



HAL
open science

Zeros of the derivative of a p-adic meromorphic function

Jean-Paul Bézivin, Kamal Boussaf, Alain Escassut

► **To cite this version:**

Jean-Paul Bézivin, Kamal Boussaf, Alain Escassut. Zeros of the derivative of a p-adic meromorphic function. Bulletin des Sciences Mathématiques, 2012. hal-01907050

HAL Id: hal-01907050

<https://uca.hal.science/hal-01907050v1>

Submitted on 28 Oct 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Zeros of the derivative of a p -adic meromorphic function

Jean-Paul Bézivin, Kamal Boussaf, Alain Escassut

2012/ 06/29

Abstract

Let \mathbb{K} be a complete algebraically closed field of characteristic 0 and let f be a transcendental meromorphic function in \mathbb{K} . A conjecture suggests that f' takes every values infinitely many times, what was proved when f has finitely many multiple poles. Here we can generalize the conclusion just by assuming that there exists positive constants c, d such that number of multiple poles inside the disk $|x| \leq r$ is less than cr^d for all $r \geq 1$. Applications are given to entire functions g in \mathbb{K} such that g' divides g , to links between residues and zeros of functions admitting primitives and finally to the p -adic Hayman conjecture in the cases that are not yet solved.

1 Introduction and results.

Throughout the paper, \mathbb{K} is an algebraically closed field of characteristic 0, complete with respect to a p -adic absolute value denoted by $|\cdot|$ (example \mathbb{C}_p). Given $a \in \mathbb{K}$ and $r > 0$ we denote by $d(0, r)$ the disk $\{x \in \mathbb{K} \mid |x - a| \leq r\}$.

We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} i.e. the set of power series with coefficients in \mathbb{K} converging in all \mathbb{K} and we denote by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} , i.e. the field of fraction of $\mathcal{A}(\mathbb{K})$. Given $f, g \in \mathcal{A}(\mathbb{K})$, we denote by $W(f, g)$ the Wronskian of f and g .

Let $f \in \mathcal{A}(\mathbb{K})$ be non-constant. We can factorize $f(x)$ in the form

$$f(x) = cx^{n_0} \prod_{\omega \in \Omega, \omega \neq 0} \left(1 - \frac{x}{\omega}\right)^{n_\omega},$$

where c is a non zero constant, Ω is the set of distinct zeros of f and $n_0 = 0$ if 0 is not a zero of f . We set $\bar{f} = x^{m_0} \prod_{\omega \in \Omega, \omega \neq 0} \left(1 - \frac{x}{\omega}\right)$, with $m_0 = 0$ if 0 is not a zero of f and $m_0 = 1$ else. So, the

function \bar{f} is an entire function admitting as zeros the distinct zeros of f , all with order 1.

We can then set $f = \bar{f}\tilde{f}$ where the function \tilde{f} is an entire function admitting for zeros the multiple zeros of f , each with order $q - 1$ when it is a zero of f of order q . Particularly, if f is constant, we set $\bar{f} = 1$ and $\tilde{f} = f$.

⁰2000 Mathematics Subject Classification: 12J25; 30D35; 30D45; 12E05.

⁰Keywords: p -adic meromorphic functions, Distribution of values, Picard's values.

Now, $f(x)$ is a power series $\sum_{n=0}^{\infty} a_n x^n$ of infinite radius of convergence. According to classical notation [7], we set $|f|(r) = \sup\{|f(x)| \mid |x| \leq r\}$. We know that

$$|f|(r) = \sup_{n \in \mathbb{N}} |a_n| r^n = \lim_{|x| \rightarrow r, |x| \neq r} |f(x)|.$$

That notation defines an absolute value on $\mathcal{A}(\mathbb{K})$ and has continuation to $\mathcal{M}(\mathbb{K})$ as

$|\frac{f}{g}|(r) = \frac{|f|(r)}{|g|(r)}$ with $f, g \in \mathcal{A}(\mathbb{K})$. In the paper [4], the following Theorems A and B are proven:

Theorem A: *Let f, g be entire functions on \mathbb{K} such that $W(f, g)$ is a non-identically zero polynomial. Then both f, g are polynomials.*

(Theorem A has been recently generalized to several functions [2])

Theorem B is an easy consequence of Theorem A:

Theorem B: *Let f be a transcendental meromorphic function on \mathbb{K} having finitely many multiple poles. Then f' takes every value infinitely many times.*

That has suggested the following conjecture:

Conjecture: *Let f be a meromorphic function on \mathbb{K} such that f' has finitely many zeros. Then f is a rational function.*

This paper is aimed at proving the following theorem which generalizes Theorem B. For this we will define new expressions.

