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# Zeros of the derivative of a $p$ -adic meromorphic function

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## Abstract

Let  $\mathbb{K}$  be a complete algebraically closed field of characteristic 0 and let  $f$  be a transcendental meromorphic function in  $\mathbb{K}$ . A conjecture suggests that  $f'$  takes every values infinitely many times, what was proved when  $f$  has finitely many multiple poles. Here we can generalize the conclusion just by assuming that there exists positive constants  $c, d$  such that number of multiple poles inside the disk  $|x| \leq r$  is less than  $cr^d$  for all  $r \geq 1$ . Applications are given to entire functions  $g$  in  $\mathbb{K}$  such that  $g'$  divides  $g$ , to links between residues and zeros of functions admitting primitives and finally to the  $p$ -adic Hayman conjecture in the cases that are not yet solved.

## 1 Introduction and results.

Throughout the paper,  $\mathbb{K}$  is an algebraically closed field of characteristic 0, complete with respect to a  $p$ -adic absolute value denoted by  $|\cdot|$  (example  $\mathbb{C}_p$ ). Given  $a \in \mathbb{K}$  and  $r > 0$  we denote by  $d(0, r)$  the disk  $\{x \in \mathbb{K} \mid |x - a| \leq r\}$ .

We denote by  $\mathcal{A}(\mathbb{K})$  the  $\mathbb{K}$ -algebra of entire functions in  $\mathbb{K}$  i.e. the set of power series with coefficients in  $\mathbb{K}$  converging in all  $\mathbb{K}$  and we denote by  $\mathcal{M}(\mathbb{K})$  the field of meromorphic functions in  $\mathbb{K}$ , i.e. the field of fraction of  $\mathcal{A}(\mathbb{K})$ . Given  $f, g \in \mathcal{A}(\mathbb{K})$ , we denote by  $W(f, g)$  the Wronskian of  $f$  and  $g$ .

Let  $f \in \mathcal{A}(\mathbb{K})$  be non-constant. We can factorize  $f(x)$  in the form

$$f(x) = cx^{n_0} \prod_{\omega \in \Omega, \omega \neq 0} \left(1 - \frac{x}{\omega}\right)^{n_\omega},$$

where  $c$  is a non zero constant,  $\Omega$  is the set of distinct zeros of  $f$  and  $n_0 = 0$  if 0 is not a zero of  $f$ . We set  $\bar{f} = x^{m_0} \prod_{\omega \in \Omega, \omega \neq 0} \left(1 - \frac{x}{\omega}\right)$ , with  $m_0 = 0$  if 0 is not a zero of  $f$  and  $m_0 = 1$  else. So, the

function  $\bar{f}$  is an entire function admitting as zeros the distinct zeros of  $f$ , all with order 1.

We can then set  $f = \bar{f}\tilde{f}$  where the function  $\tilde{f}$  is an entire function admitting for zeros the multiple zeros of  $f$ , each with order  $q - 1$  when it is a zero of  $f$  of order  $q$ . Particularly, if  $f$  is constant, we set  $\bar{f} = 1$  and  $\tilde{f} = f$ .

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Now,  $f(x)$  is a power series  $\sum_{n=0}^{\infty} a_n x^n$  of infinite radius of convergence. According to classical notation [7], we set  $|f|(r) = \sup\{|f(x)| \mid |x| \leq r\}$ . We know that

$$|f|(r) = \sup_{n \in \mathbb{N}} |a_n| r^n = \lim_{|x| \rightarrow r, |x| \neq r} |f(x)|.$$

That notation defines an absolute value on  $\mathcal{A}(\mathbb{K})$  and has continuation to  $\mathcal{M}(\mathbb{K})$  as

$|\frac{f}{g}|(r) = \frac{|f|(r)}{|g|(r)}$  with  $f, g \in \mathcal{A}(\mathbb{K})$ . In the paper [4], the following Theorems A and B are proven:

**Theorem A:** *Let  $f, g$  be entire functions on  $\mathbb{K}$  such that  $W(f, g)$  is a non-identically zero polynomial. Then both  $f, g$  are polynomials.*

(Theorem A has been recently generalized to several functions [2])

Theorem B is an easy consequence of Theorem A:

**Theorem B:** *Let  $f$  be a transcendental meromorphic function on  $\mathbb{K}$  having finitely many multiple poles. Then  $f'$  takes every value infinitely many times.*

That has suggested the following conjecture:

**Conjecture:** *Let  $f$  be a meromorphic function on  $\mathbb{K}$  such that  $f'$  has finitely many zeros. Then  $f$  is a rational function.*

This paper is aimed at proving the following theorem which generalizes Theorem B. For this we will define new expressions.

