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Kamal Boussaf, Alain Escassut, Jacqueline Ojeda. Survey and additional properties on p-adic meromorphic functions  $f \in \mathcal{P}^*(f)$ ,  $g \in \mathcal{P}^*(g)$  sharing a small function. Contemporary mathematics, 2013. hal-01907057

**HAL Id: hal-01907057**

**<https://uca.hal.science/hal-01907057v1>**

Submitted on 28 Oct 2018

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# Survey and additional properties on $p$ -adic meromorphic functions $f'P'(f)$ , $g'P'(g)$ sharing a small function

Kamal Boussaf, Alain Escassut and Jacqueline Ojeda<sup>1</sup>

2012/ 08/14

## Abstract

Let  $\mathbb{K}$  be a complete algebraically closed  $p$ -adic field of characteristic zero. Let  $f, g$  be two transcendental meromorphic functions in the whole field  $\mathbb{K}$  or meromorphic functions in an open disk that are not quotients of bounded analytic functions. Let  $P$  be a polynomial of uniqueness for meromorphic functions in  $\mathbb{K}$  or in an open disk and let  $\alpha$  be a small meromorphic function with regards to  $f$  and  $g$ . Here we present the following results: if  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  counting multiplicity, then we show that  $f = g$  provided that the multiplicity order of zeros of  $P'$  satisfy certain inequalities. If  $\alpha$  is a Moebius function or a non-zero constant, we can obtain more general results on  $P$ . Further, when  $f, g$  are entire analytic functions or analytic functions inside an open disk, we can obtain a new result improving that published by the third author

## 1 Introduction and Main Results

Let  $f, g$  be two meromorphic functions in a  $p$ -adic field. Here we study polynomials  $P$  such that, when  $f'P'(f)$  and  $g'P'(g)$  share a small function  $\alpha$ , then  $f = g$ . Problems of uniqueness on meromorphic functions were examined first in  $\mathbb{C}$  [10], [11], [13], [16], [17], [18], [19], [20], [25], [26], [27] and next in a  $p$ -adic field [1], [5], [6], [7], [14], [15], [21], [22], [23], [24]. After examining problems of the form  $P(f) = P(g)$ , several studies were made on the equality  $f'P'(f) = g'P'(g)$ , or value sharing questions: if  $f'P'(f)$  and  $g'P'(g)$  share a value, or a small function, do we have  $f = g$ ? Here we recall results previously obtained no matter what the number of zeros of  $P'$ . Results also apply to meromorphic functions inside an open disk. They were published in [2]. Moreover, here we will examine the particular case of analytic functions and will improve results of [23].

Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero, complete for an ultrametric absolute value denoted by  $|\cdot|$ . We denote by  $\mathcal{A}(\mathbb{K})$  the  $\mathbb{K}$ -algebra of entire functions in  $\mathbb{K}$ , by  $\mathcal{M}(\mathbb{K})$  the field of meromorphic functions in  $\mathbb{K}$ , i.e. the field of fractions of  $\mathcal{A}(\mathbb{K})$  and by  $\mathbb{K}(x)$  the field of rational functions.

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<sup>0</sup>2000 Mathematics Subject Classification: 12J25; 30D35; 30G06.

<sup>0</sup>Keywords: Meromorphic, Nevanlinna, Ultrametric, Sharing Value, Unicity, Distribution of values.

<sup>1</sup>Partially supported by CONICYT N°9090014 (Inserción de Capital Humano a la Academia)

Let  $a \in \mathbb{K}$  and  $R \in ]0, +\infty[$ . We denote by  $d(a, R)$  the closed disk  $\{x \in \mathbb{K} : |x - a| \leq R\}$  and by  $d(a, R^-)$  the "open" disk  $\{x \in \mathbb{K} : |x - a| < R\}$ . We denote by  $\mathcal{A}(d(a, R^-))$  the set of analytic functions in  $d(a, R^-)$ , i.e. the  $\mathbb{K}$ -algebra of power series  $\sum_{n=0}^{\infty} a_n(x - a)^n$  converging in  $d(a, R^-)$  and by  $\mathcal{M}(d(a, R^-))$  the field of meromorphic functions inside  $d(a, R^-)$ , i.e. the field of fractions of  $\mathcal{A}(d(a, R^-))$ . Moreover, we denote by  $\mathcal{A}_b(d(a, R^-))$  the  $\mathbb{K}$ -subalgebra of  $\mathcal{A}(d(a, R^-))$  consisting of the bounded analytic functions in  $d(a, R^-)$ , i.e. which satisfy  $\sup_{n \in \mathbb{N}} |a_n|R^n < +\infty$ . And we denote by  $\mathcal{M}_b(d(a, R^-))$  the field of fractions of  $\mathcal{A}_b(d(a, R^-))$ . Finally, we denote by  $\mathcal{A}_u(d(a, R^-))$  the set of unbounded analytic functions in  $d(a, R^-)$ , i.e.  $\mathcal{A}(d(a, R^-)) \setminus \mathcal{A}_b(d(a, R^-))$ . Similarly, we set  $\mathcal{M}_u(d(a, R^-)) = \mathcal{M}(d(a, R^-)) \setminus \mathcal{M}_b(d(a, R^-))$ .