Notation: Let $f \in \mathcal{M}(\mathbb{K})$. For each $r > 0$, we denote by $\psi_f(r)$ the number of multiple zeros of f in $d(0, r)$, each counted with its multiplicity and we set $\phi_f(r) = \psi_{\frac{1}{f}}(r)$. Similarly, we denote by $\theta_f(r)$ the number of zeros of f in $d(0, r)$, taking multiplicity into account and set $\tau_f(r) = \theta_{\frac{1}{f}}(r)$.

Theorem 1: *Let f be a meromorphic function on \mathbb{K} such that, for some $c, d \in]0, +\infty[$, ϕ_f satisfies $\phi_f(r) \leq cr^d$ in $[1, +\infty[$. If f' has finitely many zeros, then f is a rational function.*

Corollary 1: *Let f be a meromorphic function on \mathbb{K} such that, for some $c, d \in]0, +\infty[$, ϕ_f satisfies $\phi_f(r) \leq cr^d$ in $[1, +\infty[$. If for some $b \in \mathbb{K}$ $f' - b$ has finitely many zeros, then f is a rational function.*

Corollary 2: *Let f be a transcendental meromorphic function on \mathbb{K} such that $\tau_f(r) \leq cr^d$ in $[1, +\infty[$ for some $c, d \in]0, +\infty[$. Then $f^{(k)}$ takes every value in \mathbb{K} infinitely many times, for each $k \in \mathbb{N}^*$.*

Corollary 3: *Let h be a transcendental entire function on \mathbb{K} and $P \in \mathbb{K}[x]$. The differential equation $y'h = yP$ admits no entire solution f different from 0, such that, f for some $c, d \in]0, +\infty[$, $\psi_f(r) \leq cr^d$ in $[1, +\infty[$.*

By Main Theorem of [5] we can derive this corollary:

Corollary 4: *Let f be a meromorphic function on \mathbb{K} such that, for certain $d \in \mathbb{N}$, the number of poles of order ≥ 3 , counted each with its multiplicity, in each disk $d(0, r)$ is bounded by r^d for all $r \in [1, +\infty[$ and such that all residues at poles are null. Then for every $b \in \mathbb{K}$, $f - b$ has infinitely many zeros.*

According to the p -adic Hayman conjecture, for every $n \in \mathbb{N}^*$ $f'f^n$ takes every non-zero value infinitely many times. Here Theorem 1 has an immediate application to that conjecture in the cases $n = 1$ or $n = 2$ which are not yet solved, except with additional hypotheses [2], [3], [4], [8].

Corollary 5: *Let f be a meromorphic function on \mathbb{K} . Suppose that there exists $c, d \in]0, +\infty[$, such that $\tau_f(r) \leq cr^d \forall r \in [1, +\infty[$. If $f'f^n - b$ has finitely many zeros for some $b \in \mathbb{K}$, with $n \in \mathbb{N}$ then f is a rational function.*

Corollary 5 may be written in another way:

Corollary 6: *Let f be a transcendental meromorphic function on \mathbb{K} . Suppose that there exists $c, d \in]0, +\infty[$, such that $\theta_f(r) \leq cr^d \forall r \in [1, +\infty[$. Then for all $m \in \mathbb{N}$, $m \geq 3$ and for all $b \in \mathbb{K}^*$, $f' - bf^m$ admits infinitely many zeros that are not zeros of f .*

Corollary 7: *Let f be a transcendental meromorphic function on \mathbb{K} . Suppose that there exists $c, d \in]0, +\infty[$, such that $\psi_f(r) \leq cr^d \forall r \in [1, +\infty[$. Then, for all $b \in \mathbb{K}$, $\frac{f'}{f^2} - b$ has infinitely many zeros.*

Remark: Using Corollary 7 to study zeros of $f' + bf^2$ that are not zeros of f is not so immediate, as done in Theorems 3, 4, 5 [3], because of residues of f at poles of order 1.

