$ and $a \in \mathbb{K} \setminus \{0\}$. If $n \geq \max\{6-k, k+1\}$, then $f = g$. Moreover, if $\alpha \in \mathbb{K} \setminus \{0\}$ and $n \geq \max\{5-k, k+1\}$, then $f = g$.

Theorem E: Let $f, g \in \mathcal{A}_u(d(0, R^-))$, let $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$ and let $a \in \mathbb{K} \setminus \{0\}$. If $f^n(f-a)^2 f'$ and $g^n(g-a)^2 g'$ share the function α C.M. and $n \geq 4$, then $f = g$. Moreover, if $f^n(f-a)f'$ and $g^n(g-a)g'$ share the function α C.M. and $n \geq 5$, then again $f = g$.

We can now state our main results and we will give a proof for those that are not published

Theorem F: Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x-a_1)^n \prod_{i=2}^l (x-a_i)^{k_i}$

with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ n &\geq k + 2, \\ \text{if } l &= 2, \text{ then } n \neq 2k, 2k+1, 3k+1, \\ \text{if } l &= 3, \text{ then } n \neq 2k+1, 3k_i - k \ \forall i = 2, 3. \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary F.1:

Corollary F.1: Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b(x-a_1)^n \prod_{i=2}^l (x-a_i)^{k_i}$ with $b \in \mathbb{K}^*$,

$l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ n &\geq k + 2, \end{aligned}$$

if $l = 3$, then $n \neq 2k + 1$, $3k_i - k \ \forall i = 2, 3$.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem A we also have Corollary F.2.

Corollary F.2: Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $\min(k, n) \geq 2$.

Suppose P satisfies the following conditions:

$$n \geq 10 + \max(0, 5 - k),$$

$$n \geq k + 2,$$

$$n \neq 2k, 2k + 1, 3k + 1.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem G: Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$n \geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

$$\text{if } l = 2, \text{ then } n \neq 2k, 2k + 1, 3k + 1,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary G.1.

Corollary G.1: Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l - 1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$n \geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem A, we have Corollary G.2.

Corollary G.2: Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $\min(k, n) \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k),$$

$$n \geq k + 2,$$

$$n \neq 2k, 2k + 1, 3k + 1,$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem H: Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2, \\ n &\geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary H.1.

Corollary H.1: Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 3$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2, \\ n &\geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem A, we have Corollary H.2

Corollary H.2: Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^k$ with $k \geq 2$ and with $b \in \mathbb{K}^*$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq 9 + \max(0, 5 - k), \\ n &\geq k + 2, \\ n &\neq 2k, 2k + 1, 3k + 1. \end{aligned}$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem I: Let $a \in K$ and $R > 0$. Let P be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$ and let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 2$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ n &\geq k + 3, \\ \text{if } l = 2, &\text{ then } n \neq 2k, 2k + 1, 3k + 1, \\ \text{if } l = 3, &\text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3. \end{aligned}$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B we can state Corollary I.1.

Corollary I.1: Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 4$, let $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$ with $b \in \mathbb{K}^*$, $l \geq 4$, $k_i \geq k_{i+1}$, $2 \leq i \leq l-1$ and let $k = \sum_{i=2}^l k_i$. Suppose P satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 3,$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

And by Theorem C we have Corollary I.2:

Corollary I.2: Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ be such that P' is of the form $b(x - a_1)^n(x - a_2)^2$ with $b \in \mathbb{K}^*$. Suppose P satisfies

$$n \geq 10 + \max(0, 5 - k).$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem J: Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form

$$b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^*, \text{ satisfying:}$$

$$n \geq l + 10,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary J.1:

Corollary J.1: Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 3$ and be such that P' is of the form

$$b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying:}$$

$$n \geq l + 10,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem K: Let $a \in K$ and $R > 0$. Let P be a polynomial of uniqueness for $\mathcal{M}_u(d(a, R^-))$

such that P' is of the form $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)$ with $l \geq 3$, $b \in \mathbb{K}^*$ satisfying:

$$n \geq l + 10,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

By Theorem B, we have Corollary K.1:

Corollary K.1: Let $a \in K$ and $R > 0$. Let $P \in \mathbb{K}[x]$ satisfy $\Phi(P) \geq 4$ and be such that P' is of

the form $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)$ with $l \geq 4$, $b \in \mathbb{K}^*$ and $n \geq l + 10$.