The problem of value sharing a small function by functions of the form  $f'P'(f)$  was examined first when  $P$  was just of the form  $x^n$  [10], [21], [27]. More recently it was examined when  $P$  was a polynomial such that  $P'$  had exactly two distinct zeros [18], [20], [23], both in complex analysis and in  $p$ -adic analysis. In [18], [20] the functions were meromorphic on  $\mathbb{C}$ , with a small function that was a constant or the identity. In [23], the problem was considered for analytic functions in the field  $\mathbb{K}$ : on one hand for entire functions and on the other hand for unbounded analytic functions in an open disk.

Actually solving a value sharing problem involving  $f'P'(f)$ ,  $g'P'(g)$  requires to know polynomials of uniqueness  $P$  for meromorphic functions.

In [23] the third author studied several problems of uniqueness and particularly the following:

*Let  $f, g \in \mathcal{A}(\mathbb{K})$  be transcendental (resp. Let  $f, g \in \mathcal{A}_u(d(0, R^-))$ ) and  $\alpha \in \mathcal{A}(\mathbb{K})$  (resp.  $\alpha \in \mathcal{A}_u(d(0, R^-))$ ) be a small function, such that  $f^n(f - a)^k f'$  and  $g^n(g - a)^k g'$  share  $\alpha$ , counting multiplicity, with  $n, k \in \mathbb{N}$  and  $a \in \mathbb{K} \setminus \{0\}$  (see Theorems D and E below).*

Here we consider functions  $f, g \in \mathcal{M}(\mathbb{K})$  or  $f, g \in \mathcal{M}(d(a, R^-))$  and ordinary polynomials  $P$ : we must only assume certain hypotheses on the multiplicity order of the zeros of  $P'$ . The method for the various theorems we will show is the following: assuming that  $f'P'(f)$  and  $g'P'(g)$  share a small function, we first prove that  $f'P'(f) = g'P'(g)$ . Next, we derive  $P(f) = P(g)$ . And then, when  $P$  is a polynomial of uniqueness for the functions we consider, we can conclude  $f = g$ .

Now, in order to define small functions, we have to briefly recall the definitions of the classical Nevanlinna theory in the field  $\mathbb{K}$  and a few specific properties of ultrametric analytic or meromorphic functions [4].

Let  $\log$  be a real logarithm function of base  $b > 1$  and let  $f \in \mathcal{M}(\mathbb{K})$  (resp.  $f \in \mathcal{M}(d(0, R^-))$ ) having no zero and no pole at 0. Let  $r \in ]0, +\infty[$  (resp.  $r \in ]0, R[$ ) and let  $\gamma \in d(0, r)$ . If  $f$  has a zero of order  $n$  at  $\gamma$ , we put  $\omega_\gamma(f) = n$ . If  $f$  has a pole of order  $n$  at  $\gamma$ , we put  $\omega_\gamma(f) = -n$  and finally, if  $f(\gamma) \neq 0, \infty$ , we set  $\omega_\gamma(f) = 0$ .

We denote by  $Z(r, f)$  the *counting function of zeros of  $f$  in  $d(0, r)$* , counting multiplicity, i.e.

$$Z(r, f) = \max(\omega_0, 0) \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by  $\bar{Z}(r, f)$  the *counting function of zeros of  $f$  in  $d(0, r)$* , ignoring multiplicity, and set

$$\bar{Z}(r, f) = u \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} (\log r - \log |\gamma|)$$

with  $u = 1$  when  $\omega_0(f) > 0$  and  $u = 0$  else.

In the same way, we set  $N(r, f) = Z\left(r, \frac{1}{f}\right)$  (resp.  $\overline{N}(r, f) = \overline{Z}\left(r, \frac{1}{f}\right)$ ) to denote the *counting function of poles of  $f$*  in  $d(0, r)$ , counting multiplicity (resp. ignoring multiplicity).

For  $f \in \mathcal{M}(d(0, R^-))$  having no zero and no pole at 0, the *Nevanlinna function* is defined by  $T(r, f) = \max\{Z(r, f), N(r, f)\}$ .

Now, we must recall the definition of a *small function* with respect to a meromorphic function and some pertinent properties.

**Definition.** Let  $f \in \mathcal{M}(\mathbb{K})$  (resp. let  $f \in \mathcal{M}(d(0, R^-))$ ) such that  $f(0) \neq 0, \infty$ . A function  $\alpha \in \mathcal{M}(\mathbb{K})$  (resp.  $\alpha \in \mathcal{M}(d(0, R^-))$ ) having no zero and no pole at 0 is called a *small function with respect to  $f$* , if it satisfies  $\lim_{r \rightarrow +\infty} \frac{T(r, \alpha)}{T(r, f)} = 0$  (resp.  $\lim_{r \rightarrow R^-} \frac{T(r, \alpha)}{T(r, f)} = 0$ ).