Theorem 2: *Let f be a transcendental meromorphic function on \mathbb{K} such that, for some $c, d \in]0, +\infty[$, we have $\theta_{f'}(r) \leq cr^d$ in $[1, +\infty[$. Then for every $b \in \mathbb{K}$, $b \neq 0$, $f' - b$ has infinitely many zeros.*

2 Preliminary results

We will need several lemmas.

Lemma 1: *Let $U, V \in \mathcal{A}(\mathbb{K})$ have no common zero and let $f = \frac{U}{V}$. If f' has finitely many zeros, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\tilde{V}$*

Proof: If V is a constant, the statement is obvious. So, we assume that V is not a constant. Now \tilde{V} divides V' and hence V' factorizes in the way $V' = \tilde{V}Y$ with $Y \in \mathcal{A}(\mathbb{K})$. Then no zero of Y can be a zero of V . Consequently, we have

$$f'(x) = \frac{U'V - UV'}{V^2} = \frac{U'\bar{V} - UY}{\bar{V}^2\tilde{V}}.$$

The two functions $U'\bar{V} - UY$ and $\bar{V}^2\tilde{V}$ have no common zero since neither have U and V . Consequently, the zeros of f' are those of $U'\bar{V} - UY$ which therefore has finitely many zeros and consequently is a polynomial.

Lemma 2 is known as the p -adic Schwarz Lemma (Lemma 23.12 [6]). Lemmas 3 and 4 are immediate corollaries:

Notation: Let $a \in \mathbb{K}$, $r', r'' \in]0, +\infty[$ with $r' < r''$. We denote by $\Gamma(a, r', r'')$ the annulus $\{x \in \mathbb{K} \mid r' < |x - a| < r''\}$.

Lemma 2: Let $r, R \in]0, +\infty[$ be such that $r < R$ and let $f \in \mathcal{M}(\mathbb{K})$ admit s zeros and t poles in $d(0, r)$ and no zero and no pole in $\Gamma(0, r, R)$. Then $\frac{|f|(R)}{|f|(r)} = \left(\frac{R}{r}\right)^{s-t}$.

Lemma 3: Let $r, R \in]0, +\infty[$ be such that $r < R$ and let $f \in \mathcal{A}(\mathbb{K})$ have q zeros in $d(0, R)$. Then

$$\frac{|f|(R)}{|f|(r)} \leq \left(\frac{R}{r}\right)^q.$$

Lemma 4: Let $f \in \mathcal{A}(\mathbb{K})$. Then f is a polynomial of degree q if and only if there exists a constant c such that $|f|(r) \leq cr^q \forall r \in [1, +\infty[$.

Notation: Let $d(a, r^-)$ be the disk $\{x \in \mathbb{K} \mid |x - a| < r\}$. We denote by $\mathcal{A}(d(a, R^-))$ the \mathbb{K} -algebra of analytic functions in $d(a, R^-)$ i.e. the set of power series in $x - a$ with coefficients in \mathbb{K} whose radius of convergence is $\geq R$ and we denote by $\mathcal{M}(d(a, R^-))$ the field of meromorphic functions in $d(a, R^-)$, i.e. the field of fraction of $\mathcal{A}(d(a, R^-))$.