**Notation:** Let  $f \in \mathcal{M}(\mathbb{K})$ . For each  $r > 0$ , we denote by  $\psi_f(r)$  the number of multiple zeros of  $f$  in  $d(0, r)$ , each counted with its multiplicity and we set  $\phi_f(r) = \psi_{\frac{1}{f}}(r)$ . Similarly, we denote by  $\theta_f(r)$  the number of zeros of  $f$  in  $d(0, r)$ , taking multiplicity into account and set  $\tau_f(r) = \theta_{\frac{1}{f}}(r)$ .

**Theorem 1:** *Let  $f$  be a meromorphic function on  $\mathbb{K}$  such that, for some  $c, d \in ]0, +\infty[$ ,  $\phi_f$  satisfies  $\phi_f(r) \leq cr^d$  in  $[1, +\infty[$ . If  $f'$  has finitely many zeros, then  $f$  is a rational function.*

**Corollary 1:** *Let  $f$  be a meromorphic function on  $\mathbb{K}$  such that, for some  $c, d \in ]0, +\infty[$ ,  $\phi_f$  satisfies  $\phi_f(r) \leq cr^d$  in  $[1, +\infty[$ . If for some  $b \in \mathbb{K}$   $f' - b$  has finitely many zeros, then  $f$  is a rational function.*

**Corollary 2:** *Let  $f$  be a transcendental meromorphic function on  $\mathbb{K}$  such that  $\tau_f(r) \leq cr^d$  in  $[1, +\infty[$  for some  $c, d \in ]0, +\infty[$ . Then  $f^{(k)}$  takes every value in  $\mathbb{K}$  infinitely many times, for each  $k \in \mathbb{N}^*$ .*

**Corollary 3:** *Let  $h$  be a transcendental entire function on  $\mathbb{K}$  and  $P \in \mathbb{K}[x]$ . The differential equation  $y'h = yP$  admits no entire solution  $f$  different from 0, such that,  $f$  for some  $c, d \in ]0, +\infty[$ ,  $\psi_f(r) \leq cr^d$  in  $[1, +\infty[$ .*

By Main Theorem of [5] we can derive this corollary:

**Corollary 4:** *Let  $f$  be a meromorphic function on  $\mathbb{K}$  such that, for certain  $d \in \mathbb{N}$ , the number of poles of order  $\geq 3$ , counted each with its multiplicity, in each disk  $d(0, r)$  is bounded by  $r^d$  for all  $r \in [1, +\infty[$  and such that all residues at poles are null. Then for every  $b \in \mathbb{K}$ ,  $f - b$  has infinitely many zeros.*

According to the  $p$ -adic Hayman conjecture, for every  $n \in \mathbb{N}^*$   $f'f^n$  takes every non-zero value infinitely many times. Here Theorem 1 has an immediate application to that conjecture in the cases  $n = 1$  or  $n = 2$  which are not yet solved, except with additional hypotheses [2], [3], [4], [8].