If 0 is a zero or a pole of  $f$  or  $\alpha$ , we can make a change of variable such that the new origin is not a zero or a pole for both  $f$  and  $\alpha$ . Thus it is easily seen that the last relation does not really depend on the origin.

We denote by  $\mathcal{M}_f(\mathbb{K})$  (resp.  $\mathcal{M}_f(d(0, R^-))$ ) the set of small meromorphic functions with respect to  $f$  in  $\mathbb{K}$  (resp. in  $d(0, R^-)$ ).

**Remark 1.** Thanks to classical properties of the Nevanlinna function  $T(r, f)$  with respect to the operations in a field of meromorphic functions, such as  $T(r, f + g) \leq T(r, f) + T(r, g) + O(1)$  and  $T(r, fg) \leq T(r, f) + T(r, g) + O(1)$ , for  $f, g \in \mathcal{M}(\mathbb{K})$  and  $r > 0$ , it is easily proved that  $\mathcal{M}_f(\mathbb{K})$  (resp.  $\mathcal{M}_f(d(0, R^-))$ ) is a subfield of  $\mathcal{M}(\mathbb{K})$  (resp.  $\mathcal{M}(d(0, R^-))$ ) and that  $\mathcal{M}(\mathbb{K})$  (resp.  $\mathcal{M}(d(0, R^-))$ ) is a transcendental extension of  $\mathcal{M}_f(\mathbb{K})$  (resp. of  $\mathcal{M}_f(d(0, R^-))$ ) [9].

Let us remember the following definition.

**Definition.** Let  $f, g, \alpha \in \mathcal{M}(\mathbb{K})$  (resp. let  $f, g, \alpha \in \mathcal{M}(d(0, R^-))$ ). We say that  $f$  and  $g$  share the function  $\alpha$  C.M., if  $f - \alpha$  and  $g - \alpha$  have the same zeros with the same multiplicity in  $\mathbb{K}$  (resp. in  $d(0, R^-)$ ).

Recall that a polynomial  $P \in \mathbb{K}[x]$  is called a *polynomial of uniqueness* for a class of functions  $\mathcal{F}$  if for any two functions  $f, g \in \mathcal{F}$  the property  $P(f) = P(g)$  implies  $f = g$ .

The definition of polynomials of uniqueness was introduced in [19] by P. Li and C. C. Yang and was studied in many papers [12], [13] for complex functions and [1], [6], [7], [14], [15], [17], [24], for  $p$ -adic functions.

Actually, in a  $p$ -adic field, we can obtain various results, not only for functions defined in the whole field  $\mathbb{K}$  but also for functions defined inside an open disk because the  $p$ -adic Nevanlinna Theory works inside a disk, for functions of  $\mathcal{M}_u(d(0, R^-))$ .

The following theorem is classical for analytic functions [26] and was proved for meromorphic functions in [7] and in [14] but it is also a consequence of Theorem 1 in [24].

**Theorem A:** Let  $P \in \mathbb{K}[x]$  be such that  $P'$  has exactly two distinct zeros  $\gamma_1$  of order  $c_1$  and  $\gamma_2$  of order  $c_2$ . Then  $P$  is a polynomial of uniqueness for  $\mathcal{A}(\mathbb{K})$ . Moreover, if  $\min\{c_1, c_2\} \geq 2$ , then  $P$  is a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$ .

Another way to obtain polynomials of uniqueness was given in [7]. It also applies to meromorphic functions inside an open disk.

**Notation:** Let  $P \in \mathbb{K}[x] \setminus \mathbb{K}$  and let  $\Xi(P)$  be the set of zeros  $c$  of  $P'$  such that  $P(c) \neq P(d)$  for every zero  $d$  of  $P'$  other than  $c$ . We denote by  $\Phi(P)$  the cardinal of  $\Xi(P)$ .

**Remark 2.** If  $\deg(P) = q$  then  $\Phi(P) \leq q - 1$ .

From [7] we have the following results:

**Theorem B:** Let  $d(a, R^-)$  be an open disk in  $\mathbb{K}$  and  $P \in \mathbb{K}[x]$ . If  $\Phi(P) \geq 2$  then  $P$  is a polynomial of uniqueness for  $\mathcal{A}(\mathbb{K})$ . If  $\Phi(P) \geq 3$  then  $P$  is a polynomial of uniqueness for both  $\mathcal{A}_u(d(a, R^-))$  and  $\mathcal{M}(\mathbb{K})$ . If  $\Phi(P) \geq 4$  then  $P$  is a polynomial of uniqueness for  $\mathcal{M}_u(d(a, R^-))$ .