Let $f, g \in \mathcal{M}_u(d(a, R^-))$ and let $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$ be non-identically zero. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let $P(x) = \frac{x^{18}}{18} - \frac{2x^{17}}{17} - \frac{x^{16}}{16} + \frac{2x^{15}}{15}$. Then $P'(x) = x^{17} - 2x^{16} - x^{15} + 2x^{14} = x^{14}(x-1)(x+1)(x-2)$. We check that:

$$P(0) = 0,$$

$$P(1) = \frac{1}{18} - \frac{2}{17} - \frac{1}{16} + \frac{2}{15},$$

$$P(-1) = \frac{1}{18} + \frac{2}{17} - \frac{1}{16} - \frac{2}{15} \neq 0, P(1), \text{ and } P(2) = \frac{2^{18}}{18} - \frac{2^{18}}{17} - \frac{2^{16}}{16} + \frac{2^{16}}{15} \neq 0, P(1), P(-1).$$

Then $\Phi(P) = 4$. So, P is a polynomial of uniqueness for both $\mathcal{M}(\mathbb{K})$ and $\mathcal{M}(d(0, R^-))$. Moreover, we have $n = 14, l = 4$, hence we can apply Corollaries J.1 and K.1.

Given $f, g \in \mathcal{M}(\mathbb{K})$ transcendental or $f, g \in \mathcal{M}_u(d(0, R^-))$ such that $f'P'(f)$ and $g'P'(g)$ share a small function α C.M., we have $f = g$.

Theorem L: Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form

$$P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying}$$

$$n \geq l + 9,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a Moebius function. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Theorem M: Let P be a polynomial of uniqueness for $\mathcal{M}(\mathbb{K})$ such that P' is of the form

$$P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying } n \geq l + 9.$$

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let α be a non-zero constant. If $f'P'(f)$ and $g'P'(g)$ share α C.M., then $f = g$.

Example: Let $P(x) = x^q - ax^{q-2} + b$ with $a \in K^*, b \in K$, with $q \geq 5$ an odd integer. Then q and $q - 2$ are relatively prime and hence by Theorem 3.21 [15] P is a uniqueness polynomial for $\mathcal{M}(\mathbb{K})$ and P' admits 0 as a zero of order $n = q - 3$ and two other zeros of order 1.

Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}(\mathbb{K})$ be a small function such that f, g share α C.M.

Suppose first $q \geq 17$. By Theorem J we have $f = g$. Now suppose $q \geq 15$ and suppose α is a Moebius function or a non-zero constant. Then by Theorem L and M, we have $f = g$.

Theorem N: Let $f, g \in \mathcal{M}(\mathbb{K})$ be transcendental and let $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f - a)$ and $g'g^n(g - a)$ share the function α C.M. and if $n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(\mathbb{K})$ such that $f = \frac{a(n+2)}{n+1} \left(\frac{h^{n+1} - 1}{h^{n+2} - 1} \right) h$ and $g = \frac{a(n+2)}{n+1} \left(\frac{h^{n+1} - 1}{h^{n+2} - 1} \right)$. Moreover, if α is a constant or a Moebius function, then the conclusion holds whenever $n \geq 11$.

Inside an open disk, we have a version similar to the general case in the whole field.

Theorem O: Let $f, g \in \mathcal{M}_u(d(0, R^-))$, and let $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$ be non-identically zero. Let $a \in \mathbb{K} \setminus \{0\}$. If $f'f^n(f-a)$ and $g'g^n(g-a)$ share the function α C.M. and $n \geq 12$, then either $f = g$ or there exists $h \in \mathcal{M}(d(0, R^-))$ such that $f = \frac{a(n+2)}{n+1} \left(\frac{h^{n+1}-1}{h^{n+2}-1} \right) h$ and $g = \frac{a(n+2)}{n+1} \left(\frac{h^{n+1}-1}{h^{n+2}-1} \right)$.

Remark 3. In Theorems N and O, the second conclusion does occur. Indeed, let $h \in \mathcal{M}(\mathbb{K})$ (resp. let $h \in \mathcal{M}_u(d(0, R^-))$). Now, let us precisely define f and g as: $g = \left(\frac{n+2}{n+1} \right) \left(\frac{h^{n+1}-1}{h^{n+2}-1} \right)$ and $f = hg$. Then by Remark 1 we can see that the polynomial $P(y) = \frac{1}{n+2} y^{n+2} - \frac{1}{n+1} y^{n+1}$ satisfies $P(f) = P(g)$, hence $f'P'(f) = g'P'(g)$, therefore $f'P'(f)$ and $g'P'(g)$ trivially share any function.

Remark 4. All theorems above have suggested results on complex meromorphic functions, with figures that are just slightly less fine [3].