**Corollary 5:** *Let  $f$  be a meromorphic function on  $\mathbb{K}$ . Suppose that there exists  $c, d \in ]0, +\infty[$ , such that  $\tau_f(r) \leq cr^d \forall r \in [1, +\infty[$ . If  $f'f^n - b$  has finitely many zeros for some  $b \in \mathbb{K}$ , with  $n \in \mathbb{N}$  then  $f$  is a rational function.*

Corollary 5 may be written in another way:

**Corollary 6:** *Let  $f$  be a transcendental meromorphic function on  $\mathbb{K}$ . Suppose that there exists  $c, d \in ]0, +\infty[$ , such that  $\theta_f(r) \leq cr^d \forall r \in [1, +\infty[$ . Then for all  $m \in \mathbb{N}$ ,  $m \geq 3$  and for all  $b \in \mathbb{K}^*$ ,  $f' - bf^m$  admits infinitely many zeros that are not zeros of  $f$ .*

**Corollary 7:** *Let  $f$  be a transcendental meromorphic function on  $\mathbb{K}$ . Suppose that there exists  $c, d \in ]0, +\infty[$ , such that  $\psi_f(r) \leq cr^d \forall r \in [1, +\infty[$ . Then, for all  $b \in \mathbb{K}$ ,  $\frac{f'}{f^2} - b$  has infinitely many zeros.*

**Remark:** Using Corollary 7 to study zeros of  $f' + bf^2$  that are not zeros of  $f$  is not so immediate, as done in Theorems 3, 4, 5 [3], because of residues of  $f$  at poles of order 1.

**Theorem 2:** *Let  $f$  be a transcendental meromorphic function on  $\mathbb{K}$  such that, for some  $c, d \in ]0, +\infty[$ , we have  $\theta_{f'}(r) \leq cr^d$  in  $[1, +\infty[$ . Then for every  $b \in \mathbb{K}$ ,  $b \neq 0$ ,  $f' - b$  has infinitely many zeros.*

## 2 Preliminary results

We will need several lemmas.

**Lemma 1:** *Let  $U, V \in \mathcal{A}(\mathbb{K})$  have no common zero and let  $f = \frac{U}{V}$ . If  $f'$  has finitely many zeros, there exists a polynomial  $P \in \mathbb{K}[x]$  such that  $U'V - UV' = P\tilde{V}$*

**Proof:** If  $V$  is a constant, the statement is obvious. So, we assume that  $V$  is not a constant. Now  $\tilde{V}$  divides  $V'$  and hence  $V'$  factorizes in the way  $V' = \tilde{V}Y$  with  $Y \in \mathcal{A}(\mathbb{K})$ . Then no zero of  $Y$  can be a zero of  $V$ . Consequently, we have

$$f'(x) = \frac{U'V - UV'}{V^2} = \frac{U'\tilde{V} - UY}{\tilde{V}^2\tilde{V}}.$$

The two functions  $U'\tilde{V} - UY$  and  $\tilde{V}^2\tilde{V}$  have no common zero since neither have  $U$  and  $V$ . Consequently, the zeros of  $f'$  are those of  $U'\tilde{V} - UY$  which therefore has finitely many zeros and consequently is a polynomial.

Lemma 2 is known as the  $p$ -adic Schwarz Lemma (Lemma 23.12 [6]). Lemmas 3 and 4 are immediate corollaries:

**Notation:** Let  $a \in \mathbb{K}$ ,  $r', r'' \in ]0, +\infty[$  with  $r' < r''$ . We denote by  $\Gamma(a, r', r'')$  the annulus  $\{x \in \mathbb{K} \mid r' < |x - a| < r''\}$ .

**Lemma 2:** Let  $r, R \in ]0, +\infty[$  be such that  $r < R$  and let  $f \in \mathcal{M}(\mathbb{K})$  admit  $s$  zeros and  $t$  poles in  $d(0, r)$  and no zero and no pole in  $\Gamma(0, r, R)$ . Then  $\frac{|f|(R)}{|f|(r)} = \left(\frac{R}{r}\right)^{s-t}$ .

**Lemma 3:** Let  $r, R \in ]0, +\infty[$  be such that  $r < R$  and let  $f \in \mathcal{A}(\mathbb{K})$  have  $q$  zeros in  $d(0, R)$ . Then

$$\frac{|f|(R)}{|f|(r)} \leq \left(\frac{R}{r}\right)^q.$$

**Lemma 4:** Let  $f \in \mathcal{A}(\mathbb{K})$ . Then  $f$  is a polynomial of degree  $q$  if and only if there exists a constant  $c$  such that  $|f|(r) \leq cr^q \forall r \in [1, +\infty[$ .