We can find the statements of Theorem C in [23]:

**Theorem C:** Let  $P \in K[x]$  such that  $P'$  admits exactly 2 distinct zeros of respective order  $n$  and  $k$ .

- i) Suppose  $k = 2$ ,  $n \geq 3$ . Then  $P$  is a polynomial of uniqueness for  $\mathcal{M}_u(d(0, R^-))$ .
- ii) Suppose  $k = 1$ ,  $n \geq 1$ . Then  $P$  is a polynomial of uniqueness for  $\mathcal{A}_u(d(0, R^-))$ .

In [23], the third author proved the following theorems D and E concerning entire functions and analytic functions in a disk when the polynomial  $P$  has only two distinct zeros:

**Theorem D:** Let  $f, g \in \mathcal{A}(\mathbb{K})$  be transcendental such that  $f^n(f-a)^k f'$  and  $g^n(g-a)^k g'$  share the function  $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$  C.M. with  $n, k \in \mathbb{N}$  and  $a \in \mathbb{K} \setminus \{0\}$ . If  $n \geq \max\{6-k, k+1\}$ , then  $f = g$ . Moreover, if  $\alpha \in \mathbb{K} \setminus \{0\}$  and  $n \geq \max\{5-k, k+1\}$ , then  $f = g$ .

**Theorem E:** Let  $f, g \in \mathcal{A}_u(d(0, R^-))$ , let  $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$  and let  $a \in \mathbb{K} \setminus \{0\}$ . If  $f^n(f-a)^2 f'$  and  $g^n(g-a)^2 g'$  share the function  $\alpha$  C.M. and  $n \geq 4$ , then  $f = g$ . Moreover, if  $f^n(f-a)f'$  and  $g^n(g-a)g'$  share the function  $\alpha$  C.M. and  $n \geq 5$ , then again  $f = g$ .

We can now state our main results and we will give a proof for those that are not published

**Theorem F:** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$ , let  $P' = b(x-a_1)^n \prod_{i=2}^l (x-a_i)^{k_i}$

with  $b \in \mathbb{K}^*$ ,  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l-1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

$$\text{if } l = 2, \text{ then } n \neq 2k, 2k + 1, 3k + 1,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem B, we have Corollary F.1:

**Corollary F.1:** Let  $P \in \mathbb{K}[x]$  satisfy  $\Phi(P) \geq 3$ , let  $P' = b(x-a_1)^n \prod_{i=2}^l (x-a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,

$l \geq 3$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l-1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

if  $l = 3$ , then  $n \neq 2k + 1$ ,  $3k_i - k \forall i = 2, 3$ .

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

And by Theorem A we also have Corollary F.2.

**Corollary F.2:** Let  $P \in \mathbb{K}[x]$  be such that  $P'$  is of the form  $b(x - a_1)^n(x - a_2)^k$  with  $\min(k, n) \geq 2$ .

Suppose  $P$  satisfies the following conditions:

$$n \geq 10 + \max(0, 5 - k),$$

$$n \geq k + 2,$$

$$n \neq 2k, 2k + 1, 3k + 1.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem G:** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l - 1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

$$\text{if } l = 2, \text{ then } n \neq 2k, 2k + 1, 3k + 1,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k \forall i = 2, 3.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a Moebius function. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem B, we have Corollary G.1.

**Corollary G.1:** Let  $P \in \mathbb{K}[x]$  satisfy  $\Phi(P) \geq 3$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 3$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l - 1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 2,$$

$$\text{if } l = 3, \text{ then } n \neq 2k + 1, 3k_i - k \forall i = 2, 3.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a Moebius function. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

And by Theorem A, we have Corollary G.2.

**Corollary G.2:** Let  $P \in \mathbb{K}[x]$  be such that  $P'$  is of the form  $b(x - a_1)^n(x - a_2)^k$  with  $\min(k, n) \geq 2$  and with  $b \in \mathbb{K}^*$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 9 + \max(0, 5 - k),$$

$$n \geq k + 2,$$

$$n \neq 2k, 2k + 1, 3k + 1,$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a Moebius function. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem H:** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l - 1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2, \\ n &\geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \end{aligned}$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a non-zero constant. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem B, we have Corollary H.1.

**Corollary H.1:** Let  $P \in \mathbb{K}[x]$  satisfy  $\Phi(P) \geq 3$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 3$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l - 1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq k + 2, \\ n &\geq 9 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2). \end{aligned}$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a non-zero constant. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

And by Theorem A, we have Corollary H.2

**Corollary H.2:** Let  $P \in \mathbb{K}[x]$  be such that  $P'$  is of the form  $b(x - a_1)^n (x - a_2)^k$  with  $k \geq 2$  and with  $b \in \mathbb{K}^*$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq 9 + \max(0, 5 - k), \\ n &\geq k + 2, \\ n &\neq 2k, 2k + 1, 3k + 1. \end{aligned}$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a non-zero constant. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem I:** Let  $a \in K$  and  $R > 0$ . Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}_u(d(a, R^-))$  and let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 2$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l - 1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$\begin{aligned} n &\geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2), \\ n &\geq k + 3, \\ \text{if } l = 2, &\text{ then } n \neq 2k, 2k + 1, 3k + 1, \\ \text{if } l = 3, &\text{ then } n \neq 2k + 1, 3k_i - k \ \forall i = 2, 3. \end{aligned}$$

Let  $f, g \in \mathcal{M}_u(d(a, R^-))$  and let  $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem B we can state Corollary I.1.