In order to prove Theorem P, we will prove the following three lemmas:

Lemma 1: Let $R \in \mathbb{R}_+^*$ and let $f, g \in \mathcal{A}(d(a, R^-))$. Then fg belongs to $\mathcal{A}_b(d(a, R^-))$ if and only if so do both f, g .

Lemma 3 is known in the Nevanlinna Theory [4], [8].

Lemma 2: Let $f \in \mathcal{M}(K)$ (resp. $f \in \mathcal{M}(d(0, R^-))$) have no zero and no pole at 0. Let $G(Y) \in \mathcal{M}_f(K)(Y)$, (resp. $G \in \mathcal{M}_f(d(0, R^-))(Y)$), let $n = \deg(G)$. Then $T(r, G(f)) = nT(r, f) + o(T(r, f))$.

Lemma 3: Let $P \in K[x]$ with $\deg(P) > 1$ and let $f, g \in \mathcal{A}(K) \setminus K$ (resp. $f, g \in \mathcal{A}_u(d(a, R^-))$) be such that $P(f) = P(g) + c$, $c \in K$. Then $c = 0$.

Proof: Let $P(x) = \sum_{k=0}^n a_k x^k$ with $a_n \neq 0$. For each $k = 1, \dots, n-1$, let $Q_k(x, y) = a_k \sum_{j=0}^k x^j y^{k-j}$. Then $P(x) - P(y) = (x-y)(\sum_{k=1}^{n-1} Q_k(x, y))$. Suppose first $f, g \in \mathcal{A}(K)$. Since $(f-g)(\sum_{k=1}^{n-1} Q_k(f, g))$ is a constant, we know that both $f-g$, $\sum_{k=1}^{n-1} Q_k(f, g)$ are constants. Thus we have $g = f + b$ with $b \in K$. Let $G(x) = \sum_{k=1}^{n-1} Q_k(x, x+b)$. We can check that G is a polynomial of degree $n-1$. And since $G(f)$ is a constant, we have $n-1 = 0$, a contradiction. Thus, $f = g$.

Similarly, suppose now $f, g \in \mathcal{A}_u(d(a, R^-))$. By Lemma 1 both $f-g, \sum_{k=1}^{n-1} Q_k(f, g)$ are bounded, so we have $g = f + h$, with $h \in \mathcal{A}_b(d(a, R^-))$. Consider the polynomial $B(x) = \sum_{k=1}^{n-1} Q_k(x, x+h) \in \mathcal{M}_b(d(a, R^-))(x)$. Then $B(x)$ is a polynomial with coefficients in $\mathcal{M}_b(d(a, R^-))$ and $\deg(B)$ is $n-1$ hence by Lemma 2 we have $T(r, B(f)) = (n-1)T(r, f) + S_f(r)$. But since $B(f)$ is bounded, it belongs to $\mathcal{M}_b(d(a, R^-))(x)$, hence $n = 1$, a contradiction again.

Theorem P: Let $P(x) \in K[x]$ be a polynomial of uniqueness for $\mathcal{A}(K)$ (resp. for $\mathcal{A}(d(a, R^-))$), let $P'(x) = \prod_{i=1}^l (x - a_i)^{k_i}$ and let $f, g \in \mathcal{A}(K)$ be transcendental (resp. let $f, g \in \mathcal{A}_u(d(a, R^-))$)

such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(K) \cap \mathcal{A}_g(K)$ (resp. $\alpha \in \mathcal{A}_f(d(, R^-)) \cap \mathcal{A}_g(d(a, R^-))$). If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$. Moreover, if f, g belong to $\mathcal{A}(K)$, if α is a constant and if $\sum_{i=1}^l k_i \geq 2l + 1$ then $f = g$.

Proof: Set $F = \frac{f'P'(f)}{\alpha}$ and $G = \frac{g'P'(g)}{\alpha}$, so F and G share the value 1 C.M. Since F, G share α C.M., the function $\frac{F - \alpha}{G - \alpha}$ has no zero and no pole, hence it is a constant $\phi \in K^*$ (resp. an invertible function $\phi \in \mathcal{A}(d(a, R^-))$). Suppose $\phi \neq 1$. We have $F - \alpha = \phi(G - \alpha)$, hence $F = \phi G + \alpha(1 - \phi)$. Now applying the Second main Theorem to F we have:

$$T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, G) + o(T(r, F)). \quad (1)$$

Now, let $m = \sum_{i=1}^l k_i$. By Lemma 2 we have $T(r, F) \geq mT(r, f) + o(T(r, f))$, $T(r, G) \geq mT(r, g) + o(T(r, g))$. Consequently, since f, g are transcendental (resp. unbounded), so are F and G .

