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# Zeroes of the derivative of a p-adic meromorphic function and applications.

by Kamal Boussaf, Jacqueline Ojeda and Alain Escassut

**Abstract** Let  $K$  be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value. We show that if the Wronskian of two entire functions in  $K$  is a polynomial, then both functions are polynomials. As a consequence, if a meromorphic function  $f$  on all  $K$  is transcendental and has finitely many multiple poles, then  $f'$  takes all values in  $K$  infinitely many times. We then study applications to a meromorphic function  $f$  such that  $f' + bf^2$  has finitely many zeroes, a problem linked to the Hayman conjecture on a p-adic field.

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## Notation and Definitions.

Let  $K$  be an algebraically closed field of characteristic 0, complete with respect to an ultrametric absolute value  $|\cdot|$ . Given  $\alpha \in K$  and  $R \in \mathbb{R}_+^*$ , we denote by  $d(\alpha, R)$  the disk  $\{x \in K \mid |x - \alpha| \leq R\}$  and by  $d(\alpha, R^-)$  the disk  $\{x \in K \mid |x - \alpha| < R\}$ , by  $\mathcal{A}(K)$  the  $K$ -algebra of analytic functions in  $K$  (i.e. the set of power series with an infinite radius of convergence), by  $\mathcal{M}(K)$  the field of meromorphic functions in  $K$  and by  $K(x)$  the field of rational functions. Given  $f, g \in \mathcal{A}(K)$ , we denote by  $W(f, g)$  the Wronskian  $f'g - fg'$ .

We know that any non-constant entire function  $f \in \mathcal{A}(K)$  takes all values in  $K$ . More precisely, a function  $f \in \mathcal{A}(K)$  other than a polynomial takes all values in  $K$  infinitely many times (see [5], [8], [9]). Next, a non-constant meromorphic function  $f \in \mathcal{M}(K)$  takes every value in  $K$ , except at most one value. And more precisely, a meromorphic function  $f \in \mathcal{M}(K) \setminus K(x)$  takes every value in  $K$  infinitely many times except at most one value (see [5], [9]).

Many previous studies were made on Picard's exceptional values for complex and p-adic functions and their derivatives (see [1], [3], [6], [7], [8]). Here we mean to examine precisely whether the derivative of a transcendental meromorphic function in  $K$  having finitely many multiple poles, may admit a value that is taken finitely many times and then we will look for applications to Hayman's problem when  $m = 2$ .

From Theorem 4 [6], we can state the following Theorem A:

**Theorem A:** *Let  $h, l \in \mathcal{A}(K)$  satisfy  $W(h, l) = c \in K$ , with  $h$  non-affine. Then  $c = 0$  and  $\frac{h}{l}$  is a constant.*

Now we can improve Theorem A:

**Theorem 1:** *Let  $f, g \in \mathcal{A}(K)$  be such that  $W(f, g)$  is a non-identically zero polynomial. Then both  $f, g$  are polynomials.*

**Remark:** Theorem 1 does not hold in characteristic  $p \neq 0$ . Indeed, suppose the characteristic of  $K$  is  $p \neq 0$ . Let  $\psi \in \mathcal{A}(K)$ . Let  $f = x(\psi)^p$  and let  $g = (x + 1)\psi^{-p}$ . Since  $p \neq 0$ , we have  $f' = (\psi)^p$ ,  $g' = \psi^{-p}$  hence  $W(f, g) = 1$  and this is true for any function  $\psi \in \mathcal{A}(K)$ .

**Theorem 2:** *Let  $f \in \mathcal{M}(K) \setminus K(x)$  have finitely many multiple poles. Then  $f'$  takes every value  $b \in K$  infinitely many times.*

We can easily show Corollary 2.1 from Theorem 2, though it is possible to get it through an expansion in simple elements.

**Corollary 2.1:** *Let  $f \in \mathcal{M}(K) \setminus K(x)$ . Then  $f'$  belongs to  $\mathcal{M}(K) \setminus K(x)$ .*

**Open question:** Do exist transcendental meromorphic functions  $f$  such that  $f'$  has finitely many zeroes? By Theorem 2, such functions should have infinitely many multiple poles.