**Corollary I.1:** Let  $a \in K$  and  $R > 0$ . Let  $P \in \mathbb{K}[x]$  satisfy  $\Phi(P) \geq 4$ , let  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)^{k_i}$  with  $b \in \mathbb{K}^*$ ,  $l \geq 4$ ,  $k_i \geq k_{i+1}$ ,  $2 \leq i \leq l-1$  and let  $k = \sum_{i=2}^l k_i$ . Suppose  $P$  satisfies the following conditions:

$$n \geq 10 + \sum_{i=3}^l \max(0, 4 - k_i) + \max(0, 5 - k_2),$$

$$n \geq k + 3,$$

Let  $f, g \in \mathcal{M}_u(d(a, R^-))$  and let  $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

And by Theorem C we have Corollary I.2:

**Corollary I.2:** Let  $a \in K$  and  $R > 0$ . Let  $P \in \mathbb{K}[x]$  be such that  $P'$  is of the form  $b(x - a_1)^n (x - a_2)^2$  with  $b \in \mathbb{K}^*$ . Suppose  $P$  satisfies

$$n \geq 10 + \max(0, 5 - k).$$

Let  $f, g \in \mathcal{M}_u(d(a, R^-))$  and let  $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem J:** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$  such that  $P'$  is of the form

$$b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^*, \text{ satisfying:}$$

$$n \geq l + 10,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem B, we have Corollary J.1:

**Corollary J.1:** Let  $P \in \mathbb{K}[x]$  satisfy  $\Phi(P) \geq 3$  and be such that  $P'$  is of the form

$$b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying:}$$

$$n \geq l + 10,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem K:** Let  $a \in K$  and  $R > 0$ . Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}_u(d(a, R^-))$

such that  $P'$  is of the form  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)$  with  $l \geq 3$ ,  $b \in \mathbb{K}^*$  satisfying:

$$n \geq l + 10,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let  $f, g \in \mathcal{M}_u(d(a, R^-))$  and let  $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

By Theorem B, we have Corollary K.1:

**Corollary K.1:** Let  $a \in K$  and  $R > 0$ . Let  $P \in \mathbb{K}[x]$  satisfy  $\Phi(P) \geq 4$  and be such that  $P'$  is of

the form  $P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i)$  with  $l \geq 4$ ,  $b \in \mathbb{K}^*$  and  $n \geq l + 10$ .



Let  $f, g \in \mathcal{M}_u(d(a, R^-))$  and let  $\alpha \in \mathcal{M}_f(d(a, R^-)) \cap \mathcal{M}_g(d(a, R^-))$  be non-identically zero. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Example:** Let  $P(x) = \frac{x^{18}}{18} - \frac{2x^{17}}{17} - \frac{x^{16}}{16} + \frac{2x^{15}}{15}$ . Then  $P'(x) = x^{17} - 2x^{16} - x^{15} + 2x^{14} = x^{14}(x-1)(x+1)(x-2)$ . We check that:

$$P(0) = 0,$$

$$P(1) = \frac{1}{18} - \frac{2}{17} - \frac{1}{16} + \frac{2}{15},$$

$$P(-1) = \frac{1}{18} + \frac{2}{17} - \frac{1}{16} - \frac{2}{15} \neq 0, P(1), \text{ and } P(2) = \frac{2^{18}}{18} - \frac{2^{18}}{17} - \frac{2^{16}}{16} + \frac{2^{16}}{15} \neq 0, P(1), P(-1).$$

Then  $\Phi(P) = 4$ . So,  $P$  is a polynomial of uniqueness for both  $\mathcal{M}(\mathbb{K})$  and  $\mathcal{M}(d(0, R^-))$ . Moreover, we have  $n = 14, l = 4$ , hence we can apply Corollaries J.1 and K.1.

Given  $f, g \in \mathcal{M}(\mathbb{K})$  transcendental or  $f, g \in \mathcal{M}_u(d(0, R^-))$  such that  $f'P'(f)$  and  $g'P'(g)$  share a small function  $\alpha$  C.M., we have  $f = g$ .

**Theorem L:** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$  such that  $P'$  is of the form

$$P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying}$$

$$n \geq l + 9,$$

$$\text{if } l = 3, \text{ then } n \neq 2l - 1.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a Moebius function. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Theorem M:** Let  $P$  be a polynomial of uniqueness for  $\mathcal{M}(\mathbb{K})$  such that  $P'$  is of the form

$$P' = b(x - a_1)^n \prod_{i=2}^l (x - a_i) \text{ with } l \geq 3, b \in \mathbb{K}^* \text{ satisfying } n \geq l + 9.$$

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha$  be a non-zero constant. If  $f'P'(f)$  and  $g'P'(g)$  share  $\alpha$  C.M., then  $f = g$ .