**Notation:** Let  $d(a, r^-)$  be the disk  $\{x \in \mathbb{K} \mid |x - a| < r\}$ . We denote by  $\mathcal{A}(d(a, R^-))$  the  $\mathbb{K}$ -algebra of analytic functions in  $d(a, R^-)$  i.e. the set of power series in  $x - a$  with coefficients in  $\mathbb{K}$  whose radius of convergence is  $\geq R$  and we denote by  $\mathcal{M}(d(a, R^-))$  the field of meromorphic functions in  $d(a, R^-)$ , i.e. the field of fraction of  $\mathcal{A}(d(a, R^-))$ .

**Lemma 5:** Let  $f \in \mathcal{M}(d(0, R^-))$ . For each  $n \in \mathbb{N}$ , and for all  $r \in ]0, R[$ , we have

$$|f^{(n)}|(r) \leq |n!| \frac{|f|(r)}{r^n}.$$

**Proof:** Suppose first  $f$  belongs to  $\mathcal{A}(d(0, R^-))$  and set  $f(x) = \sum_{k=0}^{\infty} a_k x^k$ . Then

$$f^{(n)}(x) = \sum_{k=n}^{\infty} (n!) \binom{k}{n-k} a_k x^{k-n}.$$

The statement then is immediate. Consider now the general case and set  $f = \frac{U}{V}$  with  $U, V \in \mathcal{A}(d(0, R^-))$ . The stated inequality is obvious when  $n = 1$ . So, we assume it holds for  $q \leq n - 1$  and consider  $f^{(n)}$ . Writing  $U = V \left(\frac{U}{V}\right)$ , by Leibniz Theorem we have

$$U^{(n)} = \sum_{q=0}^n \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)}$$

and hence

$$V \left(\frac{U}{V}\right)^{(n)} = U^{(n)} - \sum_{q=0}^{n-1} \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)}.$$

Now,  $|U^{(n)}|(R) \leq |n!| \frac{|U|(R)}{R^n}$  and for each  $q \leq n - 1$ , we have  $|V^{(n-q)}|(R) \leq |(n-q)!| \frac{|V|(R)}{R^{n-q}}$  and

$\left|\left(\frac{U}{V}\right)^{(q)}\right|(R) \leq |q!| \frac{|U|(R)}{|V|(R)R^q}$ . Therefore, we can derive that terms on the right hand side are upper

bounded by  $|n!| \frac{|U|(R)}{|V|(R)R^n}$  and hence the conclusion holds for  $q = n$ .

**Notation:** For each  $n \in \mathbb{N}^*$ , we set  $\lambda_n = \max\{\frac{1}{|k|}, 1 \leq k \leq n\}$ .

**Remark:** For every  $n \in \mathbb{N}^*$ , we have  $\lambda_n \leq n$  because  $k|k| \geq 1 \ \forall k \in \mathbb{N}$ . The equality holds for all  $n$  of the form  $p^h$ .

**Lemma 6:** Let  $U, V \in \mathcal{A}(d(0, R^-))$ . Then for all  $r \in ]0, R[$  and  $n \geq 1$  we have

$$|U^{(n)}V - UV^{(n)}|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{r^{n-1}}.$$

More generally, given  $j, l \in \mathbb{N}$ , we have

$$|U^{(j)}V^{(l)} - U^{(l)}V^{(j)}|(r) \leq |(j!)(l!)|\lambda_{j+l} \frac{|U'V - UV'|(r)}{r^{j+l-1}}.$$

**Proof:** Set  $g = \frac{U}{V}$  and  $f = g'$ . Applying Lemma 5 to  $f$  for  $k-1$ , we obtain

$$|g^{(k)}|(r) = |f^{(k-1)}|(r) \leq |(k-1)!| \frac{|f|(r)}{r^{k-1}} = |(k-1)!| \frac{|U'V - UV'|(r)}{|V^2|(r)r^{k-1}}.$$

