

The p -adic Hayman conjecture when $n = 2$

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Abstract

Let \mathbb{K} be a complete ultrametric algebraically closed field of characteristic 0. According to the p -adic Hayman conjecture, given a transcendental meromorphic function f in \mathbb{K} , for each $n \in \mathbb{N}^*$, $f^n f'$ takes every value $b \neq 0$ infinitely many times. It was proven by the second author for $n \geq 3$. Here we prove it for $n = 2$ by using properties of meromorphic functions having finitely many multiple poles.

1 Introduction

Let \mathbb{K} be a complete ultrametric algebraically closed field of characteristic 0. We denote by $\mathcal{A}(\mathbb{K})$ the \mathbb{K} -algebra of entire functions in \mathbb{K} and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in \mathbb{K} , i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$. Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and let $b \in \mathbb{K}$. Similarly to classical definitions in complex analysis [15], b is called *an exceptional value for f* or *a Picard value for f* if $f - b$ has no zero in \mathbb{K} and b is called *a quasi-exceptional value for f* if $f - b$ has finitely many zeros in \mathbb{K} . By classical results [13], [10], we know that f has at most one quasi-exceptional value and if $f \in \mathcal{A}(\mathbb{K})$, then f has no quasi-exceptional value.

On \mathbb{C} , considering a transcendental meromorphic function f in the whole field, W. Hayman showed that for every $n \geq 3$, the function $f^n f'$ has no quasi-exceptional value different from 0 and he conjectured that the statement remains true for $n = 2$ and $n = 1$ [12]. That was proved for $n = 2$ by E. Mues [14] and for $n = 1$ by W. Bergweiler and Eremenko [1] and separately by H. Chen and M. Fang [8].

On the field \mathbb{K} , the same question makes sense too and similarly, J. Ojeda proved that for every $n \geq 3$, $f^n f'$ has no quasi-exceptional value different from 0 [16]. For $n = 2$ and $n = 1$, several particular solutions were given concerning subclasses of meromorphic functions. In [5] it was shown that if f has infinitely many multiple zeros or infinitely many multiple poles, then $f^2 f'$ has no quasi-exceptional value $b \neq 0$. On the other hand, on [2], it was shown that if the number of multiple poles of f in disks of diameter r is upper bounded by a power of r , then f' has infinitely many zeros and this implies that $f^n f'$ takes every value infinitely many times. In [16], it was also shown that the same conclusion holds if f is optimal, i.e. if there exists a sequence $(r_n)_{n \in \mathbb{N}}$ such that $\lim_{n \rightarrow +\infty} r_n = +\infty$ and such that the difference between the number of zeros and the number of poles inside the disk $|x| < r_n$ is prime to the residue characteristic of \mathbb{K} . In [4], it was shown that a

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function f violating the conjecture should satisfy other conditions. Several other works were made on close topics [11], [17].

But so far, no solution has been given yet to prove the conjecture for $n = 1$ and $n = 2$ in the general case. Here we mean to solve the conjecture for $n = 2$, without restriction.

Theorem : *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental. Then for every $b \in \mathbb{K}$ different from 0, $f^2 f' - b$ has infinitely many zeros.*

2 The proof

We will use the techniques of the Nevanlinna Theory. Let us recall the definition of the counting functions and the Nevanlinna function.

Let $f \in \mathcal{M}(\mathbb{K})$, let $a \in \mathbb{K}$ and let $r \in]0, +\infty[$. By classical results [9], [10] we know that $|f(x)|$ has a limit when $|x - a|$ tends to r , while being different from r . We set

$$|f|(r) = \lim_{|x-a| \rightarrow r, |x-a| \neq r} |f(x)|.$$

Let \log be a real logarithm function of base $b > 1$. Let $f \in \mathcal{M}(\mathbb{K})$ and let $\gamma \in \mathbb{K}$. If γ is a zero of f of order n we set $\omega_\gamma(f) = n$. If f has a pole of order n at γ , we set $\omega_\gamma(f) = -n$ and finally, if $f(\gamma) \neq 0, \infty$, we set $\omega_\gamma(f) = 0$.

We denote by $Z(r, f)$ the *counting function of zeros of f in $d(0, r)$* , counting multiplicities, i.e.

$$Z(r, f) = \max(\omega_0(f), 0) \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} \omega_\gamma(f) (\log r - \log |\gamma|).$$

Similarly, we denote by $\bar{Z}(r, f)$ the counting function of zeros of f in $d(0, r)$, ignoring multiplicities, and set

$$\bar{Z}(r, f) = u \log r + \sum_{\omega_\gamma(f) > 0, 0 < |\gamma| \leq r} (\log r - \log |\gamma|)$$

with $u = 1$ when $\omega_0(f) > 0$ and $u = 0$ else. In the same way, we set $N(r, f) = Z\left(r, \frac{1}{f}\right)$ (resp.