**Example:** Let  $P(x) = x^q - ax^{q-2} + b$  with  $a \in K^*, b \in K$ , with  $q \geq 5$  an odd integer. Then  $q$  and  $q - 2$  are relatively prime and hence by Theorem 3.21 [15]  $P$  is a uniqueness polynomial for  $\mathcal{M}(\mathbb{K})$  and  $P'$  admits 0 as a zero of order  $n = q - 3$  and two other zeros of order 1.

Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}(\mathbb{K})$  be a small function such that  $f, g$  share  $\alpha$  C.M.

Suppose first  $q \geq 17$ . By Theorem J we have  $f = g$ . Now suppose  $q \geq 15$  and suppose  $\alpha$  is a Moebius function or a non-zero constant. Then by Theorem L and M, we have  $f = g$ .

**Theorem N:** Let  $f, g \in \mathcal{M}(\mathbb{K})$  be transcendental and let  $\alpha \in \mathcal{M}_f(\mathbb{K}) \cap \mathcal{M}_g(\mathbb{K})$  be non-identically zero. Let  $a \in \mathbb{K} \setminus \{0\}$ . If  $f'f^n(f - a)$  and  $g'g^n(g - a)$  share the function  $\alpha$  C.M. and if  $n \geq 12$ ,

then either  $f = g$  or there exists  $h \in \mathcal{M}(\mathbb{K})$  such that  $f = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right) h$  and  $g = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1} - 1}{h^{n+2} - 1} \right)$ . Moreover, if  $\alpha$  is a constant or a Moebius function, then the conclusion holds whenever  $n \geq 11$ .

Inside an open disk, we have a version similar to the general case in the whole field.

**Theorem O:** Let  $f, g \in \mathcal{M}_u(d(0, R^-))$ , and let  $\alpha \in \mathcal{M}_f(d(0, R^-)) \cap \mathcal{M}_g(d(0, R^-))$  be non-identically zero. Let  $a \in \mathbb{K} \setminus \{0\}$ . If  $f'f^n(f-a)$  and  $g'g^n(g-a)$  share the function  $\alpha$  C.M. and  $n \geq 12$ , then either  $f = g$  or there exists  $h \in \mathcal{M}(d(0, R^-))$  such that  $f = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1}-1}{h^{n+2}-1} \right) h$  and  $g = \frac{a(n+2)}{n+1} \left( \frac{h^{n+1}-1}{h^{n+2}-1} \right)$ .

**Remark 3.** In Theorems N and O, the second conclusion does occur. Indeed, let  $h \in \mathcal{M}(\mathbb{K})$  (resp. let  $h \in \mathcal{M}_u(d(0, R^-))$ ). Now, let us precisely define  $f$  and  $g$  as:  $g = \left( \frac{n+2}{n+1} \right) \left( \frac{h^{n+1}-1}{h^{n+2}-1} \right)$  and  $f = hg$ . Then by Remark 1 we can see that the polynomial  $P(y) = \frac{1}{n+2} y^{n+2} - \frac{1}{n+1} y^{n+1}$  satisfies  $P(f) = P(g)$ , hence  $f'P'(f) = g'P'(g)$ , therefore  $f'P'(f)$  and  $g'P'(g)$  trivially share any function.

**Remark 4.** All theorems above have suggested results on complex meromorphic functions, with figures that are just slightly less fine [3].

In order to prove Theorem P, we will prove the following three lemmas:

**Lemma 1:** Let  $R \in \mathbb{R}_+^*$  and let  $f, g \in \mathcal{A}(d(a, R^-))$ . Then  $fg$  belongs to  $\mathcal{A}_b(d(a, R^-))$  if and only if so do both  $f, g$ .

Lemma 3 is known in the Nevanlinna Theory [4], [8].

**Lemma 2:** Let  $f \in \mathcal{M}(K)$  (resp.  $f \in \mathcal{M}(d(0, R^-))$ ) have no zero and no pole at 0. Let  $G(Y) \in \mathcal{M}_f(K)(Y)$ , (resp.  $G \in \mathcal{M}_f(d(0, R^-))(Y)$ ), let  $n = \deg(G)$ . Then  $T(r, G(f)) = nT(r, f) + o(T(r, f))$ .

**Lemma 3:** Let  $P \in K[x]$  with  $\deg(P) > 1$  and let  $f, g \in \mathcal{A}(K) \setminus K$  (resp.  $f, g \in \mathcal{A}_u(d(a, R^-))$ ) be such that  $P(f) = P(g) + c$ ,  $c \in K$ . Then  $c = 0$ .