As in the proof of Lemma 5, we set  $U = V\left(\frac{U}{V}\right)$ . By Leibniz formula again, now we can obtain

$$U^{(n)} = \sum_{q=1}^n \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)} + V^{(n)} \left(\frac{U}{V}\right)$$

hence

$$(1) \quad U^{(n)} - V^{(n)} \left(\frac{U}{V}\right) = \sum_{q=1}^n \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)}$$

Now we have

$$\left| \left(\frac{U}{V}\right)^{(q)} \right|(r) = |g^{(q)}|(r) \leq |(q-1)!| \frac{|U'V - UV'|(r)}{|V^2|(r)r^{q-1}}$$

and

$$|V^{(n-q)}|(r) \leq |(n-q)!| \frac{|V|(r)}{r^{n-q}}.$$

Consequently, the general term in (1) is upper bounded as

$$\left| \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)} \right|(r) \leq \frac{|(n!)((n-q)!)((q-1)!)|}{|(q!)((n-q)!)|} \frac{|U'V - UV'|(r)}{|V|(r)r^{n-1}} \leq \lambda_n \frac{|n!||U'V - UV'|(r)}{|V|(r)r^{n-1}}.$$

Therefore by (1) we obtain

$$\left| U^{(n)} - V^{(n)} \left(\frac{U}{V}\right) \right|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{|V|(r)r^{n-1}}$$

and finally

$$\left| U^{(n)}V - V^{(n)}U \right|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{r^{n-1}}.$$

We can now generalize the first statement. Set  $P_j = U^{(j)}V - UV^{(j)}$ . By induction, we can show the following equality that already holds for  $l \leq j$ :

$$U^{(j)}V^{(l)} - U^{(l)}V^{(j)} = \sum_{h=0}^l \binom{l}{h} (-1)^h P_{j+h}^{(l-h)}$$

Now, the second statement gets just an application of the first.

**Remark:** Suppose the residue characteristic is  $p \neq 0$  and  $U(x) = 1, V(x) = x^p, n = p$ . Then, for all  $R > 0$ , the inequality  $|U^{(p)}V - UV^{(p)}|(R) \leq |p!| \frac{|U'V - UV'| (R)}{R^{p-1}}$  is not satisfied. So, we can't eliminate the factor  $\lambda_n$ .

**Lemma 7:** Let  $U, V \in \mathcal{A}(\mathbb{K})$  and let  $r, R \in ]0, +\infty[$  satisfy  $r < R$ . For all  $x, y \in \mathbb{K}$  with  $|x| \leq R$  and  $|y| \leq r$ , we have the inequality:

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'| (R)}{e(\log R - \log r)}$$

**Proof:** By Taylor's formula at the point  $x$ , we have

$$U(x+y)V(x) - U(x)V(x+y) = \sum_{n \geq 0} \frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!} y^n$$

Now,  $\left| \frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!} y^n \right| \leq \lambda_n \frac{|U'V - UV'| (R)}{R^{n-1}} r^n$ . But as remarked above, we have  $\lambda_n \leq n$ , hence

$$\left| \frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!} y^n \right| \leq nR|U'V - UV'| (R) \left(\frac{r}{R}\right)^n.$$

And we notice that  $\lim_{n \rightarrow +\infty} n \left(\frac{r}{R}\right)^n = 0$ . Consequently, we can define  $B = \max_{n \geq 1} \left(n \left(\frac{r}{R}\right)^n\right)$  and we have  $|U(x+y)V(x) - U(x)V(x+y)| \leq BR|U'V - UV'| (R) \forall x \in d(0, R), y \in d(0, r)$ .