Now, we can look for some applications to Hayman's problem in a p-adic field. Let  $f \in \mathcal{M}(K)$ . Recall that in [9], [10] it was shown that if  $m$  is an integer  $\geq 5$  or  $m = 1$ , then  $f' + f^m$  has infinitely many zeroes that are not zeroes of  $f$ . If  $m = 3$  or  $m = 4$ , for many functions  $f \in \mathcal{M}(K)$ ,  $f' + f^m$  has infinitely many zeroes that are not zeroes of  $f$  (see [2], [10]) but there remain some cases where it is impossible to conclude, except when the field has residue characteristic equal to zero (see [10]). When  $m = 2$ , few results are known. Recall also that as far as complex meromorphic functions  $f$  are concerned,  $f' + f^m$  has infinitely many zeroes that are not zeroes of  $f$  for every  $m \geq 3$ , but obvious counter-examples show this is wrong for  $m = 1$  (with  $f(x) = e^x$ ) and for  $m = 2$  (with  $f(x) = \tan(-x)$ ). Here we will particularly examine functions  $f' + bf^2$ , with  $b \in K^*$ .

**Theorem 3:** *Let  $b \in K^*$  and let  $f \in \mathcal{M}(K)$  have finitely many zeroes and finitely many residues at its simple poles equal to  $\frac{1}{b}$  and be such that  $f' + bf^2$  has finitely many zeroes. Then  $f$  belongs to  $K(x)$ .*

**Remark:** If  $f(x) = \frac{1}{x}$ , the function  $f' + bf^2$  has no zero whenever  $b \neq 1$ .

**Theorem 4:** *Let  $f \in \mathcal{M}(K) \setminus K(x)$  have finitely many multiple zeroes and let  $b \in K$ . Then  $\frac{f'}{f^2} + b$  has infinitely many zeroes. Moreover, if  $b \neq 0$ , every zero  $\alpha$  of  $\frac{f'}{f^2} + b$  that is not a zero of  $f' + bf^2$  is a simple pole of  $f$  such that the residue of  $f$  at  $\alpha$  is equal to  $\frac{1}{b}$ .*

**Corollary 4.1 :** *Let  $b \in K^*$  and let  $f \in \mathcal{M}(K) \setminus K(x)$  have finitely many multiple zeroes and finitely many simple poles. Then  $f' + bf^2$  has infinitely many zeroes that are not zeroes of  $f$ .*

**Remark:** In Archimedean analysis, the typical example of a meromorphic function  $f$  such that  $f' + f^2$  has no zero is  $\tan(-x)$  and its residue is 1 at each pole of  $f$ . Here we find the same implication but we can't find an example satisfying such properties.

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## The Proofs.

**Notation:** Given  $f \in \mathcal{A}(K)$  and  $r > 0$ , we denote by  $|f|(r)$  the norm of uniform convergence on the disk  $d(0, r)$ . This norm is known to be multiplicative (see [4], [5]).

Lemma 1 is well known (see Theorem 13.5 [4]) :

**Lemma 1:** *Let  $f \in \mathcal{M}(K)$ . Then  $|f^{(k-1)}|(r) \leq \frac{|f|(r)}{r^{k-1}} \forall r > 0, \forall k \in \mathbb{N}^*$ .*

**Proof of Theorem 1:** First, by Theorem A, we check that the claim is satisfied when  $W(f, g)$  is a polynomial of degree 0. Now, suppose the claim holds when  $W(f, g)$  is a polynomial of certain degree  $d$ . We will show it for  $d + 1$ . Let  $f, g \in \mathcal{A}(K)$  be such that  $W(f, g)$  is a non-identically zero polynomial  $P$  of degree  $d + 1$ .

By hypothesis, we have  $f'g - fg' = P$ , hence  $f''g - fg'' = P'$ . We can extract  $g'$  and get  $g' = \frac{f'g - P}{f}$ . Now, consider the function  $Q = f''g' - f'g''$  and replace  $g'$  by what we just found: we can get  $Q = f'(\frac{f''g - fg''}{f}) - \frac{Pf''}{f}$ .

Now, we can replace  $f''g - fg''$  by  $P'$  and obtain  $Q = \frac{f'P' - Pf''}{f}$ . Thus, in that expression of  $Q$ , we can write  $|Q|(R) \leq \frac{|f|(R)|P|(R)}{R^2|f|(R)}$ , hence  $|Q|(R) \leq \frac{|P|(R)}{R^2} \forall R > 0$ . But by definition,  $Q$  belongs to  $\mathcal{A}(K)$  and further,  $\deg(Q) \leq \deg(P) - 2$ . Consequently,  $Q$  is a polynomial of degree at most  $d - 2$ .

Now, suppose  $Q$  is not identically zero. Since  $Q = W(f', g')$  and since  $\deg(Q) < d$ , by induction  $f'$  and  $g'$  are polynomials and so are  $f$  and  $g$ . Finally, suppose  $Q = 0$ . Then  $P'f' - Pf'' = 0$  and therefore  $f'$  and  $P$  are two solutions of the differential equation of order 1 for meromorphic functions in  $K : (\mathcal{E}) y' = \psi y$  with  $\psi = \frac{P'}{P}$ , whereas  $y$  belongs to  $\mathcal{A}(K)$ . The space of solutions of  $(\mathcal{E})$  is known to be of dimension 0 or 1 (see for instance Lemma 60.1 in [4]). Consequently, there exists  $\lambda \in K$  such that  $f' = \lambda P$ , hence  $f$  is a polynomial. The same holds for  $g$ .