**Proof:** Let  $P(x) = \sum_{k=0}^n a_k x^k$  with  $a_n \neq 0$ . For each  $k = 1, \dots, n-1$ , let  $Q_k(x, y) = a_k \sum_{j=0}^k x^j y^{k-j}$ . Then  $P(x) - P(y) = (x-y) \left( \sum_{k=1}^{n-1} Q_k(x, y) \right)$ . Suppose first  $f, g \in \mathcal{A}(K)$ . Since  $(f-g) \left( \sum_{k=1}^{n-1} Q_k(f, g) \right)$  is a constant, we know that both  $f-g$ ,  $\sum_{k=1}^{n-1} Q_k(f, g)$  are constants. Thus we have  $g = f + b$  with  $b \in K$ . Let  $G(x) = \sum_{k=1}^{n-1} Q_k(x, x+b)$ . We can check that  $G$  is a polynomial of degree  $n-1$ . And since  $G(f)$  is a constant, we have  $n-1 = 0$ , a contradiction. Thus,  $f = g$ .

Similarly, suppose now  $f, g \in \mathcal{A}_u(d(a, R^-))$ . By Lemma 1 both  $f-g$ ,  $\sum_{k=1}^{n-1} Q_k(f, g)$  are bounded, so we have  $g = f + h$ , with  $h \in \mathcal{A}_b(d(a, R^-))$ . Consider the polynomial  $B(x) = \sum_{k=1}^{n-1} Q_k(x, x+h) \in \mathcal{M}_b(d(a, R^-))(x)$ . Then  $B(x)$  is a polynomial with coefficients in  $\mathcal{M}_b(d(a, R^-))$  and  $\deg(B)$  is  $n-1$  hence by Lemma 2 we have  $T(r, B(f)) = (n-1)T(r, f) + S_f(r)$ . But since  $B(f)$  is bounded, it belongs to  $\mathcal{M}_b(d(a, R^-))(x)$ , hence  $n = 1$ , a contradiction again.

**Theorem P:** Let  $P(x) \in K[x]$  be a polynomial of uniqueness for  $\mathcal{A}(K)$  (resp. for  $\mathcal{A}(d(a, R^-))$ ), let  $P'(x) = \prod_{i=1}^l (x - a_i)^{k_i}$  and let  $f, g \in \mathcal{A}(K)$  be transcendental (resp. let  $f, g \in \mathcal{A}_u(d(a, R^-))$ )

such that  $f'P'(f)$  and  $g'P'(g)$  share a small function  $\alpha \in \mathcal{A}_f(K) \cap \mathcal{A}_g(K)$  (resp.  $\alpha \in \mathcal{A}_f(d(, R^-)) \cap \mathcal{A}_g(d(a, R^-))$ ). If  $\sum_{i=1}^l k_i \geq 2l + 2$  then  $f = g$ . Moreover, if  $f, g$  belong to  $\mathcal{A}(K)$ , if  $\alpha$  is a constant and if  $\sum_{i=1}^l k_i \geq 2l + 1$  then  $f = g$ .

**Proof:** Set  $F = \frac{f'P'(f)}{\alpha}$  and  $G = \frac{g'P'(g)}{\alpha}$ , so  $F$  and  $G$  share the value 1 C.M. Since  $F, G$  share  $\alpha$  C.M., the function  $\frac{F - \alpha}{G - \alpha}$  has no zero and no pole, hence it is a constant  $\phi \in K^*$  (resp. an invertible function  $\phi \in \mathcal{A}(d(a, R^-))$ ). Suppose  $\phi \neq 1$  We have  $F - \alpha = \phi(G - \alpha)$ , hence  $F = \phi G + \alpha(1 - \phi)$ . Now applying the Second main Theorem to  $F$  we have:

$$T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, G) + o(T(r, F)). \quad (1)$$

Now, let  $m = \sum_{i=1}^l k_i$ . By Lemma 2 we have  $T(r, F) \geq mT(r, f) + o(T(r, f))$ ,  $T(r, G) \geq mT(r, g) + o(T(r, g))$ . Consequently, since  $f, g$  are transcendental (resp. unbounded), so are  $F$  and  $G$ .

On the other hand, since  $T(r, f') \leq T(r, f) - \log r + O(1)$ ,  $T(r, g') \leq T(r, g) - \log r + O(1)$ , we have  $\overline{Z}(r, F) + \overline{Z}(r, G) \leq (l + 1)(T(r, f) + T(r, g)) - 2 \log r + o(T(r, f))$ . We also notice that  $o(T(r, F)) = o(T(r, f))$ ,  $o(T(r, G)) = o(T(r, g))$ . Consequently, by (1), we obtain  $mT(r, f) \leq l(T(r, f) + T(r, g)) + o(T(r, f))$  and similarly,  $mT(r, g) \leq l(T(r, f) + T(r, g)) + o(T(r, g))$ . So,

$$m(T(r, f) + T(r, g)) \leq (2l + 1)(T(r, f) + T(r, g)) + o(T(r, f) + T(r, g)). \quad (2)$$

Thus, by (2) we have  $m \leq 2l + 2$ . Moreover, we notice that if  $f, g \in \mathcal{A}(K)$  and if  $\alpha$  is a constant, (1) gets

$$T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, G) - 2 \log r + O(1)$$

hence  $m \leq 2l$ .