$\bar{N}(r, f) = \bar{Z}\left(r, \frac{1}{f}\right)$) to denote the *counting function of poles of f in $d(0, r)$* , counting multiplicities (resp. ignoring multiplicities).

Let S be a finite subset of \mathbb{K} . We denote by $Z_0^S(r, f')$ the counting function of zeros c of f' when $f(c) \notin S$ i.e.

$$Z_0^S(r, f') = \max(\omega_0(f'), 0) \log r + \sum_{\substack{\omega_\gamma(f') > 0, 0 < |\gamma| \leq r \\ f(\gamma) \notin S}} \omega_\gamma(f') (\log r - \log |\gamma|).$$

For $f \in \mathcal{M}(\mathbb{K})$ or $f \in \mathcal{M}(d(0, R^-))$, we call *Nevanlinna function of f or characteristic function of f* the function $T(r, f) = \max \{Z(r, f), N(r, f)\}$.

We will use these classical lemmas [10]:

Lemma 1: *Let $f \in \mathcal{M}(\mathbb{K})$. Then $\log(|f|(r)) = Z(r, f) - N(r, f) + O(1)$. Moreover, $Z(r, f') \leq Z(r, f) + \bar{N}(r, f) - \log r + O(1)$ and $N(r, f') = N(r, f) + \bar{N}(r, f)$.*

We must recall the p -adic Main Fundamental Theorem [7], [13], [10]:

Theorem N: Let $f \in \mathcal{M}(\mathbb{K})$ (resp. $f \in \mathcal{M}(d(0, R^-))$) and let $a_1, \dots, a_q \in \mathbb{K}$ be distinct and let $S = \{a_1, \dots, a_q\}$. Then

$$(q-1)T(r, f) \leq \sum_{j=1}^q \overline{Z}(r, f - a_j) + \overline{N}(r, f) - Z_0^S(r, f') - \log r + O(1)$$

We will also need Theorem A given in [6] and Theorem B given in [5]:

Theorem A: Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and have finitely many multiple poles. Then f' has no quasi-exceptional value.

Theorem B: Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and have infinitely many multiple poles or infinitely many multiple zeros. Then $f^2 f'$ takes every value infinitely many times.

Lemma 2: Let $f \in \mathcal{M}(\mathbb{K})$ be such that f' has finitely many multiple zeros. Then $\frac{f''f}{(f')^2}$ has no quasi-exceptional value.

Proof: Let $g = \frac{f}{f'}$. A pole of g is a zero of f' , hence by hypothesis, g has finitely many multiple poles. Consequently, by Theorem A, g' has no quasi-exceptional value. And hence neither has $1 - g'$. But $g' = \frac{(f')^2 - f''f}{(f')^2} = 1 - \frac{f''f}{(f')^2}$. Therefore $\frac{f''f}{(f')^2}$ has no quasi-exceptional value.

Lemma 3: Let $f \in \mathcal{M}(\mathbb{K})$ have finitely many multiple zeros. Then $f''f + 2(f')^2$ has infinitely many zeros that are not zeros of f .

Proof: Suppose first that f' has infinitely many multiple zeros. Since f has finitely many multiple zeros, the zeros of f' are not zeros of f except at most finitely many. Hence f' has infinitely many multiple zeros that are not zeros of f . And then, they are zeros of f'' , hence of $f''f + 2(f')^2$, which proves the statement.

So we are now led to assume that f' has finitely many multiple zeros. By Lemma 2, $\frac{f''f + 2(f')^2}{(f')^2}$ has infinitely many zeros. Let $c \in \mathbb{K}$ be a pole of order q of f . Without loss of generality, we can suppose $c = 0$. The beginning of the Laurent developpement of f at 0 is of the form $\frac{a_{-q}}{x^q} + \frac{\varphi(x)}{x^{q-1}}$ whereas $\varphi \in \mathcal{M}(\mathbb{K})$ has no pole at 0. Consequently, $\frac{f''f + 2(f')^2}{(f')^2}$ is of the form

$$\frac{(a_{-q})^2(3q^2 + q) + x\phi(x)}{(a_{-q})^2(q^2) + x\psi(x)}$$

whereas $\phi, \psi \in \mathcal{M}(\mathbb{K})$ have no pole at 0. So, the function $\frac{f''f + 2(f')^2}{(f')^2}$ has no zero at 0. Therefore, each zero of $\frac{f''f + 2(f')^2}{(f')^2}$ is a zero of $f''f + 2(f')^2$ and hence $f''f + 2(f')^2$ has infinitely many zeros.