**Proof of Theorem 2:** Suppose  $f'$  has finitely many zeroes. By classical results (see [4], [5]) we can write  $f$  in the form  $\frac{h}{l}$  with  $h, l \in \mathcal{A}(K)$ , having no common zero. Consequently, each zero of  $W(h, l)$  is a zero of  $f'$  except if it is a multiple zero of  $l$ . But since  $l$  only has finitely many multiple zeroes, it appears that  $W(h, l)$  has finitely many zeroes and therefore is a polynomial. Consequently, by Theorem 1, both  $h$  and  $l$  are polynomials, a contradiction because  $f$  does not belong to  $K(x)$ .

Now, consider  $f' - b$  with  $b \in K$ . It is the derivative of  $f - bx$  whose poles are exactly those of  $f$ , taking multiplicity into account. Consequently,  $f' - b$  also has infinitely many zeroes.

**Notation:** Given  $f \in \mathcal{M}(K)$ , we will denote by  $\text{res}_a(f)$  the residue of  $f$  at  $a$ .

**Lemma 2:** Let  $f = \frac{h}{l} \in \mathcal{M}(K)$  with  $h, l \in \mathcal{A}(K)$  having no common zero, let  $b \in K^*$  and let  $a \in K$  be a zero of  $h'l - hl' + bh^2$  that is not a zero of  $f' + bf^2$ . Then  $a$  is a simple pole of  $f$  and  $\text{res}_a(f) = \frac{1}{b}$ .

**Proof:** Clearly, if  $l(a) \neq 0$ ,  $a$  is a zero of  $f' + bf^2$ . Hence, a zero  $a$  of  $h'l - hl' + bh^2$  that is not a zero of  $f' + bf^2$  is a pole of  $f$ . Now, when  $l(a) = 0$ , we have  $h(a) \neq 0$  hence  $l'(a) = bh(a) \neq 0$  and therefore  $a$  is a simple pole of  $f$  such that  $\frac{h(a)}{l'(a)} = \frac{1}{b}$ . But since  $a$  is a simple pole of  $f$ , we have  $\text{res}_a(f) = \frac{h(a)}{l'(a)}$  which ends the proof.

**Proof of Theorem 3:** Let  $f = \frac{P}{l}$  with  $P$  a polynomial,  $l \in \mathcal{A}(K)$  having no common zero with  $P$ . Then  $f' + bf^2 = \frac{P'l - l'P + bP^2}{l^2}$ . By hypothesis, this function has finitely many zeroes. Moreover, if  $a$  is a zero of  $P'l - l'P + bP^2$  but is not a zero of  $f' + bf^2$  then, by Lemma 2,  $a$  is a simple pole of  $f$  such that  $\text{res}_a(f) = \frac{1}{b}$ . Consequently,  $P'l - l'P + bP^2$  has finitely many zeroes and so we may write  $\frac{P'l - l'P + bP^2}{l^2} = \frac{Q}{l^2}$  with  $Q \in K[x]$ , hence  $P'l - l'P = -bP^2 + Q$ . But then, by Theorem 1,  $l$  is a polynomial, which ends the proof.

**Proof of Theorem 4:** Let  $g = \frac{f'}{f^2} + b$ . Suppose  $b = 0$ . Since all zeroes of  $f$  are simple zeroes except maybe finitely many,  $g$  has finitely many poles of order  $\geq 3$ , hence a primitive  $G$  of  $g$  has finitely many multiple poles. Consequently, by Theorem 2,  $g$  has infinitely many zeroes.

Now, suppose  $b \neq 0$ . Let  $\alpha$  be a zero of  $g$  and let  $f = \frac{h}{l}$  with  $h, l \in \mathcal{A}(K)$  having no common zero. Then  $\frac{f'}{f^2} + b = \frac{h'l - hl' + bh^2}{h^2}$ . Since  $\alpha$  is a zero of  $\frac{f'}{f^2} + b$ , it is not a zero of  $h$  and hence it is a zero of  $h'l - hl' + bh^2$ . Then by Lemma 2, if it is not a zero of  $f' + bf^2$ , it is a simple pole of  $f$  such that  $\text{res}_\alpha(f) = \frac{1}{b}$ , which ends the proof of Theorem 4.

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