Lemma 5: Let $f \in \mathcal{M}(d(0, R^-))$. For each $n \in \mathbb{N}$, and for all $r \in]0, R[$, we have

$$|f^{(n)}|(r) \leq |n!| \frac{|f|(r)}{r^n}.$$

Proof: Suppose first f belongs to $\mathcal{A}(d(0, R^-))$ and set $f(x) = \sum_{k=0}^{\infty} a_k x^k$. Then

$$f^{(n)}(x) = \sum_{k=n}^{\infty} (n!) \binom{k}{n-k} a_k x^{k-n}.$$

The statement then is immediate. Consider now the general case and set $f = \frac{U}{V}$ with $U, V \in \mathcal{A}(d(0, R^-))$. The stated inequality is obvious when $n = 1$. So, we assume it holds for $q \leq n - 1$ and consider $f^{(n)}$. Writing $U = V \left(\frac{U}{V}\right)$, by Leibniz Theorem we have

$$U^{(n)} = \sum_{q=0}^n \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)}$$

and hence

$$V \left(\frac{U}{V}\right)^{(n)} = U^{(n)} - \sum_{q=0}^{n-1} \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)}.$$

Now, $|U^{(n)}|(R) \leq |n!| \frac{|U|(R)}{R^n}$ and for each $q \leq n - 1$, we have $|V^{(n-q)}|(R) \leq |(n - q)!| \frac{|V|(R)}{R^{n-q}}$ and

$\left|\left(\frac{U}{V}\right)^{(q)}\right|(R) \leq |q!| \frac{|U|(R)}{|V|(R)R^q}$. Therefore, we can derive that terms on the right hand side are upper

bounded by $|n!| \frac{|U|(R)}{|V|(R)R^n}$ and hence the conclusion holds for $q = n$.

Notation: For each $n \in \mathbb{N}^*$, we set $\lambda_n = \max\{\frac{1}{|k|}, 1 \leq k \leq n\}$.

Remark: For every $n \in \mathbb{N}^*$, we have $\lambda_n \leq n$ because $k|k| \geq 1 \forall k \in \mathbb{N}$. The equality holds for all n of the form p^h .

Lemma 6: Let $U, V \in \mathcal{A}(d(0, R^-))$. Then for all $r \in]0, R[$ and $n \geq 1$ we have

$$|U^{(n)}V - UV^{(n)}|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{r^{n-1}}.$$

More generally, given $j, l \in \mathbb{N}$, we have

$$|U^{(j)}V^{(l)} - U^{(l)}V^{(j)}|(r) \leq |(j!)(l!)|\lambda_{j+l} \frac{|U'V - UV'|(r)}{r^{j+l-1}}.$$

Proof: Set $g = \frac{U}{V}$ and $f = g'$. Applying Lemma 5 to f for $k-1$, we obtain

$$|g^{(k)}|(r) = |f^{(k-1)}|(r) \leq |(k-1)!| \frac{|f|(r)}{r^{k-1}} = |(k-1)!| \frac{|U'V - UV'|(r)}{|V^2|(r)r^{k-1}}.$$

As in the proof of Lemma 5, we set $U = V\left(\frac{U}{V}\right)$. By Leibniz formula again, now we can obtain

$$U^{(n)} = \sum_{q=1}^n \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)} + V^{(n)} \left(\frac{U}{V}\right)$$

hence

$$(1) \quad U^{(n)} - V^{(n)} \left(\frac{U}{V}\right) = \sum_{q=1}^n \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)}$$

Now we have

$$\left| \left(\frac{U}{V}\right)^{(q)} \right|(r) = |g^{(q)}|(r) \leq |(q-1)!| \frac{|U'V - UV'|(r)}{|V^2|(r)r^{q-1}}$$

and

$$|V^{(n-q)}|(r) \leq |(n-q)!| \frac{|V|(r)}{r^{n-q}}.$$

Consequently, the general term in (1) is upper bounded as

$$\left| \binom{n}{q} V^{(n-q)} \left(\frac{U}{V}\right)^{(q)} \right|(r) \leq \frac{|(n!)((n-q)!)((q-1)!|}{|(q!)((n-q)!|} \frac{|U'V - UV'|(r)}{|V|(r)r^{n-1}} \leq \lambda_n \frac{|n!||U'V - UV'|(r)}{|V|(r)r^{n-1}}.$$