We can check that the function  $h$  defined in  $]0, +\infty[$  as  $h(t) = t \left(\frac{r}{R}\right)^t$  reaches its maximum at the point  $u = \frac{1}{(\log R - \log r)}$ . Consequently,  $B \leq \frac{1}{e(\log R - \log r)}$  and therefore

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'| (R)}{e(\log R - \log r)}.$$

**Proof of Theorem 1:** Suppose  $f'$  has finitely many zeros. If  $V$  is a constant, the statement is immediate. So, we suppose  $V$  is not a constant and hence it admits at least one zero  $a$ . By Lemma 1, there exists a polynomial  $P \in \mathbb{K}[x]$  such that  $U'V - UV' = P\tilde{V}$ . Next, we take  $r, R \in [1, +\infty[$  such that  $|a| < r < R$  and  $x \in d(0, R), y \in d(0, r)$ . By Lemma 7 we have

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'| (R)}{e(\log R - \log r)}.$$

Notice that  $U(a) \neq 0$  because  $U$  and  $V$  have no common zero. Now set  $l = \max(1, |a|)$  and take  $r \geq l$ . Setting  $c_1 = \frac{1}{e|U(a)|}$ , we have

$$|V(a+y)| \leq c_1 \frac{R|P|(R)|\tilde{V}|(R)}{\log R - \log r}.$$

Then taking the supremum of  $|V(a+y)|$  inside the disk  $d(0, r)$ , we can derive

$$(1) \quad |V|(r) \leq c_1 \frac{R|P|(R)|\tilde{V}|(R)}{\log R - \log r}.$$

Let us apply Lemma 3, by taking  $R = r + \frac{1}{r^d}$ , after noticing that the number of zeros of  $\tilde{V}(R)$  is bounded by  $\psi_V(R)$ . So, we have

$$(2) \quad |\tilde{V}|(R) \leq \left(1 + \frac{1}{r^{d+1}}\right)^{\psi_V(r + \frac{1}{r^d})} |\tilde{V}|(r).$$

Now, due to the hypothesis:  $\psi_V(r) = \phi_f(r) \leq cr^d$  in  $[1, +\infty[$ , we have

$$(3) \quad \left(1 + \frac{1}{r^{d+1}}\right)^{\psi_V(r + \frac{1}{r^d})} \leq \left(1 + \frac{1}{r^{d+1}}\right)^{[c(r + \frac{1}{r^d})^d]} = \exp \left[ c \left(r + \frac{1}{r^d}\right)^d \log \left(1 + \frac{1}{r^{d+1}}\right) \right].$$

The function  $h(r) = c \left(r + \frac{1}{r^d}\right)^d \log \left(1 + \frac{1}{r^{d+1}}\right)$  is continuous on  $]0, +\infty[$  and equivalent to  $\frac{c}{r}$  when  $r$  tends to  $+\infty$ . Consequently, it is bounded on  $[l, +\infty[$ . Therefore, by (2) and (3) there exists a constant  $M > 0$  such that, for all  $r$ ,  $R \in [l, +\infty[$ ,  $r < R$  by (3) we obtain

$$(4) \quad |\tilde{V}|(r + \frac{1}{r^d}) \leq M |\tilde{V}|(r).$$

On the other hand,  $\log \left(r + \frac{1}{r^d}\right) - \log r = \log \left(1 + \frac{1}{r^{d+1}}\right)$  clearly satisfies an inequality of the form  $\log \left(1 + \frac{1}{r^{d+1}}\right) \geq \frac{c_2}{r^{d+1}}$  in  $[l, +\infty[$  with  $c_2 > 0$ . Moreover, we can obviously find positive constants  $c_3, c_4$  such that

$$\left(r + \frac{1}{r^d}\right) |P| \left(r + \frac{1}{r^d}\right) \leq c_3 r^{c_4}.$$

Consequently, by (1) and (4) we can find positive constants  $c_5, c_6$  such that  $|V|(r) \leq c_5 r^{c_6} |\tilde{V}|(r) \forall r \in [l, +\infty[$ . Thus, writing again  $V = \bar{V} \tilde{V}$ , we have  $|\bar{V}|(r) |\tilde{V}|(r) \leq c_5 r^{c_6} |\tilde{V}|(r)$  and hence  $|\bar{V}|(r) \leq c_5 r^{c_6} \forall r \in [l, +\infty[$ . Consequently, by Lemma 4,  $\bar{V}$  is a polynomial of degree  $\leq c_6$  and hence it has finitely many zeros and so does  $V$ . And then, by Theorem B,  $f$  must be a rational function.