Now, let us show that the zeros of $f''f + 2(f')^2$ are not zeros of f , except maybe finitely many. Let c be a zero of $f''f + 2(f')^2$ and suppose that c is a zero of f . Then, it is a zero of f' and hence it is a multiple zero of f . But by hypotheses, f has finitely many multiple zeros, hence the zeros of $f''f + 2(f')^2$ are not zeros of f , except at most finitely many. That finishes proving the claim.

Lemma 4: *Let $f \in \mathcal{M}(\mathbb{K})$ be transcendental and let $b \in \mathbb{K}^*$ be such that $f^2f' - b$ has finitely many zeros. Then, $N(r, f) \leq Z(r, f) + O(1)$.*

Proof: Let $F = f^2f'$. Since $F - b$ is transcendental and has finitely many zeros, it is of the form $\frac{P(x)}{h(x)}$ with $h \in \mathcal{A}(\mathbb{K}) \setminus \mathbb{K}[x]$. Consequently, $|F|(r)$ is a constant when r is big enough and therefore, by Lemma 1 we have $Z(r, F) = N(r, F) + O(1)$ when r is big enough. Now, $Z(r, F) = 2Z(r, f) + Z(r, f')$ and, by Lemma 1, $Z(r, f') \leq Z(r, f) + \overline{N}(r, f) - \log r + O(1)$. On the other hand, by Lemma 1 again, we have $N(r, F) = 3N(r, f) + \overline{N}(r, f)$. Consequently, $3N(r, f) + \overline{N}(r, f) \leq 3Z(r, f) + \overline{N}(r, f) - \log r + O(1)$, which proves the claim.

Proof of the Theorem: Let $b \in \mathbb{K}^*$ and suppose that $f^2f' - b$ has q zeros, taking multiplicity into account. By Theorem B, we may assume that f has finitely many multiple zeros and finitely multiple poles. Set $F = f^2f'$. Then $F' = f(f''f + 2(f')^2)$. Consequently, F' admits for zeros: the zeros of f and the zeros of $f''f + 2(f')^2$. And by Lemma 3, there exists a sequence of zeros of $f''f + 2(f')^2$ that are not zeros of f .

Let $S = \{0, b\}$ and let $Z_0^S(r, F')$ be the counting function of zeros of F' when $F(x)$ is different from 0 and b . Since $F - b$ has finitely many zeros, the zeros c of F' which are not zeros of f cannot satisfy $F(c) = b$ except at most finitely many. Consequently, there are infinitely many zeros of F' counted by the counting function $Z_0^S(r, F')$ and hence for every fixed integer $t \in \mathbb{N}$, we have

$$(1) \quad Z_0^S(r, F') \geq t \log r + O(1).$$

Let us apply Theorem N to F . We have

$$(2) \quad T(r, F) \leq \overline{Z}(r, F) + \overline{Z}(r, F - b) + \overline{N}(r, F) - Z_0^S(r, F') - \log(r) + O(1).$$

Now, we have

$$(3) \quad \overline{Z}(r, F) \leq Z(r, f) + Z(r, f')$$

$$(4) \quad \overline{N}(r, F) = \overline{N}(r, f)$$

and since the number of zeros of $F - b$ is q , taking multiplicity into account, we have

$$(5) \quad \overline{Z}(r, F - b) \leq q \log r + O(1).$$

Consequently, by (2), (3), (4), (5) we obtain

$$(6) \quad T(r, F) \leq Z(r, f) + Z(r, f') + \overline{N}(r, f) - Z_0^S(r, F') + (q - 1) \log r + O(1).$$

On the other hand, by construction, $T(r, F) \geq Z(r, F) = 2Z(r, f) + Z(r, f')$ hence by (6) we obtain (7):

$$(7) \quad Z(r, f) \leq \overline{N}(r, f) - Z_0^S(r, F') + (q - 1) \log r + O(1).$$

Now, by Lemma 4, we have $N(r, f) \leq Z(r, f) + O(1)$ hence by (7) we obtain $0 \leq (q-1) \log r - Z_0^S(r, F') + O(1)$ and hence by (1), fixing $t > q-1$ we can derive $0 \leq (q-1) \log r - t \log r + O(1)$, a contradiction. That finishes the proof of the Theorem.

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