Therefore by (1) we obtain

$$\left| U^{(n)} - V^{(n)} \left(\frac{U}{V}\right) \right|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{|V|(r)r^{n-1}}$$

and finally

$$\left| U^{(n)}V - V^{(n)}U \right|(r) \leq |n!|\lambda_n \frac{|U'V - UV'|(r)}{r^{n-1}}.$$

We can now generalize the first statement. Set $P_j = U^{(j)}V - UV^{(j)}$. By induction, we can show the following equality that already holds for $l \leq j$:

$$U^{(j)}V^{(l)} - U^{(l)}V^{(j)} = \sum_{h=0}^l \binom{l}{h} (-1)^h P_{j+h}^{(l-h)}$$

Now, the second statement gets just an application of the first.

Remark: Suppose the residue characteristic is $p \neq 0$ and $U(x) = 1, V(x) = x^p, n = p$. Then, for all $R > 0$, the inequality $|U^{(p)}V - UV^{(p)}|(R) \leq |(p!)| \frac{|U'V - UV'|}{R^{p-1}}(R)$ is not satisfied. So, we can't eliminate the factor λ_n .

Lemma 7: Let $U, V \in \mathcal{A}(\mathbb{K})$ and let $r, R \in]0, +\infty[$ satisfy $r < R$. For all $x, y \in \mathbb{K}$ with $|x| \leq R$ and $|y| \leq r$, we have the inequality:

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|}{e(\log R - \log r)}(R)$$

Proof: By Taylor's formula at the point x , we have

$$U(x+y)V(x) - U(x)V(x+y) = \sum_{n \geq 0} \frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!} y^n$$

Now, $\left| \frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!} y^n \right| \leq \lambda_n \frac{|U'V - UV'|}{R^{n-1}}(R) r^n$. But as remarked above, we have $\lambda_n \leq n$, hence

$$\left| \frac{U^{(n)}(x)V(x) - U(x)V^{(n)}(x)}{n!} y^n \right| \leq nR|U'V - UV'| \left(\frac{r}{R}\right)^n.$$

And we notice that $\lim_{n \rightarrow +\infty} n \left(\frac{r}{R}\right)^n = 0$. Consequently, we can define $B = \max_{n \geq 1} \left(n \left(\frac{r}{R}\right)^n\right)$ and we have $|U(x+y)V(x) - U(x)V(x+y)| \leq BR|U'V - UV'| \forall x \in d(0, R), y \in d(0, r)$.

We can check that the function h defined in $]0, +\infty[$ as $h(t) = t \left(\frac{r}{R}\right)^t$ reaches its maximum at the point $u = \frac{1}{(\log R - \log r)}$. Consequently, $B \leq \frac{1}{e(\log R - \log r)}$ and therefore

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|}{e(\log R - \log r)}(R).$$

Proof of Theorem 1: Suppose f' has finitely many zeros. If V is a constant, the statement is immediate. So, we suppose V is not a constant and hence it admits at least one zero a . By Lemma 1, there exists a polynomial $P \in \mathbb{K}[x]$ such that $U'V - UV' = P\tilde{V}$. Next, we take $r, R \in [1, +\infty[$ such that $|a| < r < R$ and $x \in d(0, R), y \in d(0, r)$. By Lemma 7 we have

$$|U(x+y)V(x) - U(x)V(x+y)| \leq \frac{R|U'V - UV'|}{e(\log R - \log r)}(R).$$

Notice that $U(a) \neq 0$ because U and V have no common zero. Now set $l = \max(1, |a|)$ and take $r \geq l$. Setting $c_1 = \frac{1}{e|U(a)|}$, we have

$$|V(a+y)| \leq c_1 \frac{R|P|(R)|\tilde{V}|(R)}{\log R - \log r}.$$

Then taking the supremum of $|V(a+y)|$ inside the disk $d(0, r)$, we can derive

$$(1) \quad |V|(r) \leq c_1 \frac{R|P|(R)|\tilde{V}|(R)}{\log R - \log r}.$$

Let us apply Lemma 3, by taking $R = r + \frac{1}{r^d}$, after noticing that the number of zeros of $\tilde{V}(R)$ is bounded by $\psi_V(R)$. So, we have

$$(2) \quad |\tilde{V}|(R) \leq \left(1 + \frac{1}{r^{d+1}}\right)^{\psi_V(r + \frac{1}{r^d})} |\tilde{V}|(r).$$