**Proof of Corollary 1:** Suppose  $f' - b$  has finitely many zeros. Then  $f - bx$  satisfies the same hypothesis as  $f$ , hence it is a rational function and so is  $f$ .

**Proof of Corollary 2:** Indeed, if  $k = 1$ , the statement just comes from Theorem 1. Now suppose  $k \geq 2$ . Each pole  $a$  of order  $n$  of  $f$  is a pole of order  $n + k$  of  $f^{(k)}$  and  $f^{(k)}$  has no other pole. Consequently, we have  $\phi_{f^{(k-1)}}(r) = \tau_{f^{(k-1)}}(r) \leq kcr^d$ . So, we can apply Theorem 1 to  $f^{(k-1)}$  to show the claim.



**Proof of Corollary 3:** Suppose  $V$  is an entire solution of the equation  $V'h = VP$  such that  $\psi_V(r) \leq cr^d \forall r \in [1, +\infty[$ . If  $V$  is a rational function, so is  $\frac{VP}{V'}$ , a contradiction because  $h$  is transcendental. Suppose now  $V$  is transcendental and apply Theorem 1: then  $(\frac{1}{V})' = -\frac{V'}{V^2} = \frac{-P}{hV}$  admits infinitely many zeros, a contradiction since  $P$  is a polynomial.

**Definition:** Given a meromorphic function in  $\mathbb{K}$ , we call *exceptional value of  $f$*  (or *Picard value of  $f$* ) a value  $b \in \mathbb{K}$  such that  $f - b$  has no zero. And, if  $f$  is transcendental, we call *quasi-exceptional value* a value  $b \in \mathbb{K}$  such that  $f - b$  has finitely many zeros.

**Proof of Corollary 4:** By the Main Theorem of [5],  $f$  admits a primitive  $F$  and then, by hypothesis, we have  $\phi_F(r) \leq r^d$ . Consequently, by Corollary 1  $F' - b = f - b$  has infinitely many zeros.

**Proof of Corollary 5:** Suppose  $f$  is transcendental. Due to hypothesis,  $f^{n+1}$  satisfies  $\theta_{\frac{1}{f^{n+1}}}(r) = \tau_{f^{n+1}}(r) \leq c(n+1)r^d \forall r \in [1, +\infty[$  hence by Theorem 1,  $f'f^n$  has no quasi-exceptional value.

**Proof of Corollary 6:** We set  $g = \frac{1}{f}$ . Then by Corollary 2,  $g'g^{m-2}$  has no quasi-exceptional value. Consequently, given  $b \in \mathbb{K}^*$ ,  $g'g^{m-2} + b$  has infinitely many zeros and hence  $f' - bf^m$  has infinitely many zeros that are not zeros of  $f$ .

**Proof of Corollary 7:** Set  $g = \frac{1}{f}$  again. Since the poles of  $g$  are the zeros of  $f$ , we have  $\phi_g(r) \leq cr^d$ . Consequently, by Theorem 1,  $g'$  has no quasi-exceptional value.

**Proof of Theorem 2:** Suppose  $f'$  admits a quasi-exceptional value  $b \in \mathbb{K}^*$ . Then  $f'$  is of the form  $\frac{P}{h}$  with  $P \in \mathbb{K}[x]$  and  $h$  a transcendental entire function. Consequently there exists  $S > 0$  such that  $\frac{|P|(r)}{|h|(r)} < |b| \forall r > S$  and hence  $|f'|(r) = |b| \forall r > S$ . Then by Lemma 2, the numbers of zeros and poles of  $f'$  in disks  $d(0, r)$  are equal when  $r$  when  $r > S$ . So, there exists  $S' \geq S$  such that for every  $r > S'$  we have

$$(1) \quad \tau_{f'}(r) = \theta_{f'}(r)$$

On the other hand, of course we have  $\tau_f(r) < \tau_{f'}(r)$ , hence by (1) and by hypothesis of Theorem 2, we have  $\tau_f(r) < r^d$ . Therefore by Theorem 1,  $f'$  has no quasi-exceptional value, a contradiction.

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