Thus, if  $m \geq 2l + 2$ , or if  $f, g \in \mathcal{A}(K)$  and if  $\alpha$  is a constant and  $m \geq 2l + 1$ , we have  $\phi = 1$ . We can then assume that  $\phi = 1$ , therefore  $f'P'(f) = g'P'(g)$  and hence  $P(f) - P(g)$  is a constant  $b \in K$ . But then, by Lemma 3,  $b = 0$ . Finally, since  $P$  is a polynomial of uniqueness for  $\mathcal{A}(K)$  (resp.  $\mathcal{A}(d(0, R^-))$ ), we can conclude  $f = g$ .

**Corollary P.1:** Let  $P(x) \in K[x]$  be such that  $\Upsilon(P) \geq 2$ , let  $P'(x) = \prod_{i=1}^l (x - a_i)^{k_i}$  and let  $f, g \in \mathcal{A}(K)$  be transcendental such that  $f'P'(f)$  and  $g'P'(g)$  share a small function  $\alpha \in \mathcal{A}_f(K) \cap \mathcal{A}_g(K)$ . If  $\sum_{i=1}^l k_i \geq 2l + 2$  then  $f = g$ . Moreover, if  $\alpha$  is a constant and if  $\sum_{i=1}^l k_i \geq 2l + 1$  then  $f = g$ .

**Corollary P.2:** Let  $P(x) \in K[x]$  be such that  $\Upsilon(P) \geq 3$ , let  $P'(x) = \prod_{i=1}^l (x - a_i)^{k_i}$  and let  $f, g \in \mathcal{A}_u(d(a, R^-))$  be such that  $f'P'(f)$  and  $g'P'(g)$  share a small function  $\alpha \in \mathcal{A}_f(d(a, R^-)) \cap \mathcal{A}_g(d(a, R^-))$ . If  $\sum_{i=1}^l k_i \geq 2l + 2$  then  $f = g$ .

**Theorem Q:** Let  $f, g \in \mathcal{A}(\mathbb{K})$  be transcendental such that  $(f - a)^n(f - b)^k f'$  and  $(g - a)^n(g - b)^k g'$  share the function  $\alpha \in \mathcal{A}_f(\mathbb{K}) \cap \mathcal{A}_g(\mathbb{K})$  C.M. with  $n, k \in \mathbb{N}^*$  and  $a \in \mathbb{K}^* \setminus \{0\}$ . Then  $f = g$ .

**Proof:** Without loss of generality, we can assume  $a = 0$  and  $b = 1$ . Let  $P$  be the primitive of  $x^n(x - 1)^k$  that admits 0 as a zero of order  $n + 1$ . By Lemma 3, we have  $P(f) = P(g)$ . Now, we can check that  $P(1) \neq 0$ . Indeed, if 1 is a zero of  $P$ , then it is a zero of order  $k + 1$  and hence  $\deg(P) = n + k + 2$ , a contradiction. Consequently,  $\Phi(P) = 2$  and therefore, by Theorem B,  $P$  is a polynomial of uniqueness for  $\mathcal{A}(K)$ . Hence  $f = g$ .

**Theorem R:** Let  $f, g \in \mathcal{A}_u(d(0, R^-))$ , let  $\alpha \in \mathcal{A}_f(d(0, R^-)) \cap \mathcal{A}_g(d(0, R^-))$  and let  $a, b \in K \setminus \{0\}$ ,  $a \neq b$ . If  $(f - a)^n(f - b)^2 f'$  and  $(g - a)^n(g - b)^2 g'$  share the function  $\alpha$  C.M. and  $n \geq 3$ , then  $f = g$ . Moreover, if  $(f - a)^n(f - b) f'$  and  $(g - a)^n(g - b) g'$  (with  $n \geq 1$ ) share the function  $\alpha$  C.M. and if  $n \geq 1$ , then again  $f = g$ .

**Proof:** Without loss of generality, we can assume  $a = 0$  and  $b = 1$ . Let  $P$  be the primitive of  $x^n(x - 1)^k$  such that  $P(0) = 0$ . If  $k = 2$  and  $n \geq 3$ , we can apply Theorem C showing that  $P$  is a polynomial of uniqueness for  $\mathcal{A}_u(d(0, R^-))$  and then, by Theorem P we have  $f = g$ .

And now suppose  $k = 1$ . Then  $P'(x)$  has two distinct zeros, one of order 1 and hence by Theorem C,  $P$  is a polynomial of uniqueness for  $\mathcal{A}_u(d(0, R^-))$ . Consequently, by Theorem P, we have  $f = g$ .

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