Now, due to the hypothesis: $\psi_V(r) = \phi_f(r) \leq cr^d$ in $[1, +\infty[$, we have

$$(3) \quad \left(1 + \frac{1}{r^{d+1}}\right)^{\psi_V(r + \frac{1}{r^d})} \leq \left(1 + \frac{1}{r^{d+1}}\right)^{[c(r + \frac{1}{r^d})^d]} = \exp \left[c \left(r + \frac{1}{r^d}\right)^d \log \left(1 + \frac{1}{r^{d+1}}\right) \right].$$

The function $h(r) = c \left(r + \frac{1}{r^d}\right)^d \log \left(1 + \frac{1}{r^{d+1}}\right)$ is continuous on $]0, +\infty[$ and equivalent to $\frac{c}{r}$ when r tends to $+\infty$. Consequently, it is bounded on $[l, +\infty[$. Therefore, by (2) and (3) there exists a constant $M > 0$ such that, for all $r, R \in [l, +\infty[$, $r < R$ by (3) we obtain

$$(4) \quad |\tilde{V}|(r + \frac{1}{r^d}) \leq M |\tilde{V}|(r).$$

On the other hand, $\log \left(r + \frac{1}{r^d}\right) - \log r = \log \left(1 + \frac{1}{r^{d+1}}\right)$ clearly satisfies an inequality of the form $\log \left(1 + \frac{1}{r^{d+1}}\right) \geq \frac{c_2}{r^{d+1}}$ in $[l, +\infty[$ with $c_2 > 0$. Moreover, we can obviously find positive constants c_3, c_4 such that

$$\left(r + \frac{1}{r^d}\right) |P| \left(r + \frac{1}{r^d}\right) \leq c_3 r^{c_4}.$$

Consequently, by (1) and (4) we can find positive constants c_5, c_6 such that $|V|(r) \leq c_5 r^{c_6} |\tilde{V}|(r) \forall r \in [l, +\infty[$. Thus, writing again $V = \bar{V}\tilde{V}$, we have $|\bar{V}|(r) |\tilde{V}|(r) \leq c_5 r^{c_6} |\tilde{V}|(r)$ and hence $|\bar{V}|(r) \leq c_5 r^{c_6} \forall r \in [l, +\infty[$. Consequently, by Lemma 4, \bar{V} is a polynomial of degree $\leq c_6$ and hence it has finitely many zeros and so does V . And then, by Theorem B, f must be a rational function.

Proof of Corollary 1: Suppose $f' - b$ has finitely many zeros. Then $f - bx$ satisfies the same hypothesis as f , hence it is a rational function and so is f .

Proof of Corollary 2: Indeed, if $k = 1$, the statement just comes from Theorem 1. Now suppose $k \geq 2$. Each pole a of order n of f is a pole of order $n + k$ of $f^{(k)}$ and $f^{(k)}$ has no other pole. Consequently, we have $\phi_{f^{(k-1)}}(r) = \tau_{f^{(k-1)}}(r) \leq kcr^d$. So, we can apply Theorem 1 to $f^{(k-1)}$ to show the claim.

Proof of Corollary 3: Suppose V is an entire solution of the equation $V'h = VP$ such that $\psi_V(r) \leq cr^d \forall r \in [1, +\infty[$. If V is a rational function, so is $\frac{VP}{V'}$, a contradiction because h is transcendental. Suppose now V is transcendental and apply Theorem 1: then $(\frac{1}{V})' = -\frac{V'}{V^2} = \frac{-P}{hV}$ admits infinitely many zeros, a contradiction since P is a polynomial.

Definition: Given a meromorphic function in \mathbb{K} , we call *exceptional value of f* (or *Picard value of f*) a value $b \in \mathbb{K}$ such that $f - b$ has no zero. And, if f is transcendental, we call *quasi-exceptional value* a value $b \in \mathbb{K}$ such that $f - b$ has finitely many zeros.