On the other hand, since $T(r, f') \leq T(r, f) - \log r + O(1)$, $T(r, g') \leq T(r, g) - \log r + O(1)$, we have $\overline{Z}(r, F) + \overline{Z}(r, G) \leq (l + 1)(T(r, f) + T(r, g)) - 2 \log r + o(T(r, f))$. We also notice that $o(T(r, F)) = o(T(r, f))$, $o(T(r, G)) = o(T(r, g))$. Consequently, by (1), we obtain $mT(r, f) \leq l(T(r, f) + T(r, g)) + o(T(r, f))$ and similarly, $mT(r, g) \leq l(T(r, f) + T(r, g)) + o(T(r, g))$. So,

$$m(T(r, f) + T(r, g)) \leq (2l + 1)(T(r, f) + T(r, g)) + o(T(r, f) + T(r, g)). \quad (2)$$

Thus, by (2) we have $m \leq 2l + 2$. Moreover, we notice that if $f, g \in \mathcal{A}(K)$ and if α is a constant, (1) gets

$$T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, G) - 2 \log r + O(1)$$

hence $m \leq 2l$.

Thus, if $m \geq 2l + 2$, or if $f, g \in \mathcal{A}(K)$ and if α is a constant and $m \geq 2l + 1$, we have $\phi = 1$. We can then assume that $\phi = 1$, therefore $f'P'(f) = g'P'(g)$ and hence $P(f) - P(g)$ is a constant $b \in K$. But then, by Lemma 3, $b = 0$. Finally, since P is a polynomial of uniqueness for $\mathcal{A}(K)$ (resp. $\mathcal{A}(d(0, R^-))$), we can conclude $f = g$.

Corollary P.1: Let $P(x) \in K[x]$ be such that $\Upsilon(P) \geq 2$, let $P'(x) = \prod_{i=1}^l (x - a_i)^{k_i}$ and let $f, g \in \mathcal{A}(K)$ be transcendental such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(K) \cap \mathcal{A}_g(K)$. If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$. Moreover, if α is a constant and if $\sum_{i=1}^l k_i \geq 2l + 1$ then $f = g$.

Corollary P.2: Let $P(x) \in K[x]$ be such that $\Upsilon(P) \geq 3$, let $P'(x) = \prod_{i=1}^l (x - a_i)^{k_i}$ and let $f, g \in \mathcal{A}_u(d(a, R^-))$ be such that $f'P'(f)$ and $g'P'(g)$ share a small function $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$. If $\sum_{i=1}^l k_i \geq 2l + 2$ then $f = g$.

Theorem Q: Let $f, g \in \mathcal{A}(\mathbb{K})$ be transcendental such that $(f - a)^n(f - b)^k f'$ and $(g - a)^n(g - b)^k g'$ share the function $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$ C.M. with $n, k \in \mathbb{N}^*$ and $a \in \mathbb{K}^* \setminus \{0\}$. Then $f = g$.

Proof: Without loss of generality, we can assume $a = 0$ and $b = 1$. Let P be the primitive of $x^n(x - 1)^k$ that admits 0 as a zero of order $n + 1$. By Lemma 3, we have $P(f) = P(g)$. Now, we can check that $P(1) \neq 0$. Indeed, if 1 is a zero of P , then it is a zero of order $k + 1$ and hence $\deg(P) = n + k + 2$, a contradiction. Consequently, $\Phi(P) = 2$ and therefore, by Theorem B, P is a polynomial of uniqueness for $\mathcal{A}(K)$. Hence $f = g$.

Theorem R: Let $f, g \in \mathcal{A}_u(d(0, R^-))$, let $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$ and let $a, b \in K \setminus \{0\}$, $a \neq b$. If $(f - a)^n(f - b)^2 f'$ and $(g - a)^n(g - b)^2 g'$ share the function α C.M. and $n \geq 3$, then $f = g$. Moreover, if $(f - a)^n(f - b) f'$ and $(g - a)^n(g - b) g'$ (with $n \geq 1$) share the function α C.M. and if $n \geq 1$, then again $f = g$.

Proof: Without loss of generality, we can assume $a = 0$ and $b = 1$. Let P be the primitive of $x^n(x - 1)^k$ such that $P(0) = 0$. If $k = 2$ and $n \geq 3$, we can apply Theorem C showing that P is a polynomial of uniqueness for $\mathcal{A}_u(d(0, R^-))$ and then, by Theorem P we have $f = g$.

And now suppose $k = 1$. Then $P'(x)$ has two distinct zeros, one of order 1 and hence by Theorem C, P is a polynomial of uniqueness for $\mathcal{A}_u(d(0, R^-))$. Consequently, by Theorem P, we have $f = g$.

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