Proof of Corollary 4: By the Main Theorem of [5], f admits a primitive F and then, by hypothesis, we have $\phi_F(r) \leq r^d$. Consequently, by Corollary 1 $F' - b = f - b$ has infinitely many zeros.

Proof of Corollary 5: Suppose f is transcendental. Due to hypothesis, f^{n+1} satisfies $\theta_{\frac{1}{f^{n+1}}}(r) = \tau_{f^{n+1}}(r) \leq c(n+1)r^d \forall r \in [1, +\infty[$ hence by Theorem 1, $f'f^n$ has no quasi-exceptional value.

Proof of Corollary 6: We set $g = \frac{1}{f}$. Then by Corollary 2, $g'g^{m-2}$ has no quasi-exceptional value. Consequently, given $b \in \mathbb{K}^*$, $g'g^{m-2} + b$ has infinitely many zeros and hence $f' - bf^m$ has infinitely many zeros that are not zeros of f .

Proof of Corollary 7: Set $g = \frac{1}{f}$ again. Since the poles of g are the zeros of f , we have $\phi_g(r) \leq cr^d$. Consequently, by Theorem 1, g' has no quasi-exceptional value.

Proof of Theorem 2: Suppose f' admits a quasi-exceptional value $b \in \mathbb{K}^*$. Then f' is of the form $\frac{P}{h}$ with $P \in \mathbb{K}[x]$ and h a transcendental entire function. Consequently there exists $S > 0$ such that $\frac{|P|(r)}{|h|(r)} < |b| \forall r > S$ and hence $|f'(r)| = |b| \forall r > S$. Then by Lemma 2, the numbers of zeros and poles of f' in disks $d(0, r)$ are equal when $r > S$. So, there exists $S' \geq S$ such that for every $r > S'$ we have

$$(1) \quad \tau_{f'}(r) = \theta_{f'}(r)$$

On the other hand, of course we have $\tau_f(r) < \tau_{f'}(r)$, hence by (1) and by hypothesis of Theorem 2, we have $\tau_f(r) < r^d$. Therefore by Theorem 1, f' has no quasi-exceptional value, a contradiction.

References

- [1] Bezin, Jean-Paul. Wronskien et équations différentielles p -adiques. (Preprint).
- [2] Boussaf, Kamal. Picard values of p -adic meromorphic functions. *p-Adic Numbers Ultrametric Anal. Appl.* 2, no. 4, p.285-292 (2010).
- [3] Boussaf, Kamal; Ojeda, Jacqueline. Value distribution of p -adic meromorphic functions. *Bull. Belg. Math. Soc. Simon Stevin*, 18, n.4, 2011, p 667-678 (2011).

- [4] Boussaf, Kamal; Escassut, Alain; Ojeda, Jacqueline. Zeros of the derivative of p -adic meromorphic functions and applications. Bull. Belg. Math. Soc. Simon Stevin, vol 19, n.2, p. 237-372 (2012).
- [5] Boussaf, Kamal; Escassut, Alain; Ojeda, Jacqueline. Primitives of p -adic meromorphic functions. Contemporary Mathematics, Vol. 551, p.51-56 (2011).
- [6] Escassut, Alain. Analytic elements in p -adic analysis. World Scientific Publishing, Co Pte Ltd, Singapore (1995).
- [7] Escassut, Alain. Topics on value distribution and differentiability: p -adic Distribution Value Scientific Press (2008).
- [8] Ojeda, Jacqueline. Hayman's conjecture in a p -adic field. Taiwanese J. Math. 12 , no. 9, p. 2295- 2313 (2008).

Jean-Paul Bézivin
1, Allée Edouard Quincey
94200 Ivry sur seine
jp.bezivin@orange.fr

Kamal Boussaf and Alain Escassut
Laboratoire de Mathématiques CNRS UMR 6620
Université Blaise Pascal
Les Cézeaux
63171 Aubière
FRANCE
kamal.boussaf@math.univ-bpclermont.fr
alain.escassut@math.univ-bpclermont.fr